NOTE ON STABILIZATION PARAMETERS

This document describes in detail the choice of the stabilization parameters in the numerical experiments of the paper "A stabilized hybridized Nitsche method for sign-changing elliptic PDEs" by Erik Burman, Alexandre Ern and Janosch Preuss.

1. The method

We recall the finite element spaces

$$\hat{V}_h := (V^k_{h,+} \times V^k_{h,-}) \times V^k_{h,\Gamma}, \qquad \hat{V}^*_h := (V^{k^*}_{h,+} \times V^{k^*}_{h,-}) \times V^{k^*_\Gamma}_{h,\Gamma}, \tag{1.1}$$

and the method:

Find $(\hat{u}_h, \hat{z}_h) \in \hat{V}_h \times \hat{V}^*_h$ such that, for all $(\hat{w}_h, \hat{y}_h) \in \hat{V}_h \times \hat{V}^*_h$,

$$B[(\hat{u}_h, \hat{z}_h); (\hat{w}_h, \hat{y}_h)] = \sum_{\pm} (f_\pm, z_{h,\pm})_{\Omega_\pm} + \sum_{\pm} \sum_{T \in \mathcal{T}_h^\pm} \gamma^{\mathrm{LS}}_\pm \frac{h^2}{|\sigma_\pm|} (f_\pm, \mathcal{L}_\pm(w_\pm))_T. \tag{1.2}$$

Here,

$$B[(\hat{v}_h, \hat{z}_h); (\hat{w}_h, \hat{y}_h)] := a[\hat{w}_h, \hat{z}_h] + a[\hat{v}_h, \hat{y}_h] + s[\hat{v}_h, \hat{w}_h] - \tilde{s}(z_h, y_h).$$

with

$$a[\hat{u}_h, \hat{z}_h] := \sum_{\pm} a_\pm[(u_{h,\pm}, u_{h,\Gamma}); (z_{h,\pm}, z_{h,\Gamma})],$$

for

$$\begin{aligned} &a_\pm[(u_{h,\pm}, u_{h,\Gamma}); (v_{h,\pm}, v_{h,\Gamma})] := (\sigma_\pm \nabla u_{h,\pm}, \nabla v_{h,\pm})_{\Omega_\pm} + (\mu_\pm u_{h,\pm}, v_{h,\pm})_{\Omega_\pm} \\ &- (\sigma_\pm \nabla u_{h,\pm} \cdot \mathbf{n}_\pm, v_{h,\pm} - v_{h,\Gamma})_\Gamma - (\sigma_\pm \nabla v_{h,\pm} \cdot \mathbf{n}_\pm, u_{h,\pm} - u_{h,\Gamma})_\Gamma \\ &+ \frac{\lambda_\pm |\sigma_\pm|}{h} (u_{h,\pm} - u_{h,\Gamma}, v_{h,\pm} - v_{h,\Gamma})_\Gamma, \end{aligned} \tag{1.3}$$

The stabilization terms are given by

$$s[\hat{v}_h, \hat{w}_h] := \sum_{\pm} s_\pm[(v_{h,\pm}, v_{h,\Gamma}); (w_{h,\pm}, w_{h,\Gamma})], \qquad \tilde{s}(z_h, y_h) := \sum_{\pm} \tilde{s}_\pm(z_{h,\pm}; y_{h,\pm}), \tag{1.4}$$

where

$$\tilde{s}_\pm(z_{h,\pm}, y_{h,\pm}) := \gamma^*_\pm |\sigma_\pm| (\nabla z_{h,\pm}, \nabla y_{h,\pm})_{\Omega_\pm} + \tilde{\gamma}^*_\pm \tilde{\mu}_\pm (z_{h,\pm}, y_{h,\pm})_{\Omega_\pm},$$

and

$$s_\pm[(v_{h,\pm}, v_{h,\Gamma}); (w_{h,\pm}, w_{h,\Gamma})] := \sum_{T \in \mathcal{T}_h^\pm} \gamma^{\mathrm{LS}}_\pm \frac{h^2}{|\sigma_\pm|} \left(\mathcal{L}_\pm(v_{h,\pm}), \mathcal{L}_\pm(w_{h,\pm})\right)_T$$

$$+\gamma_\pm^{\mathrm{CIP}} \sum_{F\in\mathcal{F}_h^\pm} |\sigma_\pm|\, h\,(\llbracket\nabla v_{h,\pm}\rrbracket_F\cdot\mathbf{n}_F, \llbracket\nabla w_{h,\pm}\rrbracket_F\cdot\mathbf{n}_F)_F$$

$$+\gamma_\pm^\Gamma \frac{|\sigma_\pm|}{h}(v_{h,\pm}-v_{h,\Gamma}, w_{h,\pm}-w_{h,\Gamma})_\Gamma, \tag{1.5}$$

2. Choice of parameters

We decribe now in detail the choice of the parameters

$$\lambda_\pm, \quad \gamma_\pm^{\mathrm{CIP}}, \quad \gamma_\pm^{\mathrm{LS}}, \quad \gamma_\pm^\Gamma, \quad \gamma_\pm^*, \quad \tilde{\gamma}_\pm^*$$

for each numerical experiment performed in the paper.

- **4.1 Symmetric cavity: Figure 2**

  We set $\tilde{\gamma}_\pm^* = 0$ and

$$\text{For } k=1: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 200,\ \gamma_\pm^* = \frac{10^{-3}}{|\sigma_\pm|}$$

$$\text{For } k=2: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 5\cdot 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 5\cdot 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 1,\ \gamma_\pm^* = \frac{5\cdot 10^{-1}}{|\sigma_\pm|}$$

$$\text{For } k=3: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 5\cdot 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 5\cdot 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 50,\ \gamma_\pm^* = \frac{10^{-1}}{|\sigma_\pm|}$$

- **4.1 Symmetric cavity: Figure 3**

  We set $\tilde{\gamma}_\pm^* = 0$ and

$$\text{For } k=1: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 200,\ \gamma_\pm^* = \frac{10^{-3}}{|\sigma_\pm|}$$

$$\text{For } k=2: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 5\cdot 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 5\cdot 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 1,\ \gamma_\pm^* = \frac{8\cdot 10^{-2}}{|\sigma_\pm|}$$

$$\text{For } k=3: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 5\cdot 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 5\cdot 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 50,\ \gamma_\pm^* = \frac{10^{-1}}{|\sigma_\pm|}$$

- **4.2 Metamaterial Figure 4, 5 and 6**

$$\text{For } k=1: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 10^{-6},\ \gamma_\pm^{\mathrm{LS}} = 10^{-6}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 10^{-2},\ \gamma_\pm^* = \frac{10^{-5}}{|\sigma_\pm|}$$

$$\text{For } k=2: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 10^{-6},\ \gamma_\pm^{\mathrm{LS}} = 10^{-6}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 10^{-2},\ \gamma_\pm^* = \frac{10^{-5}}{|\sigma_\pm|}$$

$$\text{For } k=3: \quad \lambda_\pm = 20\cdot k^2,\ \gamma_\pm^{\mathrm{CIP}} = 10^{-5},\ \gamma_\pm^{\mathrm{LS}} = 10^{-5}|\sigma_\pm|,\ \gamma_\pm^\Gamma = 10^{-2},\ \gamma_\pm^* = \frac{10^{-6}}{|\sigma_\pm|}$$

  and

$$\tilde{\gamma}_\pm^* = \begin{cases} 0 & \text{if } \tilde{\mu}_\pm = 0, \\ \frac{\gamma_\pm^* |\sigma_\pm|}{\tilde{\mu}_\pm} & \text{else.} \end{cases}$$

  The term

$$\frac{h}{\sigma_\sharp}\left(\tilde{\Pi}_\Gamma^{h,l}(\llbracket\sigma\nabla\tilde{u}_h\rrbracket\cdot\tilde{\mathbf{n}}_{\Gamma,h}) - \llbracket\sigma\nabla\tilde{u}_h\rrbracket\cdot\tilde{\mathbf{n}}_{\Gamma,h}, \tilde{\Pi}_\Gamma^{h,l}(\llbracket\sigma\nabla\tilde{v}_h\rrbracket\cdot\tilde{\mathbf{n}}_{\Gamma,h}) - \llbracket\sigma\nabla\tilde{v}_h\rrbracket\cdot\tilde{\mathbf{n}}_{\Gamma,h}\right)_{\tilde{\Gamma}_h}.$$

is added without additional scaling.

- **4.3 Non-symmetric cavity: Figure 7**

  We set $\tilde{\gamma}_{\pm}^{*} = 0$ and

  $$\text{For } k = 2: \quad \lambda_{\pm} = 20 \cdot k^2,\ \gamma_{\pm}^{\text{CIP}} = 5 \cdot 10^{-3},\ \gamma_{\pm}^{\text{LS}} = 5 \cdot 10^{-3} |\sigma_{\pm}|,\ \gamma_{\pm}^{\Gamma} = 200,\ \gamma_{\pm}^{*} = \frac{10^{-3}}{|\sigma_{\pm}|}.$$

  As stated in the article, we also perform an experiment with the additional stabilization term

  $$5 \cdot 10^{-2} \sum_{F \in \mathcal{F}_h^{\pm}} h^3 \left([\![D^2 u_{\pm}]\!]_F, [\![D^2 v_{\pm}]\!]_F \right)_F .$$