STAT 201A Lab (11/20)

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Problem 1a

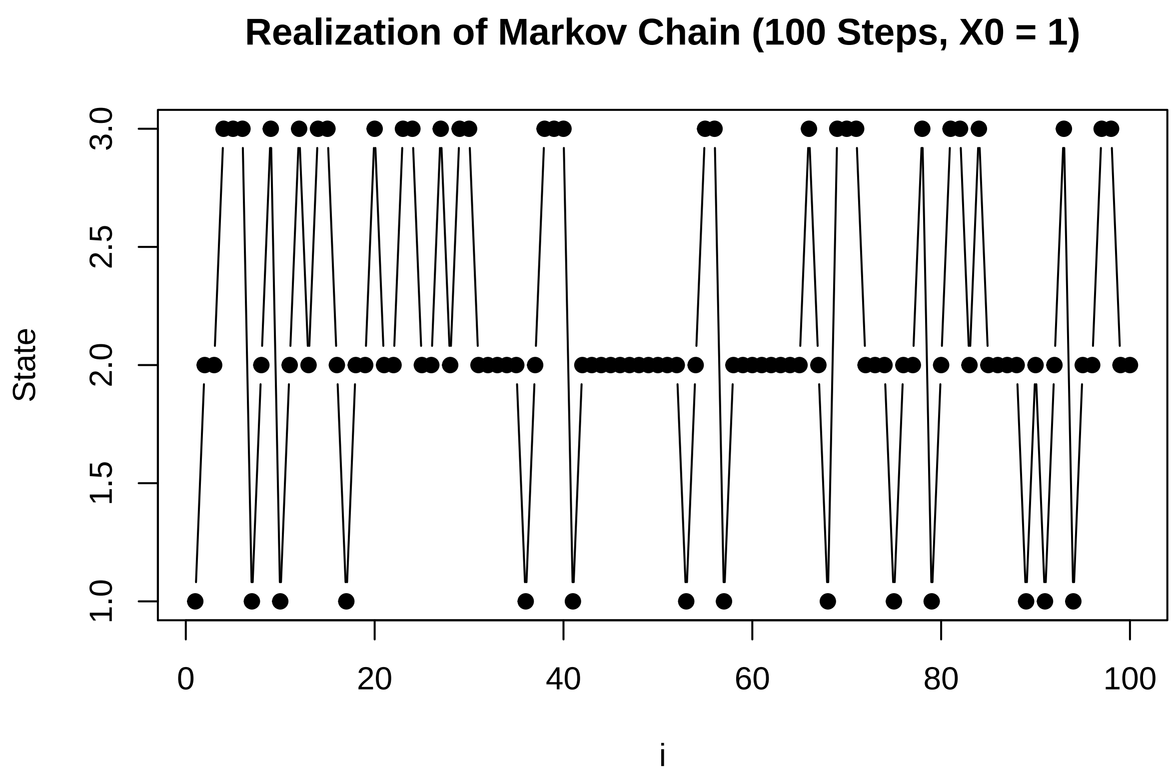
$$P = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Problem 1b

```
set.seed(24)
P <- matrix(c(0.2, 0.7, 0.1, 0.2, 0.5, 0.3, 0.2, 0.4, 0.4), nrow = 3, byrow = T)
P
```

```
##      [,1] [,2] [,3]
## [1,]  0.2  0.7  0.1
## [2,]  0.2  0.5  0.3
## [3,]  0.2  0.4  0.4
```

```
simulate_mc <- function(n, P, X0) {
    # Generates sequences of length n (including X0)
    #   P: transition probability matrix
    #   X0: initial state
    cur_X <- X0
    states <- 1:3
    MC_seq <- numeric(n)
    MC_seq[1] <- X0
    for (i in 2:n) {
        transition_probs <- P[cur_X, ]
        next_X <- sample(states, size = 1, prob = transition_probs)
        MC_seq[i] <- next_X
        cur_X <- next_X
    }
    MC_seq
}
plot(1:100, simulate_mc(100, P, 1),
     xlab = "i", ylab = "State",
     type = "b", pch = 19,
     main = "Realization of Markov Chain (100 Steps, X0 = 1)")
```



Problem 2a

```
eigs <- eigen(t(P))
stationary_state <- eigs$vectors[, 1]
stationary_state_norm <- stationary_state / sum(stationary_state)
eigs
```

```
## eigen() decomposition
## $values
## [1] 1.000000e+00 1.000000e-01 1.363926e-16
##
## $vectors
##           [,1]          [,2]       [,3]
## [1,] 0.3224585  1.073314e-16  0.2672612
## [2,] 0.8240605 -7.071068e-01 -0.8017837
## [3,] 0.4657733  7.071068e-01  0.5345225
```

Solving numerically, we see that 0.3224585, 0.8240605, 0.4657733 is the solution to $(P^T - I)\pi_\infty = 0$. Normalized, we have $\pi_\infty = 0.2, 0.5111111, 0.2888889$.

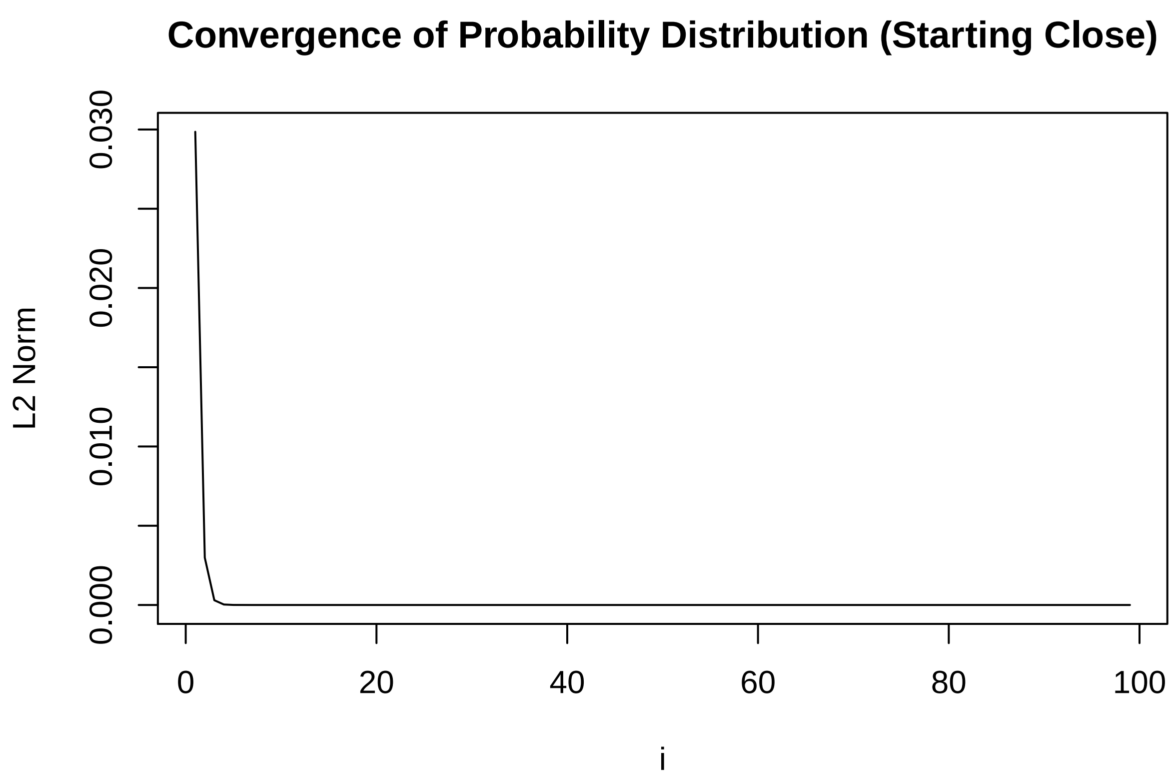
Problem 2b

We plot the convergence for an initial π_0 close to π_∞:

```
# initial distn close to pi_infinity
close_initial <- c(0.1, 0.6, 0.3)
pis <- list()
```

```r
pis[[1]] <- close_initial
n <- 100
# calculates L2 norm
l2norm <- function(u, v) {
    sqrt(sum((u - v)^2))
}
l2_norms <- numeric(n)
for (i in 2:n) {
    pis[[i]] <- pis[[i - 1]] %*% P
    l2_norms[i] <- l2norm(pis[[i]], stationary_state_norm)
}

plot(1:(n - 1), l2_norms[2:n], type = "l",
     xlab = "i", ylab = "L2 Norm",
     main = "Convergence of Probability Distribution (Starting Close)")
```

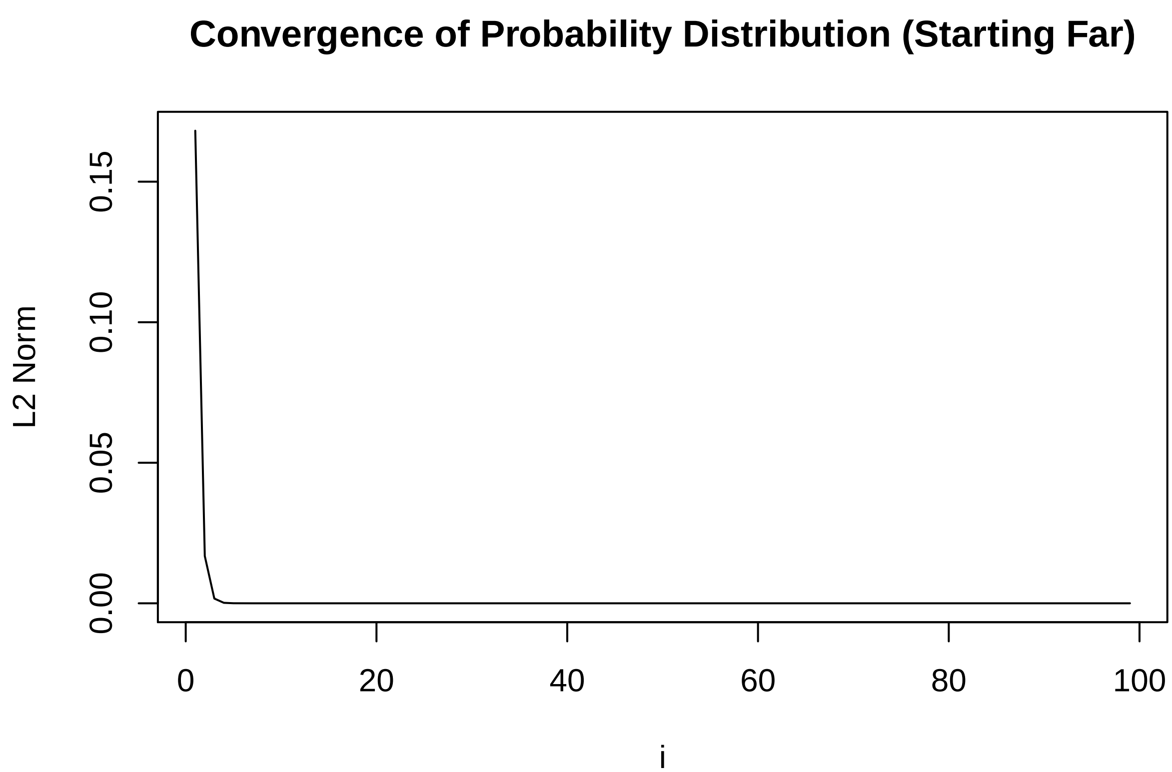


We plot the convergence for an initial π_0 far from π_∞:

```r
# initial distn far from pi_infinity
far_initial <- c(0.7, 0.2, 0.1)
pis <- list()
pis[[1]] <- far_initial
n <- 100
# calculates L2 norm
l2norm <- function(u, v) {
    sqrt(sum((u - v)^2))
```

```
}
l2_norms <- numeric(n)
for (i in 2:n) {
    pis[[i]] <- pis[[i - 1]] %*% P
    l2_norms[i] <- l2norm(pis[[i]], stationary_state_norm)
}

plot(1:(n - 1), l2_norms[2:n], type = "l",
     xlab = "i", ylab = "L2 Norm",
     main = "Convergence of Probability Distribution (Starting Far)")
```

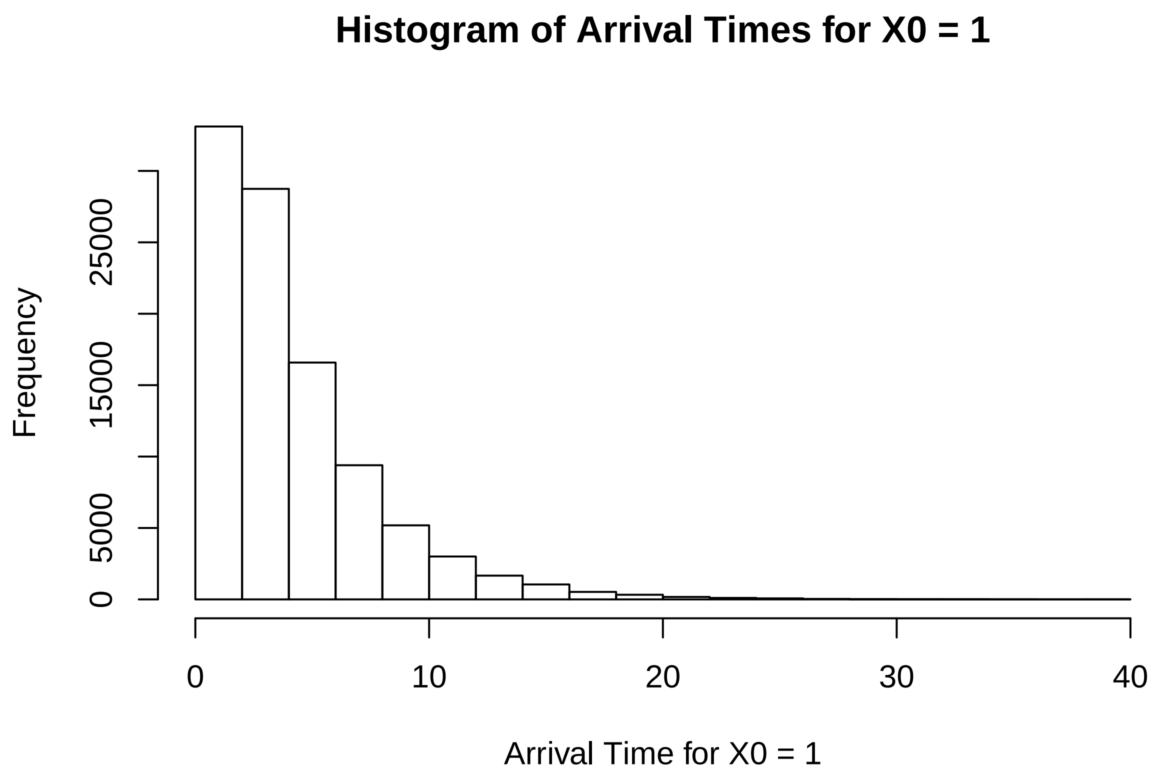


We see π_i converges to π_∞ ($\pi_i \to \pi_\infty$) quite quickly regardless of the specific π_0 value.

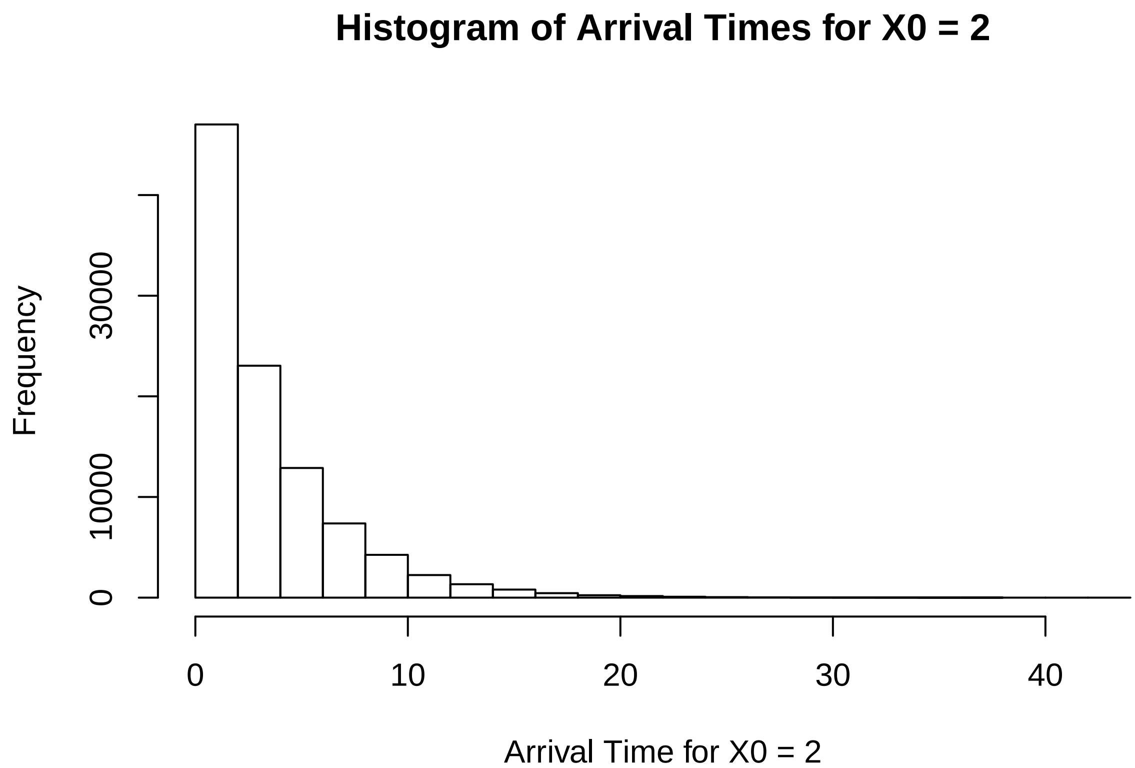
Problem 3a

```
# Function that returns arrival time
until_X3 <- function(X0) {
    t <- 0
    cur_X <- X0
    while (cur_X != 3) {
        cur_X <- simulate_mc(2, P, cur_X)[2]
        t <- t + 1
    }
    t
}
```

```r
# Simulate for X0 = 1
N <- 1e5
arrivals_1 <- numeric(N)
for (i in 1:N) {
    arrivals_1[i] <- until_X3(1)
}
hist(arrivals_1, xlab = "Arrival Time for X0 = 1",
     main = "Histogram of Arrival Times for X0 = 1")
```




```r
# Simulate for X0 = 2
arrivals_2 <- numeric(N)
for (i in 1:N) {
    arrivals_2[i] <- until_X3(2)
}
hist(arrivals_2, xlab = "Arrival Time for X0 = 2",
     main = "Histogram of Arrival Times for X0 = 2")
```




```r
# mean arrival times
mean_arrival_time1 <- mean(arrivals_1)
mean_arrival_time2 <- mean(arrivals_2)
mean_arrival_time1
```

```
## [1] 4.61894
```

```r
mean_arrival_time2
```

```
## [1] 3.84996
```

We see that starting from $X_0 = 1$, we have a mean arrival time of 4.61894 and starting from $X_0 = 2$, we have a mean arrival time of 3.84996.

3. **Absorbing state.** Consider now that node 3 is an absorbing state and we want to estimate the waiting time until the process arrives at $X_i = 3$ from any other node.

b) Compute theoretically the mean arrival time to the absorbing state and compare it with part a. To do so, notice that if T_i denotes the random variable associated to the arrival time starting from $X_0 = i$, then

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j, \tag{1}$$

with $\mu_i = \mathbb{E}[T_i]$. This is a linear system of equations that you can solve. Notice $T_3 = 0$.

$$\mu_1 = 1 + 0.2\mu_1 + 0.7\mu_2$$

$$\mu_2 = 1 + 0.2\mu_1 + 0.5\mu_2$$

$$\Downarrow$$

$$0.5\mu_2 = 1 + 0.2\mu_1 \Rightarrow \mu_2 = 2 + 0.4\mu_1$$

$$\Downarrow$$

$$\mu_1 = 1 + 0.2\mu_1 + 0.7(2 + 0.4\mu_1)$$

$$= 1 + 0.2\mu_1 + 1.4 + 0.28\mu_1$$

$$\Rightarrow 0.52\mu_1 = 2.4 \Rightarrow \boxed{\mu_1 \approx 4.615}$$

$$\Rightarrow \mu_2 \approx 2 + 0.4(4.615)$$

$$\Rightarrow \boxed{\mu_2 \approx 3.846}$$