```r
library(latex2exp)
```

Q1

a

$$P = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$

b

```r
P <- matrix(c(0.2, 0.7, 0.1, 0.2, 0.5, 0.3, 0.2, 0.4, 0.4), nrow = 3, byrow = T)
P
```

```
##      [,1] [,2] [,3]
## [1,]  0.2  0.7  0.1
## [2,]  0.2  0.5  0.3
## [3,]  0.2  0.4  0.4
```

```r
set.seed(42)

n <- 10
rec <- numeric(length = n)
rec[1] <- 1
for (i in 2:n){
  rec[i] <- sample(c(1,2,3), size = 1, prob = P[rec[i-1],])
}
```

We get one simulation, starting from state 1, showing below:

```r
rec
```

```
##  [1] 1 3 1 2 1 2 3 3 2 3
```

Q2

a

$$
\begin{aligned}
&(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = (\pi_1, \pi_2, \pi_3) \\
&\Rightarrow \begin{cases} -0.8\pi_1 + 0.2\pi_2 + 0.2\pi_3 = 0 \\ 0.7\pi_1 - 0.5\pi_2 + 0.4\pi_3 = 0 \\ 0.1\pi_1 + 0.3\pi_2 - 0.6\pi_3 = 0 \end{cases} \\
&\Rightarrow \begin{cases} \pi_2 = \frac{23}{9}\pi_1 \\ \pi_3 = \frac{13}{9}\pi_1 \end{cases}
\end{aligned}
$$

Given $\pi_1 + \pi_2 + \pi_3 = 1$. We have

$$
\pi_1 = \frac{1}{5} \quad \pi_2 = \frac{23}{45} \quad \pi_3 = \frac{13}{45}
$$

b

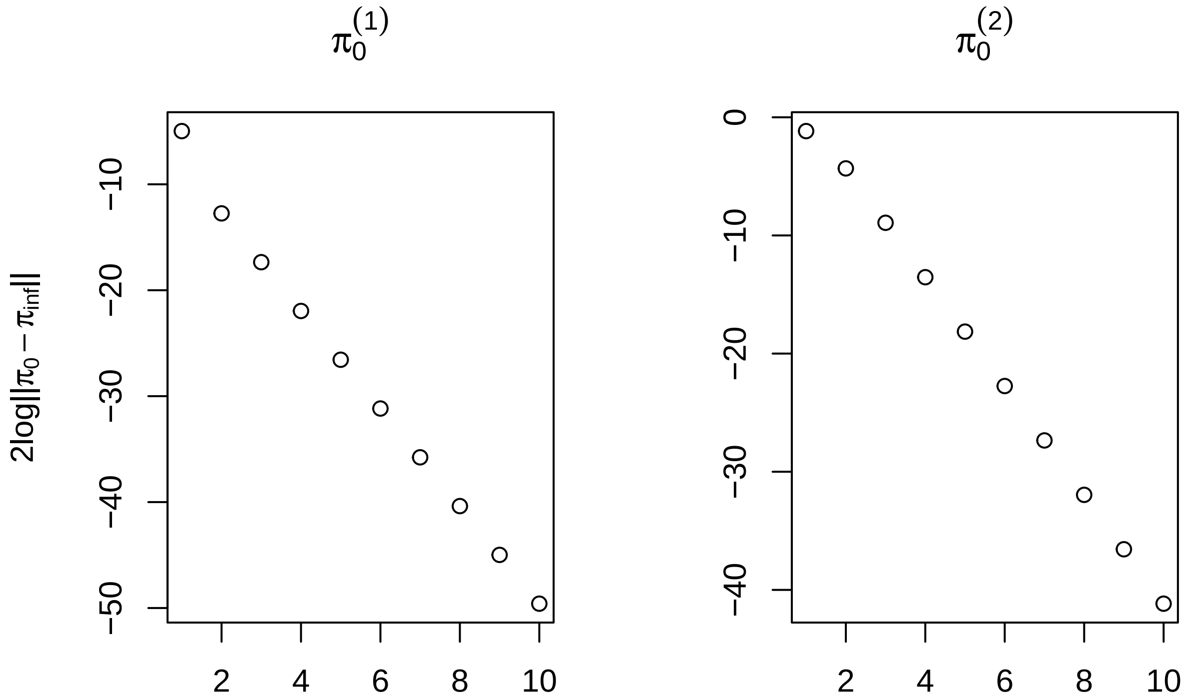
```r
set.seed(42)
k <- 2 # number of simulation
m <- 10 # number of running process in each simulation
dist <- matrix(nrow = k, ncol = m)
pi.inf <- c(1/5, 23/45, 13/45)
pi0.rec <- matrix(nrow = k, ncol = 3)

for (rep in 1:k){
  # generate three exponential random number and normalize to get pi_0
  pi0 <- rexp(3)
  pi0 <- pi0 / sum(pi0)
  pi0.rec[rep,] <- pi0

  pi.rec <- matrix(nrow = m, ncol = 3)
  pi.rec[1,] <- pi0

  for (i in 2:m){
    pi.rec[i,] <- pi.rec[i-1,] %*% P
  }
  for (i in 1:m){
    dist[rep,i] <- log(sum((pi.rec[i,] - pi.inf)^2))
  }
}
```

```r
pi0.rec
```

```
##            [,1]      [,2]      [,3]
## [1,] 0.17356507 0.5783512 0.2480838
## [2,] 0.01933771 0.2395836 0.7410787
```

Given two different initial probability: $(0.17, 0.58, 0.25)$ and $(0.02, 0.24, 0.74)$, the $\log \|\pi_i - \pi_\infty\|_2^2$ shows in the following graphs:

```r
par(mfrow=c(1,2))
plot(dist[1,], xlab = "", ylab = TeX(r"($2\log\| \pi_0 - \pi_\inf \|$)"),
     main = TeX(r"($\pi_0^{(1)}$)"))
plot(dist[2,], xlab = "", ylab = "", main =  TeX(r"($\pi_0^{(2)}$)"))
```



$\pi_0^{(1)}$
$\pi_0^{(2)}$
$2\log\|\pi_0 - \pi_{\text{inf}}\|$
−10
−20
−30
−40
−50
0
2
4
6
8
10

Q3

a

```r
set.seed(42)
m <- 2
x.0 <- c(1, 2)
n <- 10000
rec <- matrix(nrow=m, ncol=n)

for (k in 1:m){
  for (i in 1:n){
    x0 <- x.0[k]
    x.prev <- x0
    count <- 0

    while (x.prev != 3){
      x <- sample(c(1,2,3), size = 1, prob = P[x.prev,])
      x.prev <- x
      count <- count + 1
    }

    rec[k, i] <- count
  }
}
```

```r
mean(rec[1,])
```

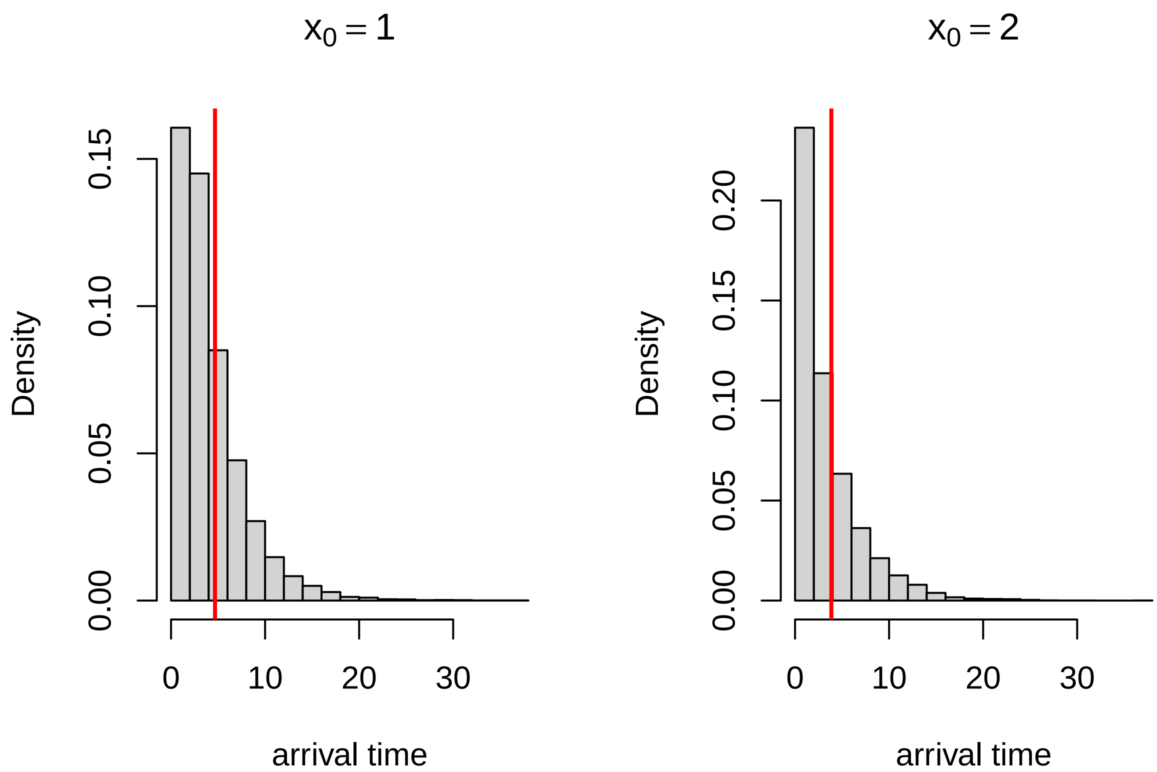
```
## [1] 4.6743
```

```r
mean(rec[2,])
```

```
## [1] 3.8608
```

The mean value of arrival time from state 1 and state 2 are 4.67 and 3.86, respectively.

```r
par(mfrow=c(1,2))
hist(rec[1,], probability = T,
     main = TeX(r"($x_0 = 1$)"),
     xlab = "arrival time")
abline(v=mean(rec[1,]), col="red", lwd=2)
hist(rec[2,], probability = T,
     main = TeX(r"($x_0 = 2$)"),
     xlab = "arrival time")
abline(v=mean(rec[2,]), col="red", lwd=2)
```



b

$$\begin{cases} \mu_1 = 1 + 0.2\mu_1 + 0.7\mu_2 \\ \mu_2 = 1 + 0.2\mu_1 + 0.5\mu_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.8\mu_1 - 0.7\mu_2 = 1 \\ -0.2\mu_1 + 0.5\mu_2 = 1 \end{cases} \Rightarrow \begin{cases} \mu_1 = \frac{60}{13} \approx 4.62 \\ \mu_2 = \frac{50}{13} \approx 3.85 \end{cases}$$