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Linear Algebra - Exam #1 - A. Terras, April 27, 2007

The exam is closed book, no calculators, no computers, no notes, no headphones .... Each problem is worth the same number of points.

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1) **Define** the following and give an example.
   a) linearly independent vectors in a vector space V
   c) linear transformation $T:V\to W$ where V and W are vector spaces

2) Given the matrix A below, find the reduced echelon form of A and then find a basis for the column space ColA and a basis for the null space NulA.

$$A=\begin{bmatrix} -3 & 9 & -2 & -6 \\ 6 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}.$$

3) **True-False.** Tell whether the following statements are true or false. If true, give a brief explanation and if false, give a counterexample.
   a) The plane consisting of vectors (x,y,z) such that x+y+z = 1 is a subspace of $\mathbb{R}^3$.

   b) Suppose W is a subspace of V and $W^\perp$ denotes the orthogonal complement of W. Then $\dim(W) + \dim(W^\perp) = \dim(V)$.

   c) Suppose A and B are nxn matrices. Then AB=BA.

4) Let $\mathbb{P}_n$ be the vector space of polynomials of degree less than or equal to n.
   a) What is the standard basis B_n for $\mathbb{P}_n$?

   b) Let $L:\mathbb{P}_2 \to \mathbb{P}_3$ be the function defined by $Lp(x) = \int_0^x p(t)dt$.

   Find the matrix of L using the basis B_2 for $\mathbb{P}_2$ and B_3 for $\mathbb{P}_3$.

5) Suppose that A is an mxm real matrix. Show that the following statements are equivalent.

   i) The columns of A span $\mathbb{R}^m$.

   ii) The equation $A\vec{x}=\vec{0}$ has a unique solution $\vec{x}=\vec{0}$.

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Math 102 Exam #1 Solutions

1) a) $\vec{v}_1, \ldots, \vec{v}_n \in V$ are linearly independent if $\sum_{j=1}^{n} c_j \vec{v}_j = \vec{0}$ for scalars $c_j \Rightarrow$ all $c_j = 0$

Example $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ in $\mathbb{R}^2$

b) linear transformation $T: V \to W$ means

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T\vec{v}_1 + c_2 T\vec{v}_2$$

for scalars c_j & vectors $\vec{v}_j \in V$

Example. $V = W = \mathbb{R}^2$

$$T(\vec{x}) = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\vec{x}$$

2)

$$A = \begin{bmatrix} -3 & 9 & -2 & -6 \\ 6 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & -2 & -6 \\ 0 & 12 & 0 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 9 & -2 & -6 \\ 0 & 12 & 0 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & -2 & -6 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & 0 & -4 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1 \end{bmatrix} = U$$

pivots in Cols 1, 2, 3

x_4 free

$$\Rightarrow \boxed{\text{Basis Col}(A) = \left\{ \begin{bmatrix} -3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}}$$

$$U\vec{x} = \vec{0} \iff \left\{ \begin{aligned} x_1 + \tfrac{1}{3}x_4 &= 0 \\ x_2 - \tfrac{1}{3}x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned} \right\} \iff \left\{ \begin{aligned} x_1 &= -\tfrac{1}{3}x_4 \\ x_2 &= \tfrac{1}{3}x_4 \\ x_3 &= -x_4 \end{aligned} \right\}$$

$$\vec{x} = x_4 \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\text{Basis Nul}(A) = \left\{ \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -1 \\ 1 \end{bmatrix} \right\}}$$

3) a) F $\quad \vec{0}$ is not in plane

b) T $\quad$ Let $\{\vec{b}_1, \dots, \vec{b}_n\}$ be basis of $W \subset \mathbb{R}^m$

Then set $A = (\vec{b}_1 \cdots \vec{b}_n)$ an $m \times n$ matrix

$W = Col(A)$ and $W^\perp = Nul(A^T) \subset \mathbb{R}^m$

$\dim W = r, \quad \dim W^\perp = m - r$

$\Longrightarrow \quad \dim W + \dim W^\perp = r + (m-r) = m$

c) F $\quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

as $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

4) a) $\{1, x, x^2, \dots, x^n\} = \mathcal{B}_n$

b) $L1 = \int_0^x 1\,dt = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0x^3$

$Lx = \int_0^1 t\,dt = \frac{1}{2}x^2 = 0 \cdot 1 + 0 \cdot x + \frac{1}{2} x^2 + 0x^3$

$Lx^2 = \int_0^1 t^2\,dt = \frac{1}{3}x^2 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \frac{1}{3}x^3$

$$\underset{\mathcal{B}_2 \to \mathcal{B}_3}{\mathrm{Mat}\, L} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

5) $Col(A) = \mathbb{R}^m \Leftrightarrow \dim Col A = m \Leftrightarrow \dim Nul A \overset{\text{Fundamental Thm I}}{=} m - \dim Col A = 0$

$\Leftrightarrow Nul A = \{0\}$