Math 102 - Winter 2013 - Final Exam

Name: ______________________________

Student ID: ______________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones.

There are 8 questions which are worth 125 points. You have 3 hours to complete the test.

Question	Score	Maximum
1		19
2		19
3		27
4		12
5		12
6		15
7		11
8		10
Total		125

Problem 1. [*19 points.*]

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}.$$

(i) [5] Find the left inverse of A.

(ii) [4] Find the matrix of the projection onto the column space of A.

(iii) [3] Find the matrix of the projection onto the left null space of A.

(iv) [7] Find the QR decomposition of A.

Problem 2. [*19 points.*]

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 5 \\ -2 & -6 & -1 & 4 \end{bmatrix}.$$

(i) [5] Find the LU decomposition of the matrix A.

(ii) [3] Find a basis for the column space of A. What is the rank of A? Fill in the blank: the column space $C(A) =$ ____.

(iii) [3] Find a basis for the null space of A.

(iv) [2] Show that the columns of A are linearly dependent by exhibiting an explicit linear relation between them.

(iv) [2] Find a basis for the row space of A.

(v) [2] Find a basis for the orthogonal complement of the column space of A.

(vi) [2] Does A admit either a left inverse or a right inverse?

Problem 3. [*27 points.*]

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

(i) [12] Write down the SVD for the matrix A.

More space provided on next page.

Continue part (i) here if needed.

(ii) [4] Find the pseudoinverse of A.

(iii) [3] Find the matrix of the projection onto the row space of A.

(iv) [4] From the SVD, write down orthonormal bases for

- the column space of A,
- the row space of A,
- the null space of A,
- the left null space of A.

(v) [4] Consider the incompatible system

$$Ax = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Write down

- all least squares solutions;
- the least squares solution of minimum length.

Problem 4. [*13 points.*]

Consider $\mathcal{P}$ the space of polynomials of degree at most equal to 2. Consider the basis $\mathcal{B}$ consisting of the polynomials $\{1, 1-x, x^2-x\}$.

(i) [4] Find the matrix of the linear transformation $\mathcal{T} : \mathcal{P} \to \mathcal{P}$,

$$\mathcal{T}(f) = x^2 f'' + x f',$$

in the basis $\mathcal{B}$.

(ii) [2] Using (i), find

- the rank of $\mathcal{T}$ and a basis for the column space of $\mathcal{T}$. (*Your basis should consist of polynomials.*)

(ii) [7] Endow $\mathcal{P}$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x).$$

Starting with the basis $\mathcal{B}$, obtain an orthogonal basis for $\mathcal{P}$ via Gram-Schmidt.

Problem 5. [*12 points.*]

Consider the quadratic form

$$Q(x, y, z) = 3x^2 + 3y^2 + 3z^2 + 4xy + 4yz + 4zx.$$

(i) [5] Discuss the definiteness of the form Q, using any method you wish.

(ii) [7] Using any method developed in this course, express Q as a sum of three squares.

Problem 6. [*15 points.*]

TRUE OR FALSE (no explanation is necessary):

(T) (F) The set of all skew symmetric $n \times n$ matrices is a vector space.

(T) (F) Any complex normal matrix is diagonalizable.

(T) (F) The product of two skew Hermitian matrices is Hermitian.

(T) (F) For any matrix A with singular values $\sigma_1, \ldots, \sigma_r$

$$\text{Trace } (AA^T) = \sigma_1^2 + \ldots + \sigma_r^2.$$

(T) (F) There exist skew-Hermitian matrices of determinant $1 + i$.

(T) (F) The LU decomposition of an invertible matrix, if it exists, must be unique.

(T) (F) The QR decomposition, if it exists, must be unique.

(T) (F) The positive decomposition $A = R^T R$ of a symmetric matrix, if it exists, must be unique.

(T) (F) If A and B are unitarily similar, then $\exp(A)$ and $\exp(B)$ are unitarily similar.

(T) (F) Any positive definite quadratic form of n variables can be written as sum of n squares of linear terms.

(T) (F) If Q is a unitary matrix, then $Q + 2I$ is invertible.

(T) (F) For any complex vector $\mathbf{v}$, the matrix

$$R = I - 2\mathbf{v} \cdot \mathbf{v}^H$$

has real eigenvalues.

(T) (F) A symmetric matrix has as many positive pivots as positive eigenvalues.

(T) (F) All positive definite symmetric matrices admit LU decompositions.

(T) (F) The rule

$$\langle f, g \rangle = f(0)\, g(0) + f(1)\, g(1)$$

defines an inner product on the space of polynomials of degree less or equal to 2.

Problem 7. [*11 points.*]

Consider the Fibonacci-type recursion

$$G_{n+2} = \frac{1}{3}G_{n+1} + \frac{2}{3}G_n,\ G_0 = 0,\ G_1 = 1.$$

(i) [2] Let $\vec{x}_n = \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}$. Write down a difference equation that $\vec{x}_n$ satisfies in the form

$$\vec{x}_{n+1} = A\vec{x}_n.$$

(*To write A down, you may need to use a self-obvious second equation* $G_{n+1} = G_{n+1}$.)

(ii) [3] Discuss the stability of the difference equation.

(iii) [6] Solve the difference equation, and write down an explicit formula for G_n. What is the limit of G_n as $n \to \infty$?

Problem 8. [*10 points.*]

Let $A = UDV^T$ be the singular value decomposition of the $n \times n$ matrix A with singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0.$$

Let $A^+ = VD^+U^T$ denote the pseudoinverse of A.

(i) [4] Explain why

- the null space of A equals the left null space of A^+;
- the column space of A equals the row space of A^+.

(ii) [2] Confirm (by direct calculation) that

$$DD^+D = D.$$

(iii) [4] Using (ii), confirm that

$$AA^+A = A.$$

(*Similarly,* $A^+AA^+ = A^+$, *but you don't need to show this.*)