```
import numpy as np
import matplotlib.pyplot as plt
```

Problem 1) Simulation of Markov Processes

Part a)

The following matrix represents the transition probabilities for the figure given in the problem set document

$$\begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

and setting up a numpy array

```
mark_arr = np.array(
    [
        [0.2, 0.7, 0.1],
        [0.2, 0.5, 0.3],
        [0.2, 0.4, 0.4],
    ]
)
```

Part b)

The following function accept a matrix representing a Markov process and some input vector of probabilities and returns the resulting vector after the Markov process is applied. The code is effectively calculating the following equation using numpy

$\pi_{t+1}{}^\top = \pi_t{}^\top X$

```
def markov_step_single(pi_0, arr_P):
    """TODO"""
    return np.random.choice([1, 2, 3], p=pi_0.T @ arr_P)


def markov_step_arbitrary(pi_0, arr_P, n):
    """TODO"""
    pi = []
    for _ in range(n):
        pi_0 = markov_step_single(pi_0, arr_P)
        pi.append(pi_0)
        pi_0 = np.zeros(3)
        pi_0[pi[-1] - 1] = 1

    return np.asarray(pi)



print("Single Steps:")
a = np.array([1, 0, 0])
print(a, "->", markov_step_single(pi_0=a, arr_P=mark_arr))

b = np.array([0, 1, 0])
print(b, "->", markov_step_single(pi_0=b, arr_P=mark_arr))

c = np.array([0, 0, 1])
print(c, "->", markov_step_single(pi_0=c, arr_P=mark_arr))


print("\nMultiple Steps")
print(a, "->", markov_step_arbitrary(pi_0=a, arr_P=mark_arr, n=10))
print(b, "->", markov_step_arbitrary(pi_0=b, arr_P=mark_arr, n=10))
print(c, "->", markov_step_arbitrary(pi_0=c, arr_P=mark_arr, n=10))
```

```
Single Steps:
[1 0 0] -> 2
[0 1 0] -> 2
[0 0 1] -> 1

Multiple Steps
[1 0 0] -> [1 2 1 2 3 1 1 1 3 3]
[0 1 0] -> [3 3 2 3 2 2 2 3 1 2]
[0 0 1] -> [2 3 2 2 2 3 3 3 3 3]
```

Problem 2) Stationary Distributions

Part a)

Here, the stationary state π_∞ for the above Markov processes is calculated analytically and the solutions checked against numerical computation. The stationary state π_∞ is one of eigenvectors to the matrix P representing the Markov process, so solving the following equation, we have

$$\pi_\infty{}^\top (P - \mathbb{I}) = 0 \implies \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = 0 \tag{1}$$

$$\implies \begin{cases} 0.2\pi_0 + 0.2\pi_1 + 0.2\pi_2 & = 0 \\ 0.7\pi_0 + 0.5\pi_1 + 0.4\pi_2 & = 0 \\ 0.1\pi_0 + 0.3\pi_1 + 0.4\pi_2 & = 0 \end{cases} \tag{2}$$

$$\implies \begin{cases} \pi_0 + \pi_1 + \pi_2 & = 0 \\ 7\pi_0 + 5\pi_1 + 4\pi_2 & = 0 \\ \pi_0 + 3\pi_1 + 4\pi_2 & = 0 \end{cases} \tag{3}$$

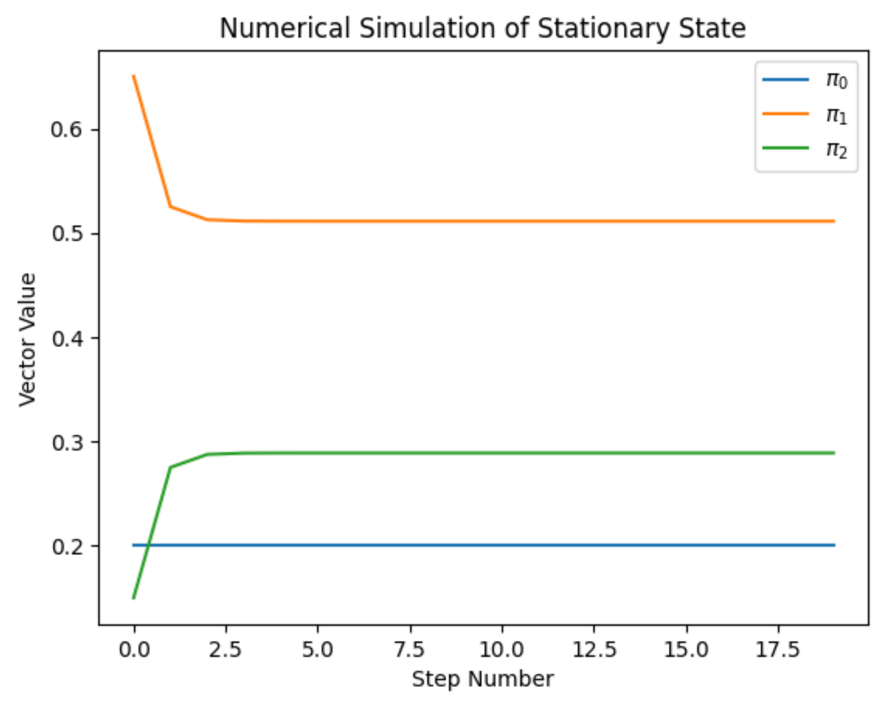
$$\implies \begin{cases} \pi_0 & = \frac{1}{5} \\ \pi_1 & = \frac{23}{45} \\ \pi_2 & = \frac{13}{45} \end{cases} \tag{4}$$

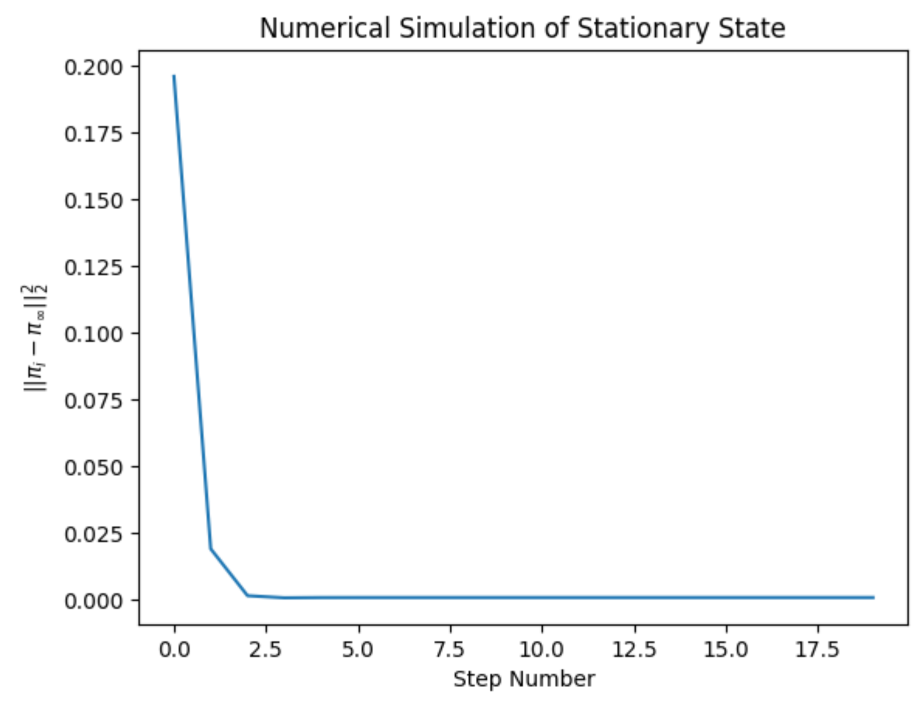
which are the fractional equivalents to the float values computed below.

Part b)

The code below simulates the convergence to the stationary state over 20 steps and generates the requested plots. The analytical stationary state agrees with the state reached by the simulation.

```
true_stat_state = np.array([])
all_pis = []

pi = np.array([0.8, 0.1, 0.1])
for _ in range(20):
    pi = pi.T @ mark_arr
    all_pis.append(pi)

all_pis = np.asarray(all_pis).T
# all_pis
plt.plot(all_pis[0], label="$\pi_0$")
plt.plot(all_pis[1], label="$\pi_1$")
plt.plot(all_pis[2], label="$\pi_2$")
plt.legend()
plt.xlabel("Step Number")
plt.ylabel("Vector Value")
plt.title("Numerical Simulation of Stationary State")
plt.show()

plt.plot(np.linalg.norm(np.array([0.2, 0.511,  0.288])[:, np.newaxis] - all_pis, axis=0))
plt.xlabel("Step Number")
plt.ylabel(r"$\vert\vert \pi_i - \pi_{\infty} \vert\vert^2_2$")
plt.title("Numerical Simulation of Stationary State")
plt.show()


print("Stationary State:", all_pis[:, -1])
```






```
Stationary State: [0.2        0.51111111 0.28888889]
```

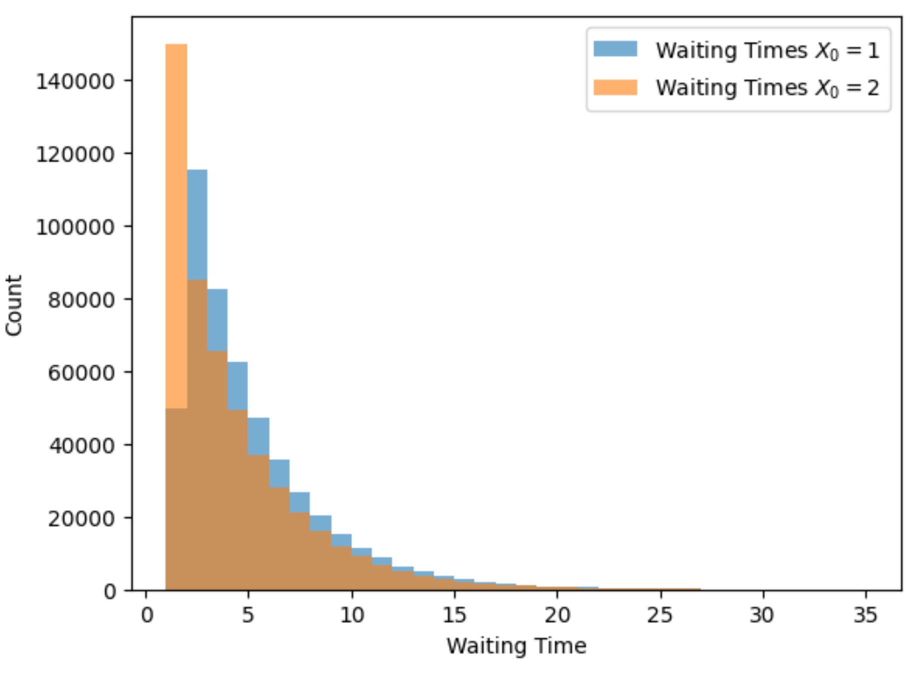
Problem 3) Absorbing State

Part a)

Below is code which simulates waiting time until reaching state 3, at which point the simulation ends.

```
def calc_waiting_time(pi0, X0, P):
    MAX_ITER = 10000
    waiting_time = 0

    while X0 != 3 and waiting_time < MAX_ITER:
        pi = np.zeros(3)
        pi[X0 - 1] = 1
        X0 = markov_step_single(pi_0=pi, arr_P=P)
        waiting_time += 1

    return waiting_time

# For X_0 = 1
waiting_times_1 = [
    calc_waiting_time(pi0=np.array([1, 0, 0]), X0=1, P=mark_arr)
    for _ in range(500000)
]


# # For X_0 = 2
waiting_times_2 = [
    calc_waiting_time(pi0=np.array([0, 1, 0]), X0=2, P=mark_arr)
    for _ in range(500000)
]
```

```
plt.hist(waiting_times_1, alpha=0.6, label="Waiting Times $X_0=1$", bins=np.arange(35) + 1)
plt.hist(waiting_times_2, alpha=0.6, label="Waiting Times $X_0=2$", bins=np.arange(35) + 1)
plt.legend()
# plt.xscale("log")
# plt.yscale("log")
plt.xlabel("Waiting Time")
plt.ylabel("Count")
plt.show()

print("Mean Arrival Time X_1:", np.mean(waiting_times_1))
print("Mean Arrival Time X_2:", np.mean(waiting_times_2))
```




```
Mean Arrival Time X_1: 4.610172
Mean Arrival Time X_2: 3.847082
```

Part b)

Here, the theoretical mean arrival time to state 3 is calculated via

$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$ where the expected arrival time to state 3 is $\mathbb{E}[T_i] = \mu_i$ where the initial state is i.

$\mu_1 = 1 + (\ 0.2\mu_1 + 0.2\mu_2 + 0.2\mu_3)$

and

$\mu_2 = 1 + (\ 0.7\mu_1 + 0.5\mu_2 + 0.4\mu_3)$

and

$\mu_3 = \mathbb{E}[T_3] = 0$

so $\begin{cases} \mu_1 = 1 + \frac{1}{5}\mu_1 + \frac{1}{5}\mu_2 \\ \mu_2 = 1 + \frac{7}{10}\mu_1 + \frac{1}{2}\mu_2 \end{cases} \implies \begin{cases} -10 = -8\mu_1 + 7\mu_2 \\ -10 = 2\mu_1 + -5\mu_2 \end{cases}$ meaning that $\begin{cases} \mathbb{E}[T_1] = \mu_1 = \frac{60}{13} \\ \mathbb{E}[T_2] = \mu_2 = \frac{50}{13} \end{cases}$ which is relatively close to the numerically calculated waiting times.