Midterm, Math 102, Fall 2011, Mark Gross, Solutions

1. Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix}.$$

Find:

(a) a basis for the column space of A.

(b) a basis for the null space of A.

(c) a basis for the row space of A.

(a) We first row-reduce A to echelon form:

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 9 & -6 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the column space of this echelon form matrix is given by the columns with pivots, i.e., the first and second columns, and hence the first and second columns of A form a basis for the column space, i.e., $(1, 1, 1)$, $(-2, 1, 7)$.

For (b), we go to row-reduced form:

$$R = \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & -2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

x_3, x_4 are free variables, and we see the general solution is $x_1 = (x_3 - 4x_4)/3$, $x_2 = (2x_3 + x_4)/3$. Thus a basis of solutions is $(1/3, 2/3, 1, 0)$, $(-4/3, 1/3, 0, 1)$.

For (c), the row space of A and the row space of R are the same, so a basis for the row space is $(1, 0, -1/3, 4/3)$, $(-4/3, 1/3, 0, 1)$.

2. Find a QR-decomposition for the matrix

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{pmatrix}.$$

We need to apply Gram-Schmidt to the vectors $v_1 = (3, 0, 4)$, $v_2 = (-1, 0, 7)$, $v_3 = (2, 9, 11)$. We have $\|v_1\| = 5$, $q_1 = (3/5, 0, 4/5)$. We then have

$$q_2' = v_2 - (v_2 \cdot q_1)q_1 = (-1, 0, 7) - (25/5)(3/5, 0, 4/5) = (-4, 0, 3),$$

so $q_2 = (-4/5, 0, 3/5)$. We have

$$q_3' = v_3 - (v_3 \cdot q_1)q_1 - (v_3 \cdot q_2)q_2 = (2, 9, 11) - (50/5)(3/5, 0, 4/5) - (25/5)(-4/5, 0, 3/5) = (0, 9, 0),$$

so $q_3 = (0, 1, 0)$. The QR decomposition is then

$$A = QR = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & 3/5 & 0 \end{pmatrix} \begin{pmatrix} q_1 \cdot v_1 & q_1 \cdot v_2 & q_1 \cdot v_3 \\ 0 & q_2 \cdot v_2 & q_2 \cdot v_3 \\ 0 & 0 & q_3 \cdot v_3 \end{pmatrix},$$

and one computes

$$R = \begin{pmatrix} 5 & 5 & 10 \\ 0 & 5 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

(a) What is the projection matrix P which gives orthogonal projection of vectors in $\mathbf{R}^3$ onto the column space of A?

(b) Find a "best fit" solution to the equation

$$A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For (a), we use the formula $P = A(A^T A)^{-1} A^T$. We calculate

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

so

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1/3 & -1/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{pmatrix}.$$

The best fit solution is

$$(A^T A)^{-1} A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1/3 & -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/3 \end{pmatrix}.$$

4. Answer each of the following questions, giving a brief reason justifying your answer.

(a) Let A be a 4×7 matrix. Can the column rank of A can be 5?

No: since the row and column ranks are the same, and the row rank can be at most 4, it is impossible for the column rank to be 5.

(b) Let A be a 4×5 matrix of rank 4. Does A have a right inverse?

Yes: the column space is dimension 4 and is a subspace of $\mathbf{R}^4$, hence the equation $Ax = b$ always has a solution. Thus if we take v_i to be the solution to the equation $Av_i = e_i$, and C to be the matrix whose columns are $v_1, \ldots, v_4$, then AC is the 4×4 identity.

(c) Let P_n denote the vector space of polynomials of degree n. Let $T : P_n \to \mathbf{R}$ be the function which associates to any polynomial f in P_n the integral

$$\int_{-1}^{1} f(x)dx.$$

Is T a linear transformation?

Yes, because

$$\int_{-1}^{1} c_1 f_1(x) + c_2 f_2(x) dx = c_1 \int_{-1}^{1} f_1(x)dx + c_2 \int_{-1}^{1} f_2(x)dx.$$