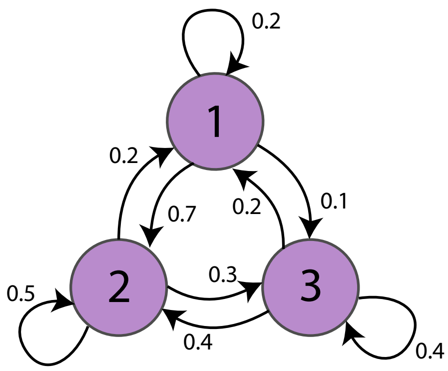
STAT 201A - Introduction to Probability at an advanced level
Problem set
Fall 2023, UC Berkeley

November 20th

Homework assignment to be delivered by December 4st in bCourses. Solutions will include math calculation and code in pdf formart.

For all this assignment, we are going to be working in the Markov process defined by the following graph. Arrows represent transition probabilities.



1. **Simulation of Markov Process.**

   a) Write the Markov process in matrix representation, that is, define the matrix $P \in \mathbb{R}^{3\times 3}$ such that P_{ij} is the probability of transitioning from the node i to j.

   b) Simulate one single realization of the chain, that is, starting from $X_0 = 1$, update the value of X_i using the probabilities defined by the process.

2. **Stationary Distribution.** The goal of this section is to show the convergence of the probability distribution of the Markov process.

   a) Calculate theoretically the stationary state of the process by finding the vector $\pi_\infty \in \mathbb{R}^3$ such that $\pi_\infty^T = \pi_\infty^T P$. Notice that this is the same as finding the eigenvector with eigenvalue equals one of the matrix P^T. This is the same as solving $(P^T - I)\pi_\infty = 0$. You can solve the linear system of equation numerically or analytically.

   b) Starting now from an initial probability distribution π_0 on the nodes, compute the value of $\pi_i^T = \pi_0^T P^i$ the probability distribution at time i. Show that $\pi_i \to \pi_\infty$

and make plot of i vs $\|\pi_i - \pi_\infty\|_2^2$. Generate this plot for at least two different initial conditions π_0 and compare.

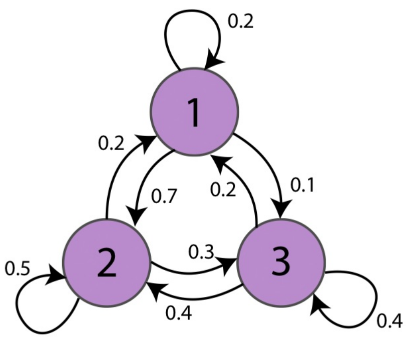
3. **Absorbing state.** Consider now that node 3 is an absorbing state and we want to estimate the waiting time until the process arrives at $X_i = 3$ from any other node.

   a) Starting from each one of $X_0 = 1$ and $X_0 = 2$, run multiple simulation of the Markov chain (Problem 1, part b) until $X_i = 3$ and store the arrival time until this happens. Make a histogram of the arrival time for both $X_0 = 1$ and $X_0 = 2$ and compute the mean.

   b) Compute theoretically the mean arrival time to the absorbing state and compare it with part a. To do so, notice that if T_i denotes the random variable associated to the arrival time starting from $X_0 = i$, then

   $$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j, \tag{1}$$

   with $\mu_i = \mathbb{E}[T_i]$. This is a linear system of equations that you can solve. Notice $T_3 = 0$.



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   a) Write the Markov process in matrix representation, that is, define the matrix $P \in \mathbb{R}^{3\times 3}$ such that P_{ij} is the probability of transitioning from the node i to j.

   b) Simulate one single realization of the chain, that is, starting from $X_0 = 1$, update the value of X_i using the probabilities defined by the process.

1a

$$\boxed{P = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}}$$

2. **Stationary Distribution.** The goal of this section is to show the convergence of the probability distribution of the Markov process.

a) Calculate theoretically the stationary state of the process by finding the vector $\pi_\infty \in \mathbb{R}^3$ such that $\pi_\infty^T = \pi_\infty^T P$. Notice that this is the same as finding the eigenvector with eigenvalue equals one of the matrix P^T. This is the same as solving $(P^T - I)\pi_\infty = 0$. You can solve the linear system of equation numerically or analytically.

$$P = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} \qquad P^T = \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.7 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{pmatrix}$$

$$P^T - I = \begin{pmatrix} 0.2-1 & 0.2 & 0.2 \\ 0.7 & 0.5-1 & 0.4 \\ 0.1 & 0.3 & 0.4-1 \end{pmatrix} = \begin{pmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.5 & 0.4 \\ 0.1 & 0.3 & -0.6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -0.8 & 0.2 & 0.2 & 0 \\ 0.7 & -0.5 & 0.4 & 0 \\ 0.1 & 0.3 & -0.6 & 0 \end{array}\right) \xrightarrow{\text{RREF calculator}} \left(\begin{array}{ccc|c} 1 & 0 & -9/13 & 0 \\ 0 & 1 & -23/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\pi_3 = t$$
$$\pi_1 = \frac{9}{13}t \qquad \pi = t\begin{pmatrix} 9/13 \\ 23/13 \\ 1 \end{pmatrix}$$
$$\pi_2 = \frac{23}{13}t$$

normalize, sum $= \frac{45}{13}$

$$\boxed{\pi_\infty = \begin{pmatrix} 9/45 \\ 23/45 \\ 13/45 \end{pmatrix} \approx \begin{pmatrix} 0.2 \\ 0.511 \\ 0.289 \end{pmatrix}}$$

b) Compute theoretically the mean arrival time to the absorbing state and compare it with part a. To do so, notice that if T_i denotes the random variable associated to the arrival time starting from $X_0 = i$, then

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j, \tag{1}$$

with $\mu_i = \mathbb{E}[T_i]$. This is a linear system of equations that you can solve. Notice $T_3 = 0$.

$$\mathbb{E}[T_1] = P_{11}(\mathbb{E}[T_1]+1) + P_{12}(\mathbb{E}[T_2]+1) + P_{13}(1)$$
$$= P_{11}\mathbb{E}[T_1] + P_{12}\mathbb{E}[T_2] + 1$$

$$\mathbb{E}[T_2] = P_{21}(\mathbb{E}[T_1]+1) + P_{22}(\mathbb{E}[T_2]+1) + P_{23}(1)$$
$$= P_{21}\mathbb{E}[T_1] + P_{22}\mathbb{E}[T_2] + 1$$

$$\mathbb{E}[T_2](1-P_{22}) = P_{21}\mathbb{E}[T_1] + 1$$

$$\mathbb{E}[T_2] = \frac{P_{21}}{1-P_{22}}\mathbb{E}[T_1] + \frac{1}{1-P_{22}}$$

$$\mathbb{E}[T_1] = P_{11}\mathbb{E}[T_1] + P_{12}\left[\frac{P_{21}}{1-P_{22}}\mathbb{E}[T_1] + \frac{1}{1-P_{22}}\right] + 1$$

$$\mathbb{E}[T_1]\left(1-P_{11}-\frac{P_{12}P_{21}}{1-P_{22}}\right) = \frac{P_{12}}{1-P_{22}} + 1$$

$$\mathbb{E}[T_1] = \frac{\frac{P_{12}}{1-P_{22}} + 1}{\left(1-P_{11}-\frac{P_{12}P_{21}}{1-P_{22}}\right)} \approx \boxed{4.615}$$

$$\mathbb{E}[T_2] = \frac{P_{21}}{1-P_{22}}\mathbb{E}[T_1] + \frac{1}{1-P_{22}} \approx \boxed{3.846}$$