CSE 114A: Fall 2021

Introduction to Functional Programming

Environments and closures

Owen Arden
UC Santa Cruz

Based on course materials developed by Nadia Polikarpova

Roadmap

Past weeks:

- How do we *use* a functional language?

Next weeks:

- How do we *implement* a functional language?
- … in a functional language (of course)

WHY??

- ***Master*** the concepts of functional languages by implementing them!
- ***Practice*** problem solving using Haskell

This week: Interpreter

- How do we *evaluate* a program given its abstract syntax tree (AST)?
- How do we *prove properties* about our interpreter (e.g. that certain programs never crash)?

The Nano Language

Features of Nano:

1. Arithmetic expressions
2. Variables and let-bindings
3. Functions
4. Recursion

Reminder: Calculator

Arithmetic expressions:

```
e ::= n
    | e1 + e2
    | e1 - e2
    | e1 * e2
```

Example:

```
4 + 13
==> 17
```


Reminder: Calculator

Haskell datatype to *represent* arithmetic expressions:

```
data Expr = Num Int
          | Add Expr Expr
          | Sub Expr Expr
          | Mul Expr Expr
```

Haskell function to *evaluate* an expression:

```
eval :: Expr -> Int
eval (Num n)     = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Sub e1 e2) = eval e1 - eval e2
eval (Mul e1 e2) = eval e1 * eval e2
```


Reminder: Calculator

Alternative representation:

```
data Binop = Add | Sub | Mul

data Expr = Num Int              -- number
          | Bin Binop Expr Expr  -- binary expression
```

Evaluator for alternative representation:

```
eval :: Expr -> Int
eval (Num n)         = n
eval (Bin Add e1 e2) = eval e1 + eval e2
eval (Bin Sub e1 e2) = eval e1 - eval e2
eval (Bin Mul e1 e2) = eval e1 * eval e2
```

The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings
3. Functions
4. Recursion

Extension: variables

Let's add variables and `let` bindings!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
```

Example:

```
let x = 4 + 13 in  -- 17
let y = 7 - 5 in   -- 2
x * y

==> 34
```


Extension: variables

Haskell representation:

```
data Expr = Num Int              -- number
          | ???                  -- variable
          | Bin Binop Expr Expr  -- binary expression
          | ???                  -- let expression
```

Extension: variables

```
type Id = String

data Expr = Num Int               -- number
          | Var Id                -- variable
          | Bin Binop Expr Expr   -- binary expression
          | Let Id Expr Expr      -- let expression
```

Haskell function to *evaluate* an expression:

```
eval :: Expr -> Int
eval (Num n)          = n
eval (Var x)          = ???
...
```


Extension: variables


```
type Id = String

data Expr = Num Int               -- number
```


How do we evaluate a variable?

We have to remember
which *value* it was bound to!


```
eval (Var x)          = ???
...
```


Environment

An expression is evaluated in an **environment**, which maps all its *free variables* to *values*

Examples:

- How should we represent the environment?
- Which operations does it support?

```
x * y
=[x:17, y:2]=> 34

x * y
=[x:17]=> Error: unbound variable y

x * (let y = 2 in y)
=[x:17]=> 34
```

Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable z

http://tiny.cc/cse116-vars-ind

Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable z

http://tiny.cc/cse116-vars-grp

Environment: API

To evaluate `let x = e1 in e2` in `env`:

- evaluate `e2` in an **extended** environment `env + [x:v]`
- where `v` is the result of evaluating `e1`

To evaluate `x` in `env`:

- **lookup** the most recently added binding for `x`

```
type Value = Int

data Env = ... -- representation not that important

-- | Add a new binding
add    :: Id -> Value -> Env -> Env

-- | Lookup the most recently added binding
lookup :: Id -> Env -> Value
```

Evaluating expressions

Back to our expressions… now with environments!

```haskell
data Expr = Num Int               -- number
          | Var Id                -- variable
          | Bin Binop Expr Expr   -- binary expression
          | Let Id Expr Expr      -- let expression
```


Evaluating expressions

Haskell function to *evaluate* an expression:

```haskell
eval :: Env -> Expr -> Value
eval env (Num n)          = n
eval env (Var x)          = lookup x env
eval env (Bin op e1 e2)   = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2)    = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```


Example evaluation

Nano expression

```
let x = 1 in
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

is represented in Haskell as:

```haskell
exp1 =   Let "x"
           (Num 1)                        exp2
           (Let "y"
             (Add                         exp3
               (Let "x" (Num 2) (Var x))
         exp4  (Var x))
             (Let "x"                     exp5
               (Num 3)
               (Add (Var x) (Var y))))
```

Example evaluation

```
eval [] exp1
=> eval []                      (Let "x" (Num 1) exp2)
=> eval [("x",eval [] (Num 1))] exp2
=> eval [("x",1)]
    (Let "y" (Add exp3 exp4) exp5)
=> eval [("y",(eval [("x",1)] (Add exp3 exp4))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",1)] (Let "x" (Num 2) (Var "x"))
          + eval [("x",1)] (Var "x"))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",2), ("x",1)] (Var "x") -- new binding for x
          + 1)), ("x",1)]
    exp5
=> eval [("y",(2  -- use latest binding for x
          + 1)), ("x",1)]
    exp5
=> eval [("y",3), ("x",1)]
    (Let "x" (Num 3) (Add (Var "x") (Var "y")))
```


Example evaluation

```
=> eval [("y",3), ("x",1)]
    (Let "x" (Num 3) (Add (Var "x") (Var “y")))
=> eval [("x",3), ("y",3), ("x",1)]              -- new binding for x
    (Add (Var "x") (Var "y"))
=> eval [("x",3), ("y",3), ("x",1)] (Var "x")
 + eval [("x",3), ("y",3), ("x",1)] (Var "y")
=> 3 + 3
=> 6
```


Example evaluation

Same evaluation in a simplified format (Haskell `Expr` terms replaced by their “pretty-printed version”):

```
   eval []
   {let x = 1 in let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:(eval [] 1)]
                {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:1]
                {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [y:(eval [x:1] {(let x = 2 in x) + x}), x:1]
                                    {let x = 3 in x + y}
=> eval [y:(eval [x:1]  {let x = 2 in x} + eval [x:1] {x}), x:1]
                                    {let x = 3 in x + y}
         -- new binding for x:
=> eval [y:(eval [x:2,x:1] {x}           + eval [x:1] {x}), x:1]
                                    {let x = 3 in x + y}
    -- use latest binding for x:
=> eval [y:(               2           + eval [x:1] {x}), x:1]
                                    {let x = 3 in x + y}
=> eval [y:(               2           + 1)            , x:1]
                                    {let x = 3 in x + y}
```

Example evaluation

```
=> eval [y:(                2           + 1)              , x:1]
                                                {let x = 3 in x + y}
=> eval [y:3, x:1]
                                                {let x = 3 in x + y}
   -- new binding for x:
=> eval [x:3, y:3, x:1]
                                                             {x + y}
=> eval [x:3, y:3, x:1] x + eval [x:3, y:3, x:1] y
   -- use latest binding for x:
=> 3 + 3
=> 6
```


Runtime errors

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value
eval env (Num n)         = n
eval env (Var x)         = lookup x env -- can fail!
eval env (Bin op e1 e2)  = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2)   = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```

How do we make sure `lookup` doesn't cause a run-time error?

Free vs bound variables

In `eval env e`, `env` must contain bindings for *all free variables* of `e`!

- an occurrence of `x` is **free** if it is not **bound**
- an occurrence of `x` is **bound** if it's inside `e2` where `let x = e1 in e2`
- evaluation succeeds when an expression is **closed**!

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

- (A) None
- (B) x
- (C) y
- (D) x y

http://tiny.cc/cse116-free-ind

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

- (A) None
- (B) x
- (C) y
- (D) x y

http://tiny.cc/cse116-free-grp

The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings **[done]**
3. Functions
4. Recursion

Extension: functions

Let's add lambda abstraction and function application!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
    | \x -> e  -- abstraction
    | e1 e2    -- application
```

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
==> 84
```


Extension: functions

Haskell representation:

```
data Expr = Num Int             -- number
          | Var Id              -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr    -- let expression
          | ???                 -- abstraction
          | ???                 -- application
```


Extension: functions

Haskell representation:

```
data Expr = Num Int             -- number
          | Var Id              -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr    -- let expression
          | Lam Id Expr         -- abstraction
          | App Expr Expr       -- application
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

represented as:

```
Let "c"
  (Num 42)
  (Let "cTimes"
    (Lam "x" (Mul (Var "c") (Var "x")))
    (App (Var "cTimes") (Num 2)))
```


Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

How should we evaluate this expression?

```
   eval []
    {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                  {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:???, c:42]
                                           {cTimes 2}
```

What is the **value** of `cTimes`???

Rethinking our values

Until now: a program *evaluates* to an integer (or fails)

```
type Value = Int

type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```

Rethinking our values

What do these programs evaluate to?

```
(1)
\x -> 2 * x
==> ???

(2)
let f = \x -> \y -> 2 * (x + y) in
f 5
==> ???
```

Conceptually, (1) evaluates to itself (not exactly, see later). while (2) evaluates to something equivalent to `\y -> 2 * (5 + y)`

Rethinking our values

Now: a program evaluates to an integer or *a lambda abstraction* (or fails)

- Remember: functions are *first-class* values

Let's change our definition of values!

```
data Value = VNum Int
           | VLam ??? -- What info do we need to store?

-- Other types stay the same
type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```


Function values

How should we represent a function value?

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

We need to store enough information about `cTimes` so that we can later evaluate any *application* of `cTimes` (like `cTimes 2`)!

First attempt:

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

Function values

Let's try this!

```
   eval []
    {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                  {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                        {cTimes 2}
    -- evaluate the function:
=> eval [cTimes:(\x -> c*x), c:42]
                                 {(\x -> c * x) 2}
    -- evaluate the argument, bind to x, evaluate body:
=> eval [x:2, cTimes:(\x -> c*x), c:42]
                                        {c * x}
=>                                      42 * 2
=>                                      84
```

Looks good… can you spot a problem?

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

- (A) 84
- (B) 10
- (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-ind

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

- (A) 84
- (B) 10
- (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-grp

Static vs Dynamic Scoping

What we want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 84
```

Lexical (or **static**) scoping:

- each occurrence of a variable refers to the most recent binding *in the program text*
- definition of each variable is unique and known *statically*
- good for readability and debugging: don't have to figure out where a variable got "assigned"

Static vs Dynamic Scoping

What we **don't** want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 10
```

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

Static vs Dynamic Scoping

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

```
let cTimes = \x -> c * x in
let c = 5 in
let res1 = cTimes 2 in -- ==> 10
let c = 10 in
let res2 = cTimes 2 in -- ==> 20!!!
res2 - res1
```

Function values

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

This representation can only implement dynamic scoping!

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
```

evaluates as:

```
   eval []
   {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
```


Function values

```
   eval []
   {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                         {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                                      {cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                               {(\x -> c * x) 2}
=> eval [x:2, c:5, cTimes:(\x -> c*x), c:42]
                                                      {c * x}
  -- latest binding for c is 5!
=>                                                     5 * 2
=>                                                     10
```

Lesson learned: need to remember what `c` was bound to when `cTimes` was defined!

- i.e. "freeze" the environment at function definition

Closures

To implement lexical scoping, we will represent function values as *closures*

Closure = *lambda abstraction* (formal + body) + *environment* at function definition

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body
```

Closures

Our example:

```
   eval []
   {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                 {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
   -- remember current env:
=> eval [cTimes:<[c:42], \x -> c*x>, c:42]
                                       {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                                    {cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                     {<[c:42], \x -> c * x> 2}
  -- restore env to the one inside the closure, then bind 2 to x:
=> eval [x:2, c:42]
                                                    {c * x}
=>                                                  42 * 2
=>                                                  84
```


QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
  in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z

http://tiny.cc/cse116-env-ind

QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
  in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z

http://tiny.cc/cse116-env-grp

Free vs bound variables

- An occurrence of `x` is **free** if it is not **bound**
- An occurrence of `x` is **bound** if it's inside
  - `e2` where `let x = e1 in e2`
  - `e` where `\x -> e`
- A closure environment has to save *all free variables* of a function definition!

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x -- a is the only free variable!
  in ...
```


Evaluator

Let's modify our evaluator to handle functions!

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body

eval :: Env -> Expr -> Value
eval env (Num n)        = VNum n -- must wrap in VNum now!
eval env (Var x)        = lookup x env
eval env (Bin op e1 e2) = VNum (f v1 v2)
  where
    (VNum v1) = eval env e1
    (VNum v2) = eval env e2
    f = ... -- as before
eval env (Let x e1 e2) = eval env' e2
  where
    v = eval env e1
    env' = add x v env
eval env (Lam x body) = ??? -- construct a closure
eval env (App fun arg) = ??? -- eval fun, then arg, then apply
```


Evaluator

Evaluating functions:

- **Construct a closure**: save environment at function definition
- **Apply a closure**: restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) = eval bodyEnv body
  where
    (VClos closEnv x body) = eval env fun -- eval function to closure
    vArg                   = eval env arg -- eval argument
    bodyEnv                = add x vArg closEnv
```

Evaluator

Evaluating functions:

- **Construct a closure:** save environment at function definition
- **Apply a closure:** restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) =
    let vArg = eval env arg in -- eval argument
    let (VClos closEnv x body) = (eval env fun) in
    let bodyEnv = add x vArg closEnv in
    eval bodyEnv body
```


Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable f

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Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable f

http://tiny.cc/cse116-enveval-grp

Evaluator

```
   eval []
     {let f = \x -> x + y in let y = 10 in f 5}
=> eval [f:<[], \x -> x + y>]
                             {let y = 10 in f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                                       {f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                           {<[], \x -> x + y> 5}
=> eval [x:5] -- env got replaced by closure env + formal!
                                    {x + y}  -- y is unbound!
```


Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

- (A) 120
- (B) Evaluation does not terminate
- (C) Error: unbound variable f

http://tiny.cc/cse116-enveval2-ind

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

- (A) 120
- (B) Evaluation does not terminate
- (C) Error: unbound variable f

http://tiny.cc/cse116-enveval2-grp

Evaluator

```
   eval []
       {let f = \n -> n * f (n - 1) in f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
                                      {f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
              {<[], \n -> n * f (n - 1)> 5}
=> eval [n:5] -- env got replaced by closure env + formal!
                         {n * f (n - 1)} -- f is unbound!
```

Lesson learned: to support recursion, we need a different way of constructing the closure environment!