UCSD Math Club Integration Bee Qualifying Exam Solutions

May 7, 2014

1. $\int x^{2014}\,dx$

Solution.

$$\int x^{2014}\,dx = \frac{x^{2015}}{2015} + C. \qquad \square$$

2. $\int (2x+1)(3x-2)\,dx$

Solution.

$$\begin{aligned}\int (2x+1)(3x-2)\,dx &= \int (6x^2 - x - 2)\,dx \\ &= 2x^3 - \frac{x^2}{2} - 2x + C.\end{aligned} \qquad \square$$

3. $\int \frac{x^5 - x + 1}{x^2}\,dx$

Solution.

$$\begin{aligned}\int \frac{x^5 - x + 1}{x^2}\,dx &= \int \left(x^3 - \frac{1}{x} + \frac{1}{x^2}\right)\,dx \\ &= \frac{1}{4}x^4 - \log(x) - \frac{1}{x} + C.\end{aligned} \qquad \square$$

4. $\int_1^3 \left|x^2 - 4\right|\,dx$

Solution.

$$\begin{aligned}\int_1^3 \left|x^2 - 4\right|\,dx &= \int_1^2 (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx \\ &= \left(4x - \frac{1}{3}x^3\right)\Bigg|_1^2 + \left(\frac{1}{3}x^3 - 4x\right)\Bigg|_2^3 \\ &= 4.\end{aligned} \qquad \square$$

5. $\displaystyle\int \frac{\cos(\pi x)}{\sin^2(\pi x)}\,dx$

Solution.

$$\begin{aligned}\int \frac{\cos(\pi x)}{\sin^2(\pi x)}\,dx &= \int \frac{d}{dx}\left(-\frac{1}{\pi\sin(\pi x)}\right)\,dx \\ &= -\frac{1}{\pi\sin(\pi x)}.\end{aligned} \qquad \square$$

6. $\displaystyle\int x^{-1/5}\log(x)\,dx$

Solution.

$$\begin{aligned}\int x^{-1/5}\log(x)\,dx &= \frac{5}{4}x^{4/5}\log(x) - \frac{5}{4}\int x^{-1/5}\,dx \\ &= \frac{5}{4}x^{4/5}\log(x) - \frac{25}{16}x^{4/5} + C.\end{aligned} \qquad \square$$

7. $\displaystyle\int x\cos(x)\sin(x)\,dx$

Solution.

$$\begin{aligned}\int x\cos(x)\sin(x)\,dx &= \frac{1}{2}\int x\sin(2x)\,dx \\ &= -\frac{1}{4}x\cos(2x) + \frac{1}{4}\int \cos(2x)\,dx \\ &= -\frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x) + C.\end{aligned} \qquad \square$$

8. $\displaystyle\int \log\left(\frac{1}{x}\right)\,dx$

Solution.

$$\begin{aligned}\int \log\left(\frac{1}{x}\right)\,dx &= -\int \log(x)\,dx \\ &= -x\log(x) + \int 1\,dx \\ &= x\log\left(\frac{1}{x}\right) + x + C.\end{aligned} \qquad \square$$

9. $\displaystyle\int_0^\infty \frac{\arctan(x)}{x^2+1}\,dx$

Solution.

$$\begin{aligned}\int_0^\infty \frac{\arctan(x)}{x^2+1}\,dx &= \int_0^\infty \frac{d}{dx}\left(\frac{1}{2}\arctan^2(x)\right)\,dx\\ &= \frac{1}{2}\arctan^2(x)\Big|_0^\infty\\ &= \frac{\pi^2}{8}.\end{aligned}$$ $\square$

10. $\displaystyle\int x\left(e^x+\cos(x)+\sin(x)\right)\,dx$

Solution.

$$\begin{aligned}\int x\left(e^x+\cos(x)+\sin(x)\right)\,dx &= x(e^x+\sin(x)-\cos(x)) - \int (e^x+\sin(x)-\cos(x))\,dx\\ &= x(e^x+\sin(x)-\cos(x)) - e^x+\cos(x)+\sin(x)+C.\end{aligned}$$ $\square$

11. $\displaystyle\int \frac{e^x}{e^{2x}+e^x}\,dx$

Solution.

$$\begin{aligned}\int \frac{e^x}{e^{2x}+e^x}\,dx &= \int \frac{e^x}{e^x(e^x+1)}\,dx\\ &= \int \left(1-\frac{e^x}{e^x+1}\right)\,dx\\ &= \int \left(1-\frac{d}{dx}\log(e^x+1)\right)\,dx\\ &= x-\log(e^x+1)+C.\end{aligned}$$ $\square$

12. $\displaystyle\int \frac{1}{\tan^2(x)}\,dx$

Solution.

$$\begin{aligned}\int \frac{1}{\tan^2(x)}\,dx &= \int \left(\frac{1}{\sin^2(x)}-1\right)\,dx\\ &= \int \left(\frac{d}{dx}\left(-\frac{1}{\tan(x)}\right)-1\right)\,dx\\ &= -\frac{1}{\tan(x)}-x+C.\end{aligned}$$ $\square$

13. $\displaystyle\int_0^1 \frac{x^2}{1+x^2}\,dx$

Solution.

$$\begin{aligned}\int_0^1 \frac{x^2}{1+x^2}\,dx &= \int_0^1 \left(1-\frac{1}{1+x^2}\right)\,dx \\ &= (x-\arctan(x))\Big|_0^1 \\ &= 1-\frac{\pi}{4}.\end{aligned}$$ □

14. $\displaystyle\int \frac{\sin^2(x)(1+\tan^2(x))}{\tan^2(x)}\,dx$

Solution.

$$\begin{aligned}\int \frac{\sin^2(x)(1+\tan^2(x))}{\tan^2(x)}\,dx &= \int \frac{\sin^2(x)(1+\frac{\sin^2(x)}{\cos^2(x)})}{\frac{\sin^2(x)}{\cos^2(x)}}\,dx \\ &= \int (\cos^2(x)+\sin^2(x))\,dx \\ &= \int 1\,dx \\ &= x+C.\end{aligned}$$ □

15. $\displaystyle\int \left(2e^{x^2}x^2+e^{x^2}\right)\,dx$

Solution.

$$\begin{aligned}\int \left(2e^{x^2}x^2+e^{x^2}\right)\,dx &= \int \frac{d}{dx}\left(e^{x^2}x\right)\,dx \\ &= e^{x^2}x+C.\end{aligned}$$ □

16. $\displaystyle\int_0^\infty \pi^{-\lfloor x\rfloor}\,dx$

Solution.

$$\begin{aligned}\int_0^\infty \pi^{-\lfloor x\rfloor}\,dx &= \sum_{n=0}^{\infty}\int_n^{n+1}\pi^{-n}\,dx \\ &= \sum_{n=0}^{\infty}\pi^{-n} \\ &= \frac{\pi}{\pi-1}.\end{aligned}$$ □

17. $\displaystyle\int_0^1 \sin(\arctan(x))\,dx$

Solution.

$$\begin{aligned}\int_0^1 \sin(\arctan(x))\,dx &= \int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx \\ &= \int_0^1 \frac{d}{dx}\sqrt{x^2+1}\,dx \\ &= \sqrt{2}-1.\end{aligned}$$

□

18. $\displaystyle\int e^x \tan(e^x)\,dx$

Solution.

$$\begin{aligned}\int e^x \tan(e^x)\,dx &= \int \frac{d}{dx}\left(-\log(\cos(e^x))\right)\,dx \\ &= -\log(\cos(e^x)) + C.\end{aligned}$$

□

19. $\displaystyle\int_0^\infty \frac{e^{-x}-e^{-\pi x}}{x}\,dx$

Solution.

$$\begin{aligned}\int_0^\infty \frac{e^{-x}-e^{-\pi x}}{x}\,dx &= \int_0^\infty \left(-\frac{e^{-xy}}{x}\right)\Bigg|_1^\pi\,dx \\ &= \int_0^\infty \int_1^\pi e^{-xy}\,dy\,dx \\ &= \int_1^\pi \int_0^\infty e^{-xy}\,dx\,dy \\ &= \int_1^\pi \left(-\frac{e^{-xy}}{y}\right)\Bigg|_0^\infty\,dy \\ &= \int_1^\pi \frac{1}{y}\,dy \\ &= \log(\pi).\end{aligned}$$

□

20. $\displaystyle\int \frac{1}{x^{1/2}+x^{1/3}}\,dx$

Solution. Let $u = x^{1/6}$, so $x = u^6$. Then $dx = 6u^5\,du$, and

$$\begin{aligned}
\int \frac{1}{x^{1/2}+x^{1/3}}\,dx &= 6\int \frac{u^5}{u^3+u^2}\,du \\
&= 6\int \frac{u^3}{u+1}\,du \\
&= 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)\,du \\
&= 2u^3 - 3u^2 + 6u - 6\log(u+1) + C \\
&= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\log(x^{1/6}+1) + C. \qquad \square
\end{aligned}$$

21. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

22. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx$

Solution.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx &= \int_{-\pi/4}^{0} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx + \int_{0}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx \\
&= \int_{0}^{\pi/4} \frac{\sin(-x)+\sin(-2x)}{\cos(-x)+\cos(-2x)}\,dx + \int_{0}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx \\
&= -\int_{0}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx + \int_{0}^{\pi/4} \frac{\sin(x)+\sin(2x)}{\cos(x)+\cos(2x)}\,dx \\
&= 0. \qquad \square
\end{aligned}$$

23. $\displaystyle\int_0^\infty \frac{\log(x)}{1+x^2}\,dx$

Solution.

$$\begin{aligned}
\int_0^\infty \frac{\log(x)}{1+x^2}\,dx &= \int_0^1 \frac{\log(x)}{1+x^2}\,dx + \int_1^\infty \frac{\log(t)}{1+t^2}\,dt \\
&= \int_0^1 \frac{\log(x)}{1+x^2}\,dx + \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1+\frac{1}{x^2}}\,\frac{dx}{x^2} \\
&= \int_0^1 \frac{\log(x)}{1+x^2}\,dx - \int_0^1 \frac{\log(x)}{x^2+1}\,dx \\
&= 0. \qquad \square
\end{aligned}$$

24. $\displaystyle\int \cos(\log(x))\,dx$

Solution. Let $x = e^u$, so $dx = e^u\,du$. Then

$$\begin{aligned}
\int \cos(\log(x))\,dx &= \int e^u \cos(u)\,du \\
&= e^u \sin(u) - \int e^u \sin(u)\,du \\
&= e^u \sin(u) + e^u \cos(u) - \int e^u \cos(u)\,du \\
&= \frac{1}{2} e^u \sin(u) + \frac{1}{2} e^u \cos(u) + C \\
&= \frac{1}{2} x \sin(\log(x)) + \frac{1}{2} x \cos(\log(x)) + C.
\end{aligned}$$

□

25. $\displaystyle\int_0^\infty \frac{\sin(x)}{\sqrt{x}}\,dx$