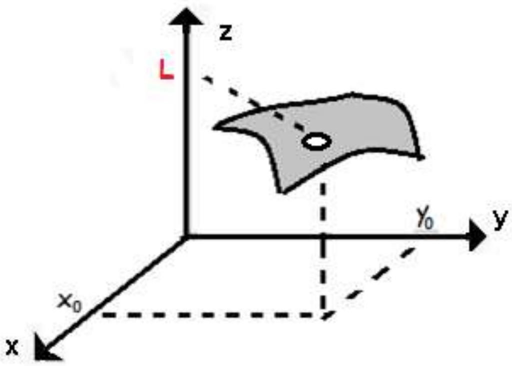
# Limite Doble




$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

- $\boxed{n^0}$ ( Es el valor del limite)
- $\boxed{\frac{0}{0}}$ ⇨ Tomamos limites Iterados

$$L_1 = \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$$

$$L_2 = \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right)$$

- Si $L_1 \neq L_2$ **no existe limite**
- Si $L_1 = L_2$ **puede existir limite (*)**

(*) Tomamos limite radial a travez del haz de rectas

$$\boxed{y = m(x - x_0) + y_0}$$

En el origen:

| | |
|---|---|
| $y = m\,x$ | $x = m\,y$ |
| $y = m\,x^2$ | $x = m\,y^2$ |
| $y = m\,x^3$ | $x = m\,y^3$ |

**Reemplazamos en el límite inicial:** $\lim\limits_{x\to x_0} f\left(x, m(x - x_0) + y_0\right)$

Si el resultado depende de "m" NO existe el límite

Si el resultado no depende de "m" Aplicamos la definición:

Definición:

$$\lim_{(x,y)\to(0,0)} f(x,y) = L \quad sii \quad \forall \xi > 0, \quad \exists\ \delta > 0 \quad /$$

$$0 < \left\|\vec{x} - \vec{x}_0\right\| < \delta \quad \left(\vec{x} \in D_f\right) \quad \Rightarrow \left|f(\vec{x}) - L\right| < \xi$$

**Ejemplos**

1) $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^2-1}{x^2+y^2+2} = \frac{(0)^3+(0)^2-1}{(0)^2+(0)^2+2} = \frac{-1}{2} \quad \exists \quad \lim$$

2) $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+3y^2} = \frac{(0)^3+(0)^2}{(0)^2+3\,(0)^2} = \frac{0}{0}$$

**Tomamos límites Iterados:**

$$L_1 = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^3+y^2}{x^2+3y^2}\right) = \lim_{x\to 0}\left(\frac{x^3+(0)^2}{x^2+3\,(0)^2}\right) = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x\to 0}(x) = 0$$

$$L_2 = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^3+y^2}{x^2+3y^2}\right) = \lim_{y\to 0}\left(\frac{(0)^3+y^2}{(0)^2+3\,y^2}\right) = \lim_{y\to 0}\left(\frac{y^2}{3\,y^2}\right) = \lim_{y\to 0}\left(\frac{1}{3}\right) = \frac{1}{3}$$

**Limites Iterados distintos,** **no existe límite**

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = \dfrac{2(0)(0)}{(0)^2+(0)^2} = \dfrac{0}{0}$

**Tomamos límites Iterados:**

$$L_1 = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2+(0)^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = 0$$

$$L_2 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{y\to 0}\left(\frac{0}{(0)^2+y^2}\right) = \lim_{y\to 0}\left(\frac{0}{y^2}\right) = 0$$

**Limites Iterados iguales, puede existir límite**

**Tomamos limite radial a través de**

$$y = m(x - x_0) + y_0$$

$$y = m(x-0)+0$$

$$y = m\,x$$

$$\lim_{x\to 0}\frac{2x\,m\,x}{x^2+(m\,x)^2} = \lim_{x\to 0}\frac{2\,mx^2}{x^2+m^2x^2} = \lim_{x\to 0}\frac{2\,m\,x^2}{x^2\left(1+m^2\right)} = \lim_{x\to 0}\frac{2m}{\left(1+m^2\right)} = \frac{2m}{\left(1+m^2\right)}$$

Depende de m no existe limite

4) $$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{2x^2+y^4} = \frac{2(0)(0)^2}{2(0)^2+(0)^4} = \frac{0}{0}$$

**Tomamos límites Iterados:**

$$L_1 = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{4xy^2}{2x^2+y^4}\right) = \lim_{x\to 0}\left(\frac{4x(0)^2}{2x^2+(0)^4}\right) \lim_{x\to 0}\left(\frac{0}{2x^2}\right) = 0$$

$$L_2 = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{4xy^2}{2x^2+y^4}\right) = \lim_{y\to 0}\left(\frac{0}{2(0)^2+y^4}\right) = \lim_{y\to 0}\left(\frac{0}{y^4}\right) = 0$$

**Limites Iterados iguales, puede existir límite**

**Tomamos limite radial a través de**

$$x = m\,y^2$$

$$\lim_{y\to 0} \frac{4m\,y^2y^2}{2(m\,y^2)^2+y^4} = \lim_{y\to 0} \frac{4m\,y^4}{2m^2y^4+y^4} = \lim_{y\to 0} \frac{4m\,y^4}{y^4(2m^2+1)} = \lim_{y\to 0} \frac{4m}{2m^2+1} = \frac{4m}{2m^2+1}$$

Depende de m no existe limite

5) $$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^2} = \frac{2(0)(0)^4}{(0)^2+(0)^2} = \frac{0}{0}$$

**Tomamos límites Iterados:**

$$L_1 = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy^4}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2+(0)^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = 0$$

$$L_2 = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy^4}{x^2+y^2}\right) = \lim_{y\to 0}\left(\frac{0}{(0)^2+y^2}\right) = \lim_{y\to 0}\left(\frac{0}{y^2}\right) = 0$$

**Limites Iterados iguales, puede existir límite**

**Tomamos limite radial a través de**

$$y = m\,x$$

$$\lim_{x\to 0} \frac{x(m\,x)^4}{x^2+(m\,x)^2} = \lim_{x\to 0} \frac{m^4x^5}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m^4x^5}{x^2\left(1+m^2\right)} = \lim_{x\to 0} \frac{m^4x^3}{\left(1+m^2\right)} = \frac{m^4(0)^3}{\left(1+m^2\right)} = 0$$

No depende de m aplicamos la definición

Definición:
$$\boxed{\begin{array}{l} \lim\limits_{(x,y)\to(0,0)} f(x,y) = L \quad sii \quad \forall \xi > 0, \;\; \exists\; \delta > 0 \quad / \\ \\ 0 < \|\vec{x} - \vec{x}_0\| < \delta \quad \left(\vec{x} \in D_f\right) \;\; \Rightarrow \left| f(\vec{x}) - L \right| < \xi \end{array}}$$

**Recordar:**

$$X = (x, y) \Rightarrow \|X\| = \sqrt{x^2 + y^2}$$

$$\|X\|^2 = \left(\sqrt{x^2 + y^2}\right)^2 \Rightarrow \boxed{x^2 + y^2 = \|X\|^2}$$

$$x^2 \le \|X\|^2 \Rightarrow \boxed{x \le \|X\|}$$

$$y^2 \le \|X\|^2 \Rightarrow \boxed{y \le \|X\|}$$

**(I)** $|f(x, y) - L| < \varepsilon$

$$\left|\frac{x y^4}{x^2 + y^2} - 0\right| \le \frac{\|X\| \, \|X\|^4}{\|X\|^2} \le \|X\|^3 < \varepsilon \Rightarrow \boxed{\|X\| < \sqrt[3]{\varepsilon}}$$

**(II)** $0 < \|X - X_0\| < \delta$

$$0 < \|X - (0,0)\| < \delta$$

$$\boxed{0 < \|X\| < \delta}$$

$\Rightarrow$ Tomando $\boxed{\delta < \sqrt[3]{\xi}}$ se cumple la definición y el limite existe

Hallar el límite doble en los puntos indicados

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x-3y+1}{x+y-2}$ Rta.: $-1/2$

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x-y}$ Rta.: limites iterados $\neq$

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ *Rta.*: limites iterados = limite radial depende de "m"

4) $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2$ *Rta*: limites iterados = limite radial depende de "m"

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$ *Rta.*: limites iterados = limite radial depende de "m"

6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4-y^2}$ *Rta.*: limites iterados = limite radial depende de "m"

7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)^2}{x^4+y^4}$ *Rta.*: limites iterados = limite radial depende de "m"

8) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$ *Rta.*: 0 *por def.* $\delta=\varepsilon$

9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y^2}{x^2+y^2}$ *Rta.*: 0 *por def.* $\delta=\sqrt{\varepsilon/2}$

**De Parciales:**

(I) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^3}{x^2+y^2}$

(II) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3|y|}{x^2+y^2}$

(III) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3y}{\sqrt{4x^2+4y^2}}$