MATH 102 - PRACTICE EXAM #2

closed book, no calculators, headphones ... each problem is worth the same number of points

1) Find the projection of $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ onto the column space of A if $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}$.

2) a) Use the Gram-Schmidt orthogonalization method to obtain an orthonomal set from the vectors $\vec{a}=(1,1,0)^T$, $\vec{b}=(1,0,1)^T$, and $\vec{c}=(0,1,1)^T$.

b) Use part a) to express the matrix $\begin{pmatrix} \vec{a} & \vec{b} & \vec{c} \end{pmatrix}$ as QR, Q orthogonal, R upper triangular and non-singular.

3) a) Give the 3 defining properties of the determinant as a function of the n rows of the nxn matrix A. These are the 1^{st} 3 properties in Strang's Section 4.2.

b) Show using these 3 properties that if 2 rows of a matrix A are the same, then Det(A)=0.

c) Find the area of the parallelogram with vertices the points (1,4), (-1,5), (3,9), (5,8).

4) a) Describe 3 methods for computing a determinant.

b) Compute the following determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & 5 \end{vmatrix}$.

c) Find $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}^{-1}$ using the cofactor formula.

5) a) Define eigenvalue λ of an nxn matrix A.

b) Show that for any nxn matrix A, the eigenvectors corresponding to 2 distinct eigenvalues of A must be linearly independent.

c) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Diagonalize this matrix, that is, write $A=PDP^{-1}$, D diagonal. Then compute A^9.

6) **True - False. Tell whether the following statements are true or false. Give a brief reason for your answer.**

a) Eigenvalues must be non-0 scalars.

b) If A is a real nxn matrix, then A has real eigenvalues.

c) The sum of any 2 eigenvectors of a matrix A is always an eigenvector of A.

d) If A is an orthogonal matrix, then A preserves length of vectors; that is $\|A\vec{v}\| = \|\vec{v}\|$, for any vector $\vec{v}$.

e) det(A+B)=det(A) + det(B).

7) There is an epidemic. Every month 1/2 of the well get sick, the rest stay well. 1/4 of the sick die and 3/4 stay sick. Write down the Markov matrix A for this system. Compute the eigenvalues and eigenvectors of A. Describe the steady state. Prove that the sytem approaches this steady state by diagonalizing A.

1) $\vec{p} = \text{Proj}_{\text{Col}A}\, \vec{b} = A(A^TA)^{-1}A^T\vec{b}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \left[\begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} \right]^{-1} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}^{-1} \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$= \frac{1}{44} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 18 \\ 74 \end{bmatrix} = \frac{1}{44} \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix} = \begin{bmatrix} \frac{23}{11} \\ -\frac{14}{11} \\ \frac{65}{11} \end{bmatrix}$$

2) a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{[1\ 0\ 1]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{[0\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[0\ 1\ 1]\,[1\ {-1}\ 2]}{(1+1+4)} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-\frac{1}{2}-\frac{1}{6} \\ 1-\frac{1}{2}+\frac{1}{6} \\ 1-0-\frac{2}{6} \end{bmatrix} = \begin{bmatrix} -\frac{4}{6} \\ +\frac{4}{6} \\ \frac{4}{6} \end{bmatrix} = \frac{4}{6} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\vec{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

2) continued

b) $A = QR = [\vec{q}_1\ \vec{q}_2\ \vec{q}_3] \begin{bmatrix} \vec{q}_1^T\vec{a} & \vec{q}_1^T\vec{b} & \vec{q}_1^T\vec{c} \\ 0 & \vec{q}_2^T\vec{b} & \vec{q}_2^T\vec{c} \\ 0 & 0 & \vec{q}_3^T\vec{c} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix} = Q \cdot R$$

$\vec{q}_1^T\vec{a} = \frac{1}{\sqrt{2}}[1\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot 2$, $\vec{q}_1^T\vec{b} = \frac{1}{\sqrt{2}}[1\ 1\ 0]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot 1$,

$\vec{q}_1^T\vec{c} = \frac{1}{\sqrt{2}}[1\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot 1$, $\vec{q}_2^T\vec{b} = \frac{1}{\sqrt{6}}[1\ {-1}\ 2]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} 3$,

$\vec{q}_2^T\vec{c} = \frac{1}{\sqrt{6}}[1\ {-1}\ 2]\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \cdot 1$, $\vec{q}_3^T\vec{c} = \frac{1}{\sqrt{3}}[-1\ 1\ 1]\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot 2$

3 a) i) det(A) is linear in each row of A holding the other rows fixed; $c \in \mathbb{R}$

$$\det \begin{bmatrix} \vec{a}_1 + c\vec{b} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} + c \det \begin{bmatrix} \vec{b} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

ii) $\det(I) = 1$ where I = identity matrix $= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

iii) the determinant changes sign if you permute 2 rows for example,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ \vdots \\ \vec{a}_n \end{bmatrix} = -\det \begin{bmatrix} \vec{a}_2 \\ \vec{a}_1 \\ \vec{a}_3 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

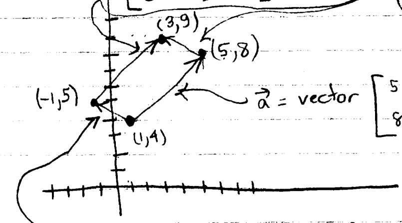
b) If you exchange the 2 equal rows, you should change the sign of the determinant – but you don't. If $-\det(A) = +\det A$, then

$$2 \det A = 0$$
$$\Rightarrow \det A = 0$$

3) continued

check that it is a parallelogram

$$\begin{bmatrix}3-(-1)\\9-5\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix} \checkmark \qquad \begin{bmatrix}3-5\\9-8\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix} \checkmark$$



$\vec{a}$ = vector $\begin{bmatrix}5-1\\8-4\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$

$\vec{b}$ = vector $\begin{bmatrix}-1-1\\5-4\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$

$$\text{area parallelogram} = \left|\det\begin{bmatrix}4 & -2\\4 & 1\end{bmatrix}\right| = |4+8| = 12$$

4) a) i) Use Gaussian elimination, remembering that if you permute 2 rows you multiply the det by -1, if you multiply a row by a scalar c you multiply the det by c. Finally if you replace row i by row i + c row j, you don't change the det.

ii) Use the big formula

$$\det A = \sum_{\sigma} \det P_\sigma \, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

σ permutation of n elements

where P_σ = det of matrix obtained by letting σ permute columns of identity matrix

iii) Expand $\det(A)$ by minors or cofactors

$C_{ij} = (-1)^{i+j}\det M_{ij}$

M_{ij} = matrix obtained from A by deleting i^{th} row & j^{th} column

(expand by i^{th} ...) $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

b)

$$\det(A) = \begin{vmatrix}1&2&3&4\\1&1&1&1\\2&2&0&0\\0&0&5&5\end{vmatrix} = \begin{vmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-6&-8\\0&0&5&5\end{vmatrix} = \begin{vmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&-2&-2\\0&0&5&5\end{vmatrix}$$

$$= (-1)\overset{-2}{2}\begin{vmatrix}1&2&3&4\\0&1&2&3\\0&0&1&1\\0&0&5&5\end{vmatrix} = 2\begin{vmatrix}1&2&3&4\\0&1&2&3\\0&0&1&1\\0&0&0&0\end{vmatrix} = 0$$

c) $A = \begin{bmatrix}1&2&0\\0&3&0\\0&4&1\end{bmatrix}$, $A^{-1} = \dfrac{1}{\det(A)} C^T$, C_{ij} = ij cofactor

$\det A = 3$, $C_{11}=3$, $C_{12}=-0$, $C_{13}=+0$

$C_{21}=-2$, $C_{22}=+1$, $C_{23}=-4$

$C_{31}=+0$, $C_{32}=-0$, $C_{33}=+3$

$$A^{-1} = \frac{1}{3}\begin{bmatrix}3&0&0\\-2&1&-4\\0&0&3\end{bmatrix}^T = \begin{bmatrix}1&-\frac{2}{3}&0\\0&\frac{1}{3}&0\\0&-\frac{4}{3}&1\end{bmatrix}$$

Check:

$$A\cdot A^{-1} = \begin{bmatrix}1&2&0\\0&3&0\\0&4&1\end{bmatrix}\begin{bmatrix}1&-\frac{2}{3}&0\\0&\frac{1}{3}&0\\0&-\frac{4}{3}&1\end{bmatrix} = \begin{bmatrix}1&-\frac{2}{3}+\frac{2}{3}&0\\0&1&0\\0&\frac{4}{3}-\frac{4}{3}&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \checkmark$$

5 a) $\lambda \in \mathbb{C}$ is an eigenvalue of A if $A\vec{v} = \lambda\vec{v}$ for some non-0 vector $\vec{v}$

b) $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$, $A\vec{w} = \mu\vec{w}$, $\vec{w} \neq 0$, $\lambda \neq \mu$

If (1) $\alpha\vec{v} + \beta\vec{w} = \vec{0}$ for $\alpha, \beta \in \mathbb{R}$ then we want to show $\alpha = \beta = 0$.

Apply A to (1) to get:

$\vec{0} = A(\alpha\vec{v} + \beta\vec{w}) = \alpha A\vec{v} + \beta A\vec{w} = \alpha\lambda\vec{v} + \beta\mu\vec{w}$.

So

$$\begin{cases} (1) & \alpha\vec{v} + \beta\vec{w} = \vec{0} \\ (2) & \alpha\lambda\vec{v} + \beta\mu\vec{w} = 0 \end{cases}$$

Multiply (1) by λ and subtract it from (2) to get

$\beta\mu\vec{w} - \beta\lambda\vec{w} = \vec{0}$

Then $\beta(\mu - \lambda)\vec{w} = \vec{0}$

Since $\mu \neq \lambda$ and $\vec{w} \neq \vec{0}$ this means $\beta = 0$.

By (1) then $\alpha\vec{v} = \vec{0}$

As $\vec{v} \neq \vec{0}$, this means $\alpha = 0$

So we've proved $\alpha = \beta = 0$.

Thus $\vec{v}, \vec{w}$ linearly independent.

c) $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$

expand by 1st row

$= (1-\lambda) \det \begin{bmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{bmatrix}$

$-3 \det \begin{bmatrix} -3 & -3 \\ 3 & 1-\lambda \end{bmatrix}$

$+3 \det \begin{bmatrix} -3 & -5-\lambda \\ 3 & 3 \end{bmatrix}$

$= (1-\lambda)\{(-5-\lambda)(1-\lambda) + 9\} - 3\{-3(1-\lambda) + 9\}$
$+3\{-9 + 3(5+\lambda)\}$

$= (1-\lambda)\{-5 + 5\lambda - \lambda + \lambda^2 + 9\} - 3\{-3 + 3\lambda + 9\}$
$+3\{-9 + 15 + 3\lambda\}$

$= (1-\lambda)\{\lambda^2 + 4\lambda + 4\} \cancel{- 3\{3\lambda + 6\} + 3\{3\lambda + 6\}}$

$= (1-\lambda)(\lambda + 2)^2$

$\Rightarrow \lambda = 1, -2, -2$ are eigenvalues

$\text{Nul}(A - I) = \text{Nul} \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + 2x_2 + x_3 = 0$
$x_2 + x_3 = 0$
$x_2 = -x_3$
$x_1 = -2x_2 - x_3$
$= 2x_3 - x_3 = x_3$

eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

5d) continued

$\lambda = -2$

$$Nul(A+2I) = Nul\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0 \quad x_1 = -x_2 - x_3 \quad x_2, x_3$ free

basis $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$\det P = 1 + 1 - 1 = 1$

cofactors $\quad C_{11} = +1 \quad C_{12} = -(-1) \quad C_{13} = +(-1)$

$C_{21} = -(-1) \quad C_{22} = +2 \quad C_{23} = -1$

$C_{31} = +(+1) \quad C_{32} = -(-1) \quad C_{33} = +0$

Check: $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$$A^9 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^9 & 0 & 0 \\ 0 & (-2)^9 & 0 \\ 0 & 0 & (-2)^9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -512 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

5c) continued

$$A^9 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -512 & -1024 & -512 \\ 512 & 512 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 513 & 513 \\ -513 & -1025 & -513 \\ 513 & 513 & 1 \end{bmatrix}$$

6) a) False $\quad$ 0 can be an eigenvalue example $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) False $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $i = \sqrt{-1}$ as an eigen value

c) False $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A\vec{v}_1 = 3\vec{v}_1, \quad A\vec{v}_2 = -\vec{v}_2$

$A(\vec{v}_1 + \vec{v}_2) = A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$

as $4 \neq 0$

d) True $\quad AA^T = I \Rightarrow$

$\|A\vec{v}\|^2 = (A\vec{v})^T(A\vec{v}) = \vec{v}^T A^T A \vec{v}$

$= \vec{v}^T \vec{v} = \|\vec{v}\|^2$

e) False $\quad \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \det\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = 6$

$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 1 + 2 = 3$

7) $d_k = \left\{\begin{matrix}\text{probability of being}\\ \text{dead}\end{matrix}\right\}$, $s_k = \left\{\begin{matrix}\text{probability of being}\\ \text{sick}\end{matrix}\right\}$, $w_k = \left\{\begin{matrix}\text{probability of being}\\ \text{well}\end{matrix}\right\}$ at time $\underset{\text{months}}{k}$

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

$\Rightarrow$ eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{3}{4}$, $\lambda_3 = \frac{1}{2}$

$\Rightarrow$ steady state solution approaches eigenvector $\vec{p}_\infty$ for $\lambda_1 = 1$ such that $\vec{p}_\infty$ is a probability vector $\quad \lim_{k\to\infty} A^k \vec{p}_0 = \vec{p}_\infty$

$$\Rightarrow \quad \vec{p}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

in the limit everyone is dead

To diagonalize A we need all the eigenvectors

$\lambda_1 = 1 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = \frac{3}{4} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{Nul} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$

$x_1 + x_2 = 0 \qquad x_3 = 0$

$\lambda_3 = \frac{1}{2} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \text{Nul} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$2x_1 + x_2 = 0 \qquad x_1 = -\frac{1}{2}x_2 = x_3$

$x_2 + 2x_3 = 0 \qquad x_2 = -2x_3$

$$A = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = S\Lambda S^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow$$

Compute this by $S^{-1} = \frac{1}{\det S} C^T$

$C_{ij} = i,j$ cofactor

$$\lim_{k\to\infty} A^k \vec{p} = \lim_{k\to\infty} (S\Lambda^k S^{-1})\vec{p}, \quad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$p_j \geq 0$

$p_1 + p_2 + p_3 = 1$

$$= \lim_{k\to\infty} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 & 0 \\ 0 & (\frac{3}{4})^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overset{=1}{p_1 + p_2 + p_3} \\ -p_2 - p_3 \\ p_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So $\lim_{k\to\infty} A^k \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.