**Name:**

Mathematics 100a
October 26, 1994

**Midterm**

*Instructions: Answer as many questions as you can. Ground rules: You may cite any theorem in the text without proof. The result of a homework problem cannot be cited; it must be reproved.*

1. Show carefully that the following defines an equivalence relation on the set $\mathbf{R}^2$, the plane: $(x, y) \sim (x', y')$ if there exists $a \in \mathbf{R}$ such that

$$(x - x', y - y') = (a, a) \ .$$

2.

( a) Let $G = \mathbf{Z}, +$, the additive group of all integers. Let $m$ be a nonzero integer and $H_m$ the subgroup of all multiples of $m$. Find all cosets of $H_m$. What is the index of $H_m$?

( b) Prove that any subgroup of an abelian group is normal.

3. Let $G = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0,\ a, b \in \mathbf{R}\}$, with the group operation given by matix multiplication. Show that $\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \neq 0\}$ is not a normal subgroup. Hint: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$.

4. Let $G$ be as in 3. and $N$ the normal subgroup $\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R}\}$. Show that the quotient group $G/N$ is isomorphic to the multiplicative group $\mathbf{R}^*$ of all nonzero real numbers.

5.

( a) Let $p$ and $q$ be prime numbers, $p \leq q$ and $1 < m < q$. Suppose $m^p \equiv 1 \bmod q$. Show that there is no $k$ with $0 < k < p$ for which $m^k \equiv 1 \bmod q$.

Remark. It is not necessary to assume that $q$ is prime, but it might make your proof simpler.

( b) Formulate (but you need not prove) a theorem for any group which would imply the conclusion of (a) as a special case.