Multiple time scale solutions of models for a pendulum

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Introduction: Non-linearity is essential in real world

Simple harmonic oscillators are ubiquitous in the physics world, however, real-world systems exhibit essential non-linearity. For example, an oscillator with a hardened spring can be described by $y'' + y + \epsilon y^3 = 0$ where $0 < \epsilon \ll 1$. Though the y^3 term is small, it does introduces a non-linearity. Also, the non-linearity term can be huge on a large scale.

Our work applied the perturbation method to give a solution to this hardened spring oscillator and refined the solution with multi-scale variable method to avoid the large-scale error. The solution produced by this method can be refined to any degree $O(\epsilon^n)$ needed.

The Method of Perturbation

As the unperturbed system can be easily solved. We define the unperturbed solution to be f_0 and the whole solution to be f. The problem can be solved as follows.

1. Set $\epsilon = 0$ and solve the resulting system(i.e. solve f_0)
2. Perturb the system by allowing ϵ to be nonzero (but small)
3. Formulate the solution as a series: $f \equiv f_0 + \epsilon^1 f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots$
4. Expand the governing equations in powers of ϵ, and solve by order(i.e. asymptotic expansion)

With this method we can obtain an asymptotic solution to the equation$y'' + y + \epsilon y^3 = 0$ with initial condition $y(0) = 1,\ y'(0) = 0$:

$$y_0 = \cos(t) \tag{1}$$

$$y_1 = \frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t) \tag{2}$$

$$y \approx \cos(t) + \epsilon\left[\frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t)\right] + \mathcal{O}\left(\epsilon^2\right) \tag{3}$$

The Method of Multi-Scale Variables

Notice that the secular term $-\epsilon\frac{3}{8}tsin(t)$ grows unbounded as t grows. Which breaks the validity of our step 4(asymptotic expansion). On that, we introduce two separate time scales:

- fast-time: $t_0 = t$ - slow-time: $t_1 = \epsilon t$

Then we can express y as:

$$y(t) = Y(t_0, t_1) = Y_0(t_0, t_1) + \epsilon Y_1(t_0, t_1) + \mathcal{O}\left(\epsilon^2\right) \tag{4}$$

by expanding terms with chain rule:

$$\frac{dy}{dt} = \frac{\partial Y}{\partial t_0} + \frac{\partial Y}{\partial t_1}\cdot\epsilon \tag{5}$$

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y}{\partial t_0^2} + 2\frac{\partial^2 Y}{t_0 t_1}\cdot\epsilon + \frac{\partial^2 Y}{\partial t_1^2}\cdot\epsilon^2 \tag{6}$$

therefore the original equation becomes:

$$\left(\frac{\partial^2 Y_0}{\partial t_0^2} + \epsilon\frac{\partial^2 Y_1}{\partial t_0^2} + 2\epsilon\frac{\partial^2 Y_0}{\partial t_0 t_1}\right) + (Y_0 + \epsilon Y_1) + \epsilon Y_0^3 + \mathcal{O}\left(\epsilon^2\right) = 0 \tag{7}$$

solving ϵ^0 terms gives:

$$Y_0(t_0, t_1) = A(t_1)\cos(t_0) + B(t_1)\sin(t_0) \tag{8}$$

then ϵ^1 terms becomes:

$$\frac{\partial^2 Y_1}{\partial t_0^2} + Y_1 = -\frac{\partial^2 Y_0}{\partial t_1 \partial t_0} - Y_0^3 \tag{9}$$

$$= \left(-2\frac{\partial B}{\partial t_1} - \frac{3}{4}A(A^2+B^2)\right)\cos(t_0) + \left(2\frac{\partial A}{\partial t_1} - \frac{3}{4}B(A^2+B^2)\right)\sin(t_0) \tag{10}$$

to avoid secular terms, $RHS = 0$

$$-2\frac{\partial B}{\partial t_1} - \frac{3}{4}A\left(A^2+B^2\right) = 0 \tag{11}$$

$$2\frac{\partial A}{\partial t_1} - \frac{3}{4}B\left(A^2+B^2\right) = 0 \tag{12}$$

with initial condition, we can have:

$$A(t_1) = \cos\left(\frac{3}{8}t_1\right) \tag{13}$$

$$B(t_1) = -\sin\left(\frac{3}{8}t_1\right) \tag{14}$$

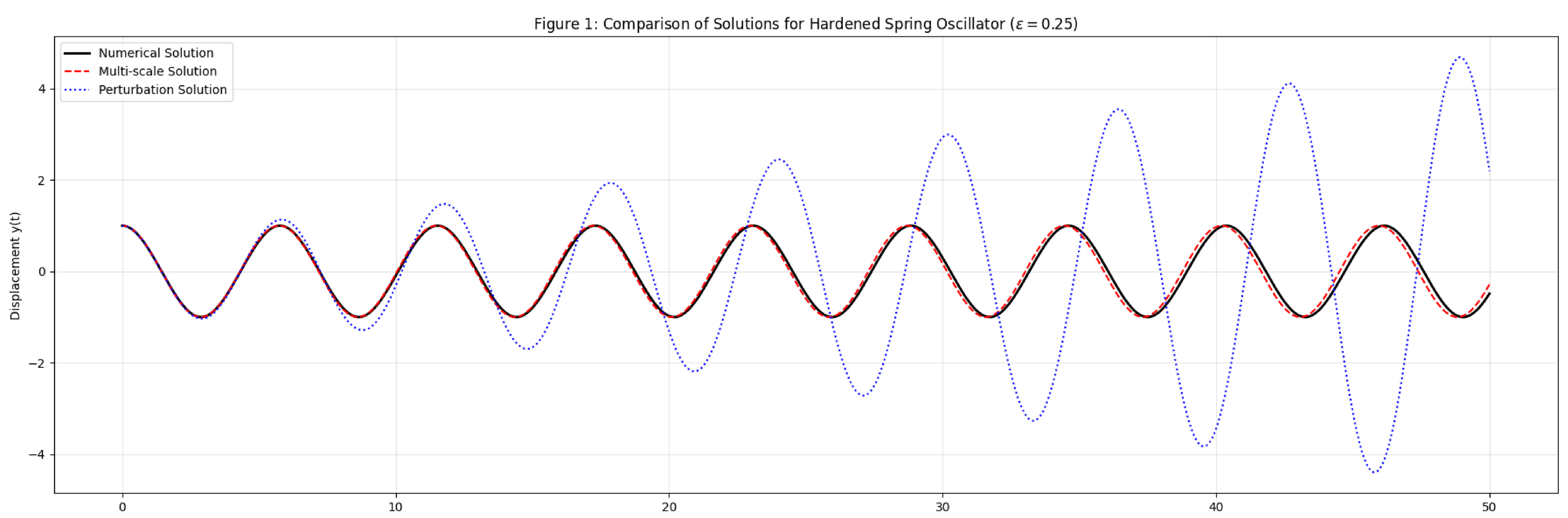
$$Y_0(t_0, t_1) = \cos\left(\frac{3}{8}t_1\right)\cos(t_0) - \sin\left(\frac{3}{8}t_1\right)\sin(t_0) = \cos\left(t_0 + \frac{3}{8}t_1\right) \tag{15}$$

substituting in $t_0 = t, t_1 = \epsilon t$:

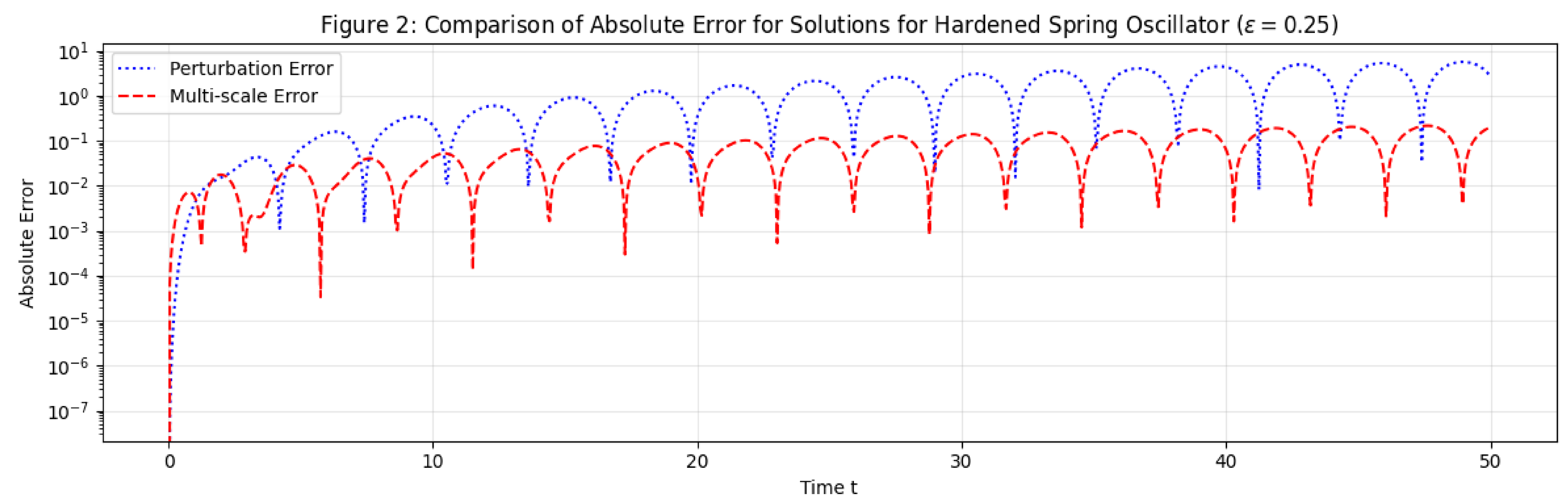
$$y \approx \cos\left((1 + \frac{3}{8}\epsilon)t\right) + \mathcal{O}(\epsilon) \tag{16}$$

with period:

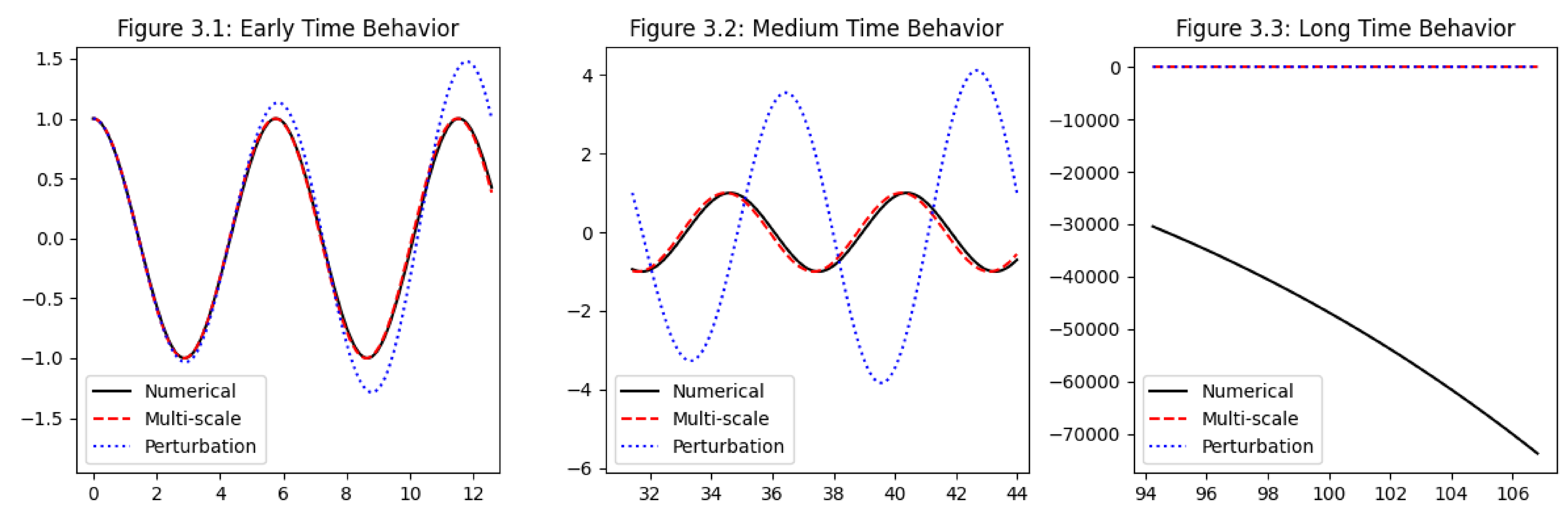
$$T = \frac{2\pi}{1 + \frac{3}{8}\epsilon} \approx 2\pi\left(1 - \frac{3}{8}\epsilon\right) \tag{17}$$



Result



The difference in the error made by these two methods is shown in the above figure. Though the error of the multi-scale variable method does exceed that of normal perturbation method occasionally, it is better in most of the time. It is also observed that the method of multi-scale variable maintains validity as needed. The phase and period difference in early, medium and long time is shown in the figure below.



Conclusion and Future Work

The perturbation method provides an asymptotic approximation to solving weakly non-linear systems like hardened spring oscillator. However, this approximation is limited to secular terms growing unbounded over time.

The method of multiple scales resolved this limitation by introduces distinct time scale. With this way, we obtained uniformly valid approximation that models the system's behavior.

Though the multi-scale method has worked well on the long-term, it makes even bigger error than the normal perturbation method in the first second. It needs further discussion to determine what caused this error. Calculating higher order of the multi-scale solution or adding another dimension of time is suggested to solve this problem.

This poster summarises a second-year project undertaken in June 2025

Reference

(1) Holmes, M.H. (2012). *Introduction to Perturbation Methods.* New York, Springer.

(2) Kevorkian, J. and Cole, J.D. (2012). *Multiple Scale and Singular Perturbation Methods.* New York, Springer.

(3) UCL LTCC note. (no date). *Introduction to perturbation methods.* Available at:`https://www.ucl.ac.uk/~ucahhwi/LTCC/section2-3-perturb-regular.pdf` (accessed 17 November 2025).