Kalman Filter with SAN

Here is a proposed Kalman filter regularization of the SAN wavefront control method. The basic idea of a Kalman filter involves iterations of predict and update. During the predict phase, we predict the evolved state before measurement, based on how we know the state changes and the control input. During the update phase, we take a measurement, and update our prediction based on the measurement. The updated state (plus the new control input) then becomes the prediction for next iteration.

First, we need to define the matrices used in the iterative filtering process. A subscript of k, indicating k-th iteration, is omitted.

	Definition	**Function**
1	$x = \begin{pmatrix} p \\ q \end{pmatrix}$	state vector (unknown). p, q are the speckle EF parameters derived in SAN.
2	$\Phi = I$	state transition matrix: models how the state evolves. Here we assume the state is constant.
3	$\hat{x}^-, \hat{x}^+$	state estimate, before and after measurement.
4	$u = -\hat{x}^+$	control input. This is the anti-speckle we apply to null the speckle.
5	$\Gamma = I$	propagation matrix: convert control input into how it changes the state vector.
6	$z = Hx + N$	measurement. N is the measurement noise.
7	$H = I$	linear observation matrix: convert measurement into how it changes the state vector.
8	K	The Kalman gain. Based on the state covariance and measurement noise, it tells the algorithm to trust the measurement more or the prediction more.
9	$P^-, P^+ = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_q^2 \end{pmatrix}$	state covariance matrix, before and after measurement.
10	Q	process noise matrix: noise from DM actuators.
11	$R = \begin{pmatrix} N_p^2 & 0 \\ 0 & N_q^2 \end{pmatrix}$	measurement noise matrix

p and q are defined, for the 5 intensity frames, as:

$$p = \frac{I_1^+ - I_1^-}{2\left(I_1^+ + I_1^- - 2I_0\right)},\ q = \frac{I_2^+ - I_2^-}{2\left(I_2^+ + I_2^- - 2I_0\right)}.$$

The Kalman gain K is defined as:

$$K = P^- H^T \left[H P^- H^T + R\right]^{-1} = P^- \left[P^- + R\right]^{-1}.$$

From *Handbook of CCD Astronomy*, the measurement noise N is defined as:

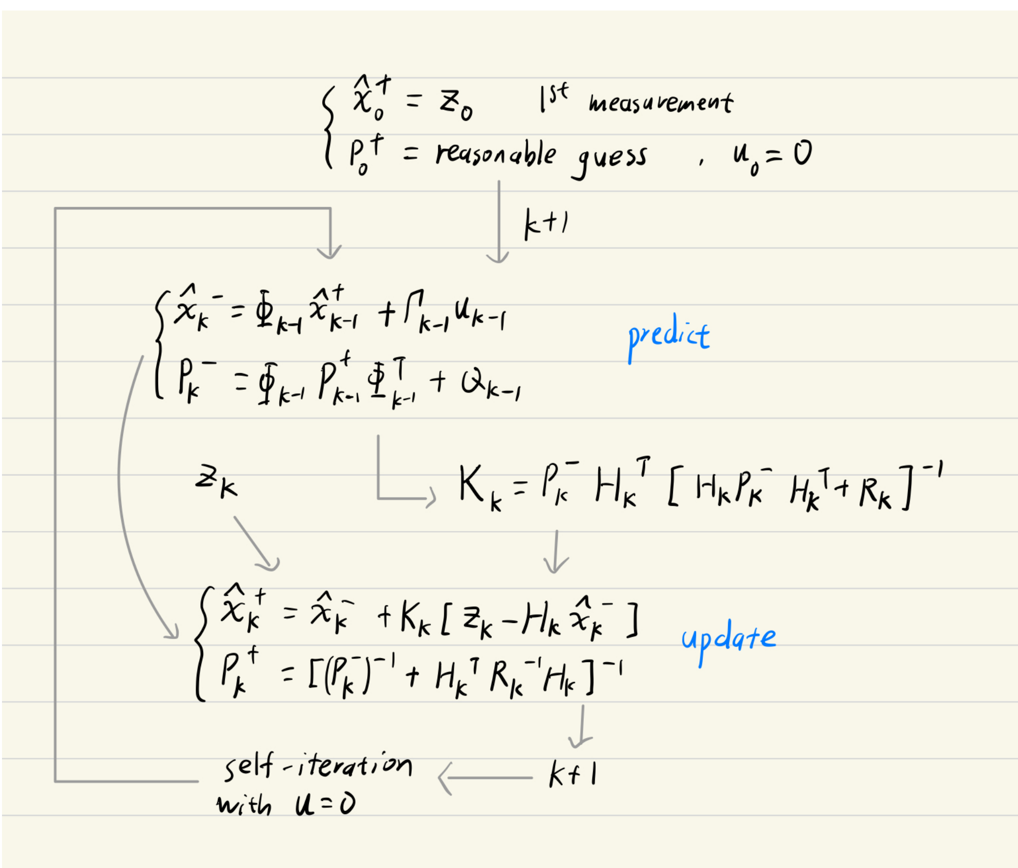
$$N = \sqrt{N_* + n_{pix}(N_S + N_D + N_R^2)}, \tag{1}$$

where N_* is the total number of photons collected from object of interest, n_{pix} is the number of pixels under consideration (in SAN this would usually be 1), N_s is the total number of photons/pixel from background/sky, N_D is the total number of dark current electrons/pixel, N_R is the total number of electrons/pixel from read noise.

For the read noise, we need two bias frames B_1, B_2 and two flat field images F_1, F_2. By measuring the standard deviation of image differences, the read noise is:

$$\text{Read noise} = \frac{\text{Gain} \cdot \sigma_{B_1 - B_2}}{\sqrt{2}},\ \text{Gain} = G = \frac{(\bar{F}_1 + \bar{F}_2) - (\bar{B}_1 + \bar{B}_2)}{\sigma^2_{F_1 - F_2} - \sigma^2_{B_1 - B_2}}$$

With everything defined, the iteration of Kalman filter can be summarized as follows:



The self-iteration step is use to deal with nonlinearity when linearizing the field to get Γ, H. It is simply repeating the iteration with $u = 0$ and z_k constant again and again, only stopping when the change of $\hat{x}$ is small. Then we can apply the control input and proceed in the cycle. Since Γ, H are identity in our case, we can ignore this step.

1 Estimating P_0 and R

The DM actuator process noise could be set to a small number since our case is dominated by the measurement noise. Thus, we are left with estimating the initial state covariance P_0 and the measurement noise R.

The simplest estimate for state covariance is $P_0 = aR,\ a \sim 10$, since the state components p and q are calculated from measured frames. a is set to inflate the number to give the filter some room for regularization. A better way would be to use the error-propagation formula. Let the numerator and denominator of p be α, β, respectively, we have:

$$\sigma_p^2 \simeq \left(\frac{\partial p}{\partial I_1^+}\right)^2 \sigma_{I_1^+}^2 + \left(\frac{\partial p}{\partial I_1^-}\right)^2 \sigma_{I_1^-}^2 + \left(\frac{\partial p}{\partial I_0}\right)^2 \sigma_{I_0}^2 \tag{2}$$

$$\frac{\partial p}{\partial I_1^+} = \frac{\beta_p - 2\alpha_p}{\beta_p^2}, \quad \frac{\partial p}{\partial I_1^-} = \frac{-\beta_p + 2\alpha_p}{\beta_p^2}, \quad \frac{\partial p}{\partial I_0} = \frac{4\alpha_p}{\beta_p^2},$$

and similarly for q.

For the variances of the intensity frames, we could either use the measurement noise N^2. But this simply makes $P_0 = R$, so we'd rather return to the simple estimate $P_0 = aR$. Or, we can calculate the variance from half-exposures as proposed in CDI-SAN:

$$\sigma_{I_1^+}^2 = \langle (I_{1a}^+ - I_{1b}^+)^2 \rangle / 4, \text{ where } \quad I_1^+ = (I_{1a}^+ + I_{1b}^+)/2.$$

For measurement noise R, it would be quite straightforward if we have dark frame, bias frame, and flat field frame. First, use equations (1) to calculate N for each image. Then plug them into (2), replacing σ_I^2 with N^2, to get N_p^2, N_q^2 and thus R.

If we only have the 5 intensity frames available, we could approximate the noise by choosing a small dark region on the corner of the image, calculate its variance V, and treat it as per pixel noise contribution $N_S + N_D + N_R^2$. Now, this noise is in the unit of ADU^2, and we still have a remaining N_* contribution in (1). Since the variance (in unit of electron squared) for the signal is N_*, the variance in ADU^2 is: $N_*/G^2 = GI/G^2 = I/G$. Thus the noise for one image is:

$$N^2 = I/G + V.$$

Now we can plug this N^2 into (2) to get R.

The main goal of SAN regularization is to deal with the case where the denominator β is small. Thus, covariance P_0 should be large when β is small, which tells the Kalman filter to trust this measurement less. The above estimates include a β^2 dependence in the denominator, which is what we want.