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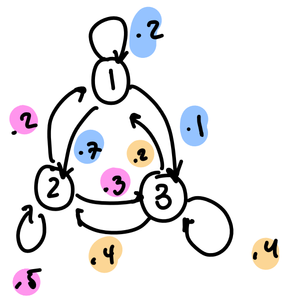
HW Due Dec 4^{th}
Stat 201A

1) Simulation of Markov Process

a) P_{ij} is probability from node i to j

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} .2 & .7 & .1 \\ .2 & .5 & .3 \\ .2 & .4 & .4 \end{bmatrix}$$



b) See Jupyter Notebook

2) Stationary Distribution

a) $(P^T - I)v = 0$ v = stationary distribution vector π_∞

$P^T v = v$

$$\left[\begin{array}{ccc|c} -.8 & .2 & .2 & 0 \\ .7 & -.5 & .4 & 0 \\ .1 & .3 & -.6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -.25 & -.25 & 0 \\ .7 & -.5 & .4 & 0 \\ .1 & .3 & -.6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -.25 & -.25 & 0 \\ 0 & -.325 & .575 & 0 \\ 0 & .325 & -.575 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -.25 & -.25 & 0 \\ 0 & 1 & 1.7692 & 0 \\ 0 & .325 & -.575 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -.6923 & 0 \\ 0 & 1 & -1.7692 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow v = \begin{bmatrix} .6923x_3 \\ 1.769x_3 \\ x_3 \end{bmatrix}$$

when $x_3 = .2\overline{88}$ we get the solution $v = \begin{bmatrix} .2 \\ .5\overline{111} \\ .2\overline{88} \end{bmatrix}$

Also see Jupyter Notebook for full solution

b) see Jupyter Notebook

3)

a) see Jupyter Notebook $\longrightarrow \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \approx \begin{bmatrix} 4.833 \\ 3.647 \end{bmatrix}$

b) show $\mu_i = 1 + \sum_{j=1}^{3} P_{ij} \cdot u_j$ $\qquad \mu_i = E[T_i]$, $T_3 = 0 \rightarrow \mu_3 = 0$

$\mu_1 = 1 + P_{11}\mu_1 + P_{12}\mu_2 + P_{13}\mu_3 \longrightarrow \mu_1 = 1 + .2\mu_1 + .7\mu_2 \longrightarrow \left[\begin{array}{cc|c} -.8 & .7 & -1 \\ .2 & -.5 & -1 \end{array}\right]$

$\mu_2 = 1 + P_{21}\mu_1 + P_{22}\mu_2 + P_{23}\mu_3 \qquad \mu_2 = 1 + .2\mu_1 + .5\mu_2$

$\mu_3 = 0$

$$\left[\begin{array}{cc|c} 1 & -.875 & 1.25 \\ .2 & -.5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -.875 & 1.25 \\ 0 & -.325 & -1.25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -.875 & 1.25 \\ 0 & 1 & 3.8461 \end{array}\right] \rightarrow \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 4.6154 \\ 3.8461 \end{bmatrix}$$

HW201A

November 27, 2023

1 b)

[37]:
```
import numpy as np
P = np.array([[.2, .7, .1],
              [.2, .5, .3],
              [.2, .4, .4]])

current_state = 0
num_steps = 1

for _ in range(num_steps):
    next_state = np.random.choice(range(len(P)), p=P[current_state])
    display_next_state = next_state+1
    display_current_state = current_state+1
    print(f"initial state: {display_current_state}, next state:␣
↪{display_next_state}")
    current_state = next_state
```

```
initial state: 1, next state: 1
initial state: 1, next state: 1
initial state: 1, next state: 2
initial state: 2, next state: 3
initial state: 3, next state: 3
initial state: 3, next state: 1
initial state: 1, next state: 2
initial state: 2, next state: 2
initial state: 2, next state: 3
initial state: 3, next state: 2
```

2 a)

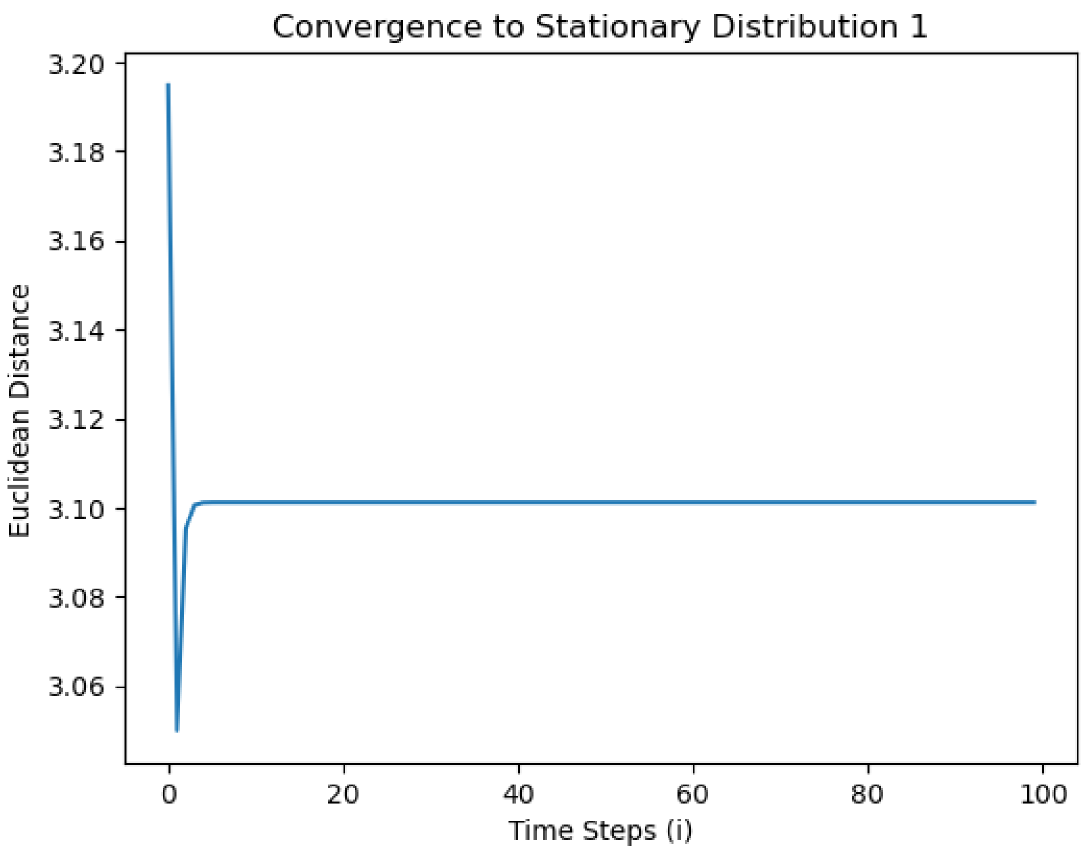
[3]:
```
PT = np.transpose(P)



eigenvalues, eigenvectors = np.linalg.eig(PT)

# find the eigenvector corresponding to eigenvalue 1
stationary_vector = eigenvectors[:, np.where(np.isclose(eigenvalues, 1))[0][0]]
```

```python
# normalize the stationary vector to make it a probability distribution
stationary_vector /= np.sum(stationary_vector)

print(stationary_vector)
```

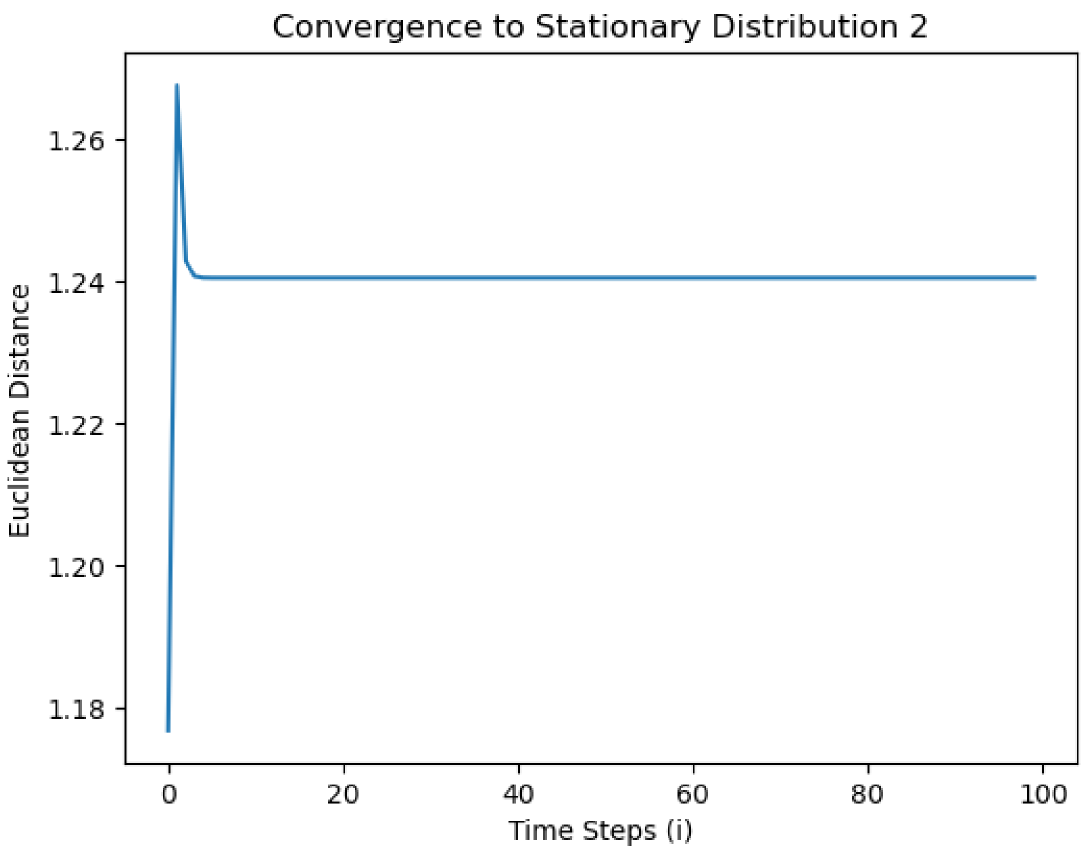
```
Stationary Distribution:
[0.2        0.51111111 0.28888889]
```

2b)

[11]:
```python
import matplotlib.pyplot as plt

# set initial values
pi0 = np.array([1, 2, 3])


num_steps = 100
distances = []
# calculate the stationary distribution
eigenvalues, eigenvectors = np.linalg.eig(PT)
stationary_vector = eigenvectors[:, np.where(np.isclose(eigenvalues, 1))[0][0]]
stationary_vector /= np.sum(stationary_vector)

# simulate the evolution of the probability distribution over time
for i in range(num_steps):
    pi_i = np.dot(pi0, np.linalg.matrix_power(P, i))

    # calculate euclidean distance between pi_i and stationary_vector
    distance = np.linalg.norm(pi_i - stationary_vector, ord=2)

    distances.append(distance)

# plot the results
plt.plot(range(num_steps), distances)
plt.xlabel('Time Steps (i)')
plt.ylabel('Euclidean Distance')
plt.title('Convergence to Stationary Distribution 1')
plt.show()
```



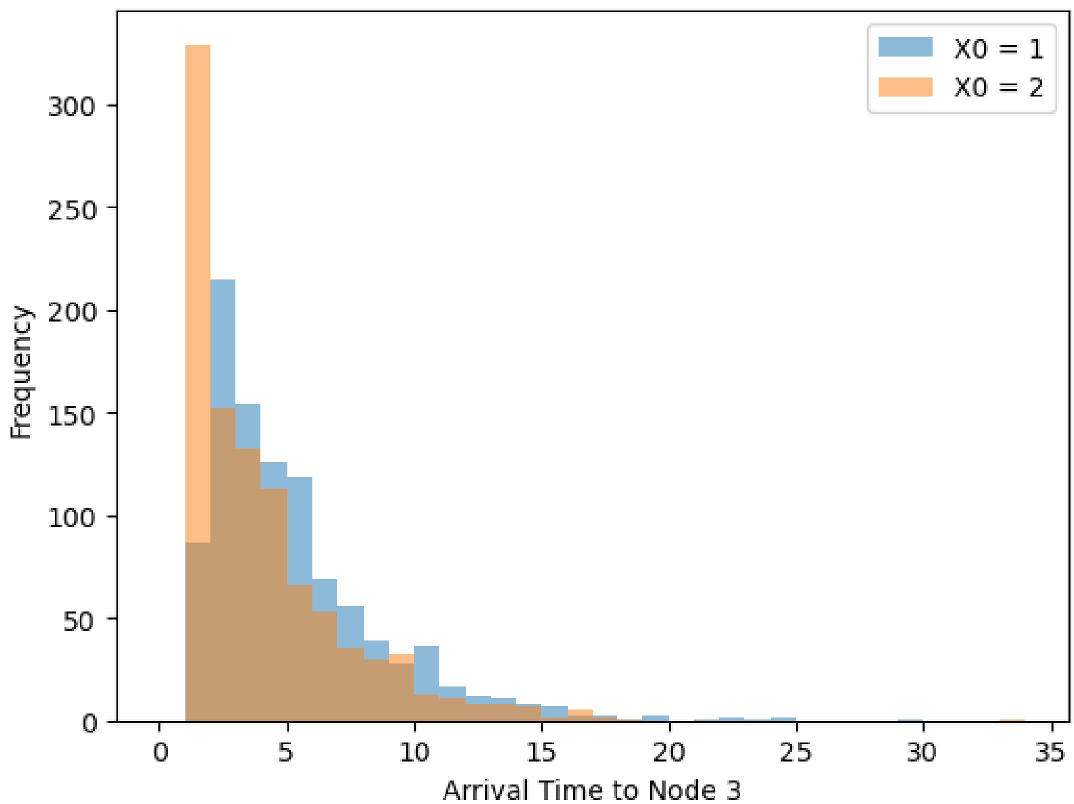
[12]:
```
# set different initial values
pi0_2 = np.array([1, 1, 1])

distances = []
# simulate the evolution of the probability distribution over time for␣
↪different initial values
for i in range(num_steps):
    pi_i = np.dot(pi0_2, np.linalg.matrix_power(P, i))
    distance = np.linalg.norm(pi_i - stationary_vector, ord=2)
    distances.append(distance)

# plot the results
plt.plot(range(num_steps), distances)
plt.xlabel('Time Steps (i)')
plt.ylabel('Euclidean Distance')
plt.title('Convergence to Stationary Distribution 2')
plt.show()
```



3 a)

[39]:
```
def simulate_until_absorption(initial_state, transition_matrix,␣
↪absorbing_state):
    current_state = initial_state
    time = 0
    while current_state != absorbing_state:
        current_state = np.random.choice(range(len(transition_matrix)),␣
↪p=transition_matrix[current_state])
        time += 1
    return time

# define the absorbing state for node 3 (indexed 0,1,2)
absorbing_state = 2

num_simulations = 1000
arrival_times_X0_1 = []
arrival_times_X0_2 = []
# X0 = 1
for _ in range(num_simulations):
```

```python
    arrival_time = simulate_until_absorption(0, P, absorbing_state)
    arrival_times_X0_1.append(arrival_time)
# X0 = 2
for _ in range(num_simulations):
    arrival_time = simulate_until_absorption(1, P, absorbing_state)
    arrival_times_X0_2.append(arrival_time)

# histograms
plt.hist(arrival_times_X0_1, bins=range(max(max(arrival_times_X0_1), max(arrival_times_X0_2)) + 1), alpha=0.5, label='X0 = 1')
plt.hist(arrival_times_X0_2, bins=range(max(max(arrival_times_X0_1), max(arrival_times_X0_2)) + 1), alpha=0.5, label='X0 = 2')
plt.xlabel('Arrival Time to Node 3')
plt.ylabel('Frequency')
plt.legend()
plt.show()

mean_arrival_time_X0_1 = np.mean(arrival_times_X0_1)
mean_arrival_time_X0_2 = np.mean(arrival_times_X0_2)
print(f'mean arrival ti me X0 = 1: {mean_arrival_time_X0_1}')
print(f'mean arrivial time X0 = 2: {mean_arrival_time_X0_2}')
```



```
mean arrival ti me X0 = 1: 4.833
mean arrivial time X0 = 2: 3.647
```