In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
```

1a) The transition matrix will be:

$$P = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

1b)

In [2]:
```
np.random.seed(42)
```

In [3]:
```
# Set up the transition matrix and possible states
P = np.array([[0.2, 0.7, 0.1],[0.2, 0.5, 0.3], [0.2, 0.4, 0.4]])
state = [1, 2, 3]

# Give a starting state and simulate one step forward
X_0 = np.array([1, 0, 0])
X_1 = np.random.choice(state, p = X_0 @ P)
X_1
```

Out[3]: 2

2a) We want to solve $(P^T - I_3)\pi_\infty = 0$,which gives us:

$$\begin{bmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.5 & 0.4 \\ 0.1 & 0.3 & -0.6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

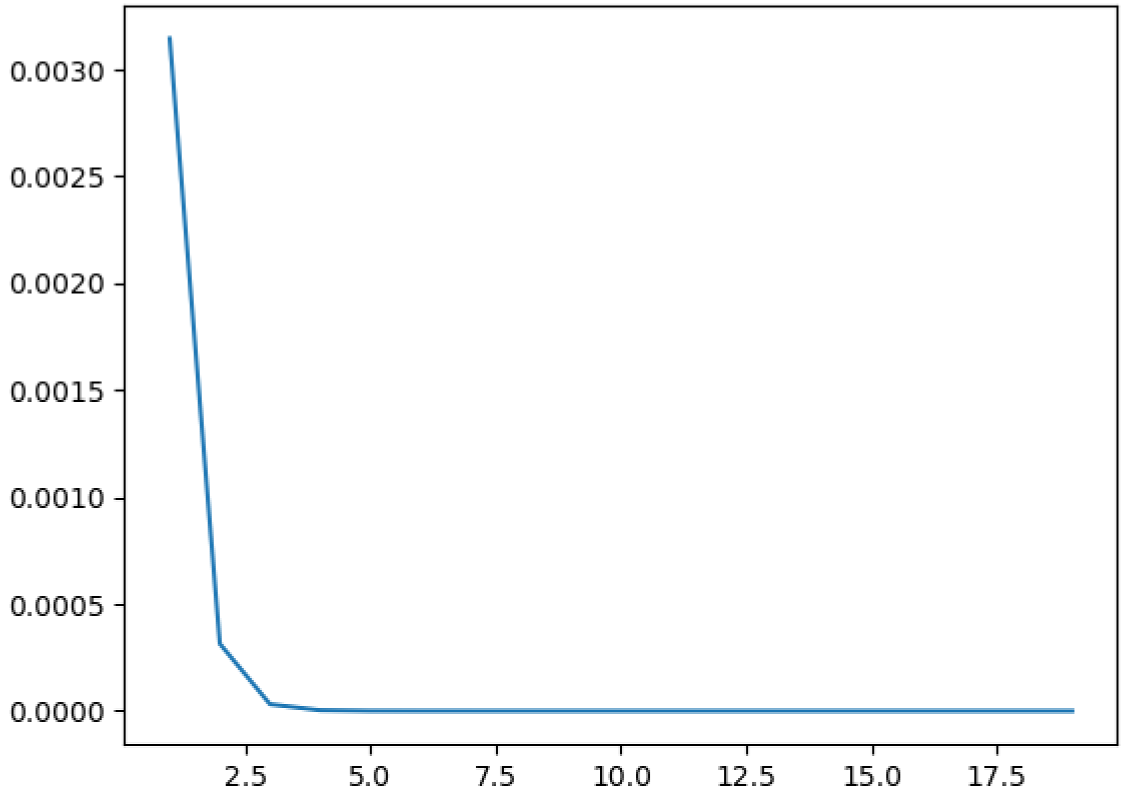
Solving this system gives us, $b = \frac{23a}{9}$ and $c = \frac{13a}{9}$. The value $a = \frac{1}{1+\frac{23}{9}+\frac{13}{9}}$ will make sure the vector π_∞ sums to 1.

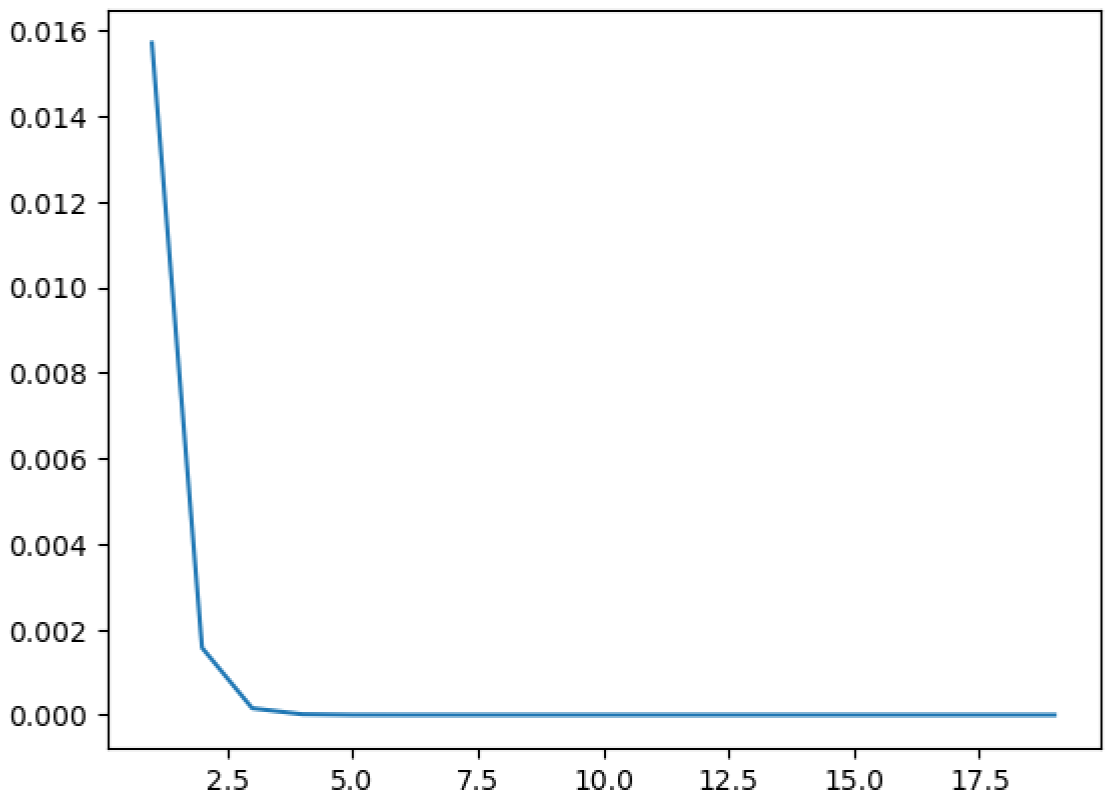
In [4]:
```
# We can find the null space of P.T - I and normalize the vector
pi_inf = sp.linalg.null_space(P.T - np.identity(3))
pi_inf = pi_inf / np.sum(pi_inf)
print(np.ndarray.flatten(pi_inf))

# Or we can use the values we obtained theoretically (gives the exact same r
a = (1 / (1 + 23/9 + 13/9))
b = a * (23/9)
c = a * (13/9)
pi_inf = np.array([a, b, c])
print(pi_inf)
```

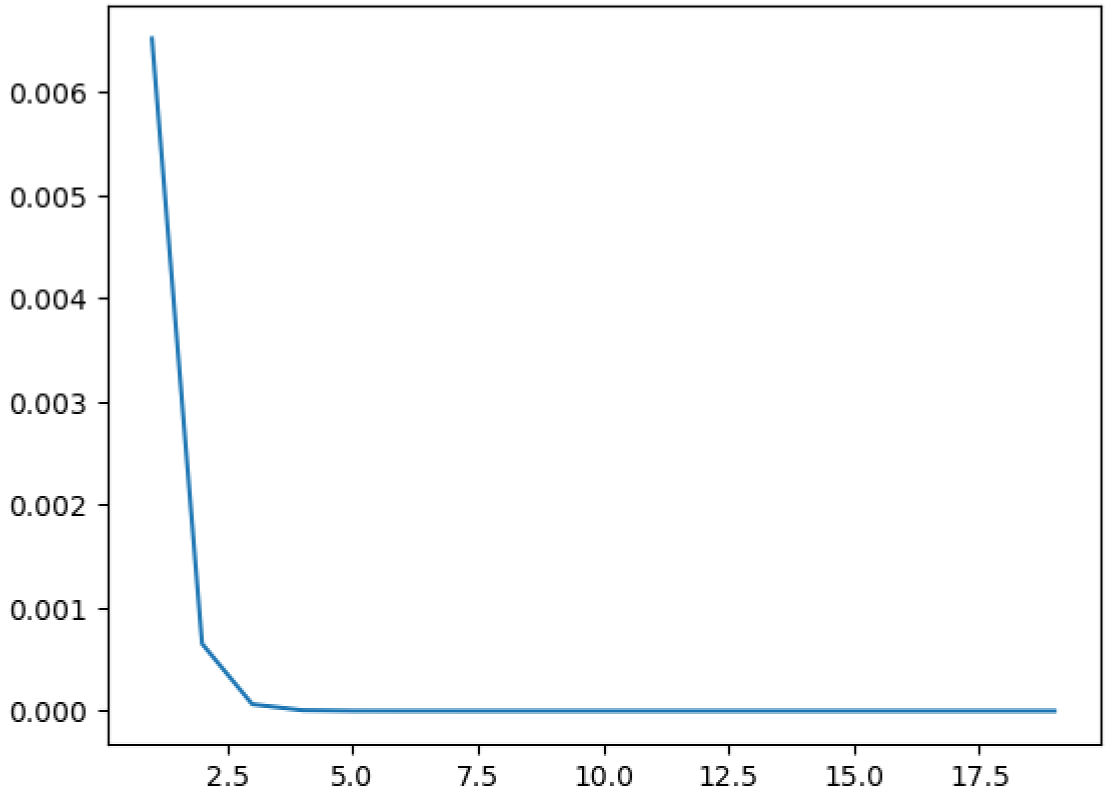
```
[0.2        0.51111111 0.28888889]
[0.2        0.51111111 0.28888889]
```

2b) The initial distribution did not seem to affect the rate of convergence to the stationary state. It would always take a few iterations to basically reach it.

In [5]:
```
def markov_process(X_0, n = 20):

    # record the state distribution and L2 norm
    X = []
    diff = np.zeros(n)
    X_i = X_0

    #iterate through
    for i in range(n):
        X_i = X_i @ P
        X.append(X_i)
        diff[i] = np.sqrt(np.sum(np.square(X_i - pi_inf)))

    return(X, diff)

# Start at any state with equal probability
X_0 = [1/3, 1/3, 1/3]
pi_1, diff_1 = markov_process(X_0)

plt.plot(range(1, 20), diff_1[1:20])
plt.show()

# Start at the third state
X_0 = [0, 0, 1]
pi_2, diff_2 = markov_process(X_0)

plt.plot(range(1, 20), diff_2[1:20])
plt.show()

# Start at the second or third state
X_0 = [0, 0.65, 0.35]
pi_3, diff_3 = markov_process(X_0)

plt.plot(range(1, 20), diff_3[1:20])
plt.show()
```


0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
2.5
5.0
7.5
10.0
12.5
15.0
17.5

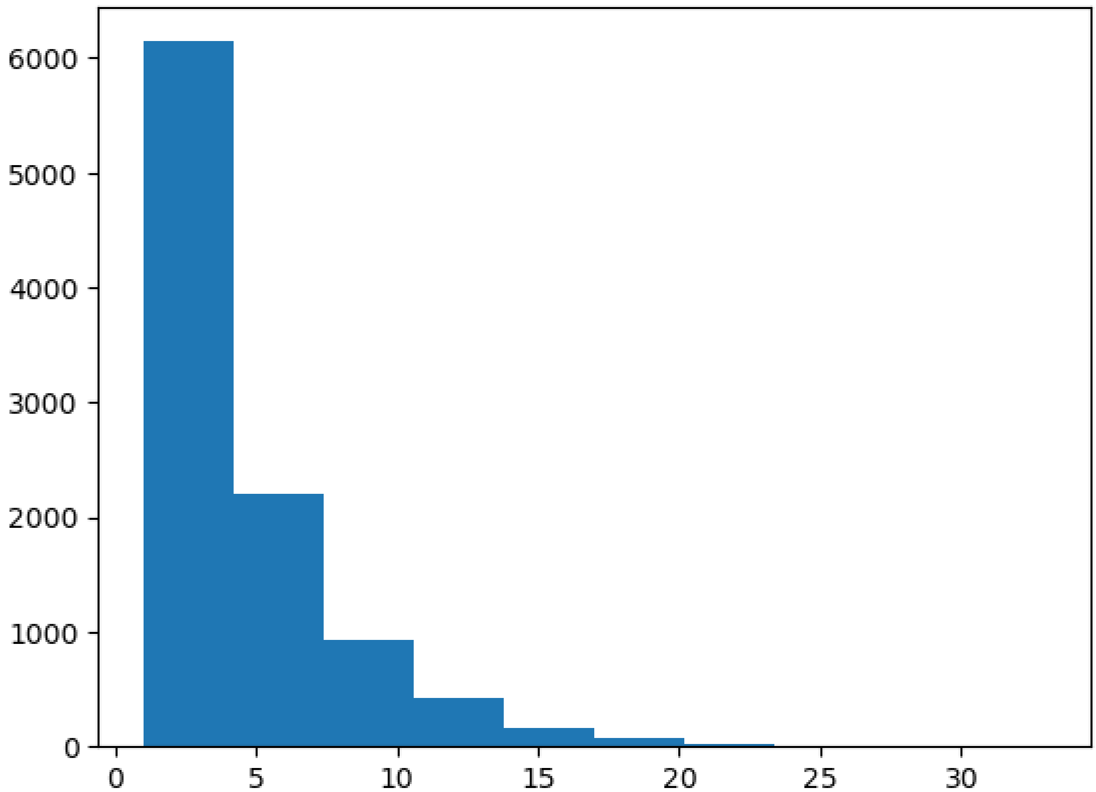

0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
2.5
5.0
7.5
10.0
12.5
15.0
17.5



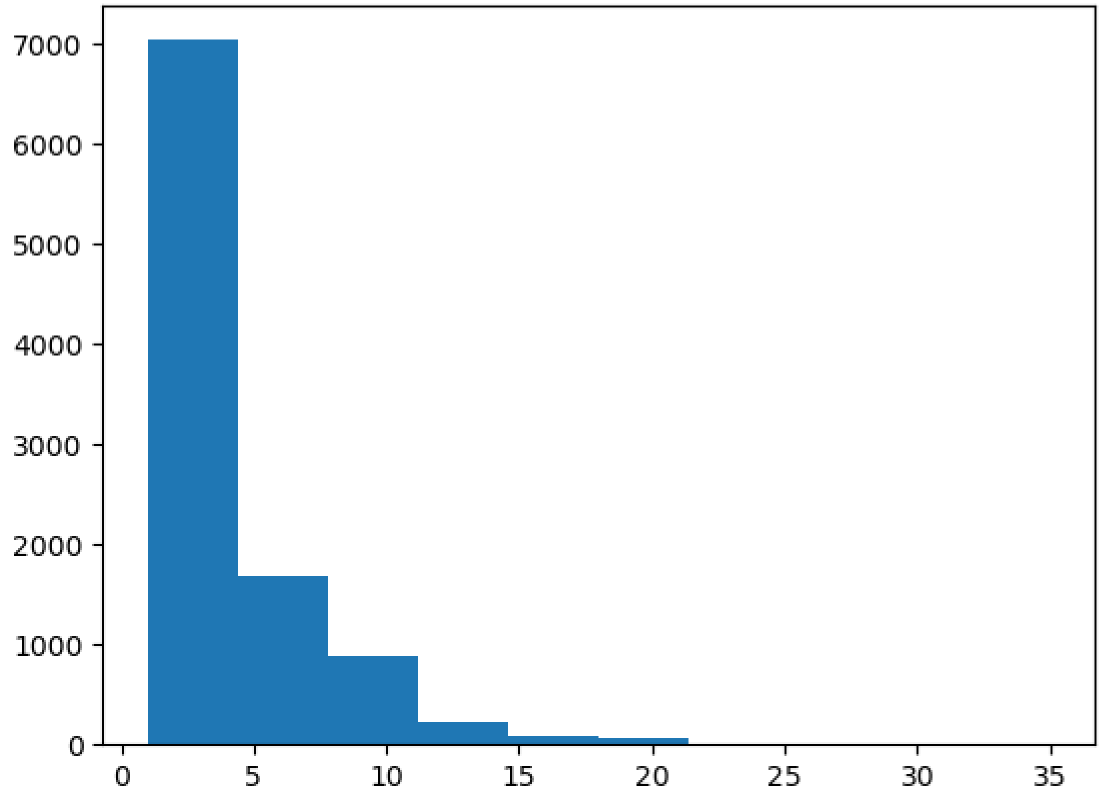
3a) Starting at state 1, on average, it takes 4.63 steps to reach state 3 for the first time. Starting at state 2, on average, it takes 3.81 steps to reach state 3 for the first time.

In [6]:
```
def start_at(start):
    # record the steps
    steps = []

    # Create a large sample
    for sample in range(10000):

        # set the starting step
        X_0 = np.zeros(3)
        X_0[start - 1] = 1
        num_steps = 1

        # Iteratively choose next step until we reach state 3
        X_i = np.random.choice(state, p = X_0 @ P)
        while X_i != 3:
            at_state = np.zeros(3)
            at_state[X_i - 1] = 1
            X_i = np.random.choice(state, p = at_state @ P)
            num_steps += 1

        steps.append(num_steps)

    return steps

steps = start_at(1)
plt.hist(steps)
```

```
plt.show()
print("Average starting at state 1:", np.mean(steps))

steps = start_at(2)
plt.hist(steps)
plt.show()
print("Average starting at state 2:", np.mean(steps))
```




```
Average starting at state 1: 4.6497
```




```
Average starting at state 2: 3.8228
```

3b)

Let T_i denote the random variable of the arrival time to the absorbing state, 3, given starting at $X_0 = i$. Note that $T_3 = 0$. We have that

$$E[T_1] = 1 + p_{11} * E[T_1] + p_{12} * E[T_2]$$

and

$$E[T_2] = 1 + p_{21} * E[T_1] + p_{22} * E[T_2].$$

Solving for $E[T_1]$ gives us $E[T_1] = \frac{1+p_{12}*E[T_2]}{1-p_{11}}$. Plugging this into the second equation,

$$E[T_2] = 1 + p_{21} * \frac{1 + p_{12} * E[T_2]}{1 - p_{11}} + p_{22} * E[T_2]$$

$$= 1 + \frac{p_{21}}{1 - p_{11}} + p_{21} * \frac{p_{12} * E[T_2]}{1 - p_{11}} + p_{22} * E[T_2]$$

$$= 1 + \frac{p_{21}}{1 - p_{11}} + (p_{21} * \frac{p_{12}}{1 - p_{11}} + p_{22}) * E[T_2]$$

So $E[T_2] = \frac{1+\frac{p_{21}}{1-p_{11}}}{(1-p_{21}*\frac{p_{12}}{1-p_{11}}+p_{22})} = \frac{1+\frac{0.2}{0.8}}{(1-0.2*\frac{0.7}{0.8}+0.5)} \approx 3.846$. Which gives us $E[T] = \frac{1+p_{12}*E[T_2]}{1-p_{11}} = \frac{1+0.7*3.846}{0.8} \approx 4.615$. Observe these are close to the values we got from running the simulation.