LABNov20

November 27, 2023

Lab November 20th

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[67]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

1 Question 1

1.1 Question 1(a)

1)

a)

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} & \\ 3 & & & \end{array} = P$$

Stochastic Matrix

b) simulate starting from

$X_0 = 1$, see code

1.2 Question 1(b)

[68]:
```
stochastic_matrix = np.array([[2/10, 7/10, 1/10], [2/10, 5/10, 3/10], [2/10,4/
 ↪10,4/10]])

nodes = np.array([1,2,3])
X = [1]
X_temp = 1

np.random.seed(123)

for i in range (20):
    if X_temp == 1:
        X_temp = np.random.choice(nodes, p = stochastic_matrix[0])
        X.append(X_temp)
    elif X_temp == 2:
        X_temp = np.random.choice(nodes, p = stochastic_matrix[1])
        X.append(X_temp)
    else:
        X_temp = np.random.choice(nodes, p = stochastic_matrix[2])
        X.append(X_temp)
```

[69]:
```
X
```

[69]: [1, 2, 2, 2, 2, 3, 2, 3, 3, 2, 2, 2, 3, 2, 1, 2, 3, 1, 1, 2, 2]

2 Question 2

2.1 Question 2(a)

2)

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} & \\ 3 & & & \end{array} = P$$

$$\begin{array}{cccc} & 1 & 2 & 3 \\ 1 & 0.2 & 0.7 & 0.1 \\ 2 & 0.2 & 0.5 & 0.3 \\ 3 & 0.2 & 0.4 & 0.4 \end{array}$$

a) find $\pi_\infty \in \mathbb{R}^3$ where $\pi_\infty^T = \pi_\infty^T P$

can solve $(P^T - I)\pi_\infty = 0$ for π_∞

$$(P^T - I) = \begin{bmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.5 & 0.4 \\ 0.1 & 0.3 & -0.6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -0.8 & 0.2 & 0.2 & 0 \\ 0.7 & -0.5 & 0.4 & 0 \\ 0.1 & 0.3 & -0.6 & 0 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & -0.25 & -0.25 & 0 \\ 0.7 & -0.5 & 0.4 & 0 \\ 0.1 & 0.3 & -0.6 & 0 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -0.325 & 0.575 & 0 \\ 0 & 0.325 & -0.575 & 0 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{13}{40} & -\frac{23}{40} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{23}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{9}{13} & 0 \\ 0 & 1 & -\frac{23}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Pi_\infty = \begin{bmatrix} \frac{9}{13} \\ \frac{23}{13} \\ 1 \end{bmatrix} X_3$$

still need to normalize

$$\det(P-\lambda I) = \left(\frac{1}{5}-\lambda\right)\det\begin{vmatrix} \frac{1}{2}-\lambda & \frac{3}{10} \\ \frac{2}{5} & \frac{2}{5}-\lambda \end{vmatrix} -$$

$$\left(\frac{1}{5}\right)\det\begin{vmatrix} \frac{7}{10} & \frac{1}{10} \\ \frac{2}{5} & \frac{2}{5}-\lambda \end{vmatrix} +$$

$$\left(\frac{1}{5}\right)\det\begin{vmatrix} \frac{7}{10} & \frac{1}{10} \\ \frac{1}{2}-\lambda & \frac{3}{10} \end{vmatrix}$$

$$= \left(\frac{1}{5}-\lambda\right)\left[\left(\frac{1}{2}-\lambda\right)\left(\frac{2}{5}-\lambda\right)-\frac{3}{25}\right] -$$

$$\frac{1}{5}\left[\frac{7}{10}\left(\frac{2}{5}-\lambda\right)-\frac{1}{25}\right] +$$

$$\frac{1}{5}\left[\frac{21}{100}-\frac{1}{10}\left(\frac{1}{2}-\lambda\right)\right] \qquad \frac{14}{50}-\frac{2}{50}$$

$$\frac{21}{100}$$

$$= \left(\frac{1}{5}-\lambda\right)\left[\lambda^2-\frac{9}{10}\lambda+\frac{2}{25}\right] -$$

$$\frac{1}{5}\left[\frac{12}{50}-\frac{7}{10}\lambda\right]+\frac{1}{5}\left[\frac{4}{25}+\frac{1}{10}\lambda\right]$$

$$= -\lambda^3 + \frac{11}{10}\lambda^2 - \frac{13}{50}\lambda + \frac{2}{125} - \frac{6}{125}$$

$$+ \frac{7}{50}\lambda + \frac{4}{125} + \frac{1}{50}\lambda$$

$$= -\lambda^3 + \frac{11}{10}\lambda^2 - \frac{1}{10}\lambda$$

set equal to 0

$$0 = -\lambda^3 + \frac{11}{10}\lambda^2 - \frac{1}{10}\lambda$$

$$= \lambda^3 - \frac{11}{10}\lambda^2 + \frac{1}{10}\lambda$$

$\lambda = [0, 1, 0.1]$ eigen values

$$\Rightarrow \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \left[\begin{array}{ccc|c} 0.2 & 0.7 & 0.1 & 0 \\ 0.2 & 0.5 & 0.3 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0.2 & 0.7 & 0.1 & 0 \\ 0 & -0.2 & 0.2 & 0 \\ 0 & -0.3 & 0.3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3.5 & 0.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigen vector

$$\vec{X}_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

$$P - I = \left[\begin{array}{ccc|c} -0.8 & 0.7 & 0.1 & 0 \\ 0.2 & -0.5 & 0.3 & 0 \\ 0.2 & 0.4 & -0.6 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & \frac{7}{8} & \frac{1}{8} & 0 \\ 0 & -\frac{13}{40} & \frac{13}{40} & 0 \\ 0 & \frac{23}{40} & -\frac{23}{40} & 0 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c} 1 & -7/8 & -1/8 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigen vector

$$\vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P - 0.1\lambda = \left[\begin{array}{ccc|c} 0.1 & 0.7 & 0.1 & 0 \\ 0.2 & 0.4 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 7 & 1 & 0 \\ 0.2 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 7 & 1 & 0 \\ 0 & 1 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigen vector

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1.7 & 0 \\ 0 & 1 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \vec{X_3} = \begin{bmatrix} -\frac{17}{10} \\ \frac{1}{10} \\ 1 \end{bmatrix}$$

So we have

$$P = V\ D\ V^{-1}$$

where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \quad \text{and}$$

$$V = \begin{bmatrix} -4 & -\frac{17}{10} & 1 \\ 1 & 1/10 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find V^{-1}

$$\left[\begin{array}{ccc|ccc} -4 & -\frac{17}{10} & 1 & 1 & 0 & 0 \\ 1 & \frac{1}{10} & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} -1 & -\frac{17}{40} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{13}{40} & \frac{5}{4} & \frac{1}{4} & 1 & 0 \\ 0 & \frac{23}{40} & \frac{5}{4} & \frac{1}{4} & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{17}{40} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{50}{13} & -\frac{10}{13} & -\frac{40}{13} & 0 \\ 0 & \frac{23}{40} & \frac{5}{4} & \frac{1}{4} & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{18}{13} & \frac{1}{13} & \frac{17}{13} & 0 \\ 0 & 1 & -\frac{50}{13} & -\frac{10}{13} & -\frac{40}{13} & 0 \\ 0 & \frac{23}{40} & \frac{5}{4} & \frac{1}{4} & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{18}{13} & \frac{1}{13} & \frac{17}{13} & 0 \\ 0 & 1 & -\frac{50}{13} & \frac{-10}{13} & -\frac{40}{13} & 0 \\ 0 & \frac{23}{40} & \frac{5}{4} & \frac{1}{4} & 0 & 1 \end{array}\right] =$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 18/13 & \frac{1}{13} & \frac{17}{13} & 0 \\ 0 & 1 & -\frac{50}{13} & \frac{-10}{13} & \frac{-40}{13} & 0 \\ 0 & 0 & \frac{45}{13} & \frac{9}{13} & \frac{23}{13} & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 18/13 & \frac{1}{13} & \frac{17}{13} & 0 \\ 0 & 1 & -\frac{50}{13} & \frac{-10}{13} & \frac{-40}{13} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{23}{45} & \frac{13}{45} \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{50}{13} & \frac{-10}{13} & \frac{-40}{13} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{23}{45} & \frac{13}{45} \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & 0 & \frac{-10}{9} & \frac{10}{9} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{23}{45} & \frac{13}{45} \end{array}\right]$$

$$\Downarrow$$

$$V^{-1}$$

So we now have

$$\pi_i^T = \pi_0^T P^i = \pi_0^T V^{-1} D^i V$$

π_0^T chose this π_0^T but should be true for all π_0

$$\pi_i^T = \overbrace{\begin{bmatrix} 1/3, & 1/3, & 1/3 \end{bmatrix}}^{\pi_0^T} *$$

$$V^{-1} \begin{bmatrix} -1/5 & 3/5 & -2/5 \\ 0 & -10/9 & 10/9 \\ 1/5 & 23/45 & 13/45 \end{bmatrix} *$$

$$D^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1^i & 0 \\ 0 & 0 & 1 \end{bmatrix} *$$

$$V \begin{bmatrix} -4 & -\frac{17}{10} & 1 \\ 1 & 1/10 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

now to find $\lim\limits_{i \to \infty} \pi_i^T$

$$\lim_{i\to\infty} \pi_i^T = \pi_0^T \lim_{i\to\infty} P^i$$

$$= \pi_0^T V^{-1} \cdot \lim_{i\to\infty} D^i V$$

$$\lim_{i\to\infty} D^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\pi_0^T \underbrace{V * \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\star} V^{-1}$$

$$\begin{bmatrix} -4 & -\frac{17}{10} & 1 \\ 1 & 1/10 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow$$

$$V \lim_{i\to\infty} D^i = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V \lim_{i\to\infty} D^i V^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 3/5 & -2/5 \\ 0 & -10/9 & 10/9 \\ 1/5 & 23/45 & 13/45 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 23/45 & 13/45 \\ 1/5 & 23/45 & 13/45 \\ 1/5 & 23/45 & 13/45 \end{bmatrix}$$

$$\pi_0^T V^{-1} \lim_{i\to\infty} D^i V = \begin{bmatrix} 1/3, & 1/3, & 1/3 \end{bmatrix} \begin{bmatrix} 1/5 & 23/45 & 13/45 \\ 1/5 & 23/45 & 13/45 \\ 1/5 & 23/45 & 13/45 \end{bmatrix}$$

$$= \left[\frac{1}{5}, \frac{23}{45}, \frac{13}{45} \right] = \pi_\infty^T$$

THIS IS TRUE FOR ALL π_0^T

I'll generate the plot next

of i against $\|\pi_i - \pi_\infty\|_2^2$

2.2 Question 2(b) code analytical solution above

[70]:
```
#defining two different  0
pi_0_1 = np.array([1/3,1/3,1/3])
pi_0_2 = np.array([1,0,0])


#defining V matrix and D matrix for diagonalization
V_matrix = np.array([[-4, 1, 1], [-17/10, 1/10, 1], [1,1,1]])
#didn't match my analytical solution unless I too transpose for some reason
V_matrix_inv = np.array([[(-1/5), 0, (1/5)], [3/5, -10/9, 23/45], [-2/5,10/9,13/
↪45]]).T
diag_matrix = np.array([[0, 0, 0], [0, 0.1, 0], [0,0,1]])

#defining  ∞
pi_inf = np.array([9/45,23/45,13/45])
```

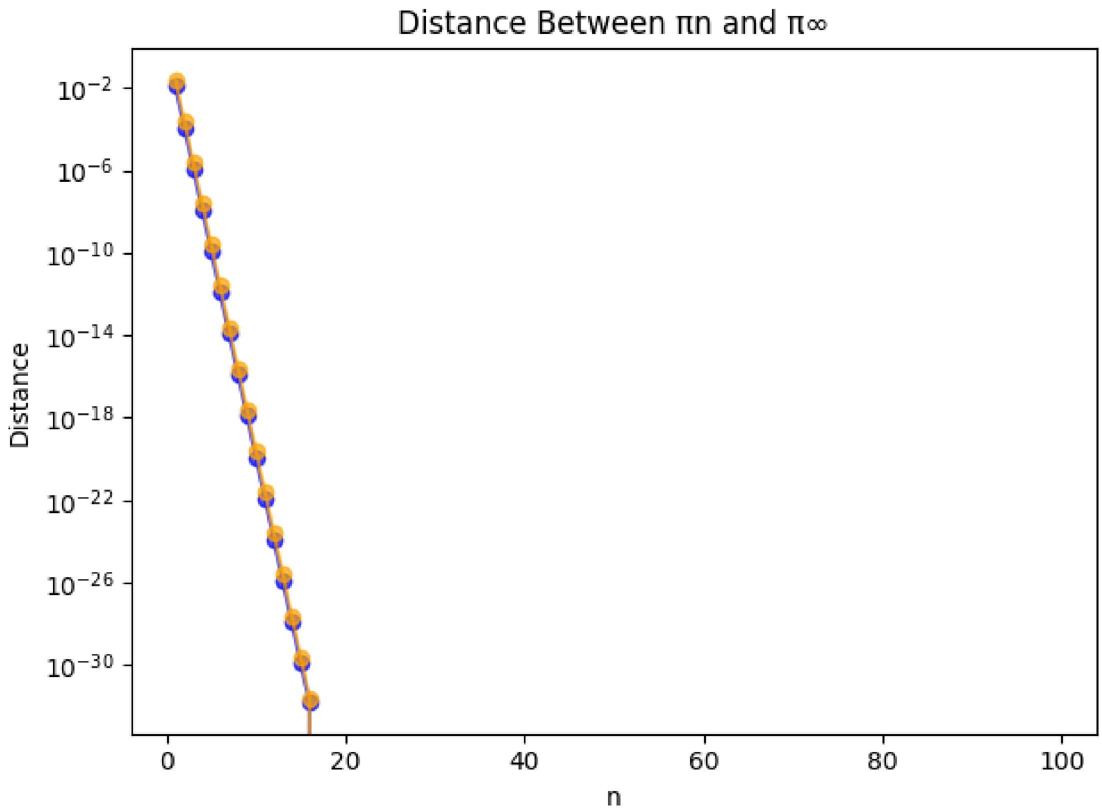
[92]:
```
#calculating the distance
def calc_diff(pi_0):
    norms_sq = []
    for i in range(1,100):
        pi_i = pi_0 @ ((V_matrix @ (diag_matrix**i)) @ V_matrix_inv)
        norm_sq = np.sum((pi_i - pi_inf)**2)
        enum_norm_sq = (i, norm_sq)
        norms_sq.append(enum_norm_sq)
    return(norms_sq)
```

[94]:
```
graphing_df1 = pd.DataFrame(calc_diff(pi_0_1))
graphing_df2 = pd.DataFrame(calc_diff(pi_0_2))
```

[95]:
```
#layering plots to see difference

plt.plot(graphing_df1[0], graphing_df1[1], 'o-', color='blue', alpha=0.7)
plt.plot(graphing_df2[0], graphing_df2[1], 'o-', color='orange', alpha=0.7)
plt.title('Distance Between  n and  ∞')
plt.xlabel('n')
plt.ylabel('Distance')
plt.yscale('log')

plt.tight_layout()
plt.show()
```



[96]:

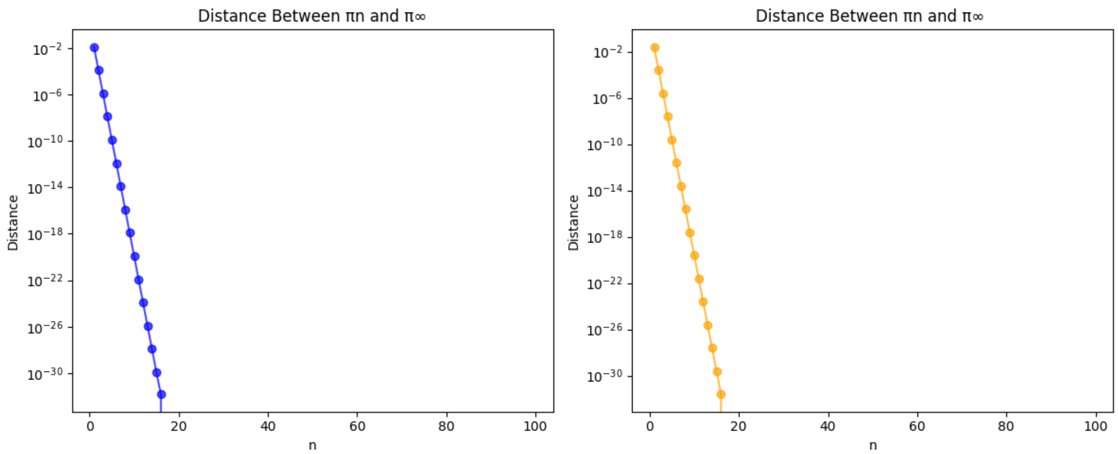
```
fig, axes = plt.subplots(1, 2, figsize=(12, 5))  # 1 row, 2 columns for side by␣
↪side histograms

# Histogram for pi_0_1
axes[0].plot(graphing_df1[0], graphing_df1[1], 'o-', color='blue', alpha=0.7)
axes[0].set_title('Distance Between  n and  ∞')
axes[0].set_xlabel('n')
axes[0].set_ylabel('Distance')
axes[0].set_yscale('log')


# Histogram for pi_0_2
axes[1].plot(graphing_df2[0], graphing_df2[1], 'o-', color='orange', alpha=0.7)
axes[1].set_title('Distance Between  n and  ∞ ')
axes[1].set_xlabel('n')
axes[1].set_ylabel('Distance')
axes[1].set_yscale('log')


# Show plot
plt.tight_layout()
```

```
plt.show()
```



They both converge regardless of the starting point!!!

3 Question 3

3.1 Question 3(a)

[76]:
```
def chain_simulator(start):
    X_temp = start[0]
    while X_temp != 3:
        if X_temp == 1:
            X_temp = np.random.choice(nodes, p = stochastic_matrix[0])
            start.append(X_temp)
        elif X_temp == 2:
            X_temp = np.random.choice(nodes, p = stochastic_matrix[1])
            start.append(X_temp)
        else:
            break
    return start, len(start)-1
```

[77]:
```
np.random.seed(123)

arrival_times_s1 = []
for i in range(100000):
    _, time = chain_simulator(start=[1])
    arrival_times_s1.append(time)
```

[78]:
```
np.random.seed(123)

arrival_times_s2 = []
```

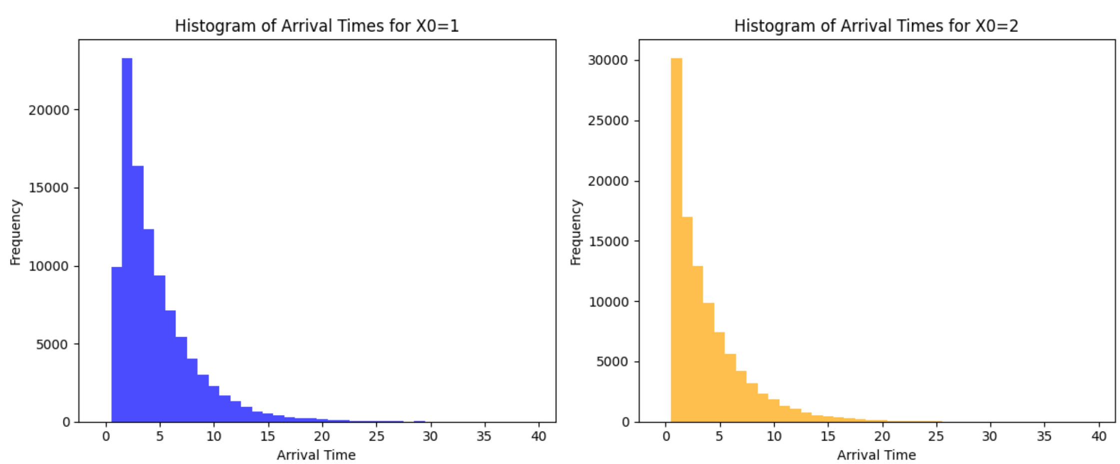
```
for i in range(100000):
    _, time = chain_simulator(start=[2])
    arrival_times_s2.append(time)
```

[79]:
```
fig, axes = plt.subplots(1, 2, figsize=(12, 5))  # 1 row, 2 columns for side by␣
 ↪side histograms

# Histogram for X0=1
axes[0].hist(arrival_times_s1, bins=np.arange(-0.5,40.5,1), color='blue',␣
 ↪alpha=0.7)
axes[0].set_title('Histogram of Arrival Times for X0=1')
axes[0].set_xlabel('Arrival Time')
axes[0].set_ylabel('Frequency')

# Histogram for X0=2
axes[1].hist(arrival_times_s2, bins=np.arange(-0.5,40.5,1), color='orange',␣
 ↪alpha=0.7)
axes[1].set_title('Histogram of Arrival Times for X0=2')
axes[1].set_xlabel('Arrival Time')
axes[1].set_ylabel('Frequency')

# Show plot
plt.tight_layout()
plt.show()
```



What are the mean arrival times calculated numerically?

[80]:
```
mu_T1 = np.mean(arrival_times_s1)
mu_T1
```

[80]: 4.61251

[81]:
```
mu_T2 = np.mean(arrival_times_s2)
mu_T2
```

[81]: 3.84046

3.2 Question 3(b)

We can see that the analyticaly solution found below matches quite closely to the numerical solution found above

3)

b) $\mu_i = \mathbb{E}[T_i] = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$

$\mu_3 = \mathbb{E}[T_3] = 0$

$$\begin{aligned}\mu_1 &= 1 + p_{11}\mu_1 + p_{12}\mu_2 + p_{13}\mu_3 \\ &= 1 + p_{11}\mu_1 + p_{12}\mu_2 \\ \mu_2 &= 1 + p_{21}\mu_1 + p_{22}\mu_2 + p_{13}\mu_3 \\ &= 1 + p_{21}\mu_1 + p_{22}\mu_2\end{aligned}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c}\begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}\end{array} = P \Rightarrow \begin{array}{l} p_{11} = 0.2 \\ p_{12} = 0.7 \\ p_{21} = 0.2 \\ p_{22} = 0.5 \end{array}$$

<u>linear system</u>

$\mu_1 = 1 + 0.2\mu_1 + 0.7\mu_2$

$\mu_2 = 1 + 0.2\mu_1 + 0.5\mu_2$

$0.8\mu_1 = 1 + 0.7\mu_2$

$\mu_1 = 1.25 + 0.875\mu_2$

$$0.5\mu_2 = 1 + 0.2(1.25 + 0.875\mu_2)$$

$$\mu_2 = 2 + 0.4(1.25 + 0.875\mu_2)$$

$$\mu_2 = 2 + 0.5 + 0.35\mu_2$$

$$\mu_2 = \frac{2.5}{0.65} = \frac{50}{13} \approx 3.846$$

$$\mu_1 = 1.25 + 0.875 \cdot \frac{50}{13}$$

$$= 1.25 + \frac{7}{8} \cdot \frac{50}{13}$$

$$= \frac{60}{13} \approx 4.6154$$