

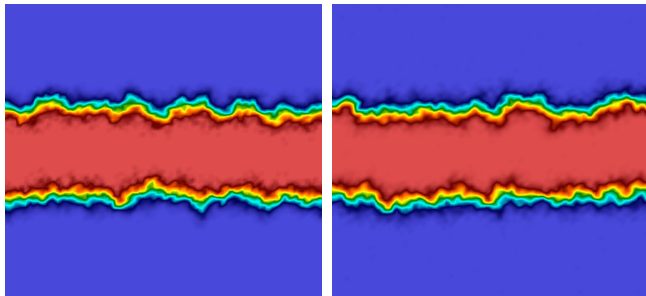
Stochastic PDEs of fluctuating hydrodynamics type

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I. Fluctuating hydrodynamics



thermal fluctuations create a rough diffusive interface in miscible fluids [Donev; 2018]

- A general conservative stochastic PDE

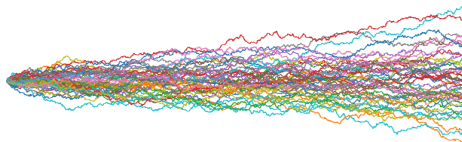
$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\sigma(\rho) d\xi) = \nabla \cdot (\Phi'(\rho) \nabla \rho - \sigma(\rho) d\xi),$$

for a d -dimensional space-time noise ξ .

- fluctuating hydrodynamics, [Spohn; 1991]
- macroscopic fluctuation theory, [Bertini et al.; 2014]

I. Fluctuating hydrodynamics

Brownian motion: continuous paths with independent, Gaussian increments



Itô integration: we have that, strongly in $L^2(\Omega)$,

$$\int_0^t f(B_s) dB_s = \lim_{|\mathcal{P}| \rightarrow 0} \sum f(B_{t_i})(B_{t_{i+1}} - B_{t_i}).$$

Alternate integration theories: for $\theta \in (0, 1)$,

$$\begin{aligned} \int_0^t f(B_s) \circ_{\theta} dB_s &= \lim_{|\mathcal{P}| \rightarrow 0} \sum (\theta f(B_{t_{i+1}}) + (1 - \theta)f(B_{t_i}))(B_{t_{i+1}} - B_{t_i}) \\ &\simeq \lim_{|\mathcal{P}| \rightarrow 0} \sum \theta f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 + \lim_{|\mathcal{P}| \rightarrow 0} \sum f(B_{t_i})(B_{t_{i+1}} - B_{t_i}) \\ &= \theta \int_0^t f'(s) ds + \int_0^t f(B_s) dB_s, \end{aligned}$$

where $\theta = 0$ is Itô, $\theta = 1/2$ is Stratonovich, and $\theta = 1$ is Klimontovich.

I. Fluctuating hydrodynamics

Itô's formula: for a Brownian motion B_t ,

$$f(B_{t_{i+1}}) - f(B_{t_i}) \simeq f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2.$$

After passing to the limit $\mathcal{P} \rightarrow 0$, for $\circ = \circ_{\frac{1}{2}}$,

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \mathrm{d}B_s + \frac{1}{2} \int_0^t f''(B_s) \mathrm{d}s = \int_0^t f'(B_s) \circ \mathrm{d}B_s,$$

or, in differential form,

$$\mathrm{d}f(B_s) = f'(B_s) \mathrm{d}B_s + \frac{1}{2}f''(B_s) \mathrm{d}s = f'(B_s) \circ \mathrm{d}B_s.$$

— Stratonovich integration satisfies the chain rule

I. Fluctuating hydrodynamics

The empirical density: let m_n denote the measure

$$m_n = \frac{1}{n} \sum_{k=1}^n \delta_{B_t^k}.$$

The derivation: for every $f \in C^\infty(\mathbb{T}^d)$,

$$\partial_t \left(\int_{\mathbb{T}^d} f(x) m_n \right) = \partial_t \left(\frac{1}{n} \sum_{k=1}^n f(B_t^k) \right) = \frac{1}{n} \sum_{k=1}^n \frac{1}{2} \Delta f(B_t^k) + \frac{1}{n} \sum_{k=1}^n \int_{\mathbb{T}^d} \nabla f(x) \delta_{B_t^k} \, dB_t^k.$$

The covariance: for points (x, t) and (y, s) ,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{n} \sum_{k=1}^n \delta_0(x - B_t^k) \, dB_t^k \cdot \frac{1}{n} \sum_{j=1}^n \delta_0(y - B_s^j) \, dB_s^j \right] &= \frac{1}{n^2} \delta_0(x - B_t^k) \delta_0(x - y) \delta_0(s - t) \\ &= \frac{1}{n} m_n(x, t) \delta_0(x - y) \delta_0(s - t). \end{aligned}$$

The Dean–Kawasaki equation: for $d\xi$ an \mathbb{R}^d -valued space-time white noise,

$$\partial_t \left(\int_{\mathbb{T}^d} f(x) m_n \right) = \frac{1}{2} \int_{\mathbb{T}^d} \Delta f m_n + \frac{1}{\sqrt{n}} \sum_{k=1}^n \int_{\mathbb{T}^d} \nabla f(x) \cdot \sqrt{m_n} \, d\xi.$$

Or, in the sense of distributions,

$$\partial_t m_n = \frac{1}{2} \Delta m_n - \frac{1}{\sqrt{n}} \nabla \cdot (\sqrt{m_n} \, d\xi).$$

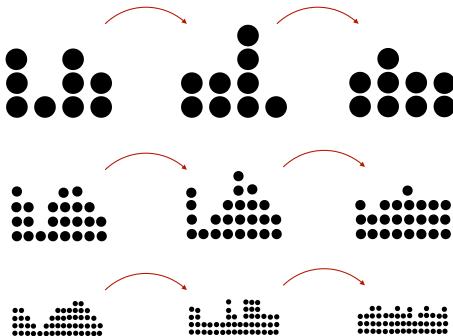
I. Fluctuating hydrodynamics

A zero range process: on $\mathbb{T}_N^d = (\mathbb{Z}^d / N\mathbb{Z}^d)$ with generator

$$(\mathcal{L}_N f)(\eta) = \sum_{x \in \mathbb{T}_N^d} \sum_{z \in \mathbb{T}_N^d} p_N(z) g(\eta(x)) (f(\eta^{x, x+z}) - f(\eta)),$$

The transition kernel: $p_N(z) = \sum_{y \in \mathbb{Z}^d} p(z + y_N)$ for a compactly supported p with zero mean $\sum_{z \in \mathbb{T}^d} zp(z) = 0$.

Parabolic rescalings: for $N = 4, 8, 15$,



I. Fluctuating hydrodynamics

Hydrodynamic limit: for a parabolically rescaled, centered particle process, as the particle number $N \rightarrow \infty$ [Ferrari et al.; 1988],

$$\mu_t^N \rightharpoonup \bar{\rho} dx \text{ for } \partial_t \bar{\rho} = \Delta \Phi(\bar{\rho}).$$

Sensitivity to fluctuations: if the jumps are biased in direction $\gamma \in \mathbb{R}^d$, the hyperbolically rescaled densities ν_t^N satisfy, as $N \rightarrow \infty$,

$$\nu_t^N \rightharpoonup \bar{v} dx \text{ for } \partial_t \bar{v} = \nabla \cdot (m(\bar{v})\gamma),$$

for the mobility $m(\bar{v})$ [Rezakhonlou; 1991].

The SPDE: the following formal SPDE, for a finite particle number N ,

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot \alpha \text{ for } \mathbb{E}[\alpha(x, t) \alpha(y, s)] = \frac{m(\rho)}{N^d} \delta_0(x - y) \delta_0(t - s).$$

Formally, for a space time white noise $d\xi$,

$$\partial_t \rho = \Delta \Phi(\rho) - N^{-\frac{d}{2}} \nabla \cdot (\sqrt{m(\rho)} d\xi).$$

Examples: for the zero range and symmetric simple exclusion processes

$$\partial_t \rho = \Delta \Phi(\rho) - N^{-\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) d\xi) \text{ and } \partial_t \rho = \Delta \rho - N^{-\frac{d}{2}} \nabla \cdot (\sqrt{\rho(1-\rho)} d\xi).$$

I. Fluctuating hydrodynamics

The zero range process: for the zero range process with finite particle number N ,

$$\partial_t \rho = \Delta \Phi(\rho) - N^{-\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) d\xi).$$

The Dean–Kawasaki equation: in the case of independent particles,

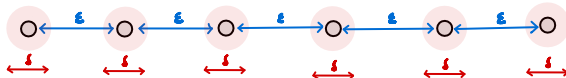
$$\partial_t \rho = \Delta \rho - N^{-\frac{d}{2}} \nabla \cdot (\sqrt{\rho} d\xi).$$

- supercritical in the language of singular SPDE
- formally derived by [Dean; 1996] and [Kawasaki; 1998] as the SPDE satisfied by

$$m_N(x, t) = \frac{1}{N^d} \sum_{i=1}^{N^d} \delta_0(B_t^i - x).$$

- ill-posedness vs. triviality [Konarovskiy, Lehmann, von Renesse; 2018]

- White noise is too singular (particles systems, course graining):



- Spatially correlated noise: $\xi^\delta = \xi * \kappa^\delta$ for convolution kernels $\{\kappa^\delta\}_{\delta \in (0,1)}$.

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The Dean–Kawasaki equation: with a spatially correlated noise ξ^δ , for ε the inverse particle number,

$$\begin{aligned}\partial_t \rho &= \Delta \rho - \varepsilon^{\frac{d}{2}} \nabla \cdot (\sqrt{\rho} d\xi^\delta) \\ &= \Delta \rho - \varepsilon^{\frac{d}{2}} (\nabla \sqrt{\rho}) \cdot d\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot d\xi^\delta) \\ &= \Delta \rho - \frac{\varepsilon^{\frac{d}{2}}}{2\sqrt{\rho}} \nabla \rho \cdot d\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot d\xi^\delta).\end{aligned}$$

Parabolicity and stochastic transport: for a Brownian motion B_t and $a, b \in \mathbb{R}$,

$$\partial_t v = a \Delta v + b \nabla v \cdot dB_t.$$

If \tilde{v} solves

$$\partial_t \tilde{v} = \left(a - \frac{b^2}{2}\right) \Delta \tilde{v} \quad \text{then} \quad v(x, t) = \tilde{v}(x + bB_t, t).$$

Parabolic and well-posed if and only if $\frac{b^2}{2} < a$.

I. Fluctuating hydrodynamics

The noise: if ξ^δ takes the form

$$\xi^\delta = \sum_{k=1}^{\infty} f_k(x) B_t^k \quad \text{then} \quad \langle \xi^\delta \rangle_1 = \sum_{k=1}^{\infty} f_k^2.$$

Probabilistic stationarity: we assume that

$$\langle \xi^\delta \rangle_1 \text{ is spatially constant with } \langle \nabla \cdot \xi^\delta \rangle_1 = \sum_{k=1}^{\infty} |\nabla f_k|^2 < \infty.$$

The Dean–Kawasaki equation: analyzing the parabolicity yields that

$$\begin{aligned} \partial_t \rho &= \Delta \rho - \frac{\varepsilon^{\frac{d}{2}}}{2\sqrt{\rho}} \nabla \rho \cdot d\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot d\xi^\delta) \\ &= \Delta \left(\rho - \frac{\varepsilon \langle \xi^\delta \rangle_1}{8} \log(\rho) \right) - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot d\xi^\delta) \end{aligned}$$

The corrected equation: for $\theta \in [0, \infty)$, we consider the corrected equation

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} d\xi^\delta) + \frac{\theta \langle \xi^\delta \rangle_1}{4} \Delta \log(\rho).$$

Itô integration is $\theta = 0$, Stratonovich is $\theta = 1/2$, and Kilmontovich is $\theta = 1$.

I. Fluctuating hydrodynamics

The Dean–Kawasaki equation with correlated Stratonovich noise:

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} d\xi^\delta) + \frac{\langle \xi^\delta \rangle_1}{8} \Delta \log(\rho).$$

The entropy formulation: for a convex S and a nonnegative ψ ,

$$\begin{aligned} \int_{\mathbb{T}^d} S(\rho) \psi \Big|_{s=0}^{s=t} &\leq - \int_0^t \int_{\mathbb{T}^d} (\nabla \rho + \frac{\langle \xi^\delta \rangle_1}{8} \nabla \log(\rho)) \cdot S'(\rho) \nabla \psi - \int_0^t \int_{\mathbb{T}^d} S''(\rho) \psi |\nabla \rho|^2 \\ &\quad - \int_0^t \int_{\mathbb{T}^d} S'(\rho) \psi \nabla \cdot (\sqrt{\rho} d\xi^\delta) + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \langle \nabla \cdot \xi^\delta \rangle_1 \rho S''(\rho) \psi. \end{aligned}$$

The kinetic formulation: for $\eta \in \mathbb{R}$ and a nonnegative measure q ,

$$\chi(x, t, \eta) = \mathbf{1}_{\{0 < \eta < \rho(x, t)\}} - \mathbf{1}_{\{\rho(x, t) < \eta < 0\}} \quad \text{and} \quad \delta_\rho |\nabla \rho|^2 \leq q,$$

for $\delta_\rho = \delta_0(\eta - \rho)$, we have the distributional equalities

$$\nabla \chi = \delta_0(\eta - \rho) \nabla \rho \quad \text{and} \quad \partial_\eta \chi = \delta_0 - \delta_\rho \quad \text{and} \quad \int_{\mathbb{R}} \chi S'(\eta) = S(\rho),$$

and we have

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi S' \psi \Big|_{s=0}^{s=t} &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} (1 + \frac{\langle \xi^\delta \rangle_1}{8\eta}) \nabla \chi \cdot \nabla_x (S' \psi) - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\eta (S' \psi) dq \\ &\quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} S'(\rho) \psi \delta_\rho \nabla \cdot (\sqrt{\rho} d\xi^\delta) + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \langle \nabla \cdot \xi^\delta \rangle_1 \eta \delta_\rho \partial_\eta (S' \psi). \end{aligned}$$

I. Fluctuating hydrodynamics

Renormalized kinetic solutions [F., Gess; 2024]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and \mathcal{F}_0 -measurable. A *renormalized kinetic solution* is a nonnegative, \mathbb{P} -a.s. continuous $L^1(\mathbb{T}^d)$ -valued, and \mathcal{F}_t -predictable function $\rho \in L^\infty(\Omega; L^1(\mathbb{T}^d \times [0, T]))$ that satisfies:

- *Preservation of mass*: \mathbb{P} -a.s. for every time t , $\|\rho(x, t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$.
- *Local regularity*: \mathbb{P} -a.s. for every $K \in [1, \infty)$,

$$((\rho \wedge K) \vee K^{-1}) \in L^2([0, T]; H^1(\mathbb{T}^d)).$$

Furthermore, there exists a nonnegative parabolic defect measure q that satisfies:

- *Regularity*: \mathbb{P} -a.s. in the sense of measures, $\delta_\rho |\nabla \rho|^2 \leq q$.
- *Vanishing at infinity*: \mathbb{P} -a.s. $\liminf_{M \rightarrow \infty} q(\mathbb{T}^d \times [0, T] \times [M, M + 1]) = 0$.
- *The equation*: \mathbb{P} -a.s. for every $\psi \in C_c^\infty(\mathbb{T}^d \times (0, \infty))$,

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \psi \Big|_{s=0}^{s=t} &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \left(1 + \frac{\langle \xi^\delta \rangle_1}{8\eta}\right) \nabla \chi \cdot \nabla_x \psi - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\eta \psi \, dq \\ &\quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \psi \delta_\rho \nabla \cdot (\sqrt{\rho} \, d\xi^\delta) + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\eta \psi \langle \nabla \cdot \xi^\delta \rangle_1 \eta \delta_\rho. \end{aligned}$$

I. Fluctuating hydrodynamics

A useful equality: for two solutions ρ_1 and ρ_2 ,

$$\begin{aligned}\int_{\mathbb{T}^d} |\rho_1 - \rho_2| &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} |\chi_1 - \chi_2|^2 = \int_{\mathbb{T}^d} \int_{\mathbb{R}} |\chi_1| + |\chi_2| - 2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} \text{sgn}(\eta)\chi_1 + \text{sgn}(\eta)\chi_2 - 2\chi_1\chi_2.\end{aligned}$$

The stochastic integral: since

$$\partial_t \rho_i = \Delta \rho_i - \nabla \cdot (\sqrt{\rho_i} d\xi^\delta) + \frac{\langle \xi^\delta \rangle_1}{8} \Delta \log(\rho_i),$$

formally differentiating the above,

$$\begin{aligned}\partial_t \left(\int_{\mathbb{T}^d} |\rho_1 - \rho_2| \right) &= \dots - \int_{\mathbb{T}^d} \text{sgn}(\rho_1 - \rho_2) \nabla \cdot ((\sqrt{\rho_1} - \sqrt{\rho_2}) d\xi^\delta) + \dots \\ &= \dots + \int_{\mathbb{T}^d} (\sqrt{\rho_1} - \sqrt{\rho_2}) \delta_0(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot d\xi^\delta + \dots \\ &\quad \text{"} \simeq \dots + \int_{\mathbb{T}^d} (\sqrt{\rho_1} - \sqrt{\rho_2}) \delta_0^{\kappa}(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot d\xi^\delta + \dots \text{"} \\ &\quad \text{"} \simeq \dots + \int_{\mathbb{T}^d} \kappa^{\frac{1}{2}} \cdot \kappa^{-1} \mathbf{1}_{\{0 < |\rho_1 - \rho_2| < \kappa\}} \nabla(\rho_1 - \rho_2) \cdot d\xi^\delta + \dots \text{"}\end{aligned}$$

I. Fluctuating hydrodynamics

Renormalized kinetic solutions [F. Gess; 2024]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and \mathcal{F}_0 -measurable. Then, there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} d\xi^\delta) + \frac{\langle \xi^\delta \rangle_1}{8} \Delta \log(\rho),$$

with initial data ρ_0 . Furthermore, two solutions ρ_1 and ρ_2 satisfy \mathbb{P} -a.s. that

$$\max_{t \in [0, T]} \|\rho_1 - \rho_t\|_{L^1(\mathbb{T}^d)} \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

Extensions: general equations of the type

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ \xi^\delta + \nu(\rho)) + \lambda(\rho) + \phi(\rho) \xi^\delta,$$

including $\Phi(\rho) = \rho^m$ for every $m \in (0, \infty)$, $\sigma(\rho) = \rho^{\frac{m}{2}}$, and $\phi(\rho) = \sqrt{\rho}$.

Flux through zero: if ρ is a nonnegative solution of the heat equation,

$$\lim_{\beta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \beta^{-1} |\nabla \rho|^2 \mathbf{1}_{\{\beta/2 < \rho < \beta\}} = 0.$$

I. Fluctuating hydrodynamics

Random dynamical system [F., Gess, Gvalani; 2024]

Let $\rho(t, \rho_0, \omega)$ denote the solution

$$\partial_t \rho = \Delta \rho - \varepsilon^{\frac{d}{2}} \nabla \cdot (\sqrt{\rho} d\xi^\delta(\omega)) + \frac{\varepsilon^d \langle \xi^\delta \rangle_1}{8} \Delta \log(\rho) \quad \text{in } \mathbb{T}^d \times (t, \infty),$$

with $\rho(t, \rho_0, \omega)(x, t) = \rho_0$. The solutions $\rho(t, \rho_0, \omega)$ generate a random dynamical system on with respect to the usual Borel σ -algebra on the space of nonnegative, integrable functions on \mathbb{T}^d .

- the pathwise contraction of solutions controls zero sets
- new estimates for the initial time continuity of the flow

Invariant measures [F., Gess, Gvalani; 2024]

The Markov process $\rho(t, \cdot, \cdot)$ is uniquely ergodic: it has a unique invariant probability measure and is strongly mixing.

- the entropy dissipation estimate:

$$\mathbb{E} \left[\max_{t \in [0, T]} \int_{\mathbb{T}^d} \rho \log(\rho) \right] + \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 \right] \leq \mathbb{E} \left[\int_{\mathbb{T}^d} \rho_0 \log(\rho_0) \right] + c(\langle \xi^\delta \rangle_1 + T \langle \nabla \cdot \xi^\delta \rangle_1).$$

I. Fluctuating hydrodynamics

Quantitative LLN and Central Limit Theorem [Clini, F.; 2025]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \rightarrow 0$,

$$\varepsilon \delta(\varepsilon)^{-(1+2/d)} \rightarrow 0 \text{ and } \delta(\varepsilon) \rightarrow 0,$$

and for every $\varepsilon \in (0, 1)$ let ρ^ε be the solution

$$\partial_t \rho^\varepsilon = \Delta \Phi(\rho^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

Law of Large Numbers: then almost surely with a quantitative rate, as $\varepsilon \rightarrow 0$,

$$\rho^\varepsilon \rightarrow \bar{\rho} \text{ for } \partial_t \bar{\rho} = \Delta \Phi(\bar{\rho}).$$

Central Limit Theorem: then as distributions, as $\varepsilon \rightarrow 0$,

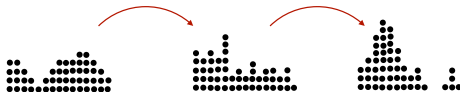
$$\varepsilon^{-\frac{d}{2}} (\rho^\varepsilon - \bar{\rho}) \rightarrow v \text{ for } \partial_t v = \Delta(\Phi'(\bar{\rho})v) - \nabla \cdot (\Phi^{\frac{1}{2}}(\bar{\rho})\xi).$$

The central limit theorem is a statement about “universal” behavior:

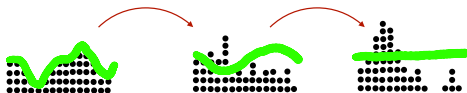
$$\partial_t \tilde{\rho}^\varepsilon = \Delta \Phi(\tilde{\rho}^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\bar{\rho})\xi^{\delta(\varepsilon)}).$$

II. Macroscopic fluctuation theory

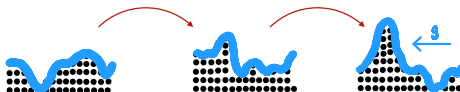
The zero range process: μ^N on $\mathbb{T}^1 \times [0, T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \bar{\rho} = \Delta \bar{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} \cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-N^d I(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf \{ \|g\|_{L^2_{t,x}}^2 : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g) \}.$$

II. Macroscopic fluctuation theory

The hydrodynamic limit: for the parabolically rescaled, mean zero particle process μ_t^N on \mathbb{T}_N^d , as $N \rightarrow \infty$, for $J(\bar{\rho}) = \nabla \sigma(\bar{\rho})$,

$$\mu_t^N \rightharpoonup \bar{\rho} dx \text{ for } \partial_t \bar{\rho} = \Delta \sigma(\bar{\rho}) = \nabla \cdot J(\bar{\rho}).$$

Macroscopic fluctuation theory: the probability of observing a space-time fluctuation (ρ, j) satisfying

$$\partial_t \rho = \nabla \cdot j,$$

satisfies the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp(-N^d I(\rho)) \text{ for } I(\rho) = \int_0^T \int_{\mathbb{T}^d} (j - J(\rho)) \cdot m(\rho)^{-1} (j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho) = \int_0^T \int_{\mathbb{T}^d} |g|^2$ and

$$\partial_t \rho = \nabla \cdot (J(\rho) + (j - J(\rho))) = \Delta \sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}g).$$

The zero range process: we have that, for $\Phi(\rho) = \rho^m$ for any $m \in [1, \infty)$,

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)g).$$

The exclusion process: we have that

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho(1-\rho)}g).$$

II. Macroscopic fluctuation theory

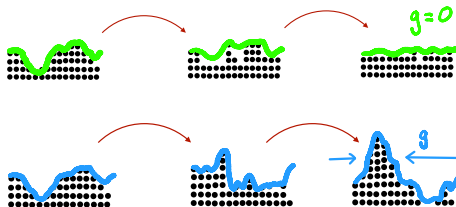
The large deviations rate function: for every $m \in L^1(\mathbb{T}^d \times [0, T])$,

$$I(m) = \frac{1}{2} \inf \{ \|g\|_{L^2(\mathbb{T}^d \times [0, T])}^2 : \partial_t m = \Delta m - \nabla \cdot (\sqrt{m} g) \}.$$

Large Deviations Principle [Benois, Kipnis, Landim; 1995], [F., Gess; 2023]

For every measurable $A \subseteq L^1([0, T]; L^1(\mathbb{T}^d))$,

$$\mathbb{P}[\mu^N \in A] \simeq e^{-N^d \left(\inf_{\pi \in A} I(\pi) \right)}.$$



II. Macroscopic fluctuation theory

The Large Deviations Principle [F., Gess; 2024]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon\delta(\varepsilon)^{-(1+2/d)} \rightarrow 0$ and $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and for every $\varepsilon \in (0, 1)$ let ρ_ε be the solution

$$\partial_t \rho_\varepsilon = \Delta \rho_\varepsilon - \varepsilon^{\frac{d}{2}} \nabla \cdot (\sqrt{\rho_\varepsilon} d\xi^{\delta(\varepsilon)}) + \frac{\varepsilon^d \langle \xi^\delta \rangle_1}{8} \Delta \log(\rho_\varepsilon).$$

The large deviations principle: the solutions ρ_ε satisfy a large deviations principle in $L^1_{t,x}$ and $C([0, T]; \mathcal{M})$ with rate function

$$I(\rho) = \frac{1}{2} \inf \{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g) \}.$$

The linear fluctuating hydrodynamics: the linear fluctuating hydrodynamics

$$\partial_t \tilde{\rho}_\varepsilon = \Delta \tilde{\rho}_\varepsilon - \sqrt{\varepsilon} \nabla \cdot (\sqrt{\tilde{\rho}} d\xi^{\delta(\varepsilon)}),$$

for the hydrodynamics limit $\partial_t \bar{\rho} = \Delta \Phi(\bar{\rho})$ satisfy an LDP with rate function

$$\tilde{I}(\rho) = \frac{1}{2} \inf \{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g) \}.$$

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