Weak solutions of the skeleton equation

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I. The kinetic formulation of the skeleton equation

The skeleton equation: in the case of the zero range process,

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T),$$

for an L^2-control $g \in (L^2_{t,x})^d$. We specialize to the case, for some $\alpha \in (0,\infty)$,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

The kinetic formulation: for $\chi = \mathbf{1}_{\{0<\xi<\rho\}}$,

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

for a locally finite, nonnegative measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ with

$$q \geq \delta_\rho (\alpha \xi^{\alpha-1} |\nabla \rho|^2).$$

We have that, for $\psi \in \mathrm{C}^\infty_c(\mathbb{T}^d \times (0,\infty))$,

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \psi(x,\xi) \chi \Big|_{r=0}^{r=t} = & -\int_0^t \int_{\mathbb{T}^d} \alpha \rho^{\alpha-1} \nabla \rho \cdot (\nabla \psi)(x,\rho) - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\xi \psi q \\ & + \int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla \psi)(x,\rho) + \int_0^t \int_{\mathbb{T}^d} (\partial_\xi \psi)(x,\rho) \rho^{\frac{\alpha}{2}} g \cdot \nabla \rho. \end{aligned}$$

I. The kinetic formulation of the skeleton equation

Well-posedness of renormalized kinetic solutions [F., Gess; 2023]

Let $\rho_0 \in L^1(\mathbb{T}^d)$ be nonnegative and $g \in (L^2_{t,x})^d$. Then, there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \text{ in } \mathbb{T}^d \times (0,T) \text{ with } \rho(\cdot,0) = \rho_0.$$

Furthermore, if ρ_1 and ρ_2 are two solutions with initial data $\rho_{1,0}$ and $\rho_{2,0}$, then

$$\max_{t\in[0,T]} \|\rho_1(x,t) - \rho_2(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

The entropy estimate: we have that

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho\log(\rho) + \int_0^T\int_{\mathbb{T}^d} |\nabla\rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_0\log(\rho_0) + \int_0^T\int_{\mathbb{T}^d} |g|^2.$$

An interpolation inequality: we have that

$$\|\rho^{\frac{\alpha}{2}}\|_{L^2_t L^2_x} \lesssim \|\rho_0\|^{\alpha}_{L^1_x} + \|\nabla\rho^{\frac{\alpha}{2}}\|_{L^2_t L^2_x}.$$

The skeleton equation: we have that

$$\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) = 2\nabla\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}) - \nabla\cdot(\rho^{\frac{\alpha}{2}}g).$$

I. The kinetic formulation of the skeleton equation

Weak solutions of the skeleton equation [F., Gess; 2023]

A weak solution is a nonnegative $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ that satisfies:

The entropy estimate: we have

$$\rho^{\frac{\alpha}{2}} \in L^2([0,T];H^1(\mathbb{T}^d)).$$

The equation: for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \rho(x,s)\psi(x)\Big|_{s=0}^{s=t} = -2\int_0^t\int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}\cdot\nabla\psi + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot\nabla\psi.$$

The kinetic formulation: for $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d\times(0,\infty))$,

$$\begin{aligned}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\psi(x,\xi)\chi\Big|_{r=0}^{r=t} = &-\int_0^t\int_{\mathbb{T}^d}\alpha\rho^{\alpha-1}\nabla\rho\cdot(\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\partial_\xi\psi q\\ &+\int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot(\nabla\psi)(x,\rho) + \int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}}g\cdot\nabla\rho.\end{aligned}$$

Weak-strong continuity: does a *weakly convergent* sequence $g_n \rightharpoonup g \in (L^2_{t,x})^d$ induce a *strongly convergent* sequence $\rho_n \to \rho \in L^1_{t,x}$?

Burger's equation: in one-dimension,

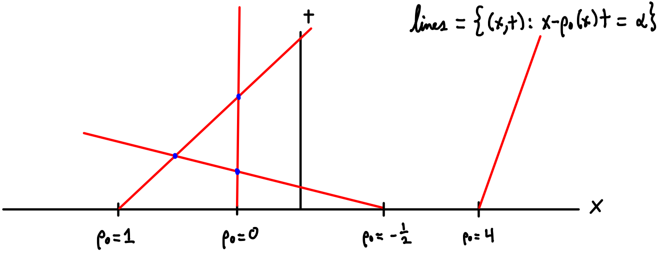
$$\partial_t \rho + \partial_x \left(\frac{1}{2} \rho^2 \right) = \partial_t \rho + \rho \partial_x \rho = 0.$$

The characteristics: In this case, $A'(\rho) = \rho$ and the characteristic equations are

$$\dot{X}_t^x = A'(\rho_0(x)) = \rho_0(x) \ \text{ with } \ X_t^x = x + \rho_0(x)t.$$

We therefore have, for the inverse characteristics Y_t^x,

$$Y_t^x = x - \rho_0(x)t \ \text{ and } \ \rho(x,t) = \rho_0(x - \rho_0(x)t).$$



Burger's equation: in one-dimension,

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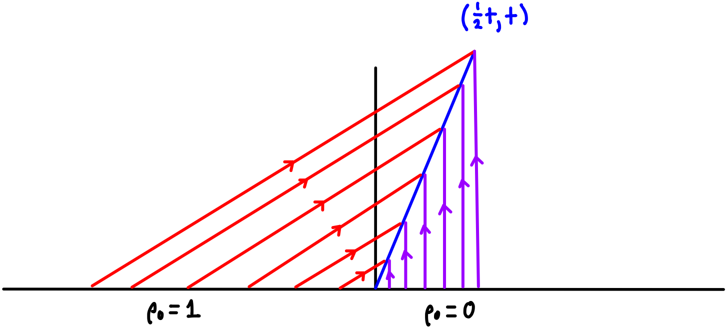
Shock vs. Rarefaction Wave:



II. Entropy solutions

The regularized equation: for $\eta \in (0,1)$, the equation

$$\partial_t \rho_\eta - \eta \Delta \rho_\eta + \frac{1}{2}\nabla \cdot (\rho_\eta)^2 = 0 \text{ in } \mathbb{T}^d \times (0,\infty) \text{ with } \rho_\eta(\cdot,0) = \rho_0.$$

A selection principle as $\eta \to 0$: if S is convex and ϕ is nonnegative,

$$\begin{aligned}\partial_t \Big(\int_{\mathbb{T}^d} \phi(x) S(\rho_\eta) \Big) &= \eta \int_{\mathbb{T}^d} \phi(x) S'(\rho_\eta) \Delta \rho_\eta - \frac{1}{2} \int_{\mathbb{T}^d} \phi(x) S'(\rho_\eta) \nabla (\rho_\eta)^2 \\ &= -\eta \int_{\mathbb{T}^d} S'(\rho_\eta) \nabla \rho_\eta \cdot \nabla \phi - \int_{\mathbb{T}^d} \phi S''(\rho_\eta) |\nabla \rho_\eta|^2 + \int_{\mathbb{T}^d} \beta(\rho_\eta) \nabla \phi,\end{aligned}$$

for $\beta(0) = 0$ and $\beta'(\xi) = S'(\xi)\xi$.

The entropy inequality: as $\eta \to 0$, if $\rho_\eta \to \rho$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \phi(x) S(\rho) \Big) \leq \int_{\mathbb{T}^d} \beta(\rho_\eta) \nabla \phi,$$

or, in the sense of distributions,

$$\partial_t S(\rho) + \nabla \cdot \beta(\rho) \leq 0.$$

— an ensemble of equations for all "entropy-flux pairs" (S, β)

Inviscid and Viscous Burger's equations: in one-dimension,

$$\partial_t \rho + \rho \partial_x \rho = 0 \;\text{ and }\; \partial_t \rho_\eta + \rho_\eta \partial_x \rho_\eta = \eta \Delta \rho_\eta,$$

and we have, for the inverse characteristics Y_t^x,

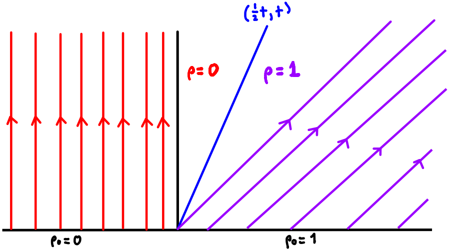
$$Y_t^x = x - \rho_0(x)t \;\text{ and }\; \rho(x,t) = \rho_0(x - \rho_0(x)t).$$

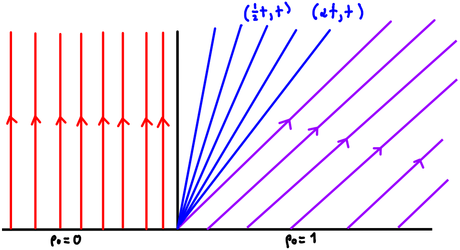
The kinetic formulation: for the kinetic function $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$,

$$\partial_t \chi_\eta + \xi \nabla_x \chi_\eta = \eta \Delta \chi_\eta + \partial_\xi q_\eta,$$

and for a *nonnegative* "defect measure" $q = \lim_{\eta\to 0} \eta |\nabla \rho_\eta|^2$ and $q_\eta = \eta |\nabla \rho_\eta|^2$.

Shock vs. Rarefaction Wave:





DiPerna–Lions theory [DiPerna, Lions; 1989], [Ambrosio; 2004]

Let $b \in \left(L^1_t L^\infty_x \cap L^1_t BV_x\right)^d$. Then, for every $\rho_0 \in L^\infty(\mathbb{T}^d)$ the continuity equation

$$\partial_t \rho = \nabla \cdot (\rho b),$$

has a unique weak solution in $(L^1 \cap L^\infty)(\mathbb{T}^d \times [0,T])$.

Relaxed assumptions: a one-sided bounded on $\nabla \cdot b$ is sufficient [Ambrosio; 2004]

Optimality: counterexamples for b failing to be BV on hyperplane [Depauw; 2003]

Commutator estimates: for a weak solution ρ, for $\rho_\varepsilon = (\rho * \kappa_\varepsilon)$,

$$\partial_t \int_{\mathbb{T}^d} S(\rho_\varepsilon)\phi(x) = \int_{\mathbb{T}^d} \phi(x) S'(\rho_\varepsilon)\nabla \cdot (\rho b)_\varepsilon \simeq \int_{\mathbb{T}^d} \phi(x) S'(\rho_\varepsilon)\nabla \cdot (\rho_\varepsilon b).$$

The skeleton equation: for $g \in (L^2_{t,x})^d$, for any $m \in (0,\infty)$,

$$\partial_t \rho = \Delta \rho^m - \nabla \cdot (\rho^{\frac{m}{2}} g).$$

Equivalence of weak and kinetic solutions: for initial data with finite entropy, is a weak solution

$$\int_{\mathbb{T}^d} \rho(x,s)\psi(x)\Big|_{s=0}^{s=t} = -2\int_0^t\int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}\cdot\nabla\psi + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot\nabla\psi,$$

a kinetic solution

$$\int_{\mathbb{T}^d}\chi\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d}\alpha\rho^{\alpha-1}\nabla\rho\cdot(\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\partial_\xi\psi)(x,\xi)q$$
$$+\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho + \int_0^T\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot(\nabla\psi)(x,\rho)?$$

Deriving the kinetic form: for $\partial_\xi\Psi(x,\xi)=\psi(x,\xi)$, for $\rho_\varepsilon=(\rho*\kappa^\varepsilon)$,

$$\begin{aligned}
&\partial_t\int\Psi(x,\rho_\varepsilon) = \int\psi(x,\rho_\varepsilon)\partial_t\rho_\varepsilon\\
&= -2\int(\nabla\psi)(x,\rho_\varepsilon)\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}})_\varepsilon - \int(\nabla\psi)(x,\rho_\varepsilon)\cdot(\rho^{\frac{\alpha}{2}} g)_\varepsilon\\
&\quad -2\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho_\varepsilon)\nabla\rho_\varepsilon\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}})_\varepsilon - \int(\partial_\xi\psi)(x,\rho_\varepsilon)\nabla\rho_\varepsilon\cdot(\rho^{\frac{\alpha}{2}} g)_\varepsilon.
\end{aligned}$$

II. Weak solutions of the skeleton equation

The equation satisfied by the convolution: for $\rho = (\rho * \kappa^{\varepsilon})$,

$$\partial_t \rho_{\varepsilon} = -(2\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} * \nabla \kappa^{\varepsilon}) + (\rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}).$$

Deriving the kinetic form: if ρ is a weak solution, for $\partial_{\xi} \Psi(x,\xi) = \psi(x,\xi)$,

$$\partial_t \int_{\mathbb{T}^d} \Psi(x,\rho_{\varepsilon}) = -2 \int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} * \nabla \kappa^{\varepsilon}) + \int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}).$$

A useful decomposition: let $\mathrm{Supp}(\psi) \subseteq \mathbb{T}^d \times [0,M]$ and let

$$A_k = \{(x,t) \colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + k\} \text{ and let } A_0 = (\mathbb{T}^d \times [0,T]) \setminus A_1.$$

We then write, for $\mathbf{1}_k = \mathbf{1}_{A_{k+1} \setminus A_k}$,

$$\begin{aligned}
&\int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}) \\
&= \int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\mathbf{1}_{A_0} \rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}) + \sum_{k=1}^{\infty} \int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\mathbf{1}_k \rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}) \\
&\lesssim \int_{\mathbb{T}^d} \psi(x,\rho_{\varepsilon})(\mathbf{1}_{A_0} \rho^{\frac{\alpha}{2}} g * \nabla \kappa^{\varepsilon}) \\
&\quad + \varepsilon^{-1} \|g\|_{L^2_{t,x}} \Big(\sum_{k=1}^{\infty} (M^{\frac{\alpha}{2}} + k + 1)^2 \int_{\mathbb{T}^d} |\psi(x,\rho_{\varepsilon})| \big(\mathbf{1}_{A_k} * |\varepsilon \nabla \kappa^{\varepsilon}|\big) \Big)^{\frac{1}{2}}.
\end{aligned}$$

II. Weak solutions of the skeleton equation

Equivalence of weak and renormalized kinetic solutions [F., Gess; 2023]

Under assumptions including $\Phi(\xi) = \xi^m$ for every $m \in [1, \infty)$, a nonnegative function $\rho \in \mathrm{C}([0,T]; L^1(\mathbb{T}^d))$ that satisfies

$$\Phi^{\frac{1}{2}}(\rho) \in L^2([0,T]; H^1(\mathbb{T}^d))$$

is a renormalized kinetic solution of the skeleton equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot, 0) = \rho_0,$$

for a nonnegative ρ_0 with finite entropy if and only if ρ is a weak solution. In particular, weak solutions exist and are unique.

— equivalence of renormalized and weak solutions [DiPerna, Lions; 1989], [Ambrosio; 2004].

— strong continuity with respect to weak convergence of the control

— for example, $\Phi^{\frac{1}{2}}$ convex or concave or Φ satisfies that $0 < \lambda \leq \Phi' \leq \Lambda$.

II. Weak solutions of the skeleton equation

Weak-strong continuity [F., Gess; 2023]

If ρ_n are solutions of the skeleton equation with controls $g_n \rightharpoonup g$ and initial data $\rho_{0,n} \rightharpoonup \rho_0$ with uniformly bounded entropy, then $\rho_n \to \rho$ for ρ the solution of the skeleton equation with control g and initial data ρ_0.

The entropy estimate: if $\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} g_n)$ then

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_n \log(\rho_n) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho_n^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_{0,n} \log(\rho_{0,n}) + \int_0^T \int_{\mathbb{T}^d} |g_n|^2.$$

Compactness since the g_n are uniformly $(L^2_{t,x})^d$-bounded,

$$\rho_n \text{ is strongly compact in } L^1_{t,x} \text{ and } \rho_n^{\frac{\alpha}{2}} \text{ is weakly compact in } L^2_t H^1_x.$$

Uniqueness of the limit: We have for some ρ that, along a subsequence,

$$\rho_n \to \rho \text{ in } L^1_{t,x} \text{ and } \rho_n^{\frac{\alpha}{2}} \rightharpoonup \rho^{\frac{\alpha}{2}} \text{ in } L^2_t H^1_x,$$

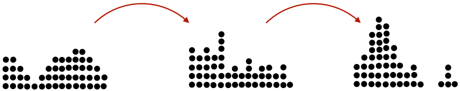
from which we conclude that

$$\partial_t \rho = 2\nabla \cdot (\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}}) - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

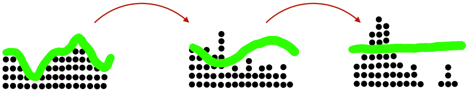
and that $\rho_n \to \rho$ along the full sequence $n \to \infty$.

III. L.s.c. envelope of the rate function

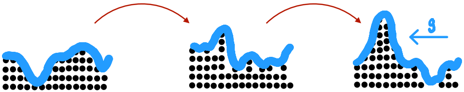
The zero range process: μ^N on $\mathbb{T}^1 \times [0,T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \overline{\rho} = \Delta \overline{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} \cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-NI(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g)\}.$$

The rate function: we have that

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2(\mathbb{T}^d \times [0,T];\mathbb{R}^d)} \colon \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} \cdot g) \right\}.$$

Large Deviations Principle [Benois, Kipnis, Landim; 1995]

For every closed $A \subseteq D([0,T]; \mathcal{M}_+(\mathbb{T}^d))$,

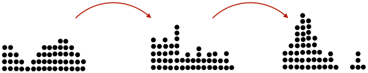
$$\limsup_{N \to \infty} \frac{1}{N} \log \left(\mathbb{P}(\mu^N \in A) \right) \leq - \inf_{m \in A} I(m).$$

For the space of smooth fluctuations

$$\mathcal{S} = \{ \partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0,T]) \},$$

for every open subset $A \subseteq D([0,T]; \mathcal{M}_+(\mathbb{T}^d))$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \left(\mathbb{P}(\mu^N \in A) \right) \geq - \inf_{\rho \in A} \overline{I(\rho)|_{\mathcal{S}}}^{\mathrm{lsc}}.$$



IV. L.s.c. envelope of the rate function

The rate function: we have

$$I(\rho) = \frac{1}{2}\inf\left\{\|g\|_{L^2}^2 : \partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g)\right\}.$$

The Hilbert space: $H^1_{\rho^\alpha}$ is the strong closure w.r.t. the inner product

$$\langle\nabla\psi, \nabla\phi\rangle = \int_0^T\int_{\mathbb{T}^d}\rho^\alpha\nabla\psi\cdot\nabla\phi \;\text{ for }\; \phi, \psi \in \mathrm{C}^\infty.$$

Unique minimizer: the equation defines

$$\partial_t\rho - \Delta\rho^\alpha = -\nabla\cdot(\rho^{\frac{\alpha}{2}}g) \in H^{-1}_{\rho^\alpha},$$

and if $I(\rho) < \infty$ then the minimizer $g = \rho^{\frac{\alpha}{2}}\nabla H$ for $H \in H^1_{\rho^\alpha}$ and

$$I(\rho) = \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\rho^\alpha|\nabla H|^2 = \frac{1}{2}\|H\|^2_{H^1_{\rho^\alpha}} = \frac{1}{2}\|\partial_t\rho - \Delta\Phi(\rho)\|^2_{H^{-1}_{\rho^\alpha}}.$$

The "ill-posed" equation: we have the formally "supercritical" equation

$$\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^\alpha\nabla H).$$

IV. L.s.c envelope of the rate function

The space of smooth fluctuations: we have that

$$\mathcal{S} = \{\partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0,T])\}.$$

The recovery sequence: given an arbitrary fluctuation

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

to show that $\overline{I|_{\mathcal{S}}}^{lsc}(\rho) = I(\rho)$ we need to find a sequence $\rho_n \in \mathcal{S}$ such that

$$\rho_n \to \rho \in L^1_{t,x} \;\text{ and }\; I(\rho_n) \to I(\rho).$$

A first attempt: there exists $H \in H^1_{\rho^\alpha}$ such that

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^\alpha \nabla H) \;\text{ and }\; I(\rho) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho^\alpha |\nabla H|^2.$$

Let ρ_ε solve

$$\partial_t \rho_\varepsilon = \Delta \rho_\varepsilon^\alpha - \nabla \cdot (\rho_\varepsilon^\alpha (\nabla H * \kappa_\varepsilon)).$$

- supercritical with no stable estimates with respect to ∇H
- the Hilbert space framework is too rigid

A second attempt: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \ \text{ with } \ \rho(\cdot, 0) = \rho_0.$$

Regularizing the data: we consider

$$\rho_{0,n} = \Big((\rho_0 \wedge n) \vee \frac{1}{n} \Big) * \kappa_x^{\frac{1}{n}} \ \text{ and } \ g_n = g * \kappa_{t,x}^{\frac{1}{n}},$$

and solve

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} g_n) \ \text{ with } \ \rho_n(\cdot, 0) = \rho_{0,n}.$$

There exists $H_n \in H^1_{\rho_n^\alpha}$ such that

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^\alpha \nabla H_n) \ \text{ with } \ \rho_n(\cdot, 0) = \rho_{0,n}.$$

Deducing the regularity of H_n: we have the elliptic equation

$$-\nabla \cdot (\rho_n^\alpha \nabla H_n) = \partial_t \rho_n - \Delta \rho_n^\alpha.$$

— is not necessarily uniformly elliptic

— how regular is ρ_n?

IV. L.s.c envelope of the rate function

The final attempt: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \ \text{ with } \ \rho(\cdot, 0) = \rho_0.$$

Regularizing the data: we consider

$$\rho_{0,n} = \Big((\rho_0 \wedge n) \vee \frac{1}{n}\Big) * \kappa_x^{\frac{1}{n}} \ \text{ and } \ g_n = g * \kappa_{t,x}^{\frac{1}{n}},$$

"Turning off" the control: for $\sigma_n(\xi) = 0$ if $\xi \leq \frac{1}{n}$ or $\xi \geq n$, solve

$$\begin{aligned} \partial_t \rho_n &= \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \sigma_n(\rho_n) g_n) \\ &= \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \tilde{g}_n), \end{aligned}$$

for the control $\tilde{g}_n = \sigma_n(\rho_n) g_n$.

Regularity of ρ_n: we have that $\frac{1}{n} \leq \rho_n \leq n$ and $\rho_n \in \mathrm{C}^\infty(\mathbb{T}^d \times [0, T])$.

Deducing the regularity of H_n: There exists $H_n \in H^1_{\rho_n^\alpha}$ such that

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^\alpha \nabla H_n) \ \text{ and } \ -\nabla \cdot (\rho_n^\alpha \nabla H_n) = \partial_t \rho_n - \Delta \rho_n^\alpha.$$

The fluctuation: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \text{ with } \rho(\cdot, 0) = \rho_0.$$

The recovery sequence: for $\sigma_n(\xi) = 0$ if $\xi \leq \frac{1}{n}$ or $\xi \geq n$, solve

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \sigma_n(\rho_n) g_n) = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \tilde{g}_n),$$

for the control $\tilde{g}_n = \sigma_n(\rho_n) g_n$ and with $\rho_n(\cdot, 0) = \rho_{0,n}$.

Compactness: the ρ_n satisfy uniformly the entropy estimate and

$$\rho_n \to \rho \text{ and } \sigma(\rho_n) g_n \mathbf{1}_{\{\rho > 0\}} \to g \mathbf{1}_{\{\rho > 0\}} \text{ and } I(\rho_n) \leq \|\sigma(\rho_n) g_n\|_2^2 \to \|g\|_2^2.$$

Large deviations of the zero range process [F., Gess; 2023]

For the space of smooth fluctuations

$$\mathcal{S} = \{\partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0, T])\},$$

we have that

$$\overline{I(\rho)\big|_{\mathcal{S}}}^{\mathrm{lsc}} = I(\rho) = \frac{1}{2} \inf\{\|g\|_2^2 \colon \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)\}.$$

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