Counting Techniques

Counting problems arise throughout mathematics and computer science. In computer science, for example, we need to count the number of operations used by an algorithm to study its time complexity. Under this topic, we will introduce the basic techniques of counting. These methods serve as the foundation for almost all counting techniques.

We will first discuss about two basic principles of counting.

Let's consider the following example.

Example 1. The menu for a particular restaurant is shown below.

Main courses:

Rice and curry

Noodles

Pasta.

Dessert:

Ice cream

Chocolate mousse

Marshmallow pudding

Bread Pudding

Beverages:

Orange Juice

Beer.

Part 1: How many different dinners consist of one main course and one dessert ?

For simplicity, let's denote each main course, dessert and beverage by its first letter. Then, for example, RI refers to the dinner consisting of rice and ice cream.

Following are the possible different dinners.

RI	NI	PI
RC	NC	PC
RM	NM	PM
RB	NB	PB.

Note that there are $3 \times 4 = 12$ possible dinners consisting of one main course and one dessert.

Part 2: How many different dinners consist of one main course, one dessert and one beverage ?

Note that there were 12 possible dinners consisting of one main course and one dessert. Let's take one such dinner. For example, consider RI. Because there are two beverages, you can complete the dinner in two different ways. You can have either an orange jusie with rice and ice cream, denoted by RIO, or a beer with rice and ice cream, denoted by RIB.

Since there are 12 possible dinners consisting of one main course and one dessert and for each such dinner, there are two ways to complete the dinner, there are $12 \times 2 = 24$ ways of completing the dinner.

These two simple examples illustrate our first principle of counting, namely the product rule.

The product rule / The multiplication rule :

Suppose that an activity can be broken down into a sequence of two tasks and that there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task. Then, there are $n_1 \cdot n_2$ ways to do the activity.

Example : The computer science class of a certain university consists of 55 girls and 65 boys. Each class of this university is supposed to have two representatives - one female representative and one male representative. In how many ways can two representatives be selected ?

Solution: Let's use the product rule to solve this problem.

Note that the activity is selecting two representatives, one female and one male. The first step (or task) is to select, say, a female (or a male) representative. This can be done in 55 ways as there are 55 girls. Let us denote the i^{th} girl, where $i = 1, \ldots, 55$, by G_i and the j^{th} boy, where $j = 1, \ldots, 65$, by B_j. WLOG assume G_1 has been selected. Then, the second step, that is, selecting a male (female) representative, can be completed by selecting one boy out of 65 boys. Thus, there will be 65 ways to select two representatives with G_1 as a member. This is true for each $1 \leq i \leq 55$.

Thus, by product rule, two representatives can be selected in $55 \times 65 = 3575$ ways.

Theorem 1: Suppose that an activity can be constructed in m successive steps $S_1, S_2, \ldots, S_m$ and that for each $i = 1, 2, \ldots, m$ the i^{th} step S_i can be done in n_i ways. Then, there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to complete the activity.

Proof: Use induction on m.

Example: Part 2 of Example 1 can be solved using this theorem. There, the activity is to select a dinner consisting of one main course, one dessert and one beverage. Let's take, as the first step, choosing a main course. This can be done in 3 ways. Let's take, as the second step, choosing a dessert. This can be done in 4 ways. The last step, choosing a beverage, can be done in 2 ways as there are only two beverages. Thus, there are $3 \times 4 \times 2 = 24$ different dinners consist of one main course, one dessert and one beverage.

Remark: Observe that for each choice of a main course, there are 4 choices for the dessert. Also, for each choice of a main course and a dessert, there are 2 choices for the beverage. Thus, the product $3 \times 4 \times 2$ gives the number of possible different dinners consist of one main course, one dessert and one beverage.

Example 4: The ID card of a certain university should consists of two uppercase English letters followed by four digits.

(i). If no letter or digit can be repeated, then there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3,276,000$ different possible ID cards.

(ii). If letters and digits can be repeated, then there are $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000$ different possible ID cards.

(iii). If the first digit should be nonzero and repetition is allowed, then there are $26 \times 26 \times 9 \times 10 \times 10 \times 10 = 6,084,000$ different possible ID cards.

Now, consider the following example.

Example 5: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative for the Epsilon-Delta society of the University of Colombo. Suppose that the mathematics faculty consists of 17 members and there are 49 mathematics majors. Assume that there is no one who is both a faculty member and a mathematics major. How many different choices are there for this representative?

Solution: Note that there are 17 ways to choose a faculty member and 49 ways to choose a mathematics major. Because no one is both a faculty member and a mathematics major, choosing a member of the mathematics faculty is never the same as choosing a mathematics major. Thus, there are $17+49=66$ possible ways to select this representative.

The above example introduces our second principle of counting, namely the sum rule.

The sum rule / The addition rule.

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.

Note that, in the above example, the task is to select a representati-ve for the society. The representative can be selected from the 17 faculty members or from the 49 mathematics majors. Because no one is both a faculty member and a mathematics major, choosing a faculty member is never the same as choosing a math major. Thus, the result follows from the sum rule.

Theorem 2: Let m be a positive integer. Suppose that a task can be done in one of n_1 ways, or in one of n_2 ways, ..., or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i, j, of integers such that $1 \le i < j \le m$. Then, the number of ways to do the task is $n_1 + n_2 + \dots + n_m$.

Example 6: A statistic major can choose a project from one of three lists. The three lists contain 13, 25, and 17 possible projects, respectively. Assume no project is on more than one list. How many possible projects are there to choose from?

Solution: Note that the student can choose a project from the first list, or from the second list or from the third list. Because no project is on more than one list, by the theorem just given, the student has $13+25+17=55$ possible project to choose from.

Example 7: Suppose one of two students has 3 text books on discrete mathematics and the other student has 6 text books on discrete mathematics. Let n denote the number of <u>different</u> book's on discrete mathematics that these two students can have. What is the value of n?

Solution: Since it is possible that these students can have copies of the same text book, $6 \leq n \leq 9$. Due to the same reason, you cannot apply the sum rule to conclude that $n=3+6=9$

The product rule and the sum rule in terms of sets.

The product rule can be phrased in terms of sets as well.

The product rule: Let m be a positive integer. Let $A_1, A_2, \dots, A_m$ be m non empty finite subsets of some set X. Then, the number of elements in the cartesian product $A_1 \times \dots \times A_m$ is the product of the numbers of elements in each set. That is, $|A_1 \times \dots \times A_m| = |A_1| \times \dots \times |A_m|$; where, for any finite set S, $|S|$ denotes the number of elements in S. If $S = \emptyset$, then $|S| = 0$.

- Recall that the cartesian product $A_1 \times \dots \times A_m$ of m nonempty sets $A_1, \dots, A_m$ is defined as the set of all ordered m-tuples $(a_1, \dots, a_m)$; where for each $i = 1, \dots, m$, $a_i \in A_i$. That is

$A_1 \times \dots \times A_m = \{(a_1, \dots, a_m) \mid a_i \in A_i \text{ for } i = 1, \dots, m\}$.

Let's see how this version of the product rule can be applied in a problem.

Consider Part 2 of Example 1 on page 2. The problem was finding the number of different dinners consist of one main course, one dessert and one beverage. Recall the notation we used there.

For instance, RIO denotes the dinner consisting of rice and curry, ice cream and orange juice. Notice that each such dinner can be regarded as an element in the cartesian product $A_1 \times A_2 \times A_3$, where $A_1 = \{$Rice and curry, Noodles, Pasta$\}$,

$A_2 = \{$Ice cream, Chocalate mousse, Marshmallow pudding, Bread pudding$\}$

and $A_3 = \{$Orange Juice, Beer$\}$. On the other hand, each $(x, y, z) \in A_1 \times A_2 \times A_3$ can be regarded as a dinner in which x is the main course, y is the dessert and z is the beverage. Thus, the number of different dinners is equal to $|A_1 \times A_2 \times A_3|$. Note that $|A_1 \times A_2 \times A_3| = |A_1| \times |A_2| \times |A_3| = 3 \times 4 \times 2 = 24$, as expected.

The sum rule also can be phrased in terms of sets.

<u>The sum rule</u>: Let m be a positive integer and let $A_1, \dots, A_m$ be m nonempty, pairwise disjoint finite subsets of some set X. Then, the number of elements in the union of these sets is equal to the sum of the numbers of elements in $A_1, \dots, A_m$. That is $|A_1 \cup \dots \cup A_m| = |A_1| + \dots + |A_m|$, where for each $i \neq j$, $1 \leq i, j \leq m$, $A_i \cap A_j = \emptyset$.

In order to apply this version of the sum rule in a problem, we have to break up the problem into mutually exclusive cases that exhaust all possibilities. In other words, if A is the set to be counted, then you have to define subsets $A_1, \ldots, A_m$ of A such that $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $A_1 \cup \ldots \cup A_m = A$. But, when doing so you should be smart enough to find A_i s which can be readily counted.

For example, consider Example 6 on page 7. Let's define A_1 to be the set consisting of the projects in list 1, A_2 to be the set consisting of the projects in list 2, and A_3 to be the set consisting of the projects in list 3. Because no project is on more than one list, $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$. In otherwords, A_1, A_2, A_3 are pairwise disjoint. Clearly $A_1 \cup A_2 \cup A_3$ is equal to the set of all projects which are available to choose. Thus the number of possible projects available is $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 13 + 25 + 17 = 55$.

In more complex counting problems, we need to use both the sum rule and the product rule in combination.

Example 8: An identity card can be made using either two upper-case English letters followed by 2 digits or one uppercase English letter followed by three digits. How many ID cards can be made if repetition of letters and digits is allowed?

Solution: By the product rule, $26 \times 26 \times 10 \times 10$ number of ID cards can be made by using two uppercase English letters and two digits. Again, by the product rule, $26 \times 10 \times 10 \times 10$ number of ID cards can be made by using one uppercase letter and three digits. Because the two ways of making ID cards are different, by the sum rule, $26^2 \times 10^2 + 26 \times 10^3 = 67,600 + 26,000 = 93,600$ number of ID cards are possible.

Example 9: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there? Assume that repetition of characters is allowed. (No need to simplify)

Solution: Let n_1, n_2, n_3 be the numbers of possible passwords of length 6, 7 and 8 respectively. Also, let n be the total number of possible passwords. Observe that by the sum rule, $n = n_1 + n_2 + n_3$. Note that by the product rule, $n_1 = 36^6 - 26^6$, $n_2 = 36^7 - 26^7$ and $n_3 = 36^8 - 26^8$. Thus, $n = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$.

Exercise 1: How many even numbers between 1000 and 9999 have distinct digits?

Exercise 2: How many different five-digit numbers can be constructed out of the digits 2, 2, 2, 5, 7?

The following two rules of counting are equaly important.

* Subtraction rule.
* Division rule.

Let U be a non empty finite set and let S be a subset of U to be counted. Sometimes it is more convenient to count the number of elements in $U \setminus S$, rather than to count the number of elements in S itself provided that the number of elements in U is known or that it can be easily counted.

You have already seen this sort of situation in Example 9 on page 11. n_1, for example, is the number of possible passwords of length 6. Did you notice the way we obtained n_1? We first found the number of all passwords of length 6 as if there were no condition on creating a password. Because we have 26 letters and 10 digits, this was equal to 36^6.

Now, the condition was having at least one digit in the password. So, there could be only 1 digit in the password, or 2 digits in the password, ..., or even all the six characters could be digits. Thus, the direct calculation of the number of passwords of length 6 subjected to the given condition is, undoubtedly, a tedious task. However, one can easily find the number of passwords of length 6 with no digits in it. This is simply 26^6. Now, you know the number of all passwords of length 6 and that of passwords which do not satisfy the condition. Therefore, the number of length six passwords satisfying the given condition is $36^6 - 26^6$.

Here, U is the set of all passwords of length 6, and S is the set of all passwords of length 6, satisfying the condition. The required answer, $|S|$, is equal to $|U| - |U \setminus S|$. This is the subtraction rule.

Subtraction Rule : Let U be a finite set and let $S \subseteq U$. Then, the number of elements in S, $|S|$ is equal to $|U| - |U \setminus S|$.

As we pointed out above, the subtraction rule make sense only if it is easier to count the number of elements in U and in $|U \setminus S|$ than to count the number of elements in S.

Exercise 3: Find the number of 6-digit numbers that can be formed using 1, 2, 3, 4, 5, 6 if 2 and 4 are not allowed to be in consecutive positions.

Division Rule: Suppose a task can be done using a procedure that can be carried out in n ways, and for each way w, exactly d of the n ways correspond to way w. Then, there are n/d different ways to do the task.

In terms of set, the division rule can be phrased as follows.

Division Rule: Let S be a finite set and suppose S is a union of n pairwise disjoint subsets of S each having d number of elements. Then $n = \frac{|S|}{d}$.

Permutations and Combinations.

This section is about numbers of arrangements of different objects, and the number of ways you can choose different objects.

Permutations.

Let's start with a simple example.
Suppose that you have the four letters A, B, C and D, one on each of four separate cards, and that you are going to arrange them in a line to form "words" (with or without a meaning). How many four-letter words are there?

By now, you know how to solve this sort of problem. But, let me explain the solution.

Note that you have four choices for the first letter: either A, B, C or D. For each choice, you have three choices for the second letter.

B, C or D if A has been used,
A, C or D if B has been used,
A, B or D if C has been used,
A, B or C if D has been used.

For each choice of the first two letters, you have two choices for the third letter.

C, or D if A and B have been used,
B or D if A " C " " ",
B or C if A " D " " ",
A or C if B " D " " ",
A or D if B " C " " ",
A or B if C " D " " ".

Finally, for each choice of the first three letters, you have only one choice for the fourth letter.

only D is left if A, B and C have been used,
only C is left if A, B and D have been used,
only B is left if A, C and D have been used
only A is left if B, C and D have been used.

So, by the product rule, there are $4 \times 3 \times 2 \times 1 = 24$ different four-letter words.

Now, suppose that, for an exhibition, an artist is going to arrange 10 paintings in a row on a wall of a gallery. In how many ways can this be done?

By invoking the product rule, you may easily conclude that there are $10 \times 9 \times \ldots \times 3 \times 2 \times 1$ number of ways to do this.

One of the most crucial observations you should make at this point is that in both problems the order of the objects does matter. That is, for example, ABCD and ABDC are two <u>different</u> words even if they have been formed using the same four letters.

You saw that there are 24 different 4-letter words that can be formed using the four letters A, B, C and D. The same answer (24) is obtained irrespective of the way you form the word. For example, if you first chose the third letter, and then the fourth letter, and then the first letter and finally the second letter you still get the answer $2 \times 1 \times 4 \times 3 = 24$.

As in the above two examples, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

Definition : For an integer $n \geq 0$, "n factorial" (denoted $n!$) is defined by

$0! = 1,$

$n! = 1 \times 2 \times 3 \times \dots \times (n-1) \times n, \text{ for } n \geq 1.$

Thus, $1! = 1$, $2! = 1 \times 2 = 2$, $3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$, and so on.

Also, for each $n \geq 0$, $(n+1)! = n! \cdot (n+1)$.

Definition: Given a collection of n distinct objects, any ordered (linear) arrangement of these objects is called a permutation of the collection.

Theorem : Given a collection of n distinct objects, there are $n!$ linear arrangements (or permutations) of these objects.

Proof: Follows from the product rule.

Suppose now that from a collection of 10 paintings, five are to be chosen and arranged in a row on a wall. How many such linear arrangements are possible?

Again, the order of the 5 paintings does matter.

Observe that, arranging 5 paintings in a row can be considered as an activity comprises of five steps (or tasks), where the i^{th} step is choosing a painting for the i^{th} place for $i = 1, 2, 3, 4, 5$.

Now, there are 10 choices for the first place in the row. Because repetitions are not possible here, having chosen the first painting, we have only 9 paintings to select from for the second place. Continuing in this way, we find only 6 paintings to choose from for the fifth and the final place. This yields a total of $10 \times 9 \times 8 \times 7 \times 6 = 30,240$ possible arrangements of 5 paintings chosen from the collection of 10.

$$\underset{1^{st}\text{ place}}{10} \times \underset{2^{nd}\text{ place}}{9} \times \underset{3^{rd}\text{ place}}{8} \times \underset{4^{th}\text{ place}}{7} \times \underset{5^{th}\text{ place.}}{6}$$

The above example is an instance of arranging not all but some of the objects of a collection.

<u>Definition</u>: Let n be a positive integer and let r be an integer such that $1 \leq r \leq n$. Then, any ordered (linear) arrangement of r distinct objects of a collection of n distinct objects is called an <u>r-permutation</u>, or a <u>permutation of size r</u> of the collection.

Example 1: 2, 3 and 3, 2 are two 2-permutations of the collection $\{-1, 1, 2, 3\}$.

Theorem: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$. Then, there are ${}^nP_r = P(n,r) = n(n-1)\dots(n-r+1)$ number of r-permutations of a collection of n distinct objects.

Proof: Follows from the product rule.

Now, let's define $P(n, 0)$ to be equal to 1.

Corollary: Let n be a positive integer and r be a nonnegative integer such that $r \leq n$. Then ${}^nP_r = P(n,r) = \dfrac{n!}{(n-r)!}$.

Proof: Suppose $r = 0$. Then, by definition, $P(n,0) = 1$. Also, $\dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$. Thus, $P(n,0) = \dfrac{n!}{(n-0)!}$.

Suppose now that $1 \leq r \leq n$. Then, $P(n,r) = n(n-1)(n-2)\dots[n-(r-1)]$

$$= \frac{n\cdot(n-1)\cdot(n-2)\dots[n-(r-1)]\cdot[n-r]\cdot[n-(r+1)]\cdot\dots\cdot3\cdot2\cdot1}{[n-r]\cdot[n-(r+1)]\cdot\dots\cdot3\cdot2\cdot1}$$

$$= \frac{n!}{(n-r)!}.$$

This completes the proof.

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Example 2: A certain flag should consist of three horizontal stripes of three different colors. The available colors are red, white, blue, green and maroon. Find the number of possible flags that can be constructed.

Solution: Observe that for each possible flag, there exists a corresponding 3-permutation of the given collection of five colors and vice versa. Thus, the number of possible flags that can be constructed is equal to the number of 3-permutations of the given collection of five colors. In this case $n=5$ and $r=3$.

Thus, $P(n,r) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = 60$. Hence, the number of possible flags is 60.

Permutations with repetition.

Note that $P(n,r)$ counts linear arrangements in which objects cannot be repeated. Counting permutations when repetition of elements is allowed can easily be done, using the product rule.

Example 3: How many strings of length 4 can be formed from the uppercase letters of the English alphabet?

Solution: The process of forming one string of length 4 consists of 4 steps, where for $i=1,2,3,4$, the i^{th} step is selecting the i^{th} letter of the string. Because there are 26 uppercase English letters and because each letter can be used repeatedly, by the product rule, there are $26\times26\times26\times26 = 26^4$ strings of length 4.

Theorem: The number of r-permutations of a collection of n distinct objects, when repetition is allowed is n^r.

Proof: Follows from the product rule.

Combinations.

In the last section we considered permutations (arrangements), for which the order of the objects is significant when we count the number of different possibilities. In some circumstances, however, the order of selection does not matter. For example, suppose a class teacher wants two students from her class consists of 25 students to be participated in a seminar. In this case, the order in which the two students are chosen is irrelavent. All we need is just two students.

Example 1. How many different groups of three students can be formed from a group of four students.

Solution: Observe that, in this problem, the order of selecting three students does not matter. Thus, $4 \times 3 \times 2 = 24$ is NOT the answer. Let's denote the four students by S_1, S_2, S_3 and S_4. You can find the number of different groups of three students in the following obvious way.

Note that, because our ultimate goal is choosing three students out of four students, choosing 3 students is exactly the same as removing one student from the group of 4 students. So, you may remove S_1 to form the group consisting of S_2, S_3, S_4, or you may remove S_2 to form the group consisting of S_1, S_3, S_4. or you may remove S_3 to form the group consisting of S_1, S_2, S_4 or you may remove S_4 to form the group consisting of S_1, S_2 and S_3. Thus, there are only 4 ways of selecting 3 students out of 4 students.

Now, let's use our knowledge about permutations to obtain the above answer.

As you know, there are ${}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4 \times 3 \times 2 \times 1 = 24$ 3-permutat-ions of the 4 students. Let's pick one of them, say, $S_2 S_3 S_4$.
Note that there are $3! = 6$ permutations, namely $S_2 S_3 S_4$, $S_2 S_4 S_3$, $S_3 S_2 S_4$, $S_3 S_4 S_2$, $S_4 S_2 S_3$, $S_4 S_3 S_2$, of the three letters S_2, S_3 and S_4.
If you took a permutation different from the six already listed above, say $S_1 S_2 S_3$, then again there are $3! = 6$ permutations, namely $S_1 S_2 S_3$, $S_1 S_3 S_2$, $S_2 S_1 S_3$, $S_2 S_3 S_1$, $S_3 S_1 S_2$, $S_3 S_2 S_1$, of that three letters S_1, S_2 and S_3.

Did you observe that even if the six permutations listed at each instance are all different, they all correspond to just one unordered selection (or particular set) of three letters. In the first case, the three letters are S_2, S_3 and S_4, whereas in the second case, the three letters are

S_1, S_2 and S_3. Thus, if x is the number of ways of selecting three students from the four students, then because each set of three letters give rise to $3! = 6$ permutations, the total number of permutations is equal to $3!\cdot x$. But, we know that $3!\cdot x = {}^4P_3 = 24$.

Thus, $x = \dfrac{{}^4P_3}{3!} = \dfrac{4!/(4-3)!}{3!} = \dfrac{4!}{3!\,(4-3)!} = \dfrac{24}{6} = 4.$

So, there are 4 ways of selecting 3 students from the group of 4 students.

If we start with n distinct objects, then each selection of r objects of these n objects, with no reference to order, corresponds to $r!$ permutations of size r. Because there are ${}^nP_r$ r-permutations, the number of possible selections of r objects taken from a collection of n distinct objects is equal to $\dfrac{{}^nP_r}{r!} = \dfrac{n!/(n-r)!}{r!} = \dfrac{n!}{r!\,(n-r)!}$.

<u>Definition 1</u>: An r-combination of elements of a set is an unordered selection of r elements from the set.

<u>Remark</u>: According to the definition, an r-combination is simply a subset with r elements of the given set. It should be clear to you that r is a nonnegative integer.

Theorem 1: The number of r-combinations of a set with n distinct elements, where n is a nonnegative integer and r is an integer such that $0 \leq r \leq n$, equals

$$^{n}C_{r} = C(n,r) = \frac{n!}{r!\,(n-r)!}.$$

Proof: Let $^{n}C_{r} = C(n,r)$ denotes the number of r-combinations of the given set. Observe that each r-permutation of the set is a particular ordering of elements in a particular r-combination and each ordering of the elements in each r-combination is an r-permutation. Note that there are $^{r}P_{r} = P(r,r)$ ways of ordering the r elements in each r-combination. Thus, by the product rule there are $^{n}C_{r} \cdot {^{r}P_{r}}$ r-permutations of the set. However, the number of r-permutations of a set with n distinct elements is $^{n}P_{r}$. Thus, $^{n}C_{r} \cdot {^{r}P_{r}} = {^{n}P_{r}}$. Hence,

$$^{n}C_{r} = \frac{^{n}P_{r}}{^{r}P_{r}} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!\,(n-r)!},$$

This completes the proof.

Note that for all $n \geq 0$, $C(n,0) = C(n,n) = 1$. For $n \geq 1$, $C(n,1) = C(n,n-1) = n$. When $0 \leq n < r$, $C(n,r) = 0$.

Important: When dealing with any counting problem, we should ask ourselves about the importance of order in the problem.

When order is relevant, we think in terms of permutations and the product rule. When order is not relevant, combinations could play a key role in solving the problem.

Example 2: A club has 13 members.

(a). How many ways are there to choose 6 members of the club for 6 offices?

(b). How many ways are there to assign the 6 offices by choosing members from the 13 club members? (Assume no person can hold more than one office).

(c). Of the 13 club members, six are women. How many ways are there to choose 6 members for the six offices if exactly 3 women must hold an office?

Solution:

(a). This is simply selecting 6 out of 13. Observe that, the order of selecting members is not relevant. Therefore, there are ${}^{13}C_6 = \frac{13!}{6!\,(13-6)!}$

$$= \frac{13!}{6!\cdot 7!} = \frac{7!\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7!} = 4\cdot 3\cdot 11\cdot 13 = 1716.$$

(b). Here, we need to assign 6 members for the six offices. Clearly, the order does matter. Moreover, these members should be selected from the 13 club members. Because no person can hold more than one office, the number of ways of assigning 6 members to the six offices is equal to ${}^{13}P_6 = \frac{13!}{(13-6)!} = \frac{13!}{7!} = 1,235,520.$

(c). Here again, the order is not relevant. We need to select 6 members from 13, but this time 3 of them must be women. Note that there are ${}^{6}C_3$ ways of selecting 3 women. Because we need to choose only 6 members and 3 of them must be women, the other 3 should be selected from the 7 men. This can be done in ${}^{7}C_3$ ways. Therefore, by the product rule, the number of ways of selecting 6 out of 13 such that 3 are women is ${}^{6}C_3 \cdot {}^{7}C_3 = \frac{6!}{3!\,(6-3)!} \cdot \frac{7!}{3!\,(7-3)!} = \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3} \cdot \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3} = 700.$

Corollary 1: Let n and r be nonnegative integers with $r \leq n$. Then .

$${}^nC_r = {}^nC_{n-r} \quad [\text{or equivalently } C(n,r) = C(n, n-r)].$$

Proof: Note that ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$ and ${}^nC_{n-r} = \dfrac{n!}{(n-r)!\,[n-(n-r)]!} = \dfrac{n!}{(n-r)!\cdot r!}$.

Hence ${}^nC_r = {}^nC_{n-r}$.

Permutations with Indistinguishable Objects.

The elements or the objects in the counting problems we encountered so far were all distinct. There are, however, some counting problems in which some of the elements or objects considered are indistinguishable. If this is the case, then you have to be careful not to count things more than once.

Let's consider few examples.

Example 3: By writing out all the possible arrangements of $D_1 E_1 E_2 D_2$, show that there are $\dfrac{4!}{2!\cdot 2!} = 6$ different arrangements of the letters of the word DEED.

Solution: Let's first write out all the possible arrangements of the letters D_1, E_1, E_2 and D_2.

$D_1 E_1 E_2 D_2$	$D_1 E_1 D_2 E_2$	$D_1 D_2 E_1 E_2$	$E_1 D_1 E_2 D_2$	$E_1 D_1 D_2 E_2$	$E_1 E_2 D_1 D_2$
$D_1 E_2 E_1 D_2$	$D_1 E_2 D_2 E_1$	$D_1 D_2 E_2 E_1$	$E_1 D_2 E_2 D_1$	$E_1 D_2 D_1 E_2$	$E_1 E_2 D_2 D_1$
$D_2 E_1 E_2 D_1$	$D_2 E_1 D_1 E_2$	$D_2 D_1 E_1 E_2$	$E_2 D_1 E_1 D_2$	$E_2 D_1 D_2 E_1$	$E_2 E_1 D_1 D_2$
$D_2 E_2 E_1 D_1$	$D_2 E_2 D_1 E_1$	$D_2 D_1 E_2 E_1$	$E_2 D_2 E_1 D_1$	$E_2 D_2 D_1 E_1$	$E_2 E_1 D_2 D_1$

Now, we have written all the possible arrangements of the four letters D_1, E_1, E_2 and D_2. There are $4! = 24$ arrangements as expected. Observe that each column has all the permutations of E_1, E_2 and D_1, D_2. For each arrangement of D_1 and D_2 there are $2!$ arrangements of E_1 and E_2 (namely E_1E_2 and E_2E_1). Because there are $2!$ arrangements of D_1 and D_2 (namely D_1D_2 and D_2D_1), by the product rule there are $2! \cdot 2!$ arrangements of D_1, D_2 and E_1, E_2.

Thus, if the number of <u>different</u> arrangements of the letters D_1, E_1, E_2, and D_2 is x, then $2! \cdot 2! \cdot x = 4!$ and hence $x = \dfrac{4!}{2! \cdot 2!}$.

Now, replace D_1, D_2 by D and E_1, E_2 by E and you have the different permutations of D, E, E, D in the top row. These are DEED, DEDE, DDEE, EDED, EDDE, EEDD.

<u>Example 4</u>: How many different arrangements can be made of the letters in the word STATISTICS?

Solution: Again, because some of the letters of STATISTICS are the same, the answer is not given by the number of permutations of 10 (note that there are 10 letters of the word STATISTICS) letters. This word contains 3 S's, 3 T's, 2 I's, one A and one C.

To determine the number of different arrangements that can be made of the letters in the word STATISTICS, first note that the three S's can be placed among the 10 positions in ${}^{10}C_3$ different ways, leaving 7 positions free. Then, the three T's can be placed in ${}^{7}C_3$ ways leaving 4 free positions. Then, two I's can be placed in ${}^{4}C_2$ ways leaving 2 free positions. The A can be placed in ${}^{2}C_1$ ways, leaving just one position free. Hence, C can be placed in ${}^{1}C_1$ way. Consequently, from the product rule, the number of different arrangements that can be made is ${}^{10}C_3 \cdot {}^{7}C_3 \cdot {}^{4}C_2 \cdot {}^{2}C_1 \cdot {}^{1}C_1 = \frac{10!}{3! \cdot \cancel{7!}} \cdot \frac{\cancel{7!}}{3! \, \cancel{4!}} \cdot \frac{\cancel{4!}}{2! \cdot \cancel{2!}} \cdot \frac{\cancel{2!}}{1! \cdot \cancel{1!}} \cdot \frac{\cancel{1!}}{1! \cdot 0!} =$

$$\frac{10!}{3! \cdot 3! \cdot 2! \cdot 1! \cdot 1!} = 50{,}400.$$

We can prove our next theorem using the same sort of reasoning as in Example 4.

Theorem 2: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k with $n = n_1 + n_2 + \cdots + n_k$ is $\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$.

Proof: To determine the number of permutations, first note that the n_1 objects of type 1 can be placed among the n positions in ${}^{n}C_{n_1}$ ways, leaving $n - n_1$ positions free. Then, the objects of type 2 can be placed in ${}^{n-n_1}C_{n_2}$ ways, leaving $n - n_1 - n_2$ positions free. Continue placing the objects of type 3, ..., type $k-1$, until at the last stage, n_k objects of type k can be placed in ${}^{n-n_1-n_2-\cdots-n_{k-1}}C_{n_k}$ ways. Hence, by the product rule, the total number of different permutations is

$${}^{n}C_{n_1} \cdot {}^{n-n_1}C_{n_2} \cdots {}^{n-n_1-n_2-\cdots-n_{k-1}}C_{n_k} = \frac{n!}{n_1! \, \cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2! \, \cancel{(n-n_1-n_2)!}} \cdots \frac{\cancel{(n-n_1-\cdots-n_{k-1})!}}{n_k! \, \underbrace{(n-n_1-\cdots-n_{k-1}-n_k)!}_{=0}}$$

$$= \frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$ This completes the proof.