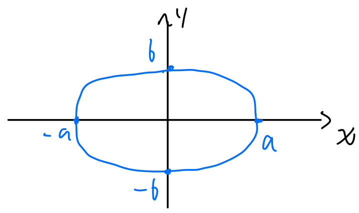
§ Ellipses and elliptic integrals



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad x=a\cos\theta \qquad y=b\sin\theta$$

Circumference $= 4\cdot$ Length of arc in the first quadrant

$$= 4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta = 4\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\cos^2\theta}\,d\theta$$

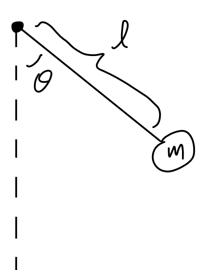
$$= 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta = 4a\int_0^1\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt \qquad k=\sqrt{1-\frac{b^2}{a^2}}$$

$$\theta=\sin^{-1}(t)$$

$$= 4a\int_0^1\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} - 4a\int_0^1\frac{k^2t^2}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt \qquad d\theta=\frac{1}{\sqrt{1-t^2}}$$

Generalization: For a rational function $R(t)$, integrals of the form $\int f(t,\sqrt{R(t)})\,dt$ is called an elliptic integral.

§ Pendulum



By conservation of energy, the motion of pendulum is periodic, and when $0<\theta_0<\pi$, $\dot\theta_0=0$ we have $\frac{1}{2}ml^2\dot\theta^2-mgl\cos\theta=-mgl\cos\theta_0$

$$\Rightarrow \quad \dot\theta^2=\frac{2g}{l}(\cos\theta-\cos\theta_0)=\frac{4g}{l}\left(\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}\right)$$

For simplicity, let $l=g$, so we have

$$\dot\theta=-2\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}} \Rightarrow dt=\frac{-1}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}\cdot\frac{d\theta}{2}$$

$$\Rightarrow \quad t=\frac{1}{2}\int_\theta^{\theta_0}\frac{d\tilde\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\tilde\theta}{2}}} \qquad u=\sin\frac{\tilde\theta}{2}\Big/\sin\frac{\theta_0}{2}$$

$$du=\frac{1}{2}\cos\frac{\tilde\theta}{2}\,d\theta\Big/\sin\frac{\theta_0}{2}=\frac{1}{2}\sqrt{1-\sin^2\frac{\theta_0}{2}u^2}\,d\tilde\theta\Big/\sin\frac{\theta_0}{2}$$

$$\Rightarrow \quad t=\int_{\sin\frac{\theta}{2}/\sin\frac{\theta_0}{2}}^{1}\frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}} \qquad 0<k=\sin\frac{\theta_0}{2}<1$$

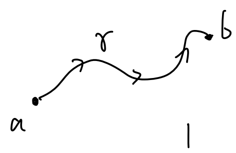
$$K = \int_0^1 \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}} \Rightarrow t = K - \int_0^{\sin\frac{\theta}{2}/\sin\frac{\theta_0}{2}} \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}$$

$$y = \int_0^x \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}$$ is increasing on $[0,1] \Rightarrow \underbrace{x = sn(y;k)}_{\text{Jacobian elliptic function.}}$ is well defined.

$$\Rightarrow \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}} = sn(K-t)$$

From conservation of energy, we see that $\theta(t)$ is $4K$ periodic, so $sn(y) = sn(y+4K)\ \forall y \in \mathbb{R}$

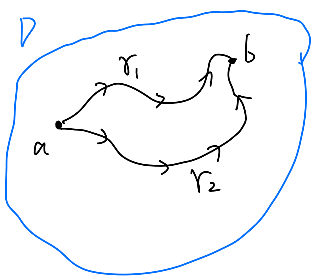
§ Elliptic integral in the complex plane

Line integral: 

$\varphi: [0,1] \mapsto \gamma$ be a C^1 parametrization. s.t. $\varphi(0)=a$, $\varphi(1)=b$

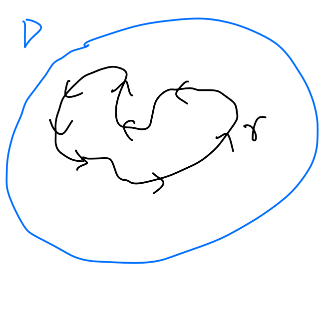
Then $$\int_\gamma f(z)dz := \int_0^1 f(\varphi(t))\varphi'(t)dt$$

In real analysis, if f is nice, the value of its integral only depends on end points. In complex analysis, we love that property too.



Let D be some connected region, we say f is analytic in D iff it's continuous in D and $\forall a,b \in D$ and $\forall$ paths γ_1, γ_2 travelling from a to b.

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

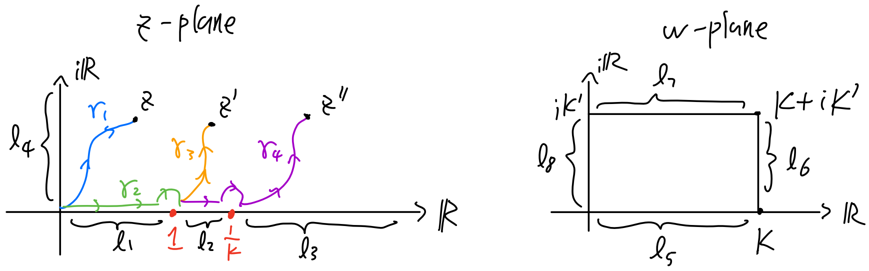


Let D be some region, we say f is analytic in D iff $\forall$ closed curve in D, $\int_\gamma f(z)dz = 0$.

γ is a simple closed curve if it's oriented counterclockwise & does not self intersect.

$$w = \int_0^z \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}} \qquad 0<k<1 \qquad K = \int_0^1 \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}, \quad K' = \int_1^{k^{-1}} \frac{du}{\sqrt{1-k^2u^2}\sqrt{u^2-1}}$$

Let z lie in the first quadrant



(i) $0 \le Re(z) \le 1$, we integrate along γ_1

$\Rightarrow l_1$ of z-plane corresponds to l_5 in w-plane

(ii) $1 < Re(z) \le k^{-1}$, we integrate along $\gamma_2 + \gamma_3$

when γ_2 is near $u=1$, $\arg(1-u)$ changes from 0 to $-\pi$

$$\Rightarrow \sqrt{1-u} \to -i\sqrt{u-1} \Rightarrow w = K + i\int_1^z \frac{du}{\sqrt{1-k^2u^2}\sqrt{u^2-1}}$$

$\Rightarrow l_2$ of z-plane corresponds to l_6 in w-plane

(iii) $Re(z) > k^{-1}$, we integrate along $\gamma_2 + \gamma_4$

$\Rightarrow \sqrt{k^{-1}-u} \to -i\sqrt{u-k^{-1}}$ as u travels near $u = k^{-1}$

$$\Rightarrow w = K + iK' - \int_{k^{-1}}^z \frac{du}{\sqrt{k^2u^2-1}\sqrt{u^2-1}}$$

$$s = \frac{1}{ku} \Rightarrow du = \frac{-ds}{ks^2}$$

$$\Rightarrow \int_{k^{-1}}^{\infty} \frac{du}{\sqrt{k^2u^2-1}\sqrt{u^2-1}} = \int_0^1 \frac{1}{\sqrt{s^{-2}-1}\sqrt{s^{-2}k^{-2}-1}} \cdot \frac{ds}{ks^2} = \int_0^1 \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} = K$$

$\Rightarrow l_3$ of z-plane corresponds to l_7 of w-plane

(iv) Imaginary axis

$$k^2s^2-k^2+1-k^2s^2$$

$$z=iy \Rightarrow w=i\int_0^y \frac{dt}{\sqrt{1+k^2t^2}\sqrt{1+t^2}}$$

$$s=\sqrt{\frac{1+t^2}{1+k^2t^2}} \qquad s^2+k^2t^2s^2=1+t^2 \Rightarrow t^2=\frac{s^2-1}{1-k^2s^2}$$

$$1+t^2=\frac{s^2(1-k^2)}{1-k^2s^2},\quad 1+k^2t^2=\frac{1-k^2}{1-k^2s^2}$$

$$t=\sqrt{\frac{s^2-1}{1-k^2s^2}} \quad \frac{dt}{ds}=\sqrt{\frac{1-k^2s^2}{s^2-1}}\cdot\frac{s(1-k^2s^2)+k^2s(s^2-1)}{(1-k^2s^2)^2}=\sqrt{\frac{1-k^2s^2}{s^2-1}}\cdot\frac{s(1-k^2)}{(1-k^2s^2)^2}$$

$$\Rightarrow \int_0^\infty \frac{dt}{\sqrt{1+k^2t^2}\sqrt{1+t^2}}=\int_1^{k^{-1}}\frac{1-k^2s^2}{s(1-k^2)}\cdot\sqrt{\frac{1-k^2s^2}{s^2-1}}\cdot\frac{s(1-k^2)}{(1-k^2s^2)^2}\,ds=K'$$

$\Rightarrow$ l_4 of z-plane corresponds to l_8 of w-plane

Thus, the boundary l_1, l_2, l_3, l_4 of the first quadrant of z-plane corresponds to the boundary l_5, l_6, l_7, l_8 of a rectangle in w-plane.

§ Further mapping properties of $\int_0^z \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}$

Integral and injective function

Argument principle: If f is analytic in D, for any simple closed curve γ,

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz=\#\text{ of zeros enclosed by }\gamma.$$

Thm 1: Suppose f is analytic in D and injective on some simple closed path γ. Then $f(z)$ maps the region enclosed by γ onto the region enclosed by $\gamma'=f(\gamma)$ injectively.

Proof. Let w_0 be any point inside γ'

Then because γ' does not self intersect,

$$\frac{1}{2\pi i}\int_{\gamma'}\frac{dw}{w-w_0}=\pm1.\ \text{However, } \frac{1}{2\pi i}\int_{\gamma'}\frac{dw}{w-w_0}=\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)-w_0}\,dz\geq 0$$

$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w_0}\,dz = 1 \Rightarrow \forall w_0$ inside γ' $\exists$ unique z_0 inside γ s.t. $f(z_0)=w_0$.

Q.E.D.

Let z_1, z_2 be distinct points from the first quadrant and $\gamma: [0,1]$ be a parametrization of a path in the first quadrant connecting them. Then we have

$$\int_\gamma \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}} = \int_0^1 \rho(t)e^{i\phi(t)}\,dt \quad \text{for some } \rho(t)>0,\ 0\leq\phi(t)<2\pi$$

$$0<\arg(u)<\frac{\pi}{2} \Rightarrow 0<\arg(u^2)<\pi \Rightarrow -\pi<\arg(1-u^2)<0$$

$$\Rightarrow 0>\arg(1-k^2u^2)>-\pi$$

$$\Rightarrow 0<\phi(t)<\pi \Rightarrow \int_0^1 \rho(t)\sin\phi(t)\,dt>0$$

$\Rightarrow \displaystyle\int_0^z \frac{du}{\sqrt{1-k^2u^2}\sqrt{1-u^2}}$ is injective on any path inside the first quadrant.

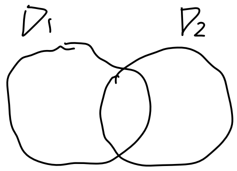
By Thm 1 and analysis on the boundary, we conclude that the first quadrant of z-plane corresponds to the rectangule of w-plane.

Since the mapping is injective, we can define an inverse mapping $z=\mathrm{sn}(w)$ on the rectangle of w-plane.

§ Analytic continuation

$\mathrm{sn}(w)$ is merely defined on a rectangular region, but we hope to extend its definition.

In other words, we need analytic continuation.



Let f_1, f_2 be analytic in D_1, D_2 resp.
We say they are analytic continuations to each other when $f_1(z)=f_2(z)\ \forall z\in D_1\cap D_2$.

Reflection principle

Motivation: If $f(z)$ is analytic in D that contains a segment of real axis and f maps reals to reals, then $f(\bar{z}) = \overline{f(z)}$.

e.g. $\bar{z}^n = \overline{z^n}$, $e^z = \sum_{n\geq 0} \frac{z^n}{n!} \Rightarrow \overline{e^z} = e^{\bar{z}}$

Thm 2: Let f be analytic in D with line l being part of its boundary s.t. $w = f(z)$ maps l of z-plane to l' of w-plane. Let $g(z), h(w)$ reflect z, w about l, l' resp. Define $f_1(z) = (h \circ f \circ g)(z)\ \forall z \in D_1$. Then f_1 is an analytic continuation of $f(z)$ in D_1.

Proof. Let $w_1 = f_1(z_1)$, $w = f_1(z)$.

$w_1' = f(g(z_1))$, $w' = f(g(z))$.

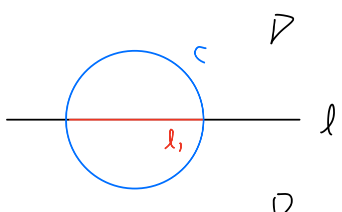
Then $|w_1 - w| = |w_1' - w'|$, $|z_1 - z| = |g(z_1) - g(z)|$

$\arg(w_1 - w) + \arg(w_1' - w') = 2\alpha'$

$\arg(z_1 - z) + \arg(g(z_1) - g(z)) = 2\alpha$

where α, α' denote the angle between l, l' and real axis resp.

$\Rightarrow \frac{f_1(z_1) - f_1(z)}{z_1 - z}$ converges as $z_1 \to z\ \forall z \in D_1 \Rightarrow f_1$ is analytic in D_1.



$f_1(z) = f(z)$ when $z \in l_1$

$\gamma = \partial(C \cap D)$ $\quad \gamma' = \partial(C \cap D_1)$

$$\phi(z) = \begin{cases} f(z) & z \in C \cap D \\ f_1(z) & z \in C \cap D_1 \end{cases}$$

When z_0 is inside C.

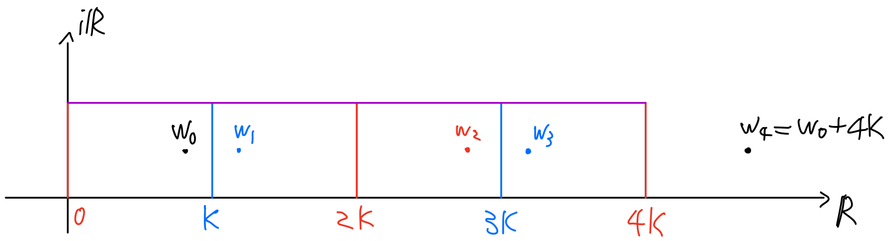
$$\frac{1}{2\pi i} \oint_{\partial C} \frac{\phi(z)}{z - z_0} dz = \underbrace{\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz}_{\substack{f(z_0) \text{ if } z_0 \in D \\ 0 \text{ if } z_0 \in D_1}} + \underbrace{\frac{1}{2\pi i} \oint_{\gamma'} \frac{f_1(z)}{z - z_0} dz}_{\substack{0 \text{ if } z_0 \in D \\ f_1(z_0) \text{ if } z_0 \in D_1}} = \phi(z_0)$$

Q.E.D.

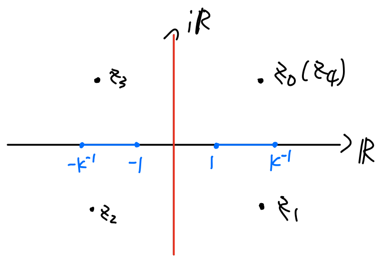
§ Application of reflection principle to $z = sn(w)$

(i) Horizontal reflections

w-plane

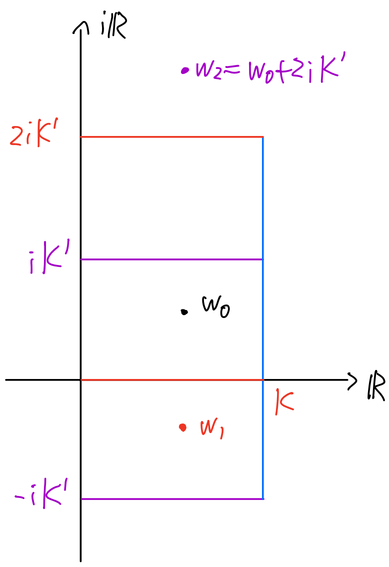


z-plane

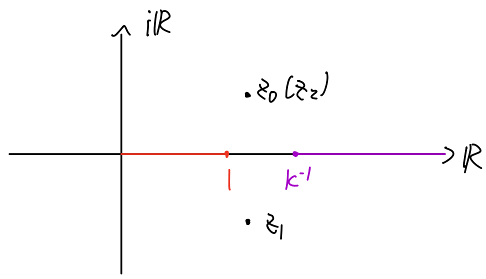


$\Rightarrow \quad sn(w_0) = sn(w_0 + 4K).$

(ii) Vertical reflections



w-plane

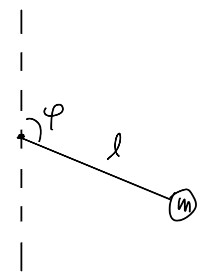


z-plane

$\Rightarrow \quad sn(w_0) = sn(w_0 + 2iK')$

§ The imaginary period of sn(•) via pendulum

(This argument is due to Paul Appell in 1878)



$$0<\varphi_0<\pi, \quad \dot{\varphi}_0=0.$$

$$\frac{1}{2}ml^2\dot{\varphi}^2+mgl\cos\varphi = mgl\cos\varphi_0$$

$$\dot{\varphi}^2 = \frac{2g}{l}(\cos\varphi_0-\cos\varphi)$$

For simplicity, assume $g=l$

$$\Rightarrow \quad \dot{\varphi}^2 = 4\left(\sin^2\frac{\varphi}{2}-\sin^2\frac{\varphi_0}{2}\right) \qquad k=\sin\frac{\varphi_0}{2}$$

$$\Rightarrow \quad \frac{dt}{d\varphi} = \frac{1}{2k}\cdot\frac{1}{\sqrt{\left(\frac{\sin\frac{\varphi}{2}}{k}\right)^2-1}} \qquad u=\frac{\sin\frac{\varphi}{2}}{k}$$

$$du = \frac{1}{2k}\cos\frac{\varphi}{2}\,d\varphi = \frac{1}{2k}\sqrt{1-k^2u^2}\,d\varphi$$

$$\Rightarrow \quad t = \int_1^{\sin\frac{\varphi}{2}/k} \frac{du}{\sqrt{u^2-1}\sqrt{1-k^2u^2}}$$

$$\Rightarrow \quad K+it = K+i\int_1^{\sin\frac{\varphi}{2}/k}\frac{du}{\sqrt{u^2-1}\sqrt{1-k^2u^2}} = \int_0^{\sin\frac{\varphi}{2}/k}\frac{du}{\sqrt{u^2-1}\sqrt{1-k^2u^2}}$$

where u travels through the first quadrant to skip $u=1$.

$$\Rightarrow \quad \frac{\sin\frac{\varphi}{2}}{k} = \mathrm{sn}(K+it)$$

K' is the time it takes for the mass to reach $\varphi=\pi$ from $\varphi=\varphi_0$.

$$\varphi(2K') = 2\pi-\varphi_0$$

$$\varphi(2K'+t) = 2\pi-\varphi(t)$$

$$\sin\frac{\varphi(2K'+t)}{2} = \sin\left(\pi-\frac{\varphi(t)}{2}\right) = \sin\frac{\varphi(t)}{2} \Rightarrow \text{sn is } 2iK' \text{ periodic.}$$

Ellipse $\Longrightarrow$ Elliptic Integral

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad w = \int_0^z \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

Pendulum $\Longrightarrow$ Elliptic Function

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \qquad z = sn(w;k)$$