stat_201a_markov

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0.1 Homework 3: Markov Process

STAT 201A

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[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

0.1.1 Simulation of Markov Process

[ ]:
```
P = np.array(
    [
        [0.2, 0.7, 0.1],
        [0.2, 0.5, 0.3],
        [0.2, 0.4, 0.4]
    ]
)
print(P)
```

```
[[0.2 0.7 0.1]
 [0.2 0.5 0.3]
 [0.2 0.4 0.4]]
```

[ ]:
```
class MarkovChain:
    def __init__(self, p, init):
        assert len(p.shape) == 2 and p.shape[0] == p.shape[1]
        self.p = p
        self.state_num = self.p.shape[0]
        self.state = init # 0, 1, 2, ...
        self.gen = 0
        self.history = [init]
    def _update(self):
        new_state = np.random.choice(self.state_num, p=self.p[self.state,:])
        self.gen += 1
        self.history.append(new_state)
        self.state = new_state
```

```
    def update(self, n):
        for _ in range(n):
            self._update()

chain = MarkovChain(P, 0)
```

[ ]:
```
chain.update(999)
print(np.unique(chain.history, return_counts=True))
```

```
(array([0, 1, 2]), array([198, 502, 300]))
```

0.1.2 Stationary Distribution

[ ]:
```
np.linalg.eig(P.T)
```

[ ]:
```
(array([1.00000000e+00, 1.36392633e-16, 1.00000000e-01]),
 array([[ 3.22458464e-01,  2.67261242e-01,  1.07331375e-16],
        [ 8.24060518e-01, -8.01783726e-01, -7.07106781e-01],
        [ 4.65773337e-01,  5.34522484e-01,  7.07106781e-01]]))
```

[ ]:
```
v = np.linalg.eig(P.T)[1][:,0]
v = v / v.sum()
print(v)
print(v.T @ P)
```

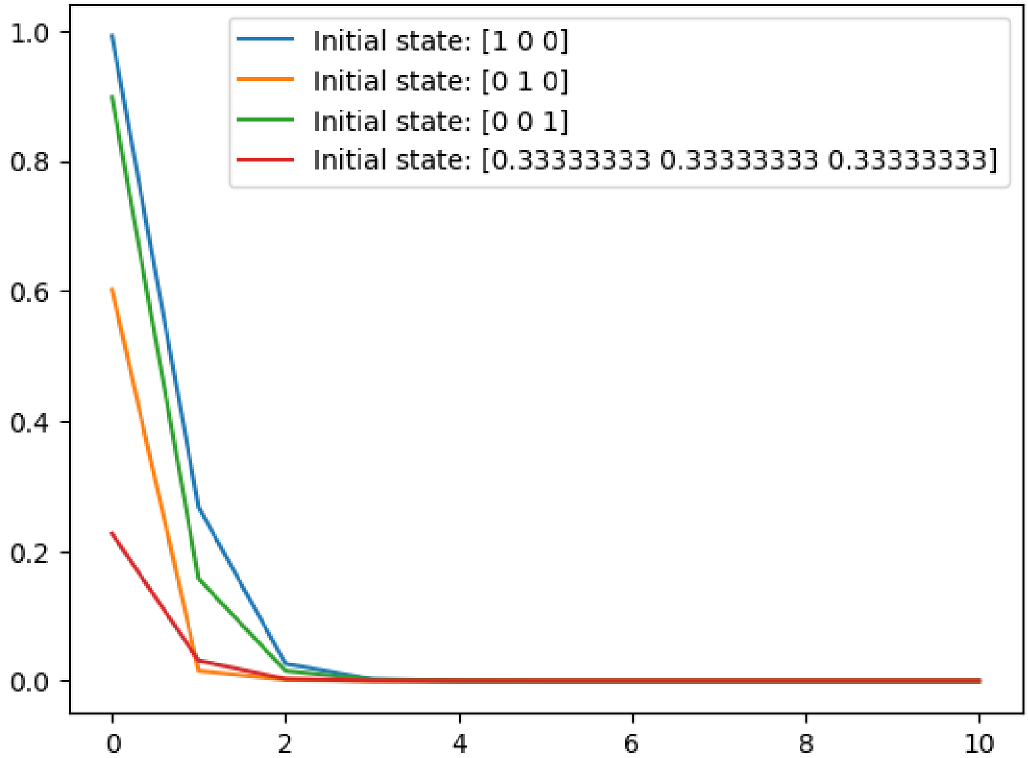
```
[0.2        0.51111111 0.28888889]
[0.2        0.51111111 0.28888889]
```

[ ]:
```
def markov_dist(p, init_dist, n):
    return init_dist.T @ np.linalg.matrix_power(p, n)

init_dists = [
    np.array([1,0,0]),
    np.array([0,1,0]),
    np.array([0,0,1]),
    np.array([1/3,1/3,1/3])
]
k = 10

for d in init_dists:
    errors = []
    for n in range(k+1):
        state = markov_dist(P, d, n)
        error = np.linalg.norm(state - v)
        errors.append(error)
    plt.plot(range(k+1), errors, label='Initial state: {}'.format(d))
plt.legend()
```

```
plt.show()
```



[ ]:
```
print(init_dists[0].T @ np.linalg.matrix_power(P, 1000))
```

```
[0.2        0.51111111 0.28888889]
```

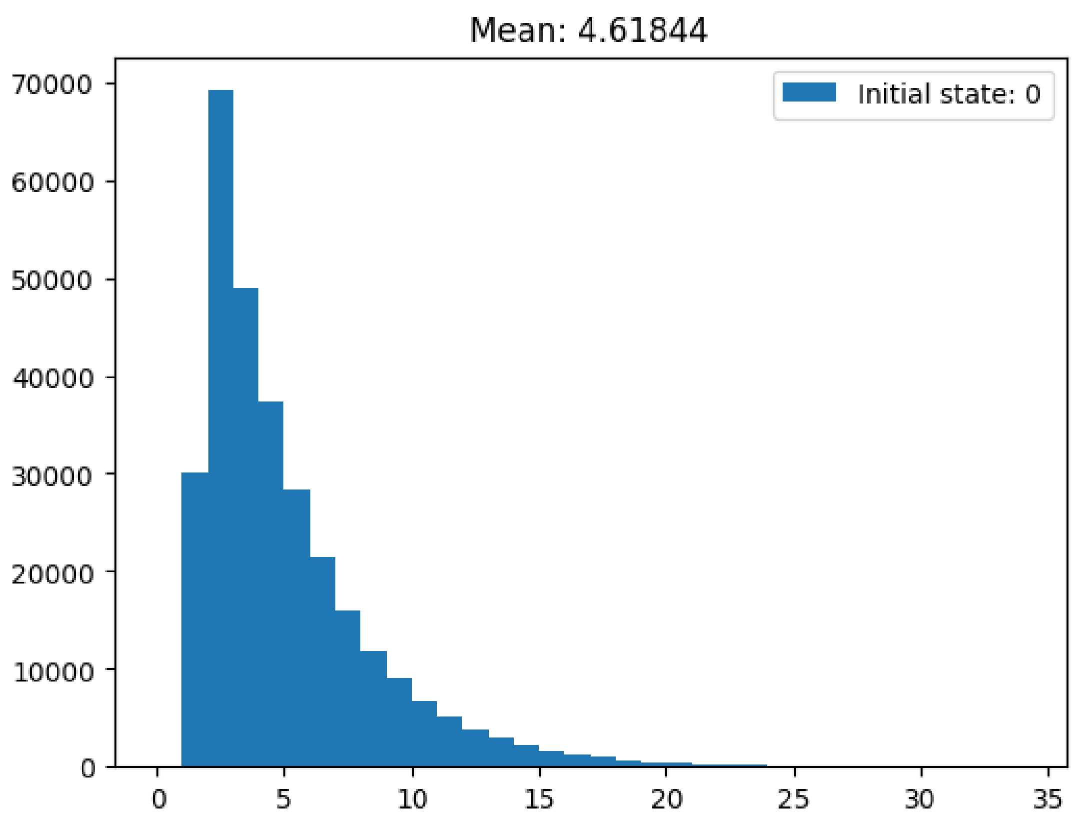
0.1.3 Absorbing State

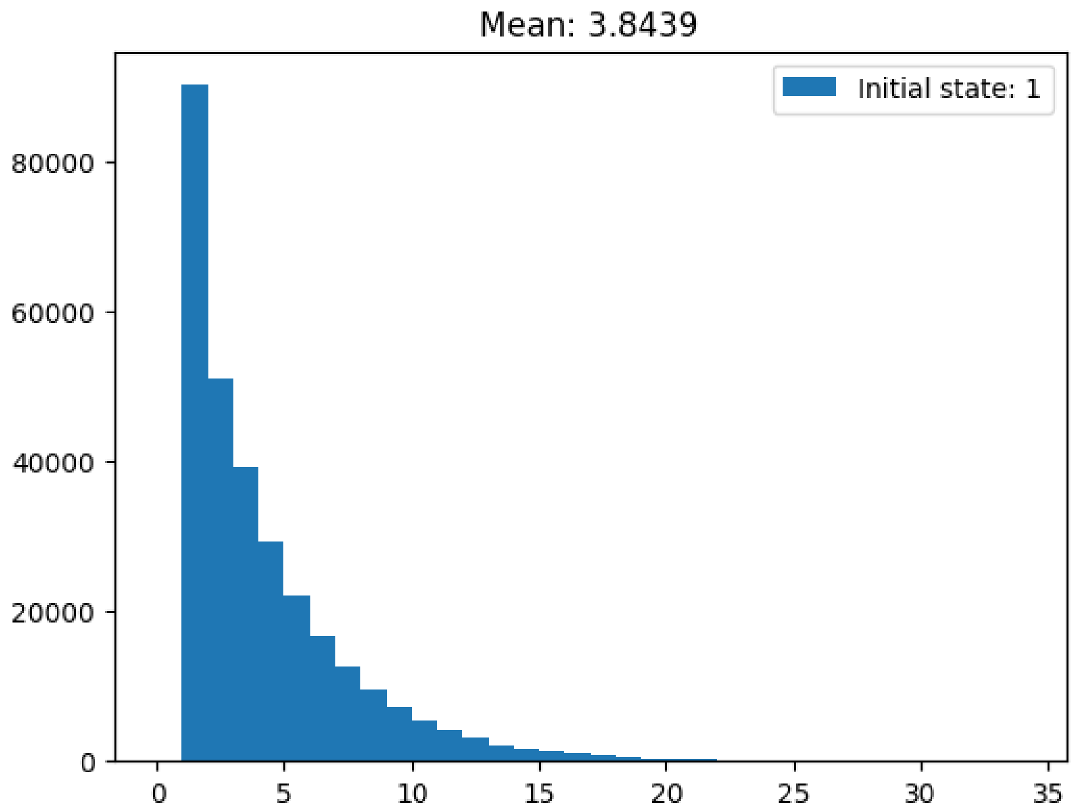
[ ]:
```
class MarkovChainAbsorbing(MarkovChain):
    def __init__(self, p, init, absorbing):
        super().__init__(p, init)
        self.absorbing = absorbing
        self.arrive = False
        self.arrive_gen = None
    def _update(self):
        if self.state != self.absorbing:
            super()._update()
        else:
            self.arrive = True
            self.arrive_gen = self.gen

def arrival_exp(p, init, absorbing, n):
```

```
    arrival_times = []
    for i in range(n):
        chain_absorb = MarkovChainAbsorbing(p, init, absorbing)
        chain_absorb.update(100)
        if chain_absorb.arrive:
            arrival_times.append(chain_absorb.arrive_gen)
        else:
            while not chain_absorb.arrive:
                chain_absorb.update(10)
                if chain_absorb.arrive:
                    arrival_times.append(chain_absorb.arrive_gen)
    return arrival_times


means = []
for init in range(2):
    arrival_times = arrival_exp(P, init, 2, 300000)
    means.append(np.mean(arrival_times))
    plt.hist(arrival_times, bins=np.arange(0,35,1), label='Initial state: {}'.
 ↪format(init))
    plt.legend()
    plt.title('Mean: {}'.format(np.mean(arrival_times)))
    plt.show()
```





If $X_i = 3$ is an absorbing state, then T_1 can be expressed as

$$T_1 = 1 + \mathbb{P}(X_1 = 1) \cdot T_1 + \mathbb{P}(X_1 = 2) \cdot T_2 + \mathbb{P}(X_1 = 3) \cdot T_3$$

Because $X_0 = 1$ for T_1 and $T_3 = 0$, the equation above can be re-written as

$$T_1 = 1 + 0.2 \cdot T_1 + 0.7 \cdot T_2$$

We have a similar expression for T_2:

$$T_2 = 1 + 0.2 \cdot T_1 + 0.5 \cdot T_2$$

Adding up the two equations we have

$$0.6 \cdot T_1 = 2 + 0.2 \cdot T_2$$

Taking expectation on both sides we have

$$2 + 0.2 \cdot \mu_2 - 0.6 \cdot \mu_1$$

Which we can evaluate using the calculated means, as shown below.

[ ]:
```
2 + 0.2 * means[1] - 0.6 * means[0]
```

[ ]:
```
-0.0022839999999995086
```

Taking in the values we have

$$\mu_1 = \frac{60}{13} = 4.6154, \quad \mu_2 = \frac{50}{13} = 3.8462$$

Which corresponds to the numerical values from (a).