EXAM 2.

1. (1) Write down the equation of the unique plane π in $\mathbb{R}^3$ containing $P_1(0,-1,1)$, $P_2(1,-1,0)$, $P_3(1,0,-1)$.

(2) Show the subset W_π of $\mathbb{R}^3$ consisting of position vectors of all points in π is a vector subspace of $\mathbb{R}^3$.

(3) Show $\mathcal{B} := \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for W_π.

(1) $\overrightarrow{P_1P_2} = (1,0,-1)$
$\overrightarrow{P_1P_3} = (1,1,-2)$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix}$$
$$= (1,1,1).$$

This is the normal vector.

So the plane is given by $x+y+z=d$.

Plug in any point to compute d:

e.g for $(0,-1,1)$, $0+(-1)+(1)=d$
$\Rightarrow d=0$.

So the equation is $\boxed{x+y+z=0.}$

(2) Let $\vec{v}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ so that $\vec{v}_1, \vec{v}_2 \in W_\pi$;

that is, ~~$x_1+x_2+x_3=0$~~ $x_1+y_1+z_1=0$ and $x_2+y_2+z_2=0$.

Closed under +:

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}.$$

Note $(x_1+x_2) + (y_1+y_2) + (z_1+z_2)$
$= \underbrace{(x_1+y_1+z_1)}_{0} + \underbrace{(x_2+y_2+z_2)}_{0}$
$= 0.$

So $v_1 + v_2 \in W_\pi$. ✓

Closed under $\bullet$:

For $c \in \mathbb{R}$, $c\vec{v}_1 = \begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}$.

Note $(cx_1) + (cy_1) + (cz_1) = c\underbrace{(x_1+y_1+z_1)}_{0}$
$= 0$

So $cv_1 \in W_\pi$ ✓

(3) It's easy to see $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are linearly independent;

e.g. If $a\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$, then $\begin{matrix} -a & & = 0 \\ & b & = 0 \\ a & - b & = 0 \end{matrix} \Rightarrow \underline{\underline{a = b = 0}}.$

Since π is a plane, it's 2 dimensional. Any two lin. independent vectors in W_π form a basis, so it suffices to show these vectors are in W_π.

But this is clear, since $(-1) + (0) + (1) = 0$
and $(0) + (1) + (-1) = 0$.

II. Let V be n-dimensional vector space over $\mathbb{R}$. Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V and let $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be a set of n vectors in V.

Let
$$A := \left[[\vec{c}_1]_{\mathcal{B}}, \dots, [\vec{c}_n]_{\mathcal{B}} \right].$$
Show if A is invertible, then $\mathcal{C}$ is a basis for V.

Proof:
If A is invertible, then its columns $[\vec{c}_1]_{\mathcal{B}}, \dots, [\vec{c}_n]_{\mathcal{B}}$ are linearly independent.

We can view "taking coordinates with respect to $\mathcal{B}$" as a linear transformation
$$V \to \mathbb{R}^n$$
$$\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$$

So if $[\vec{c}_1]_{\mathcal{B}}, \dots, [\vec{c}_n]_{\mathcal{B}}$ are linearly independent, so are $\vec{c}_1, \dots, \vec{c}_n$.

Since any collection of n linearly independent vectors in an n-dimensional vector space form a basis, we conclude that $\mathcal{C}$ is a basis.

III. (40 points)

Let $\mathcal{P}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of polynomials in variable X, of degree at most 2, with coefficients in the field $\mathbb{R}$ of real numbers.

(1) Show that $\mathcal{B} := \{1+X, 1+2X, 1+X+X^2\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

(2) Write down a linear transformation

$$T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$$

such that $T(1+X) = 1$, $T(1+2X) = 2$, $T(1+X+X^2) = 0$.

(3) Is there more than one linear transformation T satisfying the above equalities? Justify your answer.

(4) If T is the linear transformation you wrote down in (2) above, then calculate $T(1-X+X^2)$.

(5) Show that $\mathcal{C} := \{1-X, 1-2X, 1-X-X^2\}$ is also a basis for $\mathcal{P}_2(\mathbb{R})$ and compute the transition matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

(1) Show linearly independent:

Suppose $c_1(1+x) + c_2(1+2x) + c_3(1+x+x^2) = 0$

$\Rightarrow$ $c_1 + c_2 + c_3 = 0$ $\quad\Rightarrow$ ① $c_1 + c_2 + 0 = 0$

$c_1 + 2c_2 + c_3 = 0$ $\quad$ ② $c_1 + 2c_2 + 0 = 0$

$c_3 = 0$ $\quad$ subtract ① from ②,

$c_2 = 0$

$\Rightarrow c_1 + 0 + 0 = 0$

Hence, $c_1 = c_2 = c_3 = 0 \Rightarrow$ linearly independent

Since B has 3 ~~vectors~~ linearly independent vectors and $\dim P_2 = 3$, B is a basis for P_2

(2) $T(x) = T((2x+1)-(x+1))$

$= T(2x+1) - T(x+1) = 2 - 1 = 1$

$T(x^2) = T(1+x+x^2 - (1+x))$

$= T(1+x+x^2) - T(1+x) = 0 - 1 = -1$

$T(1) = T((1+x)-x) =$

$T(1+x) - T(x) = 1 - 1 = 0$

Hence, $T(a+bx+cx^2)$

$= aT(1) + bT(x) + cT(x^2)$

$= 0 + b \cdot 1 - c = b-c$

$\boxed{T(a+bx+cx^2) = b-c}$

(3) No, the linear transformation with the above properties is unique; since B is a basis for P_2, once we know what T does to the basis, what T does to all of P_2 is uniquely determined //

(4) $T(1-x+x^2) = T(1) - T(x) + T(x^2)$

$= -1 - 1 = -2$

(5) $C = \{1-x,\ 1-2x,\ 1-x-x^2\}$

Linear independence:

Suppose $c_1(1-x) + c_2(1-2x) + c_3(1-x-x^2) = 0$

then
$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 - 2c_2 - c_3 &= 0 \\ -c_3 &= 0 \quad \Rightarrow \quad c_3 = 0 \end{aligned}$$

So ① $c_1 + c_2 = 0$ Adding ① + ② get

② $-c_1 - 2c_2 = 0$ $\quad -c_2 = 0 \Rightarrow c_2 = 0$

Then $c_1 + 0 = 0 \Rightarrow c_1 = 0$

Hence C is a set of 3 linearly independent vectors, $\dim P_2 = 3 \Rightarrow C$ is a basis for P_2

To find $P_{C\leftarrow B}$: $[C \mid B] \xrightarrow{\text{row reduce}} [I \mid P_{C\leftarrow B}]$

$$\begin{array}{cc} C & B \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \end{array} \xrightarrow{\text{row reduce}} \cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & 4 \\ 0 & 1 & 0 & -2 & -3 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$P_{C\leftarrow B} = \begin{bmatrix} 3 & 4 & 4 \\ -2 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$