Non-equilibrium Fluctuations of Interacting Particle Systems

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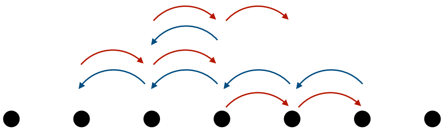
Let $\{X_i\}_{i\in\mathbb{N}}$ be independent coin flips.

That is, $\{X_i\}_{i\in\mathbb{N}}$ are independent random variables with

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = {}^1\!/_2.$$

The simple random walk $(S_n)_{n\in\mathbb{N}_0}$ is defined by $S_0 = 0$ and

$$S_n = X_1 + \ldots + X_n.$$



A realization of S_{11}.

I. The simple random walk

If T is the random time

$$T = \inf\{n \in \mathbb{N} \colon S_n = 1\},$$

then $T < \infty$ almost surely but

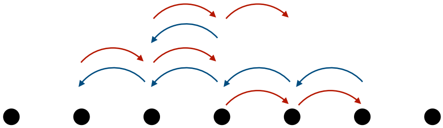
$$\mathbb{E}[T] = \infty.$$

If T_N is the stopping time

$$T_N = \inf\{n \in \mathbb{N} \colon S_n = 1 \;\text{ or }\; S_n = -N\},$$

then

$$\mathbb{P}[S_{T_N} = -N] = \frac{1}{N+1}.$$



A simple random walk.

The Law of Large Numbers: the large scale limit

$$\lim_{n\to\infty} \frac{X_1+\ldots+X_n}{n} = \mathbb{E}[X_1] = 0.$$

The Central Limit Theorem: for every $a \leq b \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\Big[a \leq \frac{X_1+\ldots+X_n}{\sqrt{n}} \leq b\Big] = \int_a^b (2\pi)^{-\frac{1}{2}} \exp\big(-\frac{x^2}{2}\big)\,\mathrm{d}x.$$

Approximate Large Deviations Principle: as $n\to\infty$, for every $\delta\in(0,\infty)$,

$$\begin{aligned}\mathbb{P}\Big[\frac{X_1+\ldots+X_n}{n} \geq \delta\Big] &= \mathbb{P}\Big[\frac{X_1+\ldots+X_n}{\sqrt{n}} \geq \sqrt{n}\delta\Big] \\ &\simeq \int_{\sqrt{n}\delta}^{\infty} (2\pi)^{-1} \exp\big(-\frac{x^2}{2}\big)\,\mathrm{d}x \\ &\simeq \exp\big(-\frac{n\delta^2}{2}\big).\end{aligned}$$

— $(X_1+\ldots+X_n)$ is expected to be of order $\sqrt{n}$

I. The simple random walk

Large deviations principle: a sequence of random variables $X_n\colon \Omega \to \mathbb{R}$ satisfy a large deviations principle with rate function $I\colon \mathbb{R} \to [0,\infty]$ if, for every $A \subseteq \mathbb{R}$,

$$-\inf_{x\in A^\circ} I(x) \leq \liminf_{n\to\infty} \frac{1}{n}\log\left(\mathbb{P}(X_n \in A)\right) \leq \limsup_{n\to\infty} \frac{1}{n}\log\left(\mathbb{P}(X_n \in A)\right) \leq -\inf_{x\in \overline{A}} I(x).$$

Informally, this means that, as $n \to \infty$,

$$\mathbb{P}(X_n \simeq x) \simeq e^{-nI(x)}.$$

The linear central limit expansion: as $n \to \infty$,

$$\begin{aligned}\frac{X_1+\ldots+X_n}{n} &\simeq \text{“law of large numbers”} + \text{“central limit correction”}\\ &= 0 + \frac{1}{\sqrt{n}}\cdot\mathcal{N}(0,1),\end{aligned}$$

for a normal random variable $\mathcal{N}(0,1)$ predicts that

$$\mathbb{P}\left[\frac{X_1+\ldots+X_n}{n} \geq \delta\right] \simeq e^{-n\tilde{I}(\delta)} \;\text{ for }\; \tilde{I}(\delta) = \frac{1}{2}\delta^2.$$

— although $\left|\frac{X_1+\ldots+X_n}{n}\right| \leq 1$!

I. The simple random walk

Cramér's theorem: for the rate function

$$I(x) = \begin{cases} \tanh^{-1}(x)x - \log\left(\frac{1}{2}\left(e^{-\tanh^{-1}(x)} + e^{\tanh^{-1}(x)}\right)\right) & \text{if } |x| \leq 1, \\ +\infty & \text{if } |x| > 1, \end{cases}$$

the random variables $\frac{X_1+\ldots+X_n}{n}$ satisfy the large deviations principle

$$\mathbb{P}\left[\frac{X_1+\ldots+X_n}{n} \geq \delta\right] \simeq e^{-nI(\delta)}.$$

The Large Deviations Principle: a Taylor expansion proves that

$$I(\delta) \simeq \frac{1}{2}\delta^2 + o(\delta^2) \text{ with } I(\pm 1) = \log(2) \text{ and } I(\delta) = \infty \text{ if } |\delta| > 1,$$

and therefore, as $n \to \infty$,

$$\mathbb{P}\left[\frac{X_1+\ldots+X_n}{n} \geq \delta\right] \simeq e^{-nI(\delta)} \simeq e^{-n\cdot\frac{\delta^2}{2}} e^{-n\cdot o(\delta^2)}.$$

— linear CLT expansion correctly predicts small fluctuations

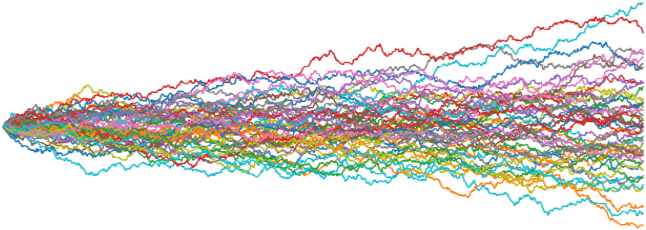
— nonlinear LDP captures large fluctuations

Brownian motion: The simple random walk

$$S_n = X_1 + \ldots + X_n \ \text{ and } \ W(t) = S_{\lfloor t \rfloor}.$$

The Brownian path

$$B(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} W(nt) \text{ in distribution on } \mathrm{C}([0,\infty)) \text{ (technically, } D([0,\infty))\text{)}.$$



Properties: (i) $B(0) = 0$, (ii) continuous sample paths, (iii) independent increments: $B(t) - B(s)$ is independent of $B(s)$, and (iv) normally distributed:

$$B(t) - B(s) \text{ has distribution } (2\pi(t-s))^{-\frac{1}{2}} \exp\Big(-\frac{|x|^2}{2(t-s)}\Big).$$

The rate function: let $I\colon\ \mathrm{C}([0,T]) \to [0,\infty]$ be defined by

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 \,\mathrm{d}t \ \text{ if } x \text{ is differentiable,}$$

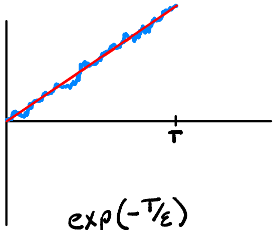
and $I(x) = \infty$ otherwise.

Schilder's theorem: for every $\varepsilon \in (0,1)$,

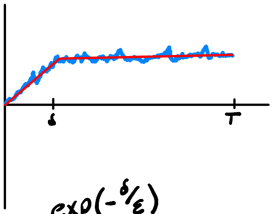
$$W^\varepsilon(t) = \sqrt{\varepsilon} B(t).$$

The paths $\{W^\varepsilon\}_{\varepsilon\in(0,1)}$ satisfy a large deviations principle on $\mathrm{C}([0,T])$:

$$\mathbb{P}(W^\varepsilon \in A) \simeq e^{-\left(\varepsilon^{-1}\inf_{x\in A} I(x)\right)}.$$



exp(−T/ε)



exp(−δ/ε)

The Ornstein–Uhlenbeck Process: We consider the solution

$$\mathrm{d}X_t^\varepsilon = -X_t^\varepsilon\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t.$$

The Controlled ODE: for a "control" $x(t) \in H^1([0,T])$, we solve

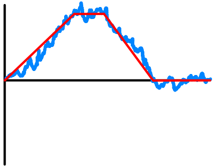
$$\mathrm{d}y_t = -y_t\,\mathrm{d}t + \dot{x}_t\,\mathrm{d}t,$$

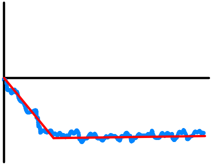
and define the large deviations rate function

$$I(y) = \frac{1}{2}\inf\Big\{\int_0^T |\dot{x}(t)|^2\,\mathrm{d}t\colon\ \mathrm{d}y_t = -y_t\,\mathrm{d}t + \dot{x}_t\,\mathrm{d}t\Big\}.$$

The Freidlin–Wentzell Theorem: we have the large deviations principle

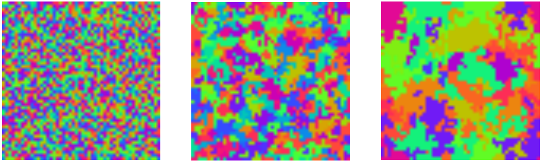
$$\mathbb{P}\big(X^\varepsilon \in A\big) \simeq e^{-\left(\varepsilon^{-1}\inf_{y\in A} I(y)\right)}.$$





III. Interacting particle systems

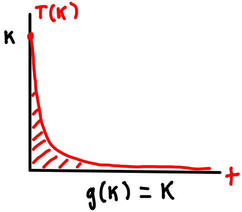
- Statistical physics
  - zero range process
  - Ising and Potts models
- Belief/infection propagation
  - voter model
  - contact process
- Traffic models
  - exclusion processes
- Neural networks as interacting particle systems

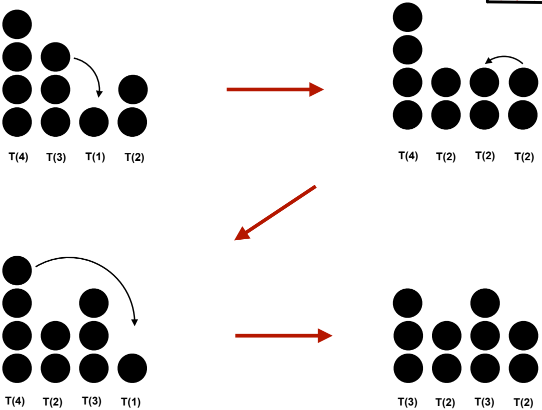


The voter model [Swart; 2020]

- Let $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ be nondecreasing
  — $g(0) = 0$ and $g(k) > 0$ if $k \neq 0$
- Independent random clocks $T(k)$ with distribution

$$T(k) \sim g(k)\exp(-g(k)t) \;\text{ on }\; [0,\infty).$$





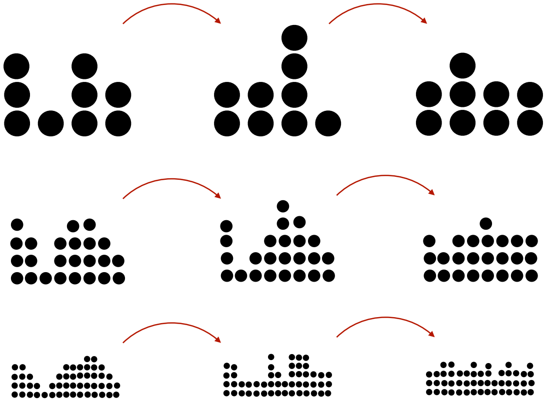
The zero range process

III. Interacting particle systems

The generator: for a compactly supported p with zero mean $\sum_{z\in\mathbb{Z}^d} zp(z) = 0$,

$$(\mathcal{L}_N f)(\eta) = \sum_{x,z\in\mathbb{T}^d_N} p(z)g(\eta(x))\big(f(\eta^{x,x+z}) - f(\eta)\big).$$

A rescaling: a zero range process defined on $(\mathbb{Z}^d/N\mathbb{Z}^d)$ rescaled in space and time,



The cases $N = 4, 8, 15$.

The zero range process η_t^N on $(\mathbb{Z}^d/N\mathbb{Z}^d)$, and the empirical density

$$\mu_t^N = \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{N^2 t}^N(x).$$

Hydrodynamic limit [Ferrari, Presutti, Vares; 1988]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N \rangle - \langle f, \overline{\rho} \rangle| > \delta \right] = 0,$$

for $\langle f, \mu^N \rangle = \int f \mu^N$, and for $\overline{\rho}$ the solution

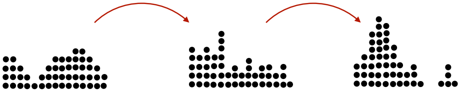
$$\partial_t \overline{\rho} = \Delta \Phi(\overline{\rho}),$$

for the mean local jump rate Φ [Kipnis, Landim; 1999].

— If $T(k) \sim e^{-t}$ then $\partial_t \overline{\rho} = \Delta\left(\frac{\overline{\rho}}{1+\overline{\rho}}\right)$.

— If $T(k) \sim k e^{-kt}$ then $\partial_t \overline{\rho} = \Delta \overline{\rho}$.

III. Interacting particle systems



A space-time white noise: a d-dimensional Gaussian noise $\mathrm{d}\xi$ satisfying

$$\mathbb{E}\left[\,\mathrm{d}\xi(x,t)\,\mathrm{d}\xi(y,s)\right] = \delta_0(x-y)\delta_0(s-t).$$

On the torus $\mathbb{T}^d$, we have the spectral representation

$$\xi = \sum_{k\in\mathbb{Z}^d}\left(\sin(2\pi k\cdot x)B^k_t + \cos(2\pi k\cdot x)W^k_t\right)$$

for independent d-dimensional Brownian motions B^k and W^k.

Analogue of Schilder's theorem: we have that, for $A\subseteq \mathrm{C}([0,T];H^{-\frac{d+1}{2}}(\mathbb{T}^d)^d)$,

$$\mathbb{P}\left[\sqrt{\varepsilon}\xi\in A\right]\simeq e^{-\left(\varepsilon^{-1}\inf_{\gamma\in A}I(\gamma)\right)}$$

for the large deviations rate function

$$I(\gamma)=\frac{1}{2}\inf\left\{\,\|g\|^2_{L^2_{t,x}}:\gamma(x,t)=\int_0^t g(x,s)\,\mathrm{d}s\right\}.$$

— formally $g=\frac{\partial\gamma}{\partial t}$ as L^2-valued processes

The zero range process with nonzero mean: let η_t^N be the zero range process on $\mathbb{T}_N^d$ with transition kernel p satisfying $\sum_{z\in\mathbb{Z}^d} zp(z) = \gamma$.

The hyperbolic rescaling: let μ_t^N be the hyperbolically rescaled density

$$\mu_t^N = \frac{1}{N^d} \sum_{x\in(\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{Nt}^N(x).$$

Hydrodynamic limit [Rezakhanlou; 1991]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N\rangle - \langle f, \overline{\rho}\rangle| > \delta \right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of the equation

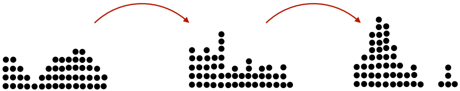
$$\partial_t \overline{\rho} = \nabla \cdot (\Phi(\overline{\rho})\gamma),$$

for the mean local jump rate Φ [Kipnis, Landim; 1999].

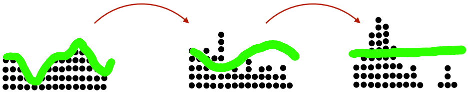
Mobility: the mobility of the zero range process is $m(\overline{\rho}) = \Phi(\overline{\rho})$

III. Interacting particle systems

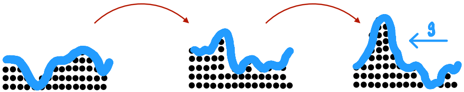
The zero range process: μ^N on $\mathbb{T}^1 \times [0,T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t\overline{p} = \Delta\overline{\rho}$,



The skeleton equation: the controlled equation $\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho}\cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-NI(\rho))$ for

$$I(\rho) = \frac{1}{2}\inf\{\|g\|^2_{L^2_{t,x}} : \partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho}g)\}.$$

The hydrodynamic limit: for the parabolically rescaled, mean zero particle process μ_t^N on $\mathbb{T}_N^d$, as $N \to \infty$, for $J(\overline{\rho}) = \nabla\sigma(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \ \text{ for } \ \partial_t\overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla\cdot J(\overline{\rho}).$$

Macroscopic fluctuation theory: for a space-time fluctuation (ρ, j) satisfying

$$\partial_t\rho = \nabla\cdot j, \quad \left(\partial_t\int_U \rho = \int_{\partial U} j\cdot\nu\right)$$

we have the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp\big(-NI(\rho)\big) \ \text{ for } \ I(\rho) = \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}(j - J(\rho))\cdot m(\rho)^{-1}(j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho) = \int_0^T\int_{\mathbb{T}^d}|g|^2$ and

$$\partial_t\rho = \nabla\cdot\big(J(\rho) + (j - J(\rho))\big) = \Delta\sigma(\rho) - \nabla\cdot(\sqrt{m(\rho)}g).$$

The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g).$$

Large Deviations Principle [Benois, Kipnis, Landim; 1995], [F., Gess; 2023]

For every measurable $A \subseteq L^1([0,T];L^1(\mathbb{T}^d))$ or $A \subseteq \mathrm{C}([0,T];\mathcal{M}_+(\mathbb{T}^d))$,

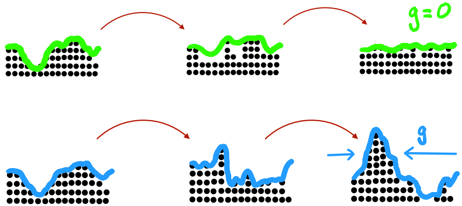
$$\mathbb{P}\left[\mu^N \in A\right] \simeq e^{-N\left(\inf_{\pi\in A} I(\pi)\right)},$$

for the large deviations rate function

$$I(\pi) = \inf\left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \pi = \Delta\Phi(\pi) - \nabla\cdot\left(\Phi^{\frac{1}{2}}(\pi)g\right)\right\}.$$

The linearized skeleton equation: for $\partial_t\overline{\rho} = \Delta\Phi(\overline{\rho})$,

$$\partial_t \pi = \Delta\Phi(\pi) - \nabla\cdot\left(\Phi^{\frac{1}{2}}(\overline{\rho})g\right).$$



The rate function: for $\rho \in L^1(\mathbb{T}^d \times [0,T])$,

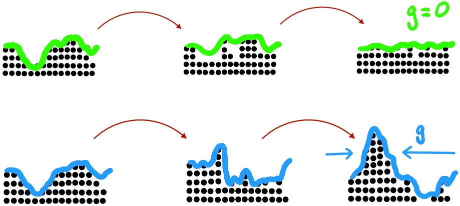
$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

The skeleton equation: for controls $g \in L^2(\mathbb{T}^d \times [0,T])^d$,

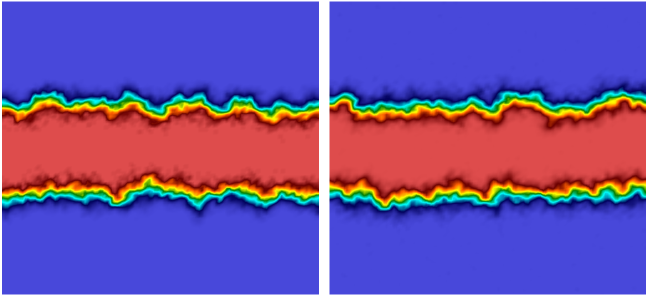
$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot, 0) = \rho_0.$$

— energy critical in L^1 and supercritical in L^p for $p \in (1, \infty)$,

— kinetic (renormalized) solutions



IV. Fluctuating hydrodynamics



a miscible mixture develops a rough diffusive interface [Donev; 2018]

Fluctuating hydrodynamics of the zero range process: the stochastic PDE [Spohn; 1991]

$$\partial_t \rho_\varepsilon = \Delta\Phi(\rho_\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho_\varepsilon) \circ \xi).$$

Large deviations: formally, the ρ_ε satisfy a large deviations principle

$$\mathbb{P}[\rho_\varepsilon \in A] \simeq e^{-\left(\varepsilon^{-1} \inf_{\rho \in A} I(\rho)\right)},$$

for $I(\rho) = \frac{1}{2}\inf\{\|g\|^2_{L^2_{t,x}} : \partial_t\rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)g)\}$.

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