Math 20E Midterm 1, Winter 2003. Lindblad.

1. Consider the vector field $\mathbf{F} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$.

(a). Sketch the vector field $\mathbf{F}$ at the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(-1, -1)$.

(b). Sketch and describe the flow line of the vector field going through $(1, 0)$. Indicate (and justify) if the flow line is described by either of the following:

$$(1)\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad (2)\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases} \quad (3)\begin{cases} x = 1 + t \\ y = 0 \end{cases} \quad (4)\begin{cases} x = 1 + t \\ y = t \end{cases} \quad -\infty < t < \infty$$

2. Consider the vector field $\mathbf{F} = (ax + by)\mathbf{i} + (cx + dy)\mathbf{j}$, where a, b, c, d are constants.

(a). Calculate the divergence $\nabla \cdot \mathbf{F}$

(b). Calculate the curl $\nabla \times \mathbf{F}$.

(c). Determine for which a, b, c, d the vector field $\mathbf{F}$ is conservative and for those values of the constants find a potential ϕ such that $\mathbf{F} = \nabla\phi$.

3. Consider the vector field $\mathbf{F} = (ax + by)\mathbf{i} + (cx + dy)\mathbf{j}$, where a, b, c, d are constants. Let C be the circle of radius r centered at the origin and going around the origin one turn in the mathematically positive direction starting from the positive x-axis. A parametrization for C is $x = r\cos t$, $y = r\sin t$, $(z = 0)$, where $0 \leq t \leq 2\pi$.

Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, for all a, b, c, d (the aswer may depend on a, b, c, d).
(You might have to use $\cos^2 t = \dfrac{1 + \cos 2t}{2}$, $\sin^2 t = \dfrac{1 - \cos 2t}{2}$, $\sin t \cos t = \dfrac{\sin 2t}{2}$.)

4. The temperature at each point (x, y, z) of space is given by $\phi(x, y, z) = x^2 + 2y^2$. A fly is flying in space and at each point (x, y, z) of its journey it flies in the direction of maximum rate of increase of the temperature.

(a). At each point (x, y, z) find the direction of maximum rate of increase of the temperature $\phi(x, y, z)$, i.e. the direction in which the directional derivative is max.

(b). Find the curve along which the fly travels if it starts at the point $(1, 2, 0)$, i.e. find the flow line for the vector field in (a) through the point.