Math 20E Final Winter 99, Lindblad.

1. Find the equation for the line of intersection of the two planes $2x-3y+z-1=0$ and $x+2y-z+3=0$.

2. The temperature of space is given by $\phi(x,y,z)=1+xy^2$. A fly is flying in space and at each point (x,y,z) of its journey it flies in the direction $\mathbf{F}(x,y,z)$ in which the rate of increase of temperature is maximum.

(a). Calculate $\mathbf{F}(x,y,z)$.

(b). Find the curve along which the fly travels if it starts at the point $(1,-1,1)$.

3. (a). Let $\mathbf{F}$ be the vector field $ze^{x-y}\mathbf{i}-ze^{x-y}\mathbf{j}+e^{x-y}\mathbf{k}$. Find ϕ with $\nabla\phi=\mathbf{F}$.

(b). Calculate $\int_C \mathbf{F}\cdot\mathbf{R}$, where C is the curve

$$y=\frac{x(1-x)}{(1+x^2)^3}, \qquad z=\frac{1-x}{(1+x^2)^3}, \qquad 0\le x\le 1,$$

oriented so that $x=0$ at the initial point.

(c). Decide whether the vector field

$$\frac{x}{x^2+y^2+z^2}\mathbf{i}+\frac{y}{x^2+y^2+z^2}\mathbf{j}+\frac{z}{x^2+y^2+z^2}\mathbf{k}$$

is conservative and **justify your answer**.

4. (a) State the change of variable theorem for the double integral over a region in the plane.

Let $x=x(u,v)$ and $y=y(u,v)$ be the change of variables given by

$$\begin{cases} x=v\cosh u \\ y=v\sinh u \end{cases} \quad \text{where} \quad \begin{cases} \cosh u=(e^u+e^{-u})/2 \\ \sinh u=(e^u-e^{-u})/2 \end{cases}$$

Let $D=\{(x,y);\ 1\le x^2-y^2\le 2,\ 2y\le x\le 3y\}$. D is the image under the mapping $(u,v)\to(x,y)$ of a rectangle in the uv-plane $R=\{(u,v);\ a\le u\le b,\ c\le v\le d\}$.

(b) Find the rectangle R. (Hint: a useful identity is $\cosh^2 u-\sinh^2 u=1$.)

(c) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(d) Find $\displaystyle\iint_D \frac{dxdy}{(1+x^2-y^2)^2}$,

5. (a) State Greens theorem for a domain D in the plane bounded by a curve C.

(b) Explain how one can use Greens theorem to find the area of the domain D.

(c) Consider the closed curve C given by $\begin{cases} x &= t^2 \\ y &= \frac{t^3}{3} - t \end{cases}$, $-\sqrt{3} \le t \le \sqrt{3}$.
Compute the area D bounded by the curve C.

(d) Find the length of the curve C.

6. (a) State the Divergence Theorem.

Find the flux $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ of the vector field $\mathbf{F} = \mathbf{i} + \mathbf{j} + z^3\mathbf{k}$ out of the sphere $S = \{(x, y, z);\ x^2 + y^2 + z^2 = a^2\}$, where $\mathbf{n}$ is the outward unit normal.

(b) By direct calculation of the surface integral.

(c) By using the Divergence Theorem to convert it to a volume integral and evaluate.

7. (a) State Stokes theorem.

Let S be the surface $S = \{(x, y, z); z = 2 - x^2 - y^2,\ (x, y) \in D\}$, where $D = \{(x, y);\ x^2 + y^2 \le 1\}$, and let $\mathbf{n}$ be the unit normal to S oriented upwards so $\mathbf{n} \cdot \mathbf{k} > 0$. Let C be the boundary of S positively oriented relative to the orientation of S. Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.
Find the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R}$.

(a) By direct calculation using the usual parameterization of the circle.

(b) By using Stokes theorem to convert it to an integral over S and evaluate.

8. A parameterization of the torus (donut)
$T = \{(x, y, z);\ \left(\sqrt{x^2 + y^2} - 4\right)^2 + z^2 \le 4\}$ is given by

$$\begin{cases} x = 4\cos\theta + 2\cos\theta\,\cos\phi \\ y = 4\sin\theta + 2\sin\theta\cos\phi \\ z = 2\sin\phi \end{cases} \quad \text{where} \quad \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le 2\pi \end{cases}$$

(a) Find a normal to T at a point with coordinates (θ, ϕ).

(b) Calculate the surface area element dA in terms of $d\theta d\phi$ and use it to calculate the area of the surface of the torus T.