FINAL EXAM MATH 110A

Due: Saturday July 30th at 7pm at Center Hall 216.

Exam Policies

- Please write all you answers in a bluebook, clearly number each solution and present them in numerical order.
- You may discuss exam problems with classmates. However all solutions you write **must entirely be your own**.
- Copying solutions from another classmate, website, or any other source is considered plagiarism and **will be reported to the office of academic integrity.**
- You may use any formulae proved in class, in the textbook or in HW problems. However, you must write the statement of the formula down explicitly, not just to say "Formula (5.3.2)."
- Show all of your work, justify each step, and state any theorems or non-trivial results used from this class. Unsupported answers will receive no credit.
- You are not allowed to ask questions directly related to this exam on Piazza.
- You are not allowed to use Theorems or Formulae from external sources (other books, websites, etc).
- Write your solutions clearly; no credit will be given for illegible solutions.
- If any question is not clear, ask for clarification.
- Ask when you are unsure if you are allowed to use a certain fact.

Problem 1 (10 pts)

Indicate whether each of the statements below are true or false. **No justification is needed.**

a) Let $u(t,x)$ be the solution to the wave equation IVP:

$$u_{tt} - u_{xx} = 0 , \qquad u(0,x) = 0 , \qquad u_t(0,x) = \begin{cases} \cos(\frac{\pi}{2} \cdot x) , & |x| \leqslant 1; \\ 0 , & |x| > 1. \end{cases}$$

Then: $\lim_{|x|\to+\infty} u(t,x) = 0$.

b) Let $u(t,x)$ be a solution to the heat equation (with convection) obeying periodic boundary conditions on the interval $[0,\pi]$:

$$u_t - u_{xx} + u_x = 0 , \qquad u(t,0) = u(t,\pi), \qquad u_x(t,0) = u_x(t,\pi) .$$

Define the *total heat* to be: $H(t) = \int_0^\pi u(t,x)\, dx$. Then, for $u(t,x)$ the total heat is conserved in the sense that: $H(t) = H(0)$.

c) Let $f, g : [a, b] \to \mathbb{R}$. The boundary conditions:

$$f(a) + f'(a) = 0 = g(a) + g'(a)$$
$$f(b) + f'(b) = 0 = g(b) + g'(b) \ ,$$

are symmetric.

d) Let $l > 0$. The orthogonal set: $\{\sin(\frac{n\pi x}{l})\}_{n=1}^{\infty}$ is complete in $L^2([0, l])$.

e) Let $l > 0$. The orthogonal set: $\{\sin(\frac{n\pi x}{l})\}_{n=1}^{\infty}$ is complete in $L^2([-l, l])$.

Problem 2 (15 pts)

Let $u(t, x)$ be a smooth solution to the wave equation on the interval $[0, \pi]$ (with wave speed $c = 1$) satisfying Neumann boundary conditions. That is, $u(t, x)$ solves:

$$u_{tt} - u_{xx} \ = \ 0 \ , \qquad u_x(t, 0) \ = \ 0 \ = \ u_x(t, \pi) \ . \qquad (t \geq 0) \tag{1}$$

Define the energy $E(t)$ associated to $u(t, x)$ at time t to be:

$$E(t) \ = \ \frac{1}{2} \int_0^{\pi} \left(u_t^2(t, x) + u_x^2(t, x) \right) \ dx \ .$$

(a) Show that the energy $E(t)$ for the function $u(t, x)$ is conserved for all time t.

(b) Suppose that $u(t, x)$ solves problem (1) with initial data (at $t = 0$):

$$u(0, x) \ = \ \cos(x) - 3\cos(2x) - 5\cos(6x) + \cos(11x) \ ,$$
$$u_t(0, x) \ = \ -\cos(x) + 4\cos(7x) + \cos(9x) - 4\cos(13x) \ .$$

Use part (a) to compute the energy at each time t for $u(t, x)$. That is, for all $t \geq 0$, compute $E(t)$ explicitly. Simplify your answer. (Hint: Orthogonality is your friend.)

Problem 3 (15 pts)

Let $f : [-2, 0] \to \mathbb{R}$ be given by $f(x) = 1 - \left|x + 1\right|$.

(a) Let $f_{per} : \mathbb{R} \to \mathbb{R}$ denote the periodic extension of $f(x)$. Show that $f_{per}(x)$ is an even function.

(b) Compute the full (classical) Fourier series of the periodic function $f_{per}(x)$. Simplify your answer.

(c) Does the series in part (b) converge to $f_{per}(x)$ in $L^2[-2, 2]$? Does it converge uniformly to $f_{per}(x)$ on $[-2, 2]$? Does it converge pointwise to $f_{per}(x)$ for all $-2 \leq x \leq 2$. Justify your answers.

Problem 4 (15 pts)

Consider the (massive) wave equation subject to Neumann boundary conditions:

$$u_{tt} = u_{xx} - u \;, \qquad 0 < x < \pi \;,$$
$$u_x(t,0) = 0 = u_x(t,\pi) \;.$$

(a) Find all the positive eigenvalues and their respective eigenfunctions.

(b) Is zero an eigenvalue for this problem? Justify your answer.

(c) Are there negative eigenvalues? Justify your answer.

(d) Write the series expansion for a solution $u(t,x)$. (You may assume all eigenvalues are real).

Problem 5 (15 pts)

Let $k > 0$. The heat equation for a solid metal ring is the following (i.e. has periodic boundary conditions):

$$\begin{aligned} u_t &= k u_{xx} \;, & -\pi < x < \pi \;, \\ u(-\pi,t) &= u(\pi,t) \;, \\ u_x(-\pi,t) &= u_x(\pi,t) \;, \\ u(x,0) &= f(x) \;. \end{aligned}$$

(a) Use separation of variables to derive the general solution $u(x,t)$ to this problem.

(b) Write down the specific solution $u(x,t)$ to the above problem with initial data:

$$f(x) = 1 + 2\cos(3x) - 5\sin(4x) \;.$$

(c) Compute the steady state temperature $T_\infty = \lim_{t\to\infty} u(x,t)$ to the solution from part (b). Show that this is equal to the average initial temperature:

$$T_{av} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \;.$$

Problem 6 (15 pts)

Consider the first order IVP:

$$t u_t(t,x) + x u_x(t,x) = 0 \;, \qquad u(0,x) = f(x) \;.$$

(a) Show that no solution exists when $f(x) = x^2$.

(b) Show that infinitely many solutions exist when $f(x) = 3$.

Problem 7 (15 pts)

Consider the solution $u(t,x)$ to the wave equation IVP:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \ , \\ u(0,x) &= \begin{cases} \sin(x) \ , & \text{if } -\pi \leqslant x \leqslant \pi \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases} \\ u_t(0,x) &= \begin{cases} \cos(x) \ , & \text{if } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases} \end{aligned}$$

(a) Compute the quantity $u(\pi, 2\pi)$ (i.e. $u(t,x)$ at $t=\pi$, $x=2\pi$).

(b) Compute the quantity $u(10,10)$. (i.e. $u(t,x)$ at $t=10$, $x=10$).

(c) Compute $\lim_{t\to+\infty} u(t,x)$.