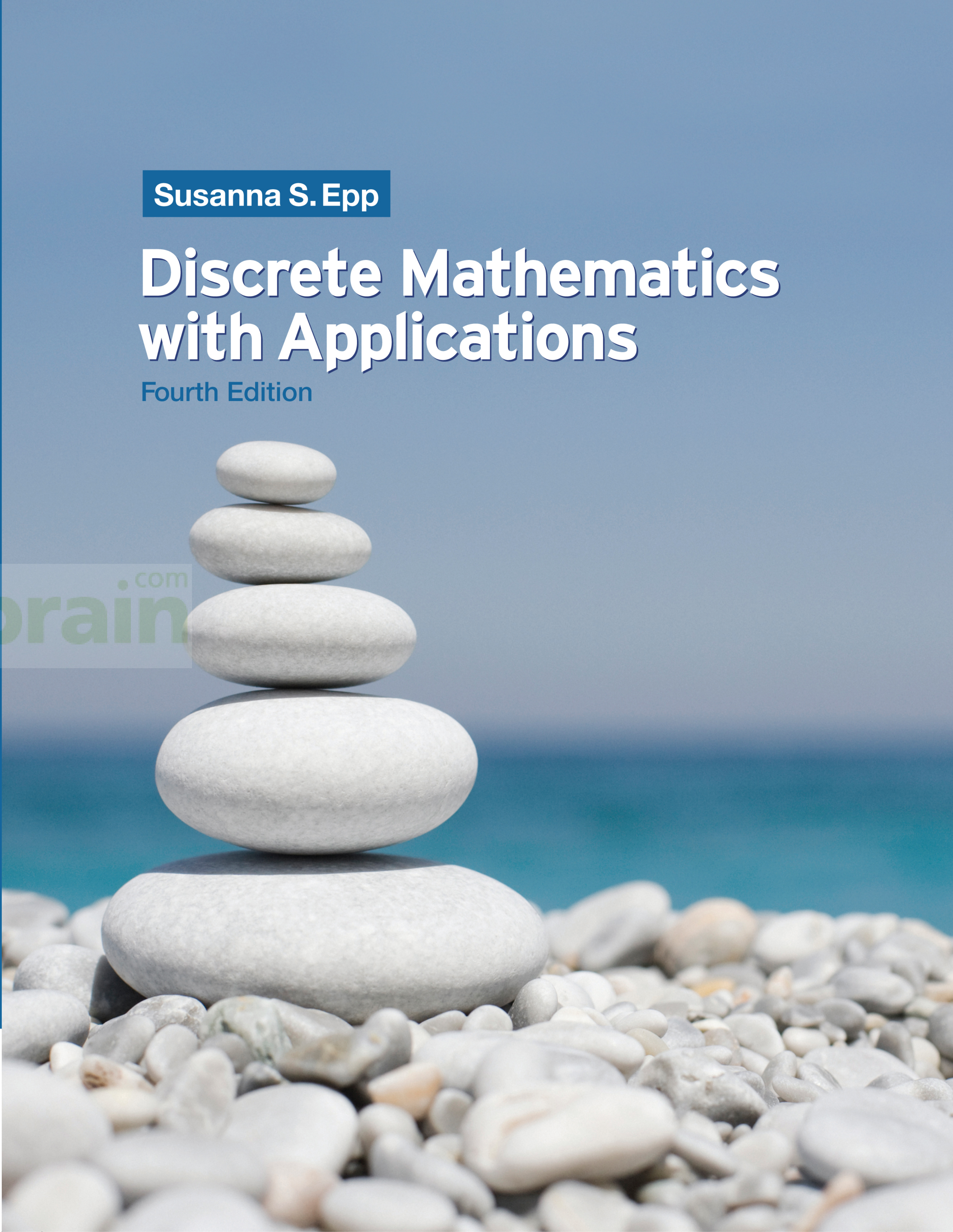
Susanna S. Epp

Discrete Mathematics with Applications

Fourth Edition



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Cover Photo: *The stones are discrete objects placed one on top of another like a chain of careful reasoning. A person who decides to build such a tower aspires to the heights and enjoys playing with a challenging problem. Choosing the stones takes both a scientific and an aesthetic sense. Getting them to balance requires patient effort and careful thought. And the tower that results is beautiful. A perfect metaphor for discrete mathematics!*

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CHAPTER 1

SPEAKING MATHEMATICALLY

Therefore O students study mathematics and do not build without foundations. —Leonardo da Vinci (1452–1519)

The aim of this book is to introduce you to a mathematical way of thinking that can serve you in a wide variety of situations. Often when you start work on a mathematical problem, you may have only a vague sense of how to proceed. You may begin by looking at examples, drawing pictures, playing around with notation, rereading the problem to focus on more of its details, and so forth. The closer you get to a solution, however, the more your thinking has to crystallize. And the more you need to understand, the more you need language that expresses mathematical ideas clearly, precisely, and unambiguously.

This chapter will introduce you to some of the special language that is a foundation for much mathematical thought, the language of variables, sets, relations, and functions. Think of the chapter like the exercises you would do before an important sporting event. Its goal is to warm up your mental muscles so that you can do your best.

1.1 Variables

A variable is sometimes thought of as a mathematical "John Doe" because you can use it as a placeholder when you want to talk about something but either (1) you imagine that it has one or more values but you don't know what they are, or (2) you want whatever you say about it to be equally true for all elements in a given set, and so you don't want to be restricted to considering only a particular, concrete value for it. To illustrate the first use, consider asking

> Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

$$\text{Is there a number } x \text{ with the property that } 2x + 3 = x^2?$$

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values. To emphasize the role of the variable as a placeholder, you might write the following:

$$\text{Is there a number } \square \text{ with the property that } 2 \cdot \square + 3 = \square^2?$$

The emptiness of the box can help you imagine filling it in with a variety of different values, some of which might make the two sides equal and others of which might not.

To illustrate the second use of variables, consider the statement:

> No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

In this case introducing a variable to give a temporary name to the (arbitrary) number you might choose enables you to maintain the generality of the statement, and replacing all instances of the word "it" by the name of the variable ensures that possible ambiguity is avoided:

> No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Example 1.1.1 Writing Sentences Using Variables

Use variables to rewrite the following sentences more formally.

a. Are there numbers with the property that the sum of their squares equals the square of their sum?

b. Given any real number, its square is nonnegative.

Solution

Note In part (a) the answer is yes. For instance, $a = 1$ and $b = 0$ would work. Can you think of other numbers that would also work?

a. Are there numbers a and b with the property that $a^2 + b^2 = (a + b)^2$?
Or: Are there numbers a and b such that $a^2 + b^2 = (a + b)^2$?
Or: Do there exist any numbers a and b such that $a^2 + b^2 = (a + b)^2$?

b. Given any real number r, r^2 is nonnegative.
Or: For any real number r, $r^2 \geq 0$.
Or: For all real numbers r, $r^2 \geq 0$. ■

Some Important Kinds of Mathematical Statements

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

> A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero*.)
>
> A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6*.)
>
> Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even*.)

In later sections we will define each kind of statement carefully and discuss all of them in detail. The aim here is for you to realize that combinations of these statements can be expressed in a variety of different ways. One way uses ordinary, everyday language and another expresses the statement using one or more variables. The exercises are designed to help you start becoming comfortable in translating from one way to another.

Universal Conditional Statements

Universal statements contain some variation of the words "for all" and conditional statements contain versions of the words "if-then." A ***universal conditional statement*** is a statement that is both universal and conditional. Here is an example:

For all animals a, if a is a dog, then a is a mammal.

One of the most important facts about universal conditional statements is that they can be rewritten in ways that make them appear to be purely universal or purely conditional. For example, the previous statement can be rewritten in a way that makes its conditional nature explicit but its universal nature implicit:

If a is a dog, then a is a mammal.
Or: If an animal is a dog, then the animal is a mammal.

The statement can also be expressed so as to make its universal nature explicit and its conditional nature implicit:

For all dogs a, a is a mammal.
Or: All dogs are mammals.

The crucial point is that the ability to translate among various ways of expressing universal conditional statements is enormously useful for doing mathematics and many parts of computer science.

Example 1.1.2 Rewriting a Universal Conditional Statement

Fill in the blanks to rewrite the following statement:

For all real numbers x, if x is nonzero then x^2 is positive.

a. If a real number is nonzero, then its square ____.

b. For all nonzero real numbers x, ____.

c. If x ____, then ____.

d. The square of any nonzero real number is ____.

e. All nonzero real numbers have ____.

Note If you introduce x in the first part of the sentence, be sure to include it in the second part of the sentence.

Solution

a. is positive
b. x^2 is positive
c. is a nonzero real number; x^2 is positive
d. positive
e. positive squares (*or*: squares that are positive) ■

Universal Existential Statements

A ***universal existential statement*** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

Note For a number b to be an additive inverse for a number a means that $a + b = 0$.

Every real number has an additive inverse.

In this statement the property "has an additive inverse" applies universally to all real numbers. "Has an additive inverse" asserts the existence of something—an additive inverse—for each real number. However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses. Knowing that an additive inverse is a real number, you can rewrite this statement in several ways, some less formal and some more formal*:

All real numbers have additive inverses.
Or: For all real numbers r, there is an additive inverse for r.
Or: For all real numbers r, there is a real number s such that s is an additive inverse for r.

Introducing names for the variables simplifies references in further discussion. For instance, after the third version of the statement you might go on to write: When r is positive, s is negative, when r is negative, s is positive, and when r is zero, s is also zero.

One of the most important reasons for using variables in mathematics is that it gives you the ability to refer to quantities unambiguously throughout a lengthy mathematical argument, while not restricting you to consider only specific values for them.

Example 1.1.3 Rewriting a Universal Existential Statement

Fill in the blanks to rewrite the following statement: Every pot has a lid.

a. All pots ____.

b. For all pots P, there is ____.

c. For all pots P, there is a lid L such that ____.

Solution

a. have lids

b. a lid for P

c. L is a lid for P ■

Existential Universal Statements

An ***existential universal statement*** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind. For example:

There is a positive integer that is less than or equal to every positive integer:

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer. We can rewrite the statement in several ways, some less formal and some more formal:

Some positive integer is less than or equal to every positive integer.
Or: There is a positive integer m that is less than or equal to every positive integer.
Or: There is a positive integer m such that every positive integer is greater than or equal to m.
Or: There is a positive integer m with the property that for all positive integers $n, m \leq n$.

*A conditional could be used to help express this statement, but we postpone the additional complexity to a later chapter.

Example 1.1.4 Rewriting an Existential Universal Statement

Fill in the blanks to rewrite the following statement in three different ways:

There is a person in my class who is at least as old as every person in my class.

a. Some ____ is at least as old as ____.

b. There is a person p in my class such that p is ____.

c. There is a person p in my class with the property that for every person q in my class, p is ____.

Solution

a. person in my class; every person in my class

b. at least as old as every person in my class

c. at least as old as q ■

Some of the most important mathematical concepts, such as the definition of limit of a sequence, can only be defined using phrases that are universal, existential, and conditional, and they require the use of all three phrases "for all," "there is," and "if-then." For example, if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, saying that

the limit of a_n as n approaches infinity is L

means that

for all positive real numbers ε, **there is** an integer N such that **for all** integers n, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.

Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. A universal statement asserts that a certain property is ____ for ____.
2. A conditional statement asserts that if one thing ____ then some other thing ____.
3. Given a property that may or may not be true, an existential statement asserts that ____ for which the property is true.

Exercise Set 1.1

Appendix B contains either full or partial solutions to all exercises with blue numbers. When the solution is not complete, the exercise number has an ***H*** next to it. A ∗ next to an exercise number signals that the exercise is more challenging than usual. Be careful not to get into the habit of turning to the solutions too quickly. Make every effort to work exercises on your own before checking your answers. See the Preface for additional sources of assistance and further study.

In each of 1–6, fill in the blanks using a variable or variables to rewrite the given statement.

1. Is there a real number whose square is -1?
   a. Is there a real number x such that ____?
   b. Does there exist ____ such that $x^2 = -1$?

2. Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?
   a. Is there an integer n such that n has ____?
   b. Does there exist ____ such that if n is divided by 5 the remainder is 2 and if ____?

Note: There are integers with this property. Can you think of one?

3. Given any two real numbers, there is a real number in between.
   a. Given any two real numbers a and b, there is a real number c such that c is ____.
   b. For any two ____, ____ such that $a < c < b$.

4. Given any real number, there is a real number that is greater.
   a. Given any real number r, there is ____ s such that s is ____.
   b. For any ____, ____ such that $s > r$.

5. The reciprocal of any positive real number is positive.
   a. Given any positive real number r, the reciprocal of ____.
   b. For any real number r, if r is ____, then ____.
   c. If a real number r ____, then ____.

6. The cube root of any negative real number is negative.
   a. Given any negative real number s, the cube root of ____.
   b. For any real number s, if s is ____, then ____.
   c. If a real number s ____, then ____.

7. Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.
   a. There are real numbers u and v with the property that $u + v < u - v$.
   b. There is a real number x such that $x^2 < x$.
   c. For all positive integers n, $n^2 \geq n$.
   d. For all real numbers a and b, $|a + b| \leq |a| + |b|$.

In each of 8–13, fill in the blanks to rewrite the given statement.

8. For all objects J, if J is a square then J has four sides.
   a. All squares ____.
   b. Every square ____.
   c. If an object is a square, then it ____.
   d. If J ____, then J ____.
   e. For all squares J, ____.

9. For all equations E, if E is quadratic then E has at most two real solutions.
   a. All quadratic equations ____.
   b. Every quadratic equation ____.
   c. If an equation is quadratic, then it ____.
   d. If E ____, then E ____.
   e. For all quadratic equations E, ____.

10. Every nonzero real number has a reciprocal.
   a. All nonzero real numbers ____.
   b. For all nonzero real numbers r, there is ____ for r.
   c. For all nonzero real numbers r, there is a real number s such that ____.

11. Every positive number has a positive square root.
   a. All positive numbers ____.
   b. For any positive number e, there is ____ for e.
   c. For all positive numbers e, there is a positive number r such that ____.

12. There is a real number whose product with every number leaves the number unchanged.
   a. Some ____ has the property that its ____.
   b. There is a real number r such that the product of r ____.
   c. There is a real number r with the property that for every real number s, ____.

13. There is a real number whose product with every real number equals zero.
   a. Some ____ has the property that its ____.
   b. There is a real number a such that the product of a ____.
   c. There is a real number a with the property that for every real number b, ____.

Answers for Test Yourself

1. true; all elements of a set 2. is true; also has to be true 3. there is at least one thing

1.2 The Language of Sets

... when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression. —George Polyá (1887–1985)

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as

Cantor did, simply as a collection of elements. For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

• Notation

If S is a set, the notation $\boldsymbol{x \in S}$ means that x is an element of S. The notation $\boldsymbol{x \notin S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . .}** to refer to the set of all positive integers. (The symbol **. . .** is called an **ellipsis** and is read "and so forth.")

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1.2.1 Using the Set-Roster Notation

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Solution

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

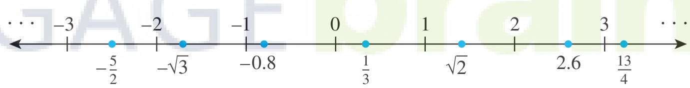
Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the table on the next page.

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

Note The **Z** is the first letter of the German word for integers, *Zahlen*. It stands for the *set* of all integers and should not be used as a shorthand for the word *integer*.

Addition of a superscript $+$ or $-$ or the letters *nonneg* indicates that only the positive or negative or nonnegative elements of the set, respectively, are to be included. Thus $\mathbf{R}^+$ denotes the set of positive real numbers, and $\mathbf{Z}^{nonneg}$ refers to the set of nonnegative integers: 0, 1, 2, 3, 4, and so forth. Some authors refer to the set of nonnegative integers as the set of **natural numbers** and denote it as **N**. Other authors call only the positive integers natural numbers. To prevent confusion, we simply avoid using the phrase *natural numbers* in this book.

The set of real numbers is usually pictured as the set of all points on a line, as shown below. The number 0 corresponds to a middle point, called the *origin*. A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. *Note that 0 is neither positive nor negative* Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes. The set of integers corresponds to a collection of points located at fixed intervals along the real number line. Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

Note We read the left-hand brace as "the set of all" and the vertical line as "such that." In all other mathematical contexts, however, we do not use a vertical line to denote the words "such that"; we abbreviate "such that" as "s. t." or "s. th." or "$\cdot \ni \cdot$."

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements *x* in *S* such that *P(x)* is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all such that

Occasionally we will write $\{x \mid P(x)\}$ without being specific about where the element x comes from. It turns out that unrestricted use of this notation can lead to genuine contradictions in set theory. We will discuss one of these in Section 6.4 and will be careful to use this notation purely as a convenience in cases where the set S could be specified if necessary.

Example 1.2.2 Using the Set-Builder Notation

Given that $\mathbf{R}$ denotes the set of all real numbers, $\mathbf{Z}$ the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

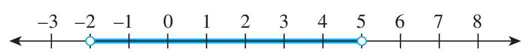
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

Solution

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between -2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between -2 and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$. ■

Subsets

A basic relation between sets is that of subset.

• Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

It follows from the definition of subset that for a set A not to be a subset of a set B means that there is at least one element of A that is not an element of B. Symbolically:

$A \nsubseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 1.2.3 Subsets

Let $A = \mathbf{Z}^+$, $B = \{n \in \mathbf{Z} \mid 0 \le n \le 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$
b. C is a proper subset of A
c. C and B have at least one element in common
d. $C \subseteq B$ e. $C \subseteq C$

Solution

a. False. Zero is not a positive integer. Thus zero is in B but zero is not in A, and so $B \nsubseteq A$.

b. True. Each element in C is a positive integer and, hence, is in A, but there are elements in A that are not in C. For instance, 1 is in A and not in C.

c. True. For example, 100 is in both C and B.

d. False. For example, 200 is in C but not in B.

e. True. Every element in C is in C. In general, the definition of subset implies that all sets are subsets of themselves.

Example 1.2.4 Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ b. $\{2\} \in \{1, 2, 3\}$ c. $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ e. $\{2\} \subseteq \{\{1\}, \{2\}\}$ f. $\{2\} \in \{\{1\}, \{2\}\}$

Solution Only (a), (d), and (f) are true.

For (b) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false.

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (c) is false.

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to either $\{1\}$ or $\{2\}$, and so (e) is false. ■

Cartesian Products

Problemy monthly, July 1959

Kazimierz Kuratowski (1896–1980)

With the introduction of Georg Cantor's set theory in the late nineteenth century, it began to seem possible to put mathematics on a firm logical foundation by developing all of its various branches from set theory and logic alone. A major stumbling block was how to use sets to define an ordered pair because the definition of a set is unaffected by the order in which its elements are listed. For example, $\{a, b\}$ and $\{b, a\}$ represent the same set, whereas in an ordered pair we want to be able to indicate which element comes first.

In 1914 crucial breakthroughs were made by Norbert Wiener (1894–1964), a young American who had recently received his Ph.D. from Harvard and the German mathematician Felix Hausdorff (1868–1942). Both gave definitions showing that an ordered pair can be defined as a certain type of set, but both definitions were somewhat awkward. Finally, in 1921, the Polish mathematician Kazimierz Kuratowski (1896–1980) published the following definition, which has since become standard. It says that an ordered pair is a set of the form

$$\{\{a\}, \{a, b\}\}.$$

This set has elements, $\{a\}$ and $\{a, b\}$. If $a \neq b$, then the two sets are distinct and a is in both sets whereas b is not. This allows us to distinguish between a and b and say that a is the first element of the ordered pair and b is the second element of the pair. If $a = b$, then we can simply say that a is both the first and the second element of the pair. In this case the set that defines the ordered pair becomes $\{\{a\}, \{a, a\}\}$, which equals $\{\{a\}\}$.

However, it was only long after ordered pairs had been used extensively in mathematics that mathematicians realized that it was possible to define them entirely in terms of sets, and, in any case, the set notation would be cumbersome to use on a regular basis. The usual notation for ordered pairs refers to $\{\{a\}, \{a, b\}\}$ more simply as (a, b).

• Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 1.2.5 Ordered Pairs

a. Is $(1, 2) = (2, 1)$?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of $(1, 1)$?

Solution

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2.1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But $1 \neq 2$, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair $(1, 1)$, the first and the second elements are both 1.

• Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 1.2.6 Cartesian Products

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

e. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Solution

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

Note This is why it makes sense to call a Cartesian product a product!

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B. $B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

e. $\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs (x, y) where both x and y are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point. The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.

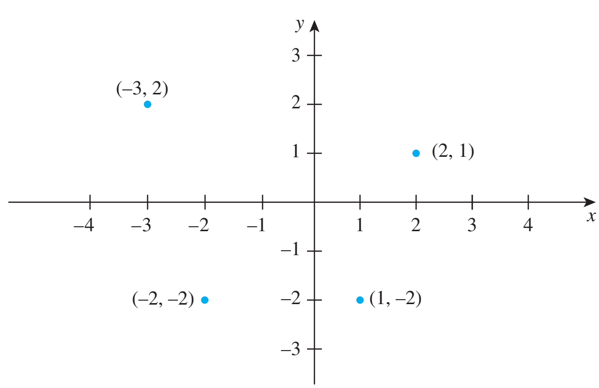


Figure 1.2.1: A Cartesian Plane

Test Yourself

1. When the elements of a set are given using the set-roster notation, the order in which they are listed ____.
2. The symbol $\mathbf{R}$ denotes ____.
3. The symbol $\mathbf{Z}$ denotes ____.
4. The symbol $\mathbf{Q}$ denotes ____.
5. The notation $\{x \mid P(x)\}$ is read ____.
6. For a set A to be a subset of a set B means that, ____.
7. Given sets A and B, the Cartesian product $A \times B$ is ____.

Exercise Set 1.2

1. Which of the following sets are equal?

$A = \{a, b, c, d\}$ $B = \{d, e, a, c\}$
$C = \{d, b, a, c\}$ $D = \{a, a, d, e, c, e\}$

2. Write in words how to read each of the following out loud.
 a. $\{x \in \mathbf{R}^+ \mid 0 < x < 1\}$
 b. $\{x \in \mathbf{R} \mid x \leq 0 \text{ or } x \geq 1\}$
 c. $\{n \in \mathbf{Z} \mid n \text{ is a factor of } 6\}$
 d. $\{n \in \mathbf{Z}^+ \mid n \text{ is a factor of } 6\}$

3. a. Is $4 = \{4\}$?
 b. How many elements are in the set $\{3, 4, 3, 5\}$?
 c. How many elements are in the set $\{1, \{1\}, \{1, \{1\}\}\}$?

4. a. Is $2 \in \{2\}$?
 b. How many elements are in the set $\{2, 2, 2, 2\}$?
 c. How many elements are in the set $\{0, \{0\}\}$?
 d. Is $\{0\} \in \{\{0\}, \{1\}\}$?
 e. Is $0 \in \{\{0\}, \{1\}\}$?

H 5. Which of the following sets are equal?

$$A = \{0, 1, 2\}$$
$$B = \{x \in \mathbf{R} \mid -1 \leq x < 3\}$$
$$C = \{x \in \mathbf{R} \mid -1 < x < 3\}$$
$$D = \{x \in \mathbf{Z} \mid -1 < x < 3\}$$
$$E = \{x \in \mathbf{Z}^+ \mid -1 < x < 3\}$$

H 6. For each integer n, let $T_n = \{n, n^2\}$. How many elements are in each of T_2, T_{-3}, T_1 and T_0? Justify your answers.

7. Use the set-roster notation to indicate the elements in each of the following sets.
 a. $S = \{n \in \mathbf{Z} \mid n = (-1)^k, \text{ for some integer } k\}$.
 b. $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^i, \text{ for some integer } i\}$.
 c. $U = \{r \in \mathbf{Z} \mid 2 \leq r \leq -2\}$
 d. $V = \{s \in \mathbf{Z} \mid s > 2 \text{ or } s < 3\}$
 e. $W = \{t \in \mathbf{Z} \mid 1 < t < -3\}$
 f. $X = \{u \in \mathbf{Z} \mid u \leq 4 \text{ or } u \geq 1\}$

8. Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
 a. Is $B \subseteq A$? b. Is $C \subseteq A$?
 b. Is $C \subseteq C$? d. Is C a proper subset of A?

9. a. Is $3 \in \{1, 2, 3\}$? b. Is $1 \subseteq \{1\}$?
 c. Is $\{2\} \in \{1, 2\}$? d. Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
 e. Is $1 \in \{1\}$? f. Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
 g. Is $\{1\} \subseteq \{1, 2\}$? h. Is $1 \in \{\{1\}, 2\}$?
 i. Is $\{1\} \subseteq \{1, \{2\}\}$? j. Is $\{1\} \subseteq \{1\}$?

10. a. Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
 b. Is $(5, -5) = (-5, 5)$?
 c. Is $\left(8 - 9, \sqrt[3]{-1}\right) = (-1, -1)$?
 d. Is $\left(\frac{-2}{-4}, (-2)^3\right) = \left(\frac{3}{6}, -8\right)$?

11. Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:
 a. $A \times B$ b. $B \times A$
 c. $A \times A$ d. $B \times B$

12. Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:
 a. $S \times T$ b. $T \times S$
 c. $S \times S$ d. $T \times T$

Answers for Test Yourself

1. does not matter 2. the set of all real numbers 3. the set of all integers 4. the set of all rational numbers 5. the set of all x such that $P(x)$ 6. every element in A is an element in B 7. the set of all ordered pairs (a, b) where a is in A and b is in B

1.3 The Language of Relations and Functions

Mathematics is a language. — Josiah Willard Gibbs (1839–1903)

There are many kinds of relationships in the world. For instance, we say that two people are related by blood if they share a common ancestor and that they are related by marriage if one shares a common ancestor with the spouse of the other. We also speak of the relationship between student and teacher, between people who work for the same employer, and between people who share a common ethnic background.

Similarly, the objects of mathematics may be related in various ways. A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common. A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$. Two identifiers in a computer program may be said to be related if they have the same first eight characters, or if the same memory location is used to store their values when the program is executed. And the list could go on!

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element x in A is related to an element y in B if, and only if, x is less than y. Let us use the notation $x\ R\ y$ as a shorthand for the sentence "x is related to y." Then

$$\begin{array}{lll} 0\ R\ 1 & \text{since} & 0 < 1, \\ 0\ R\ 2 & \text{since} & 0 < 2, \\ 0\ R\ 3 & \text{since} & 0 < 3, \\ 1\ R\ 2 & \text{since} & 1 < 2, \\ 1\ R\ 3 & \text{since} & 1 < 3, \quad \text{and} \\ 2\ R\ 3 & \text{since} & 2 < 3. \end{array}$$

On the other hand, if the notation $x \not R\ y$ represents the sentence "x is not related to y," then

$$\begin{array}{lll} 1 \not R\ 1 & \text{since} & 1 \not< 1, \\ 2 \not R\ 1 & \text{since} & 2 \not< 1, \qquad \text{and} \\ 2 \not R\ 2 & \text{since} & 2 \not< 2. \end{array}$$

Recall that the Cartesian product of A and B, $A \times B$, consists of all ordered pairs whose first element is in A and whose second element is in B:

$$A \times B = \left\{(x, y) \mid x \in A \text{ and } y \in B\right\}.$$

In this case,

$$A \times B = \left\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\right\}.$$

The elements of some ordered pairs in $A \times B$ are related, whereas the elements of other ordered pairs are not. Consider the set of all ordered pairs in $A \times B$ whose elements are related

$$\left\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\right\}.$$

Observe that knowing which ordered pairs lie in this set is equivalent to knowing which elements are related to which. The relation itself can therefore be thought of as the totality of ordered pairs whose elements are related by the given condition. The formal mathematical definition of relation, based on this idea, was introduced by the American mathematician and logician C. S. Peirce in the nineteenth century.

• Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain.

The notation for a relation R may be written symbolically as follows:

$$x\ R\ y \quad \text{means that} \quad (x, y) \in R.$$

The notation $x \not R\ y$ means that x is not related to y by R:

$$x \not R\ y \quad \text{means that} \quad (x, y) \notin R.$$

Example 1.3.1 A Relation as a Subset

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$

a. State explicitly which ordered pairs are in $A \times B$ and which are in R.

b. Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?

c. What are the domain and co-domain of R?

Solution

a. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$. To determine explicitly the composition of R, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for R.

$$(1, 1) \in R \text{ because } \tfrac{1-1}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$
$$(1, 2) \notin R \text{ because } \tfrac{1-2}{2} = \tfrac{-1}{2}, \text{ which is not an integer.}$$
$$(1, 3) \in R \text{ because } \tfrac{1-3}{2} = \tfrac{-2}{2} = -1, \text{ which is an integer.}$$
$$(2, 1) \notin R \text{ because } \tfrac{2-1}{2} = \tfrac{1}{2}, \text{ which is not an integer.}$$
$$(2, 2) \in R \text{ because } \tfrac{2-2}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$
$$(2, 3) \notin R \text{ because } \tfrac{2-3}{2} = \tfrac{-1}{2}, \text{ which is an integer.}$$

Thus

$$R = \{(1, 1), (1, 3), (2, 2)\}$$

b. Yes, $1\ R\ 3$ because $(1, 3) \in R$.
No, $2 \not R\ 3$ because $(2, 3) \notin R$.
Yes, $2\ R\ 2$ because $(2, 2) \in R$.

c. The domain of R is $\{1, 2\}$ and the co-domain is $\{1, 2, 3\}$. ■

Example 1.3.2 The Circle Relation

Define a relation C from $\mathbf{R}$ to $\mathbf{R}$ as follows: For any $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in C \quad \text{means that} \quad x^2 + y^2 = 1.$$

a. Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$? Is $-2\ C\ 0$? Is $0\ C\ (-1)$? Is $1\ C\ 1$?

b. What are the domain and co-domain of C?

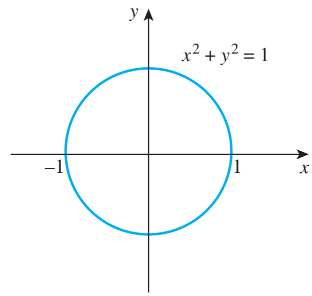
c. Draw a graph for C by plotting the points of C in the Cartesian plane.

Solution

a. Yes, $(1, 0) \in C$ because $1^2 + 0^2 = 1$.
No, $(0, 0) \notin C$ because $0^2 + 0^2 = 0 \neq 1$.
Yes, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$.
No, $-2 \not C\ 0$ because $(-2)^2 + 0^2 = 4 \neq 1$.
Yes, $0\ C\ (-1)$ because $0^2 + (-1)^2 = 1$.
No, $1 \not C\ 1$ because $1^2 + 1^2 = 2 \neq 1$.

b. The domain and co-domain of C are both $\mathbf{R}$, the set of all real numbers.

c.



■

Arrow Diagram of a Relation

Suppose R is a relation from a set A to a set B. The **arrow diagram for *R*** is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

Draw an arrow from x to y
if, and only if, $\quad \boldsymbol{x\ R\ y}$
if, and only if, $\quad \boldsymbol{(x, y) \in R}$.

Example 1.3.3 Arrow Diagrams of Relations

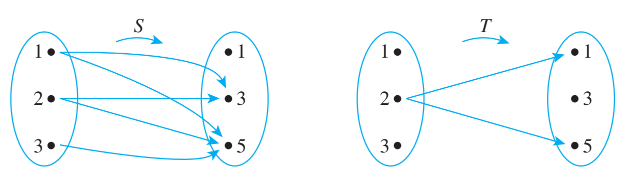
Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in S \quad \text{means that} \quad x < y \qquad \text{S is a “less than” relation.}$$

$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T.

Solution



These example relations illustrate that it is possible to have several arrows coming out of the same element of A pointing in different directions. Also, it is quite possible to have an element of A that does not have an arrow coming out of it. ■

Functions

In Section 1.2 we showed that ordered pairs can be defined in terms of sets and we defined Cartesian products in terms of ordered pairs. In this section we introduced relations as subsets of Cartesian products. Thus we can now define functions in a way that depends only on the concept of set. Although this definition is not obviously related to the way we usually work with functions in mathematics, it is satisfying from a theoretical point of view and computer scientists like it because it is particularly well suited for operating with functions on a computer.

> **• Definition**
>
> A **function F from a set A to a set B** is a relation with domain A and co-domain B that satisfies the following two properties:
>
> 1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
> 2. For all elements x in A and y and z in B,
>
> $$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Properties (1) and (2) can be stated less formally as follows: A relation F from A to B is a function if, and only if:

1. Every element of A is the first element of an ordered pair of F.
2. No two distinct ordered pairs in F have the same first element.

In most mathematical situations we think of a function as sending elements from one set, the domain, to elements of another set, the co-domain. Because of the definition of function, each element in the domain corresponds to one and only one element of the co-domain.

More precisely, if F is a function from a set A to a set B, then given any element x in A, property (1) from the function definition guarantees that there is at least one element of B that is related to x by F and property (2) guarantees that there is at most one such element. This makes it possible to give the element that corresponds to x a special name.

> **• Notation**
>
> If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "***F* of *x***."

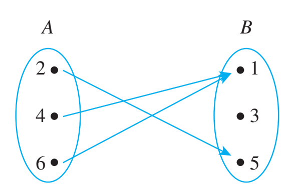
Example 1.3.4 Functions and Relations on Finite Sets

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

a. $R = \{(2, 5),\ (4, 1),\ (4, 3),\ (6, 5)\}$

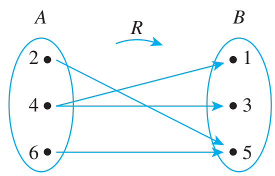
b. For all $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.

c. T is defined by the arrow diagram

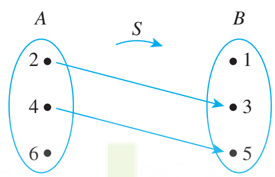


Solution

a. R is not a function because it does not satisfy property (2). The ordered pairs (4, 1) and (4, 3) have the same first element but different second elements. You can see this graphically if you draw the arrow diagram for R. There are two arrows coming out of 4: One points to 1 and the other points to 3.



b. S is not a function because it does not satisfy property (1). It is not true that every element of A is the first element of an ordered pair in S. For example, $6 \in A$ but there is no y in B such that $y = 6 + 1 = 7$. You can also see this graphically by drawing the arrow diagram for S.



Note In part (c), $T(4) = T(6)$. This illustrates the fact that although no element of the domain of a function can be related to more than one element of the co-domain, several elements in the domain can be related to the same element in the co-domain.

c. T is a function: Each element in $\{2, 4, 6\}$ is related to some element in $\{1, 3, 5\}$ and no element in $\{2, 4, 6\}$ is related to more than one element in $\{1, 3, 5\}$. When these properties are stated in terms of the arrow diagram, they become (1) there is an arrow coming out of each element of the domain, and (2) no element of the domain has more than one arrow coming out of it. So you can write $T(2) = 5$, $T(4) = 1$, and $T(6) = 1$. ■

Example 1.3.5 Functions and Relations on Sets of Real Numbers

a. In Example 1.3.2 the circle relation C was defined as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R}, \quad (x, y) \in C \quad \text{means that} \quad x^2 + y^2 = 1.$$

Is C a function? If it is, find $C(0)$ and $C(1)$.

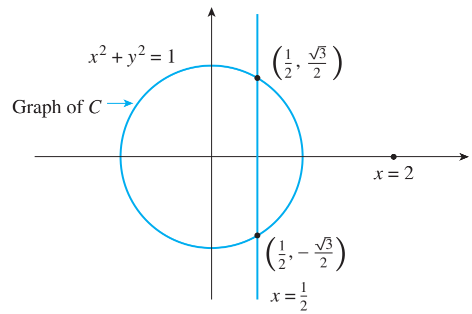
b. Define a relation from $\mathbf{R}$ to $\mathbf{R}$ as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R}, \quad (x, y) \in L \quad \text{means that} \quad y = x - 1.$$

Is L a function? If it is, find $L(0)$ and $L(1)$.

Solution

a. The graph of C, shown on the next page, indicates that C does not satisfy either function property. To see why C does not satisfy property (1), observe that there are many real numbers x such that $(x, y) \notin C$ for any y.



For instance, when $x = 2$, there is no real number y so that

$$x^2 + y^2 = 2^2 + y^2 = 4 + y^2 = 1$$

because if there were, then it would have to be true that

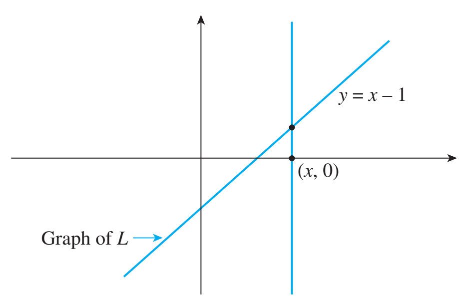
$$y^2 = -3.$$

which is not the case for any real number y.

To see why C does not satisfy property (2), note that for some values of x there are two distinct values of y so that $(x, y) \in C$. One way to see this graphically is to observe that there are vertical lines, such as $x = \frac{1}{2}$, that intersect the graph of C at two separate points: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

b. L is a function. For each real number x, $y = x - 1$ is a real number, and so there is a real number y with $(x, y) \in L$. Also if $(x, y) \in L$ and $(x, z) \in L$, then $y = x - 1$ and $z = x - 1$, and so $y = z$. In particular, $L(0) = 0 - 1 = -1$ and $L(1) = 1 - 1 = 0$.

You can also check these results by inspecting the graph of L, shown below. Note that for every real number x, the vertical line through $(x, 0)$ passes through the graph of L exactly once. This indicates both that every real number x is the first element of an ordered pair in L and also that no two distinct ordered pairs in L have the same first element.



Function Machines

Another useful way to think of a function is as a machine. Suppose f is a function from X to Y and an input x of X is given. Imagine f to be a machine that processes x in a certain way to produce the output $f(x)$. This is illustrated in Figure 1.3.1 on the next page.

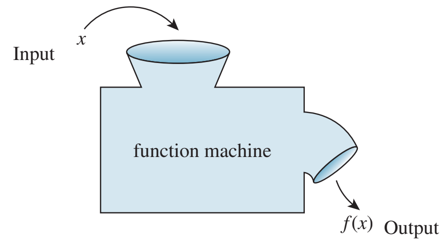


Figure 1.3.1

Example 1.3.6 Functions Defined by Formulas

The **squaring function** f from **R** to **R** is defined by the formula $f(x) = x^2$ for all real numbers x. This means that no matter what real number input is substituted for x, the output of f will be the square of that number. This idea can be represented by writing $f(\square) = \square^2$. In other words, f sends each real number x to x^2, or, symbolically, $f: x \to x^2$. Note that the variable x is a dummy variable; any other symbol could replace it, as long as the replacement is made everywhere the x appears.

The **successor function** g from **Z** to **Z** is defined by the formula $g(n) = n + 1$. Thus, no matter what integer is substituted for n, the output of g will be that number plus one: $g(\square) = \square + 1$. In other words, g sends each integer n to $n + 1$, or, symbolically, $g: n \to n + 1$.

An example of a **constant function** is the function h from **Q** to **Z** defined by the formula $h(r) = 2$ for all rational numbers r. This function sends each rational number r to 2. In other words, no matter what the input, the output is always 2: $h(\square) = 2$ or $h: r \to 2$.

The functions f, g, and h are represented by the function machines in Figure 1.3.2.

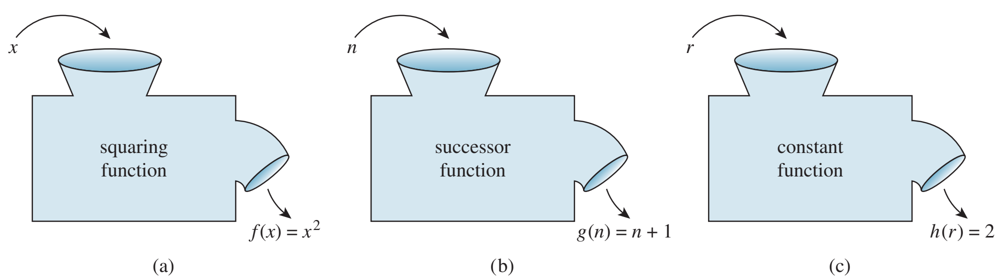


Figure 1.3.2

A function is an entity in its own right. It can be thought of as a certain relationship between sets or as an input/output machine that operates according to a certain rule. This is the reason why a function is generally denoted by a single symbol or string of symbols, such as f, G, of log, or sin.

A relation is a subset of a Cartesian product and a function is a special kind of relation. Specifically, if f and g are functions from a set A to a set B, then

$$f = \{(x, y) \in A \times B \mid y = f(x)\} \quad \text{and} \quad g = \{(x, y) \in A \times B \mid y = g(x)\}.$$

It follows that

$$\boxed{\boldsymbol{f} \textbf{ equals } \boldsymbol{g}, \quad \text{written } \boldsymbol{f = g}, \quad \text{if, and only if,} \quad \boldsymbol{f(x) = g(x)} \text{ for all } x \text{ in } A.}$$

Example 1.3.7 Equality of Functions

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ by the following formulas:

$$f(x) = |x| \quad \text{for all } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for all } x \in \mathbf{R}.$$

Does $f = g$?

Solution

Yes. Because the absolute value of any real number equals the square root of its square, $|x| = \sqrt{x^2}$ for all $x \in \mathbf{R}$. Hence $f = g$. ■

Test Yourself

1. Given sets A and B, a relation from A to B is ____.
2. A function F from A to B is a relation from A to B that satisfies the following two properties:
   a. for every element x of A, there is ____.
   b. for all elements x in A and y and z in B, if ____ then ____.
3. If F is a function from A to B and x is an element of A, then $F(x)$ is ____.

Exercise Set 1.3

1. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation R from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{y}{x} \text{ is an integer.}$$

a. Is $4\,R\,6$? Is $4\,R\,8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?
b. Write R as a set of ordered pairs.
c. Write the domain and co-domain of R.
d. Draw an arrow diagram for R.

2. Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation S from C to D as follows: For all $(x, y) \in C \times D$,

$$(x, y) \in S \quad \text{means that} \quad \frac{1}{x} - \frac{1}{y} \text{ is an integer.}$$

a. Is $2\,S\,2$? Is $-1 S - 1$? Is $(3, 3) \in S$? Is $(3, -3) \in S$?
b. Write S as a set of ordered pairs.
c. Write the domain and co-domain of S.
d. Draw an arrow diagram for S.

3. Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation T from E to F as follows: For all $(x, y) \in E \times F$,

$$(x, y) \in T \quad \text{means that} \quad \frac{x - y}{3} \text{ is an integer.}$$

a. Is $3\,T\,0$? Is $1T(-1)$? Is $(2, -1) \in T$? Is $(3, -2) \in T$?
b. Write T as a set of ordered pairs.
c. Write the domain and co-domain of T.
d. Draw an arrow diagram for T.

4. Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation V from G to H as follows: For all $(x, y) \in G \times H$,

$$(x, y) \in V \quad \text{means that} \quad \frac{x - y}{4} \text{ is an integer.}$$

a. Is $2\,V\,6$? Is $(-2)V(-6)$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?
b. Write V as a set of ordered pairs.
c. Write the domain and co-domain of V.
d. Draw an arrow diagram for V.

5. Define a relation S from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in S \quad \text{means that} \quad x \geq y.$$

a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\,S\,3$? Is $(-1)\,S\,(-2)$?
b. Draw the graph of S in the Cartesian plane.

6. Define a relation R from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in R \quad \text{means that} \quad y = x^2.$$

a. Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\,R\,9$? Is $9\,R\,(-3)$?
b. Draw the graph of R in the Cartesian plane.

7. Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define relations R, S, and T from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad x \geq y.$$
$$(x, y) \in S \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$
$$T = \{(4, 7), (6, 5), (6, 7)\}.$$

a. Draw arrow diagrams for R, S, and T.
b. Indicate whether any of the relations R, S, and T are functions.

8. Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations U, V, and W from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in U \quad \text{means that} \quad y - x > 2.$$
$$(x, y) \in V \quad \text{means that} \quad y - 1 = \frac{x}{2}.$$
$$W = \{(2, 5), (4, 1), (2, 3)\}.$$

a. Draw arrow diagrams for U, V, and W.
b. Indicate whether any of the relations U, V, and W are functions.

9. a. Find all relations from $\{0,1\}$ to $\{1\}$.
b. Find all functions from $\{0,1\}$ to $\{1\}$.
c. What fraction of the relations from $\{0,1\}$ to $\{1\}$ are functions?

10. Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

11. Define a relation P from $\mathbf{R}^+$ to $\mathbf{R}$ as follows: For all real numbers x and y with $x > 0$,

$$(x, y) \in P \quad \text{means that} \quad x = y^2.$$

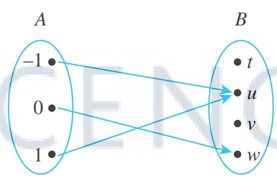
Is P a function? Explain.

12. Define a relation T from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers x and y,

$$(x, y) \in T \quad \text{means that} \quad y^2 - x^2 = 1.$$

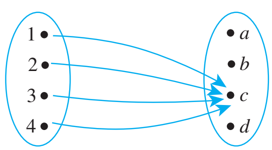
Is T a function? Explain.

13. Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F: A \to B$ by the following arrow diagram:



a. Write the domain and co-domain of F.
b. Find $F(-1)$, $F(0)$, and $F(1)$.

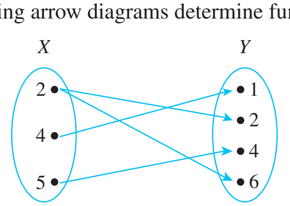
14. Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G: C \to D$ by the following arrow diagram:



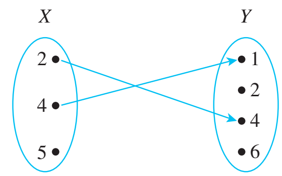
a. Write the domain and co-domain of G.
b. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

15. Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from X to Y?

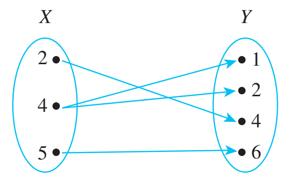
a.



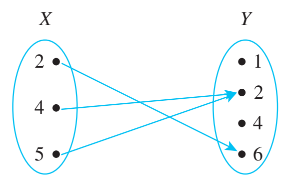
b.



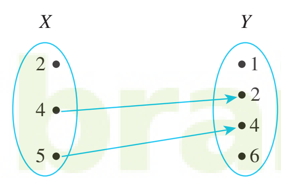
c.



d.



e.



16. Let f be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f\left(\frac{1}{2}\right)$.

17. Let g be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

18. Let h be the constant function defined in Example 1.3.6. Find $h\left(-\frac{12}{5}\right)$, $h\left(\frac{0}{1}\right)$, and $h\left(\frac{9}{17}\right)$.

19. Define functions f and g from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

20. Define functions H and K from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1.$$

Does $H = K$? Explain.

Answers for Test Yourself

1. a subset of the Cartesian product $A \times B$ 2. a. an element y of B such that $(x, y) \in F$ (i.e., such that x is related to y by F) b. $(x, y) \in F$ and $(x, z) \in F$; $y = z$ 3. the unique element of B that is related to x by F

APPENDIX B

SOLUTIONS AND HINTS TO SELECTED EXERCISES

Section 1.1

1. a. $x^2 = -1$ (*Or*: the square of x is -1)

b. A real number x

3. a. Between a and b

b. Real numbers a and b; there is a real number c

5. a. r is positive

b. Positive; the reciprocal of r is positive (*Or*: positive; $1/r$ is positive)

c. Is positive; $1/r$ is positive (*Or*: is positive; the reciprocal of r is positive)

7. a. There are real numbers whose sum is less than their difference.

True. For example, $1 + (-1) = 0$, $1 - (-1) = 1 + 1 = 2$, and $0 < 2$.

c. The square of any positive integer is greater than the integer.

True. If n is any positive integer, then $n \geq 1$. Multiplying both sides by the positive number n does not change the direction of the inequality (see Appendix A, T20), and so $n^2 \geq n$.

8. a. Have four sides

b. Has four sides

c. Has four sides

d. Is a square; J has four sides

e. J has four sides

10. a. Have a reciprocal

b. A reciprocal

c. s is a reciprocal for r

12. a. Real number; product with every number leaves the number unchanged

b. With every number leaves the number unchanged

c. $rs = s$

Section 1.2

1. $A = C$ and $B = D$

2. a. The set of all positive real numbers x such that 0 is less than x and x is less than 1

c. The set of all integers n such that n is a factor of 6

3. a. No, $\{4\}$ is a set with one element, namely 4, whereas 4 is just a symbol that represents the number 4

b. Three: the elements of the set are 3, 4, and 5.

c. Three: the elements are the symbol 1, the set $\{1\}$, and the set $\{1, \{1\}\}$

5. *Hint:* $\mathbf{R}$ is the set of all real numbers, $\mathbf{Z}$ is the set of all integers, and $\mathbf{Z}^+$ is the set of all positive integers

6. *Hint:* T_0 and T_1 do not have the same number of elements as T_2 and T_{-3}.

7. a. $\{1, -1\}$

c. $\emptyset$ (the set has no elements)

d. $\mathbf{Z}$ (every integer is in the set)

8. a. No, $B \nsubseteq A \therefore j \in B$ and $j \notin A$

d. Yes, C is a proper subset of A. Both elements of C are in A, but A contains elements (namely c and f) that are not in C.

9. a. Yes

b. No

f. No

i. Yes

10. a. No. Observe that $(-2)^2 = (-2)(-2) = 4$ whereas $-2^2 = -(2^2) = -4$. So $((-2)^2, -2^2) = (4, -4)$, $(-2^2, (-2)^2) = (-4, 4)$, and $(4, -4) \neq (-4, 4)$ because $-4 \neq 4$.

c. Yes. Note that $8 - 9 = -1$ and $\sqrt[3]{-1} = -1$, and so $(8 - 9, \sqrt[3]{-1}) = (-1, -1)$.

11. a. $\{(w, a), (w, b), (x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}$

b. $\{(a, w), (b, w), (a, x), (b, x), (a, y), (b, y), (a, z), (b, z)\}$

c. $\{(w, w), (w, x), (w, y), (w, z), (x, w), (x, x), (x, y), (x, z), (y, w), (y, x), (y, y), (y, z), (z, w), (z, x), (z, y), (z, z)\}$

d. $\{(a, a), (a, b), (b, a), (b, b)\}$

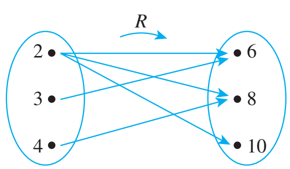
Section 1.3

1. a. No. Yes. No. Yes.

b. $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8)\}$

c. Domain of $R = A = \{2, 3, 4\}$, co-domain of $R = B = \{6, 8, 10\}$

d.



3. a. $3\ T\ 0$ because $\frac{3-0}{3} = \frac{3}{3} = 1$, which is an integer.

$1 \not{T} (-1)$ because $\frac{1-(-1)}{3} = \frac{2}{3}$, which is not an integer.

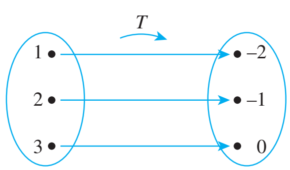
$(2, -1) \in T$ because $\frac{2-(-1)}{3} = \frac{3}{3} = 1$, which is an integer.

$(3, -2) \notin T$ because $\frac{3-(-2)}{3} = \frac{5}{3}$, which is not an integer.

b. $T = \{(1, -2), (2, -1), (3, 0)\}$

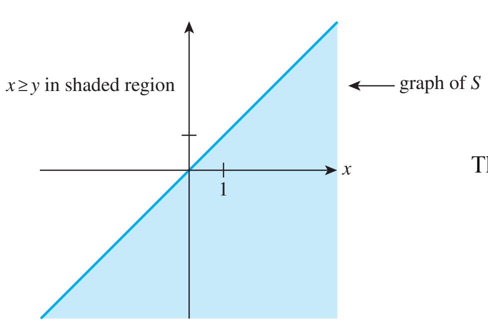
c. Domain of $T = E = \{1, 2, 3\}$, co-domain of $T = F = \{-2, -1, 0\}$

d.

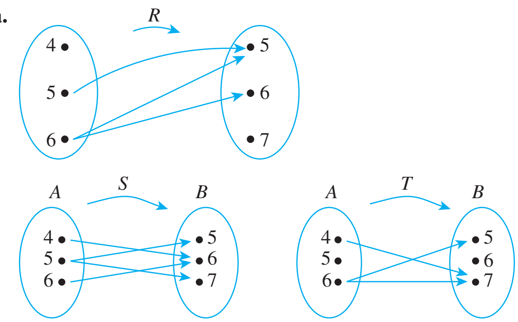


5. a. $(2, 1) \in S$ because $2 \geq 1$. $(2, 2) \in S$ because $2 \geq 2$.
$2 \not{S}\ 3$ because $2 \not\geq 3$. $(-1) \not{S} (-2)$ because $(-1) \not\geq (-2)$.

b.



7. a.



b. R is not a function because it satisfies neither property (1) nor property (2) of the definition. It fails property (1) because $(4, y) \notin R$, for any y in B. It fails property (2) because $(6, 5) \in R$ and $(6, 6) \in R$ and $5 \neq 6$.

S is not a function because $(5, 5) \in S$ and $(5, 7) \in S$ and $5 \neq 7$. So S does not satisfy property (2) of the definition of function.

T is not a function both because $(5, x) \notin T$ for any x in B and because $(6, 5) \in T$ and $(6, 7) \in T$ and $5 \neq 7$. So T does not satisfy either property (1) or property (2) of the definition of function.

9. a. $\emptyset$, $\{(0, 1)\}$, $\{(1, 1)\}$, $\{(0, 1), (1, 1)\}$

b. $\{(0, 1), (1, 1)\}$

c. 1/4

11. No, P is not a function because, for example, $(4, 2) \in P$ and $(4, -2) \in P$ but $2 \neq -2$.

13. a. Domain $= A = \{-1, 0, 1\}$, co-domain $= B = \{t, u, v, w\}$

b. $F(-1) = u,\ F(0) = w,\ F(1) = u$

15. a. This diagram does not determine a function because 2 is related to both 2 and 6.

b. This diagram does not determine a function because 5 is in the domain but it is not related to any element in the co-domain.

16. $f(-1) = (-1)^2 = 1,\ f(0) = 0^2 = 0,\ f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

19. For all $x \in \mathbf{R}$, $g(x) = \frac{2x^3+2x}{x^2+1} = \frac{2x(x^2+1)}{x^2+1} = 2x = f(x)$. Therefore, by definition of equality of functions, $f = g$.

This page contains answers for this chapter only

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