

MATH 102 - PRACTICE PROBLEMS FOR MIDTERM I

1. Find the LU -decomposition of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 3 & 1 \\ 3 & 1 & -2 \end{pmatrix}.$$

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ -1 & 0 & -1 & -2 \end{pmatrix}$$

- (i) Give a basis for the null space of A . What is the nullity of A ?
- (ii) Show that the columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ of A are linearly dependent by exhibiting explicit relations between them.
- (iii) Give the equation for $C(A)$ as a subspace of \mathbb{R}^3 .
- (iv) Give a basis for $C(A)$. What is the rank of A ?
- (v) Give a basis for the row space of A .
- (vi) What is the dimension of the left null space of A ?
- (vii) Write down the general solution to the following system of equations

$$Ax = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}.$$

- (viii) Does A admit a left inverse? How about a right inverse?

3. If the vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis of \mathbb{R}^3 , show that $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ also form a basis of \mathbb{R}^3 .

4. Which of the following are vector spaces?

- (i) The set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1 \right\}.$
- (ii) The set $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ in } \mathbb{R}^4 : x - 2y + 3z - 4w = 0 \text{ and } x - y - z + w = 0 \right\}.$
- (iii) The set of vectors in \mathbb{R}^4 equidistant from the three points $(3, -4, 0, 0)$, $(0, -3, 4, 0)$ and $(0, 4, 3, 0)$.
- (iv) The set of polynomials P of degree less or equal to 4 such that

$$P(0) = P'(0) = P''(0) = 0.$$

- (v) The set of skew-symmetric matrices.

(vi) The set of solutions to the differential equation

$$y'' + 4y = \sin t.$$

5. A matrix A and its row reduced form are shown below. What is the second column of A ?

$$A = \begin{bmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{bmatrix}, \text{ rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

Calculate $T\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right)$.

7. Consider \mathcal{P}_2 the vector space of polynomials of degree at most equal to 2 and let

$$T : \mathcal{P}_2 \rightarrow \mathcal{P}_2$$

be the linear transformation

$$T(f) = xf'' + f.$$

(i) Show that $\{1, x-1, x^2\}$ is a basis of \mathcal{P}_2 .

(ii) Find the matrix of the linear transformation T in the basis in (i).

8. Assume A is a square $k \times k$ matrix, B is a $k \times (k-1)$ matrix and C is a $(k-1) \times k$ matrix. Furthermore, assume that

$$A = BC.$$

Is it true that $N(A)$ always contains a non-zero vector?

9. Let A be a 2×2 matrix. Explain why there must exist non-zero constants c_0, c_1, c_2, c_3, c_4 such that

$$c_0I + c_1A + c_2A^2 + c_3A^3 + c_4A^4 = 0.$$

MATH 102 - SOLUTIONS TO PRACTICE PROBLEMS - MIDTERM I

1. We carry out row reduction. We begin with the row operations

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$$

yielding the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

This is already upper triangular hence

$$U = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

The lower triangular matrix equals

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

2. For further reference, we begin by row-reducing the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & b_1 \\ 2 & 3 & 5 & 7 & b_2 \\ -1 & 0 & -1 & -2 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & b_1 \\ 0 & -1 & -1 & -1 & b_2 - 2b_1 \\ 0 & 2 & 2 & 2 & b_1 + b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 & 2b_2 - 3b_1 \\ 0 & 1 & 1 & 1 & -b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 2b_2 + b_3 - 3b_1 \end{bmatrix}.$$

- (i) The null space of A is also the null space of the row-reduced matrix

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two variables x, y are pivots, the last two variables z, w are free. We obtain the system

$$x + z + 2w = 0 \implies x = -z - 2w$$

$$y + z + w = 0 \implies y = -z - w.$$

We conclude

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $N(A)$ has the basis

$$N(A) = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The nullity of A equals 2.

- (ii) Each vector in the null space gives a relation between the columns of A . For instance, the vector

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in N(A) \implies -\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3 = 0$$

and

$$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in N(A) \implies -2\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_4 = 0.$$

- (iii) The last row of zeros in the row-reduced augmented matrix gives the equation

$$2b_2 + b_3 - 3b_1 = 0$$

that must be satisfied by all vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the column space.

- (iv) The pivot columns, namely the first and second columns of A , give a basis for $C(A)$:

$$C(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

The rank of A equals 2.

- (v) The pivots are in the first and second row, so a basis for the row space of A is

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- (vi) The dimension of the left-null space of A is the number of rows minus the rank which is $3 - 2 = 1$.

- (vii) We can easily find a particular solution

$$x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The general solution is $x = x_p + x_h$ hence

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

- (viii) The rank of A equals 2 which not equal to the number of rows or columns of A . Hence A does not admit a left inverse nor a right inverse.

3. Assume first that the vectors $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ are linearly dependent. There exist constants a, b, c not all zero such that

$$a\mathbf{u} + b(\mathbf{v} - \mathbf{w}) + c(\mathbf{u} + \mathbf{v} - 2\mathbf{w}) = \mathbf{0}.$$

Rearranging, we obtain

$$(a + c)\mathbf{u} + (b + c)\mathbf{v} + (b - 2c)\mathbf{w} = \mathbf{0}.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis, the vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ must be independent. This implies that

$$a + c = 0, \quad b + c = 0, \quad b - 2c = 0.$$

Solving for a, b, c we find

$$a = b = c = 0$$

which is impossible by assumption that not all (a, b, c) are zero.

It follows that $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ are three linearly independent vectors in \mathbb{R}^3 hence they must form a basis of \mathbb{R}^3 .

4.

- (i) Not a vector space. For instance $(1, 0, \dots, 0)$ is in the set, but twice the vector $2(1, 0, \dots, 0)$ is not in the set.
- (ii) Vector space. This is the null space of the matrix

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Null spaces are always vector spaces.

- (iii) Vector space. This is a bit harder to see. First, we determine the set a bit more explicitly. The square of the distance to $(3, -4, 0, 0)$ equals

$$(x - 3)^2 + (y + 4)^2 + z^2 + w^2$$

while the square distance to $(0, -3, 4, 0)$ equals

$$x^2 + (y + 3)^2 + (z - 4)^2 + w^2.$$

Since the two distances must be equal, we have

$$(x - 3)^2 + (y + 4)^2 + z^2 + w^2 = x^2 + (y + 3)^2 + (z - 4)^2 + w^2.$$

Expanding out and canceling, we obtain the linear equation

$$-3x + y + 4z = 0.$$

The requirement that the distances from $(0, -3, 4, 0)$ and $(0, 4, 3, 0)$ be equal can be worked out in a similar fashion. We obtain the equation

$$-7y + z = 0.$$

Thus the set in question can be described by the two linear equations

$$-3x + y + 4z = 0, \quad -7y + z = 0.$$

This is a null space of the matrix

$$\begin{bmatrix} -3 & 1 & 4 & 0 \\ 0 & -7 & 1 & 0 \end{bmatrix}$$

hence it is a subspace.

(iv) Vector space. Indeed, if P and Q are two polynomials with

$$P(0) = P'(0) = P''(0) = 0, Q(0) = Q'(0) = Q''(0) = 0$$

then their sum $P + Q$ also satisfies

$$(P + Q)(0) = (P + Q)'(0) = (P + Q)''(0) = 0.$$

Similarly, scalar multiplication is preserved since cP satisfies

$$(cP)(0) = (cP)'(0) = (cP)''(0) = 0.$$

(v) Vector space. This is the set of matrices

$$A^T = -A.$$

This is a subspace since if A is skew symmetric so is cA since

$$(cA)^T = cA^T = -cA.$$

Similarly, if A, B are skew symmetric so is their sum $A + B$ since

$$(A + B)^T = A^T + B^T = -A - B = -(A + B).$$

(vi) Not a vector space. If y is a solution of the equation, then $y'' + 4y = \sin t$. But $2y$ is not a solution since

$$(2y)'' + 4(2y) = 2 \sin t.$$

Since the equation changes, the solution set is not a vector space.

5. We calculate the null space of $\text{rref}(A)$. This is the space spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. Elements in the

null space of A give relations between the columns of A . Writing v_1, v_2, v_3, v_4 for the columns of A we must have

$$v_2 = v_4 - v_1 - v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

6. Note that

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$T \left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) = 2T \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) - T \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) + T \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) = 2 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

7.

(i) Clearly, $\{1, x, x^2\}$ is a basis of \mathcal{P}_2 hence

$$\dim \mathcal{P}_2 = 3.$$

To show that $\{1, x-1, x^2\}$ is also a basis, it suffices to prove that the three vectors are linearly independent. Assume otherwise, so that

$$a + b(x-1) + cx^2 = 0$$

for some constants a, b, c . This rewrites as

$$(a-b) + bx + cx^2 = 0 \implies a-b=0, b=0, c=0 \implies a=b=c=0$$

proving the independence.

(ii) We calculate

$$T(1) = 1, T(x-1) = (x-1), T(x^2) = 2x + x^2 = x^2 + 2(x-1) + 2 \cdot 1.$$

For the last equality, we needed to express $T(x^2) = 2x + x^2$ in terms of the basis elements $1, x-1, x^2$. The matrix of T in this basis is given by collecting the coefficients of the basis elements $1, x-1, x^2$, yielding the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

8. True. To explain this fact, note that C has fewer rows than columns. Then C must have free variables. This in turn means that the null space $N(C)$ contains a non-zero vector \mathbf{x} :

$$C\mathbf{x} = 0.$$

We claim that \mathbf{x} is also in the null space of A . Indeed,

$$A\mathbf{x} = BC\mathbf{x} = B \cdot 0 = 0.$$

This shows that $N(A)$ contains a nonzero vector.

9. Consider Mat_2 the space of 2×2 matrices. This is a 4-dimensional vector space. The 5 matrices I, A, A^2, A^3, A^4 are vectors in Mat_2 . Since 5 vectors in a 4-dimensional vector space must be linearly dependent, we can therefore find constants c_0, c_1, c_2, c_3, c_4 not all zero such that

$$c_0I + c_1A + c_2A^2 + c_3A^3 + c_4A^4 = 0.$$