# CSE 114A: Fall 2021

# Introduction to Functional Programming

*Lambda Calculus*

Owen Arden
UC Santa Cruz

*Based on course materials developed by Ranjit Jhala*

## Your favorite language

- Probably has lots of features:
  - Assignment (x = x + 1)
  - Booleans, integers, characters, strings,...
  - Conditionals
  - Loops, return, break, continue
  - Functions
  - Recursion
  - References / pointers
  - Objects and classes
  - Inheritance
  - … and more



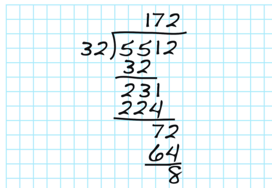
## Your favorite language

- Probably has lots of features:
  - Assignment (x = x + 1)
  - Booleans, integers, characters, strings,...
  - Conditionals
  - Loops, return, break, continue
  - F
  - R
  - References / pointers
  - Objects and classes
  - Inheritance
  - … and more

Which ones can we do without?
What is the smallest universal language?

## What is computable?

- Prior to 1930s
  - Informal notion of an effectively calculable function:

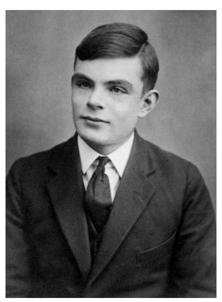



One that can be computed by a human with pen and paper, following an algorithm

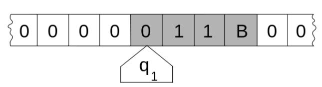


## What is computable?

- 1936: Formalization



Alan Turing: Turing machines






## What is computable?

- 1936: Formalization

Alonzo Church: lambda calculus

```
e ::= x
    | \x -> e
    | e1 e2
```

## The Next 700 Languages

- Big impact on language design!

*Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus.*

Peter Landin, 1966



## Your favorite language

- Probably has lots of features:
  - Assignment (x = x + 1)
  - Booleans, integers, characters, strings,...
  - Conditionals
  - Loops, return, break, continue
  - Functions
  - Recursion
  - References / pointers
  - Objects and classes
  - Inheritance
  - … and more



## The Lambda Calculus

- Features
  - Functions
  - (that's it)

## The Lambda Calculus

- Seriously…
  - ~~Assignment (x = x + 1)~~
  - ~~Booleans, integers, characters, strings,…~~
  - ~~Conditionals~~
  - ~~Loops, return, break, continue~~
  - **Functions**
  - ~~Recursion~~
  - ~~References / pointers~~
  - ~~Objects and classes~~
  - ~~Inheritance~~
  - ~~… and more~~

The only thing you can do is:
**Define** a function
**Call** a function



## Describing a Programming Language

- **Syntax**
  - What do programs ***look like***?
- **Semantics**
  - What do programs ***mean***?
  - **Operational semantics:**
    - How do programs ***execute step-by-step?***



## Syntax: What programs look like

```
e ::= x
    | \x -> e
    | e1 e2
```

- Programs are ***expressions*** e (also called $\lambda$-terms)
- **Variable**: x, y, z
- **Abstraction** (aka nameless function definition):
  - \x -> e *"for any* x, compute e"
  - x is the *formal parameter,* e is the *body*
- **Application** (aka function call):
  - e1 e2 *"apply e1 to e2"*
  - e1 is the *function*, e2 is the *argument*

## Examples

```
-- The identity function ("for any x compute x")
\x -> x

-- A function that returns the identity function
\x -> (\y -> y)

-- A function that applies its argument to
-- the identity function
\f -> f (\x -> x)
```



## Examples

```
-- The identity function ("for any x compute x")
\x -> x

-- A function that returns the identity function
\x -> (\y -> y)

-- A function that applies its argument to
-- the identity function
\f -> f (\x -> x)
```

- How do I define a function with two arguments?
  - e.g. a function that takes x and y and returns y



## Examples

```
-- A function that returns the identity function
\x -> (\y -> y)
```

OR: a function that takes two arguments
and returns the second one!

- How do I define a function with two arguments?
  - e.g. a function that takes x and y and returns y

# Examples

- How do I apply a function to two arguments?
  - e.g. apply \x -> (\y -> y) to apple and banana?

```
-- first apply to apple, then apply the result to banana

(((\x -> (\y -> y)) apple) banana)
```



# Syntactic Sugar

- Convenient notation used as a shorthand for valid syntax

| instead of | we write |
|---|---|
| `\x -> (\y -> (\z -> e))` | `\x -> \y -> \z -> e` |
| `\x -> \y -> \z -> e` | `\x y z -> e` |
| `(((e1 e2) e3) e4)` | `e1 e2 e3 e4` |

```
\x y -> y     -- A function that that takes two arguments
              -- and returns the second one...

(\x y -> y) apple banana -- ... applied to two arguments
```



# Semantics: What programs mean

- How do I "run" or "execute" a $\lambda$-term?

- Think of middle-school algebra:

```
-- Simplify expression:
  (x + 2)*(3*x - 1)
 =
  ???
```

- **Execute** = rewrite step-by-step following simple rules until no more rules apply

## Rewrite rules of lambda calculus

1. α-step (aka renaming formals)
2. β-step (aka function call)

But first we have to talk about **scope**



## Semantics: Scope of a Variable

- The part of a program where a **variable is visible**
- In the expression \x -> e
  - x is the newly introduced variable
  - e **is the scope** of x
  - any **occurrence** of x in \x -> e is **bound** (by the **binder** \x)



## Semantics: Scope of a Variable

- For example, x is **bound** in:

```
\x -> x
\x -> (\y -> x)
```

- An occurrence of x in e is **free** if it's *not bound* by an enclosing abstraction
- For example, x is **free** in:

```
x y                -- no binders at all!
\y -> x y          -- no \x binder
(\x -> \y -> y) x  -- x is outside the scope
                   --  of the \x binder;
                   -- intuition: it's not "the same" x
```

## Free Variables

- An variable x is **free** in **e** if there exists a free occurrence of **x** in **e**
- We can formally define the set of all free variables in a term like so:

```
FV(x)       = ???
FV(\x -> e) = ???
FV(e1 e2)   = ???
```



## Free Variables

- An variable x is **free** in **e** if there exists a free occurrence of **x** in **e**
- We can formally define the set of all free variables in a term like so:

```
FV(x)       = {x}
FV(\x -> e) = FV(e) \ {x}
FV(e1 e2)   = FV(e1) ∪ FV(e2)
```



## Closed Expressions

- If e has no free variables it is said to be closed
- Closed expressions are also called **combinators**
  - **Q:** What is the *shortest* closed expression?

## Closed Expressions

- If e has no free variables it is said to be closed
- Closed expressions are also called **combinators**
  - **Q:** What is the *shortest* closed expression?
  - **A:** \x -> x



## Rewrite rules of lambda calculus

1. α-step (aka renaming formals)
2. β-step (aka function call)



## Semantics: β-Reduction

```
(\x -> e1) e2  =b>  e1[x := e2]
```

where e1[x := e2] means "e1 with all free occurrences of x replaced with e2"

- Computation by *search-and-replace*:
  - If you see an *abstraction* applied to an argument, take the *body* of the abstraction and replace all free occurrences of the *formal* by that argument
  - We say that `(\x -> e1) e2` *β-steps* to `e1[x := e2]`

## Examples

```
(\x -> x) apple
=b> apple
```

Is this right? Ask Elsa!

```
(\f -> f (\x -> x)) (give apple)
=b> ???
```



## Examples

```
(\x -> x) apple
=b> apple
```

Is this right? Ask Elsa!

```
(\f -> f (\x -> x)) (give apple)
=b> give apple (\x -> x)
```



## A Tricky One

```
(\x -> (\y -> x)) y
=b> \y -> y
```

Is this right?

**Problem**: the free `y` in the argument has been *captured* by `\y`!

**Solution**: make sure that all *free variables* of the argument are different from the *binders* in the body.

## Capture-Avoiding Substitution

- We have to fix our definition of β-reduction:

```
(\x -> e1) e2  =b>  e1[x := e2]
```

where e1[x := e2] means ~~"e1 with all free occurrences of x replaced with e2"~~

- e1 with all *free* occurrences of x replaced with e2, **as long as** no free variables of e2 get captured
- undefined otherwise



## Capture-Avoiding Substitution

Formally:

```
x[x := e]            = e
y[x := e]            = y    -- assuming x /= y
(e1 e2)[x := e]      = (e1[x := e]) (e2[x := e])
(\x -> e1)[x := e]   = \x -> e1 -- why just `e1`?

(\y -> e1)[x := e]
  | not (y in FV(e)) = \y -> e1[x := e]
  | otherwise         = undefined -- but what then???
```



## Rewrite rules of lambda calculus

1. α-step (aka renaming formals)
2. β-step (aka function call)

## Semantics: α-Reduction

```
\x -> e   =a>   \y -> e[x := y]
  where not (y in FV(e))
```

- We can rename a formal parameter and replace all its occurrences in the body
- We say that `(\x -> e)` *α-steps* to `(\y -> e[x := y])`



## Semantics: α-Reduction

```
\x -> e   =a>   \y -> e[x := y]
  where not (y in FV(e))
```

- Example:

  ```
  \x -> x   =a>   \y -> y   =a>    \z -> z
  ```

  - All these expressions are α-equivalent



## Example

What's wrong with these?

```
-- (A)
\f -> f x   =a>  \x -> x x

-- (B)
(\x -> \y -> y) y   =a>  (\x -> \z -> z) z

-- (C)
\x -> \y -> x y  =a>  \apple -> \orange -> apple orange
```

## The Tricky One

```
(\x -> (\y -> x)) y
=a> ???
```

To avoid getting confused, you can always rename formals, so that different variables have different names!



## The Tricky One

```
(\x -> (\y -> x)) y
=a> (\x -> (\z -> x)) y
=b> \z -> y
```

To avoid getting confused, you can always rename formals, so that different variables have different names!



## Normal Forms

A **redex** is a λ-term of the form

```
(\x -> e1) e2
```

A λ-term is in **normal form** if it contains no redexes.

## Semantics: Evaluation

- A λ-term e evaluates to e' if

  1. There is a sequence of stops

     ```
     e =?> e_1 =?> ... =?> e_N =?> e'
     ```

  where each =?> is either =a> or =b> and N >= 0

  2. e' is in *normal form*



## Example of evaluation

```
(\x -> x) apple
  =b> apple


(\f -> f (\x -> x)) (\x -> x)
  =?> ???



(\x -> x x) (\x -> x)
  =?> ???
```



## Example of evaluation

```
(\x -> x) apple
  =b> apple


(\f -> f (\x -> x)) (\x -> x)
  =b> (\x -> x) (\x -> x)
  =b> \x -> x


(\x -> x x) (\x -> x)
  =?> ???
```

## Example of evaluation

```
(\x -> x) apple
  =b> apple


(\f -> f (\x -> x)) (\x -> x)
  =b> (\x -> x) (\x -> x)
  =b> \x -> x


(\x -> x x) (\x -> x)
  =b> (\x -> x) (\x -> x)
  =b> \x -> x
```



## Elsa shortcuts

- Named λ-terms

```
let ID = \x -> x  -- abbreviation for \x -> x
```

- To substitute a name with its definition, use a =d> step:

```
ID apple
  =d> (\x -> x) apple  -- expand definition
  =b> apple              -- beta-reduce
```



## Elsa shortcuts

- Evaluation
  - e1 =*> e2: e1 reduces to e2 in 0 or more steps
    - where each step is =a>, =b>, or =d>
  - e1 =~> e2: e1 evaluates to e2
- *What is the difference?*

## Non-Terminating Evaluation

```
(\x -> x x) (\x -> x x)
  =b> (\x -> x x) (\x -> x x)
```

- Oh no... we can write programs that loop back to themselves
- And never reduce to normal form!
- This combinator is called Ω



## Non-Terminating Evaluation

- What if we pass Ω as an argument to another function?

```
let OMEGA = (\x -> x x) (\x -> x x)

(\x -> \y -> y) OMEGA
```

- Does this reduce to a normal form? Try it at home!



## Programming in λ-calculus

- Real languages have lots of features
  - Booleans
  - Records (structs, tuples)
  - Numbers
  - **Functions** [we got those]
  - Recursion
- Let's see how to encode all of these features with the λ-calculus.

# λ-calculus: Booleans

- How can we encode Boolean values (TRUE and FALSE) as functions?
- Well, what do we **do** with a Boolean **b**?
  - We make a ***binary choice***

```
if b then e1 else e2
```



# Booleans: API

- We need to define three functions

```
let TRUE  = ???
let FALSE = ???
let ITE   = \b x y -> ???  -- if b then x else y
```

*such that*

```
ITE TRUE apple banana =~> apple
ITE FALSE apple banana =~> banana
```

(Here, `let NAME = e` means `NAME` is an *abbreviation* for `e`)



# Booleans: Implementation

```
let TRUE  = \x y -> x        -- Returns first argument
let FALSE = \x y -> y        -- Returns second argument
let ITE   = \b x y -> b x y  -- Applies cond. to branches
                             -- (redundant, but
                             --  improves readability)
```

## Example: Branches step-by-step

```
eval ite_true:
  ITE TRUE e1 e2
  =d> (\b x y -> b    x  y) TRUE e1 e2  -- expand def ITE
  =b>   (\x y -> TRUE x  y)      e1 e2  -- beta-step
  =b>     (\y -> TRUE e1 y)         e2  -- beta-step
  =b>            TRUE e1 e2             -- expand def TRUE
  =d>     (\x y -> x) e1 e2             -- beta-step
  =b>       (\y -> e1)   e2             -- beta-step
  =b> e1
```



## Example: Branches step-by-step

- Now you try it!
- Can you fill in the blanks to make it happen?
  - http://goto.ucsd.edu:8095/index.html#?demo=ite.lc

```
eval ite_false:
  ITE FALSE e1 e2

  -- fill the steps in!

  =b> e2
```



## Example: Branches step-by-step

```
eval ite_false:
  ITE FALSE e1 e2
  =d> (\b x y -> b    x  y) FALSE e1 e2  -- expand def ITE
  =b>   (\x y -> FALSE x  y)      e1 e2  -- beta-step
  =b>     (\y -> FALSE e1 y)         e2  -- beta-step
  =b>            FALSE e1 e2             -- expand def TRUE
  =d>     (\x y -> y) e1 e2              -- beta-step
  =b>       (\y -> y)    e2              -- beta-step
  =b> e2
```

## Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b     -> ???

let AND = \b1 b2 -> ???

let OR  = \b1 b2 -> ???
```



## Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b     -> ITE b FALSE TRUE

let AND = \b1 b2 -> ITE b1 b2 FALSE

let OR  = \b1 b2 -> ITE b1 TRUE b2
```



## Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b     -> b FALSE TRUE

let AND = \b1 b2 -> b1 b2 FALSE

let OR  = \b1 b2 -> b1 TRUE b2
```

  - (since ITE is redundant)
  - *Which definition to do you prefer and why?*

## Programming in λ-calculus

- Real languages have lots of features
  - **Booleans** [done]
  - Records (structs, tuples)
  - Numbers
  - **Functions** [we got those]
  - Recursion



## λ-calculus: Records

- Let's start with records with two fields (aka pairs)?
- Well, what do we **do** with a pair?

1. **Pack two** items into a pair, then
2. **Get first** item, or
3. **Get second** item.



## Pairs: API

- We need to define three functions

```
let PAIR = \x y -> ???    -- Make a pair with x and y
                          -- { fst : x, snd : y }
let FST  = \p -> ???      -- Return first element
                          -- p.fst
let SND  = \p -> ???      -- Return second element
                          -- p.snd
```

*such that*

```
FST (PAIR apple banana) =~> apple
SND (PAIR apple banana) =~> banana
```

## Pairs: Implementation

- A pair of x and y is just something that lets you pick between x and y! (I.e. a function that takes a boolean and returns either x or y)

```
let PAIR = \x y -> (\b -> ITE b x y)
let FST  = \p -> p TRUE   -- call w/ TRUE, get 1st value
let SND  = \p -> p FALSE  -- call w/ FALSE, get 2nd value
```



## Exercise: Triples?

- How can we implement a record that contains **three** values?

```
let TRIPLE = \x y z -> ???
let FST3  = \t -> ???
let SND3  = \t -> ???
let TRD3  = \t -> ???
```



## Exercise: Triples?

- How can we implement a record that contains **three** values?

```
let TRIPLE = \x y z -> PAIR x (PAIR y z)
let FST3  = \t -> FST t
let SND3  = \t -> FST (SND t)
let TRD3  = \t -> SND (SND t)
```

## Programming in λ-calculus

- Real languages have lots of features
  - **Booleans** [done]
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## λ-calculus: Numbers

- Let's start with **natural numbers** (0, 1, 2, …)
- What do we do with natural numbers?

1. **Count**: 0, inc
2. **Arithmetic**: dec, +, -, *
3. **Comparisons**: ==, <=, etc



## Natural Numbers: API

- We need to define:
  - A family of **numerals**: ZERO, ONE, TWO, THREE, ...
  - Arithmetic functions: INC, DEC, ADD, SUB, MULT
  - Comparisons: IS_ZERO, EQ

Such that they respect all regular laws of arithmetic, e.g.

```
IS_ZERO ZERO       =~> TRUE
IS_ZERO (INC ZERO) =~> FALSE
INC ONE            =~> TWO
...
```

# Pairs: Implementation

- **Church numerals:** a *number* N is encoded as a combinator that *calls a function on an argument* N *times*

```
let ONE   = \f x -> f x
let TWO   = \f x -> f (f x)
let THREE = \f x -> f (f (f x))
let FOUR  = \f x -> f (f (f (f x)))
let FIVE  = \f x -> f (f (f (f (f x))))
let SIX   = \f x -> f (f (f (f (f (f x)))))
...
```



# λ-calculus: Increment

```
-- Call `f` on `x` one more time than `n` does
let INC   = \n -> (\f x -> ???)
```

- Example

```
eval inc_zero :
  INC ZERO
  =d> (\n f x -> f (n f x)) ZERO
  =b> \f x -> f (ZERO f x)
  =*> \f x -> f x
  =d> ONE
```



# λ-calculus: Addition

```
--  Call `f` on `x` exactly `n + m` times
let ADD = \n m -> n INC m
```

- Example

```
eval add_one_zero :
  ADD ONE ZERO
  =~> ONE
```

## λ-calculus: Multiplication

```
--  Call `f` on `x` exactly `n * m` times
let MULT = \n m -> n (ADD m) ZERO
```

- Example

```
eval two_times_one :
  MULT TWO ONE
  =~> TWO
```



## Programming in λ-calculus

- Real languages have lots of features
  - **Booleans** [done]
  - **Records (structs, tuples)** [done]
  - **Numbers** [done]
  - **Functions** [we got those]
  - Recursion



## λ-calculus: Recursion

- I want to write a function that sums up natural numbers up to n:

```
\n -> ...          -- 1 + 2 + ... + n
```

## λ-calculus: Recursion

- No! Named terms in Elsa are just syntactic sugar
- To translate an Elsa term to λ-calculus: replace each name with its definition

```
\n -> ITE (ISZ n)
       ZERO
       (ADD n (SUM (DEC n))) -- But SUM is
                             -- not a thing!
```

- **Recursion**: Inside this function I want to call the same function on DEC n
- Looks like we can't do recursion, because it requires being able to refer to functions *by name*, but in λ-calculus functions are *anonymous*.
- ***Right***?



## λ-calculus: Recursion

- Think again!
- Recursion: ~~Inside this function I want to call the same function on DEC n~~
  - Inside this function I want to call a function on DEC n
  - And BTW, I want it to be the same function
- Step 1: Pass in the function to call "recursively"

```
let STEP =
  \rec ->
   \n -> ITE (ISZ n)
          ZERO
          (ADD n (rec (DEC n))) -- Call some rec
```



## λ-calculus: Recursion

- Step 1: Pass in the function to call "recursively"

```
let STEP =
  \rec ->
   \n -> ITE (ISZ n)
          ZERO
          (ADD n (rec (DEC n))) -- Call some rec
```

- Step 2: Do something clever to `STEP`, so that the function passed as `rec` itself becomes

```
\n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))
```

# λ-calculus: Fixpoint Combinator

- **Wanted**: a combinator `FIX` such that `FIX STEP` calls `STEP` with itself as the first argument:

```
FIX STEP
=*> STEP (FIX STEP)
```

(In math: a *fixpoint* of a function $f(x)$ is a point $x$, such that $f(x) = x$)

- Once we have it, we can define:

```
let SUM = FIX STEP
```

- Then by property of `FIX` we have:

```
SUM =*> STEP SUM -- (1)
```



# λ-calculus: Fixpoint Combinator

```
eval sum_one:
  SUM ONE
  =*> STEP SUM ONE                 -- (1)
  =d> (\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ONE
  =b> (\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ONE
                                   -- ^^^ the magic happened!
  =b> ITE (ISZ ONE) ZERO (ADD ONE (SUM (DEC ONE)))
  =*> ADD ONE (SUM ZERO)           -- def of ISZ, ITE, DEC, ...
  =*> ADD ONE (STEP SUM ZERO)      -- (1)
  =d> ADD ONE
        ((\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ZERO)
  =b> ADD ONE ((\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ZERO)
  =b> ADD ONE (ITE (ISZ ZERO) ZERO (ADD ZERO (SUM (DEC ZERO))))
  =b> ADD ONE ZERO
  =~> ONE
```



# λ-calculus: Fixpoint Combinator

- So how do we define `FIX`?

- Remember Ω? It *replicates itself!*

```
(\x -> x x) (\x -> x x)
=b> (\x -> x x) (\x -> x x)
```

- We need something similar but more involved.

# λ-calculus: Fixpoint Combinator

- The Y combinator discovered by Haskell Curry:

```
let FIX = \stp -> (\x -> stp (x x)) (\x -> stp (x x))
```

- How does it work?

```
eval fix_step:
  FIX STEP
  =d> (\stp -> (\x -> stp (x x)) (\x -> stp (x x))) STEP
  =b> (\x -> STEP (x x)) (\x -> STEP (x x))
  =b> STEP ((\x -> STEP (x x)) (\x -> STEP (x x)))
  --       ^^^^^^^^^^ this is FIX STEP ^^^^^^^^^^^
```



# Programming in λ-calculus

- Real languages have lots of features
  - **Booleans** [done]
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  - **Functions** [we got those]
  - **Recursion** [done]



# Next time: Intro to Haskell