(a) To obtain the weak formulation, multiply both sides of the equation by a test function $v \in C_0^\infty(\Omega)$ and integrate:

$$-\int_\Omega (\Delta u)v\,dxdy + \int_\Omega (b(x,y)\cdot\nabla u)v\,dxdy = \int_\Omega f(x,y)v\,dxdy$$

Let's simplify the first term by using integration by parts:

$$\begin{aligned}\int_\Omega (\Delta u)v\,dxdy &= \int_\Omega ((\partial_x^2 u + \partial_y^2)u)v\,dxdy\\ &= -\int_\Omega ((\partial_x + \partial_y)u) + ((\partial_x + \partial_y)v)\,dxdy\\ &= -\int_\Omega \nabla u\cdot\nabla v\,dxdy\end{aligned}$$

In the second equality we used that $u = 0$ on the boundary $\partial\Omega$ of Ω.
Hence we get

$$\int_\Omega (\Delta u)(\Delta v)\,dxdy + \int_\Omega b(\nabla u)v\,dxdy = \int_\Omega fv\,dxdy$$

with $u \in H^1(\Omega)$ and $v \in H_0^1(\Omega)$.
This is the weak formulation of the PDE.
The associated bilinear form is

$$a : H_0^1(\Omega)\times H_0^1(\Omega) \to R$$

and we have

$$a(w,v) = \int_\Omega \nabla w\nabla v\,dxdy + \int_\Omega b(\nabla w)v\,dxdy.$$

and the linear functional is

$$\ell : H_0^1(\Omega) \to R$$
$$\ell(v) = \int_\Omega fv\,dxdy$$

We want to prove both ℓ and a are bounded:

- ℓ bounded: need to show that $\exists M > 0$ s.t.

  $$\|\ell(v)\| \leq M\|v\|_{H_1^1(\Omega)} \quad \forall w \in H_0^1(\Omega).$$

  Using the Cauchy-Schwarz inequality we have

  $$|\ell(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

  $$\leq \|f\|_{L^2(\Omega)}\left(\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\ (*)$$

$$\Rightarrow (*) = \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$$

Since by assumption, $f \in L^2(\Omega)$, $M = \|f\|_{L^2(\Omega)} < \infty$.

• a bounded: need to show that $\exists C > 0$ s.t.

$$|a(w,v)| \le C\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

We have

$$|a(w,v)| \le \left|\int_\Omega \nabla w \cdot \nabla v \, dxdy\right| + \left|\int_\Omega b(\nabla v) w \, dxdy\right|$$

where using Cauchy-Schwarz:

$$\begin{aligned}
\left|\int_\Omega \nabla w \cdot \nabla v \, dxdy\right| &\le \sum_{i,j=x,y} \int_\Omega |\partial_i w||\partial_j v| \, dxdy \\
&\le \sum_{i,j=x,y} \|\partial_i w\|_{L^2(\Omega)}\|\partial_j v\|_{L^2(\Omega)} \\
&\le 2\left(\sum_i \|\partial_i w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\sum_j \|\partial_j v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&= 2\|\nabla w\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\le 2\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.
\end{aligned}$$

Similarly we bound the second term:

$$\left|\int_\Omega b(\nabla w) v \, dxdy\right| \le \int_\Omega |b||(\nabla w)v| \, dxdy$$

Since $b \in L^\infty(\Omega)$, $c_1 = \max_{i=1,2} \|b_i\|_{L^\infty(\Omega)} < \infty$, so

$$\begin{aligned}
\int_\Omega |b||(\nabla w)v| \, dxdy &\le c_1 \sum_{i=x,y} \int_\Omega |\partial_i w||v| \, dxdy \\
&\le c_1 \sum_i \|\partial_i w\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le c_1\sqrt{2}\left(\sum_i \|\partial_i w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \|v\|_{L^2(\Omega)} \\
&= c_1\sqrt{2}\|\nabla w\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le c_1\sqrt{2}\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.
\end{aligned}$$

Hence we can take $C = 2 + c_1\sqrt{2}$

(b) Now we let $b(x,y) = 6\begin{pmatrix} -y \\ x \end{pmatrix}$. To prove that the bilinear form $a(\cdot,\cdot)$ is coercive we need to show that there exists a constant $c_0 > 0$ such that

$$c_0\|v\|^2_{H^1(\Omega)} \leq a(v,v) \quad \forall v \in H^1_0(\Omega).$$

We have

$$\begin{aligned} a(v,v) &= \int_\Omega (\nabla v)^2\, dxdy + \int_\Omega (b\cdot\nabla v)v\, dxdy \\ &= \|\nabla v\|^2_{L^2(\Omega)} + \int_\Omega (b\cdot\nabla v)v\, dxdy \end{aligned}$$

where

$$\begin{aligned} \int_\Omega (b\nabla v)v\, dxdy &= \frac{1}{2}\int_\Omega b\cdot\nabla(v^2)\, dxdy \\ &= \frac{1}{2}\int_{\partial\Omega}(b\cdot\vec{n})v^2\, ds - \frac{1}{2}\int_\Omega (\nabla b)v^2\, dx \\ &= -\frac{1}{2}\int_\Omega (\nabla b)v^2\, dxdy \\ &= -\frac{1}{2}\int_\Omega (6\partial_x b_x + 6\partial_y b_y)v^2\, dxdy = 0 \end{aligned}$$

Hence

$$a(v,v) = \|\nabla v\|^2_{L^2(\Omega)}$$

and using Poincaré's inequality $\|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}$ we get

$$\|v\|^2_{H^1(\Omega)} \leq (1+C^2)\|\nabla v\|^2_{L^2(\Omega)}$$

so

$$a(v,v) \geq \frac{1}{1+C^2}\|v\|^2_{H^1(\Omega)}$$

Hence $a(\cdot,\cdot)$ is coercive on $H^1_0(\Omega)$. Finally, an application of the Lax-Milgram theorem shows that the problem has a unique weak solution $u \in H^1_0(\Omega)$.

(c) Now suppose $b(x,y) = 6\begin{pmatrix} x \\ y \end{pmatrix}$. We have to show that $a(\cdot,\cdot)$ is not coercive on $H^1_0(\Omega)$. Consider the test function

$$v(x,y) = 1 - \sqrt{x^2+y^2}$$

which has $\|v\|_{H^1(\Omega)} \neq 0$. To prove that $a(\cdot,\cdot)$ is not coercive it is enough to show that $a(v,v) = 0$, since in this case it's not true that

$$a(v,v) > C_1\|v\|^2_{H^1_0(\Omega)} \quad \forall v$$

We have

$$a(v, v) = \int_\Omega (\nabla v)^2 \, dxdy + \int_\Omega (b \cdot \nabla v) v \, dxdy$$

where

$$\begin{aligned}
\int_\Omega (\nabla v)^2 \, dxdy &= \int_\Omega \left((\partial_x v)^2 + (\partial_y v)^2 \right) \, dxdy \\
&= \int_\Omega \left(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} \right) \, dxdy \\
&= \int_\Omega \, dxdy = \mathrm{vol}(\Omega) = \pi
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega (b \cdot \nabla v) v \, dxdy &= 6 \int_\Omega \left(\frac{-x^2}{\sqrt{x^2 + y^2}} - \frac{y^2}{\sqrt{x^2 + y^2}} \right) (1 - \sqrt{x^2 + y^2}) \, dxdy \\
&= 6 \int_\Omega \left(x^2 + y^2 - \sqrt{x^2 + y^2} \right) \, dxdy \\
&= 6 \int_0^1 \int_0^{2\pi} (r^2 - r) r \, dr d\theta \\
&= 12\pi \int_0^1 (r^3 - r^2) dr = -\pi
\end{aligned}$$

Hence

$$a(v, v) = \pi - \pi = 0$$

as wanted.

(a) We have to check that $u(r, \theta) = -r^2(\log r \sin 2\theta)$ is a solution to the problem

$$-\Delta u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

We start checking that $u = 0$ on $\partial\Omega$. The boundary of Ω is the union of the sets:

- $\{r = 1, 0 < \theta < \frac{\pi}{2}\}$: $u = 0$ here since $\log 1 = 0$
- $\{0 \leq r \leq 1, \theta = 0\}$: $u = 0$ since $\sin\theta = 0$
- $\{0 \leq r \leq 1, \theta = \frac{\pi}{2}\}$: $u = 0$ since $\sin 2\theta = 0$

Now let us check that $-\Delta u = f$. The Laplacian in polar coordinates is

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

We have:

$$\begin{aligned}
\partial_r u &= \partial_r(-r^2 \log r \sin 2\theta) \\
&= (-2r\log r - r)\sin 2\theta \\
&= -(2\log r + 1)r \sin 2\theta \\
\partial_r^2 u &= \partial_r(-(2\log r + 1)r\sin 2\theta) \\
&= -((2\log r + 1) + 2)\sin 2\theta \\
&= -(2\log r + 3)\sin 2\theta \\
\partial_\theta^2 u &= -r^2 \log r \partial_\theta^2 \sin 2\theta \\
&= 4r^2 \log r \sin 2\theta
\end{aligned}$$

So

$$\begin{aligned}
-\Delta u &= (2\log r + 3 + 2\log r + 1 - 4\log r)\sin 2\theta \\
&= 4\sin 2\theta = f(r, \theta).
\end{aligned}$$

(b) We have to show that $u_{rr} \in L^2(\Omega)$. We have:

$$u_{rr} = (3 + 2\log r)\sin 2\theta$$

So

$$
\begin{aligned}
\|u_{rr}\|^2_{L^2(\Omega)} &= \int_\Omega (3+2\log r)^2 \sin^2 2\theta \, r dr d\theta \\
&= \int_0^1 (3+2\log r)^2 r dr \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \\
&= \frac{\pi}{4} \int_0^1 (3+2\log r)^2 r dr \\
&= \frac{\pi}{4} \left(\frac{r^2}{2} + 2r^2 \log r \right)\Big|_0^1 \\
&= \frac{\pi}{4} \left(\frac{1}{2} - \lim_{r\to 0} \left(\frac{r^2}{2} + 2r \log r \right) \right) \qquad (*) \\
&= \frac{\pi}{8} \\
&< \infty
\end{aligned}
$$

Note that the second of $(*)$ is 0 as $r\text{log}r \to 0$ as $r \to 0 \Rightarrow u_{rr} \in L^2(\Omega)$

(c) Let us show that $u_{rrr} \notin L^2(\Omega)$, so in particular $u \notin H^3(\Omega)$. We have;

$$
u_{rrr} = \frac{2\sin 2\theta}{r}
$$

So

$$
\begin{aligned}
\|u_{rrr}\|^2_{L^2(\Omega)} &= 4\int_\Omega \frac{\sin^2 2\theta}{r^2} r dr d\theta \\
&= 4\int_0^1 \frac{dr}{r} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta \\
&= \pi \int_0^1 \frac{dr}{r} \quad \text{(the integral diverges)} \\
&= \infty
\end{aligned}
$$

$\Rightarrow u_{rrr} \notin L^2(\Omega)$.

(a)
The weak formulation of the problem is

$$\int_0^1 (u'v' + \beta u'v)\,dx = \int_0^1 fv\,dx \quad (*) \qquad \forall v \in H_0^1(0,1) \quad \text{with } u \in H_0^1(0,1)$$

$l.h.s.\ of\ (*) = a(u,v) \qquad r.h.s.\ of\ (*) = l(v)$

First, we have

$$\begin{aligned}
|a(w,v)| &= \left| \int_0^1 (w'v' + \beta w'v)\,dx \right| \\
&\le \int_0^1 (|w'v'| + |\beta w'v|)\,dx \\
&= \int_0^1 |w'||v'|\,dx + |\beta| \int_0^1 |w'||v|\,dx \\
(CS) \Rightarrow &\le \left(\int_0^1 |w'|^2 \int_0^1 |v'|^2\,dx \right)^{\frac{1}{2}} + |\beta| \left(\int_0^1 |w'|^2\,dx \right)^{\frac{1}{2}} \left(\int_0^1 |v|^2\,dx \right)^{\frac{1}{2}} \\
&= \|w'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)} + |\beta| \|w'\|_{L^2(0,1)} \|v\|_{L^2(0,1)} \\
&= \|w'\|_{L^2(0,1)} (\|v'\|_{L^2(0,1)} + |\beta| \|v\|_{L^2(0,1)}) \\
&\le \max(1,|\beta|) \|w'\|_{L^2(0,1)} (\|v'\|_{L^2(0,1)} + \|v\|_{L^2(0,1)}) \\
&= 2\max(1,|\beta|) \|w\|_{H^1(0,1)} \|v\|_{H^1(0,1)} \qquad (\|v'\|_{L^2} \le \|v\|_{H^1} \text{ and } \|v\|_{L^2} \le \|v\|_{H^1} \Rightarrow 2)
\end{aligned}$$

Hence we can take $C_1 = 2\ \max(1,|\beta|)$.

Second, we have

$$\begin{aligned}
a(v,v) &= \int_0^1 \left((v')^2 + \beta v v' \right)\,dx \\
&= \int_0^1 (v')^2\,dx + \frac{1}{2}\beta \int_0^1 \frac{d}{dx}(v^2)dx \\
&= \int_0^1 (v')^2\,dx + \frac{1}{2}\beta [v^2]_0^1 \qquad \text{(2nd term = 0 by boundary conditions)} \\
&= \|v'\|_{L^2(0,1)}^2
\end{aligned}$$

Using Poincaré's inequality

$$\|v\|_{L^2(0,1)} \le \frac{1}{\pi} \|v'\|_{L^2(0,1)}$$

we have

$$\begin{aligned}
\|v\|_{H^1(0,1)}^2 &= (\|v\|_{L^2(0,1)}^2 + \|v'\|_{L^2(0,1)}^2)^{\frac{1}{2}} \\
&\le \left(\frac{1}{\pi^2} + 1 \right) \|v'\|_{L^2(0,1)}^2
\end{aligned}$$

So

$$a(v,v) \ge \left(\frac{1}{\pi^2} + 1 \right) \|v\|_{H^1(0,1)}^2$$

Thus we may take

$$C_0 = \left(\frac{1}{\pi^2} + 1 \right)^2$$

(b)
Since $u \in H^2(0,1)$, from the fact that

$$\int_0^1 u\phi''\,dx \overset{parts}{=} -\int_0^1 u'\phi'\,dx$$

$$weak\ \ formulation \Rightarrow \quad = \int_0^1 (\beta u' - f)\phi\,dx \qquad \forall \phi \in C_0^\infty(0,1)$$

We deduce that

$$u'' = \beta u' - f$$

Hence

$$\begin{aligned} \|u''\|_{L^2} &= \|\beta u' - f\|_{L^2(0,1)} \\ &\leq (|\beta|^2 \|u'\|^2_{L^2(0,1)} + \|f\|^2_{L^2(0,1)})^{\frac{1}{2}} \end{aligned}$$

To bound $\|u'\|_{L^2(0,1)}$, we use that

$$a(u,u) = \|u'\|^2_{L^2(0,1)}$$

(see part (a))

We have

$$\begin{aligned} \|u'\|^2_{L^2(0,1)} &= a(u,u) \\ &= \ell(u) \\ &= \int_0^1 fu\,dx \\ &\leq \int_0^1 |fu|\,dx \\ (CS) \Rightarrow &\leq \left(\int_0^1 |f|^2\,dx\right)^{\frac{1}{2}} \left(\int_0^1 |u|^2\,dx\right)^{\frac{1}{2}} \\ &= \|f\|_{L^2(0,1)} \|u\|_{L^2(0,1)} \\ (Poincar\acute{e}'s\ \ inequality) \Rightarrow &\leq \|f\|_{L^2(0,1)} \left(\frac{1}{\pi}\|u'\|_{L^2(0,1)}\right) \end{aligned}$$

So

$$\|u'\|_{L^2(0,1)} \leq \frac{1}{\pi}\|f\|_{L^2(0,1)}$$

Hence

$$\|u''\|_{L^2(0,1)} = \left(|\beta|^2 + 1\right)^{\frac{1}{2}} \|f\|_{L^2(0,1)}$$

Therefore we could take $C_r = (|\beta|^2 + 1)^{1/2}$.

(c)
From the lectures we know the bound

$$\|u - u_h\|_{H^1(0,1)} \leq Ch\|u''\|_{L^2(0,1)}$$

where the constant C is

$$C = \frac{C_1(1 + \frac{1}{\pi})}{C_0 \pi}$$

Since the constant C_0 does not depend on β and $C_1 = 2\max(1, |\beta|)$, we have

$$\|u - u_h\|_{H^1(0,1)} \leq \overline{C}\max(1, |\beta|)\|u''\|_{L^2(0,1)}$$

where $\overline{C}$ does not depend on β.

Combining this with the bound of part (b) we get

$$\|u - u_h\|_{H^1(0,1)} \leq \overline{C}(1 + |\beta|^2)^{\frac{1}{2}}\max(1, |\beta|)h\|f\|_{L^2(0,1)}$$

In particular, when $\beta \to \infty$, the constant grows quadratically in β.

(a)
The finite element space is the span of all hat functions that belong to the solution space of the weak formulation of the problem.

Integrating by parts the 1st integral in

$$\int_0^1 (-u'' + u)v\,dx = \int_0^1 fv\,dx$$

We obtain the weak formulation

$$\int_0^1 (u'v' + uv)\,dx = \int_0^1 fv\,dx + gv(1)$$

with $v \in H^1(0,1)$ such that $v(0) = 0$.

The solution space is thus

$$H^1_{E_0}(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$$

The hat functions

$$\phi_i = \max\left(1 - \frac{|x - x_i|}{n}, 0\right)$$

that are in $H^1_{E_0}$ are

$$\{\phi_1, \dots, \phi_{N+1}\}$$

Note: $\phi_0 \notin H^1_0$ since $\phi_0(0) \neq 0$

Hence the finite element space is spanned by

$$\{\phi_i\}_{i=1}^{N+1}$$

and has dimension $N + 1$.

(b)
The Galerkin approximation

$$u_h = \sum_{j=1}^{N} U_j \phi_j$$

is the solution to

$$A\underline{u} = \underline{f}$$

where

$$A = (A_{ij}) \qquad A_{ij} = a(\phi_j, \phi_i)$$
$$\underline{f} = (f_i) \qquad f_i = \ell(\phi_i)$$
$$\underline{u} = (U_j)$$

The bilinear form $a(\cdot,\cdot)$ is

$$a(u, v) = \int_0^1 (u'v' + uv)\,dx$$

$\Rightarrow$

$$A_{ij} = \int_0^1 (\phi_j'\phi_i' + \phi_j\phi_i)\,dx$$

When $i = j$, we have

$$A_{ii} = \int_0^1 |\phi_i'|^2\,dx + \int_0^1 (\phi_i)^2\,dx$$

$$= \frac{2}{h} + \int_{x_i-h}^{x_i+h} \left(1 - \frac{|x-x_i|}{h}\right)^2 dx$$

$$= \frac{2}{h} + \int_{x_i-h}^{x_i} \left(1 - \frac{x_i - x}{h}\right)^2 dx + \int_{x_i}^{x_i+h} \left(1 - \frac{x - x_i}{h}\right)^2 dx$$

$$= \frac{2}{h} + \frac{h}{3} + \frac{h}{3} = \frac{2}{h} + \frac{2h}{3}$$

When $j = i - 1$, we have

$$A_{i,i-1} = \int_0^1 \phi_i' \phi_{i-1}' \, dx + \int_0^1 \phi_i \phi_{i-1} \, dx$$

$$= -\frac{1}{h} + \int_{x_i-h}^{x_i} (1 - \frac{x_i - x}{h})(1 - \frac{x - (x_i - h)}{h}) dx = -\frac{1}{h} + \frac{h}{6}$$

When $j = i + 1$, we find again

$$A_{i,i+1} = -\frac{1}{h} + \frac{h}{6}$$

Hence the i-th component of $A\underline{u}$ is

$$(A\underline{u})_i = \sum_{j=1}^{N+1} A_{ij} U_j$$

$$= \left(-\frac{1}{h} + \frac{h}{6}\right)(U_{i-1} + U_{i+1}) + \left(\frac{2}{h} + \frac{2h}{3}\right) U_i$$

$$= \frac{-U_{i-1} + 2U_i - U_{i+1}}{h} + \frac{h}{6}(U_{i-1} + 4U_i + U_{i+1})$$

On the other hand

$$\ell(v) = \int_0^1 fv \, dx + gv(1)$$

So for $j = 1, \ldots, N$ we have

$$\ell(\phi_j) = \int_0^1 f\phi_j \, dx + g\phi_j(1) \qquad \phi_j(1) : \text{nonzero only iff } j = N+1$$

$$= \int_{x_{i-h}}^{x_{i+h}} f\phi_j \, dx$$

Hence the system of equations for $i = 1, \ldots, N$ is

$$\frac{-U_{i-1} + 2U_i - U_{i+1}}{h} + \frac{U_{i+1} - U_{i-1}}{2} = \int_{x_{i-h}}^{x_{i+h}} f\phi_i \, dx$$

(with the convention $U_0 = 0$)

For $i = N + 1$ we have

$$A_{N+1,N+1} = \int_0^1 (\phi_{N+1}')^2 \, dx + \int_0^1 (\phi_{N+1})^2 \, dx$$

$$= \frac{1}{h} + \int_{1-h}^1 \left(1 - \frac{1-x}{h}\right)^2 dx$$

$$= \frac{1}{h} + \frac{h}{3}$$

$$
\begin{aligned}
A_{N+1,N} &= \int_0^1 \phi'_{N+1}\phi'_N \, dx + \int_0^1 \phi_{N+1}\phi_N \, dx \\
&= -\frac{1}{h} + \int_{1-h}^1 \left(1 - \frac{1-x}{h}\right)\left(1 - \frac{x-(1-h)}{h}\right) dx \\
&= -\frac{1}{h} + \frac{h}{6}
\end{aligned}
$$

So the corresponding equation is

$$\frac{U_{N+1} - U_N}{h} + \frac{h}{6}(2U_{N+1} + U_N) = \int_0^1 f\phi_{N+1} \, dx + g$$

(c)
The general solution to the equation

$$-u'' + u = 1$$

is

$$u(x) = ae^x + be^{-x} + 1$$

So imposing the initial conditions we get

$$\begin{cases} 0 = u(0) = a + b + 1 \\ 0 = u'(1) = ae - ce^{-1} \end{cases}$$

So

$$a = -\frac{1}{1+e^2} \qquad b = -\frac{e^2}{1+e^2}$$

Thus the solution is

$$
\begin{aligned}
u(x) &= 1 - \frac{e}{1+e^2}\left(e^{x-1} + e^{1-x}\right) \\
&= 1 - \frac{2e}{1+e^2}\cosh(x-1)
\end{aligned}
$$

If $h = \frac{1}{2}$, then $N = 1$, so we have to solve the system of 2 equations

$$\frac{2U_1 - U_2}{h} + \frac{h}{6}(4U_1 + U_2) = \int_0^1 \phi_1 \, dx = h$$

$$\frac{U_2 - U_1}{h} + \frac{h}{6}(2U_2 + U_1) = \int_0^1 \phi_2 \, dx = \frac{h}{2}$$

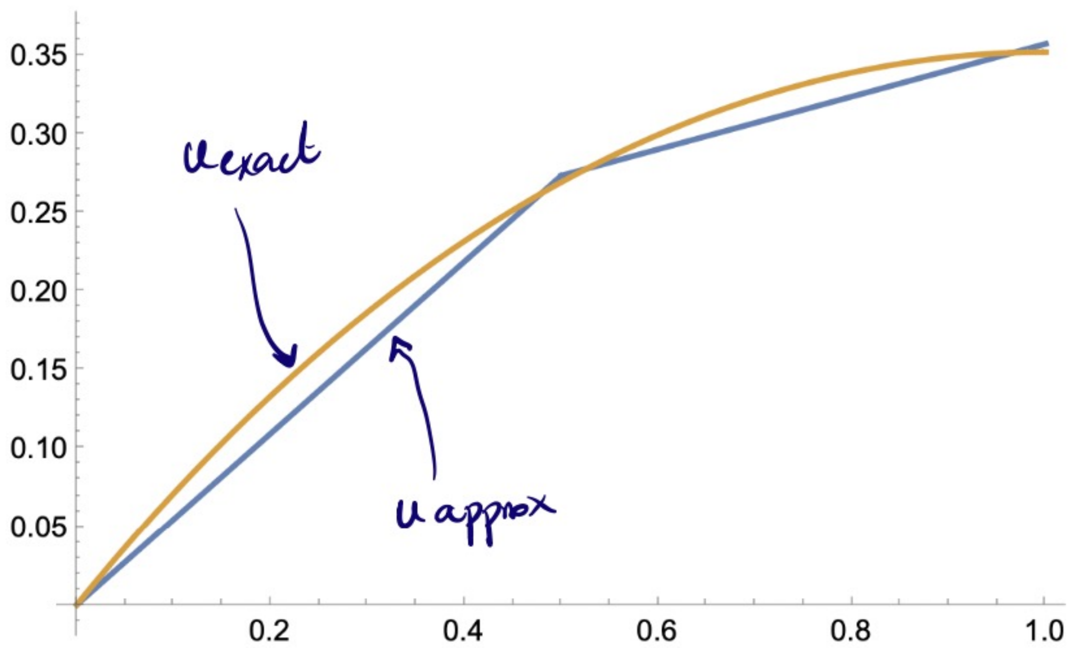
(with $h = \frac{1}{2}$)
Solving for U_1, U_2 we get

$$U_1 = \frac{225}{823} \qquad U_2 = \frac{294}{823}$$

So

$$U(x) = U_1\phi_1 + U_2\phi_2$$

is the approximation to the exact solution.



0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.2
0.4
0.6
0.8
1.0
uexact
u approx