Math Club Brainteaser: Matrix Bound

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Suppose that each entry in a 2k-1 by 2k-1 matrix M (k is a natural number) is a real number of magnitude not exceeding 1 (that is, absolute value of each entry is less than or equal to 1). Suppose also that these entries have been carefully chosen and arranged so that the four entries in every 2 by 2 submatrix add up to zero. Prove, then, that the sum of all the entries in M cannot exceed 2k-1.

Solution

We prove by induction on k. Define M_k to be the set of all real $(2k-1)\times(2k-1)$ matrices with the magnitude of each entry not exceeding 1, and such that the four entries of each 2×2 submatrix sums to zero. Let $P(k)$ be the statement "all elements of a matrix $\mathcal{M}\in M_k$ sum to at most $2k-1$".

Consider the case for $k=1$. Here we have a 1×1 matrix, say $[a]\in M_1$, $a\in\mathbb{R}$. By definition, $a\leq 1=2\cdot 1-1$. Therefore $P(1)$ is true. (Note that all the elements of each 2×2 submatrix of $[a]$ summing to zero is vacuously true.)

Now suppose $P(k)$ is true. Any element $\mathcal{M}\in M_{k+1}$ can be written in the form

$$\mathcal{M}=\begin{pmatrix} & & & a_{1,2k} & a_{1,2k+1} \\ & & & a_{2,2k} & a_{2,2k+1} \\ & \mathcal{N} & & a_{3,2k} & a_{3,2k+1} \\ & & & \vdots & \vdots \\ a_{2k,1} & a_{2k,2} & \cdots & a_{2k,2k} & a_{2k,2k+1} \\ a_{2k+1,1} & a_{2k+1,2} & \cdots & a_{2k+1,2k} & a_{2k+1,2k+1} \end{pmatrix}$$

where $\mathcal{N}\in M_k$. Since, by assumption, the the elements of $\mathcal{N}$ sum to at most $2k-1$, we focus on the remaining elements of $\mathcal{M}$.We can sum the elements in the first $2k$ columns and last 2 rows of $\mathcal{M}$ by grouping them into 2×2 matrices first, each yielding zero. Their sum is thus $k\cdot 0=0$. Similarly for the elements of the last 2 columns and the first $2k-2$ rows of $\mathcal{M}$. Thus, the sum of the elements of $\mathcal{M}$ can be reduced to the sum of the elements of $\mathcal{N}$ plus the sum of the terms $a_{2k-1,2k}, a_{2k-1,2k+1}, a_{2k,2k+1}$, and $a_{2k+1,2k+1}$. However, $a_{2k-1,2k}+a_{2k-1,2k+1}+a_{2k,2k+1}=-a_{2k,2k}$. Thus,

$$\begin{aligned}\sum\{\text{ elements of }\mathcal{M}\} &= \sum\{\text{ elements of }\mathcal{N}\}-a_{2k,2k}+a_{2k+1,2k+1} \\ &\leq 2k-1+1+1 \\ &= 2(k+1)-1\end{aligned}$$

Hence, $P(k+1)$ is true. Therefore, by induction, $P(k)$ is true for all integers $k\geq 1$. ■