MATH 102 - PRACTICE PROBLEMS FOR MIDTERM II

1. Consider the vector subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(i) Find a basis for the orthogonal complement $V^{\perp}$.
(ii) Using Gramm-Schmidt, find an orthonormal basis for $V^{\perp}$.
(iii) Find the matrix of the orthogonal projection onto $V^{\perp}$ using the basis you found in (ii).
(iv) Find the projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V^{\perp}$. Derive from here the projection of the same vector onto V.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 1 \\ 1 & -4 \end{bmatrix}, \text{ and the vector } b = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}.$$

(i) Find the left inverse of the matrix A.
(ii) Using the calculation in (i), find the matrix of the orthogonal projection onto the column space of A.
(iii) Find the least squares solution to the system

$$Ax = b$$

using the left inverse you calculated in (i).
(iv) Find the QR decomposition of A.
(v) Now redo part (iii). That is, find the least squares solution to the system

$$Ax = b$$

using the QR decomposition you found in (iv).

3. Calculate the determinant of the matrix

$$\begin{pmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{pmatrix}$$

(i) using either row or column operations;
(ii) using the method of cofactors.

4. Find the inverse of the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

using the method of cofactors.

5. Let A be a square matrix, and let λ be an eigenvalue for A.

(i) Show that λ^n is an eigenvalue for A^n.

(ii) More generally, consider a polynomial

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0.$$

Explain that $p(\lambda)$ is an eigenvalue for the matrix

$$p(A) = c_n A^n + c_{n-1}A^{n-1} + \ldots + c_0 I.$$

6. Which of the following matrices are diagonalizable:

(i) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & -9 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & 2 & 8 & 19 & 1 & 1 \\ 0 & 1 & 3 & 2 & -11 & 3 & 0 \\ 0 & 0 & 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 3 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}.$

7. Company XYZ monitors the dollars spent each year by its customers on apples and oranges. With $a(k)$ representing the number of dollars spent (in millions) on apples in year k, and $o(k)$ the number of dollars (in millions) on oranges in year k, they determine that

$$a(k+1) = \frac{2}{5}a(k) + \frac{1}{5}o(k)$$
$$o(k+1) = \frac{3}{5}a(k) + \frac{4}{5}o(k).$$

Let $\vec{v}_k = \begin{bmatrix} a(k) \\ o(k) \end{bmatrix}$.

(i) Find a matrix A such that $A\vec{v}_k = \vec{v}_{k+1}$. Explain that

$$\vec{v}_k = A^k \vec{v}_0.$$

(ii) Find the eigenvalues and eigenvectors of A.

(iii) Express the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as a linear combination of the eigenvectors of A.

(iv) Assume that the customers of XYZ spend 3 million dollars on apples in year 0, and 1 million dollars on oranges. What is a good estimate for the sum (in millions) spent on apples in year 2013? How about dollars spent on oranges in year 2013?

8.

(i) Let R be the reflection in a subspace $V \subset \mathbb{R}^n$. What are the eigenvalues of R?

(ii) Assume that T is a linear transformation with $T^5 = T$. What real numbers can be eigenvalues for T?

9. Is the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \pi & 1 & 4 & -1 & 1 & 0 & 0 \\ -19 & -\frac{1}{23} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 9 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 12 & -\frac{1}{2} & -\frac{1}{3} & 0 & 0 & -1 & 0 \\ -5 & 0 & -\frac{1}{3} & 0 & 1 & 0 & 3 \end{pmatrix}$$

invertible?

10. Assume that A, B are invertible square matrices. Explain why AB and BA have the same characteristic polynomial. *Hint: Think about similar matrices.*

11. If A is a symmetric matrix with $\exp(A) = I$ show that $A = 0$. *Hint: diagonalization.*

12. Consider the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$

(i) Diagonalize the matrix A.

(ii) Using the diagonalization in (i), calculate A^k for all k.

(iii) Calculate the matrix exponential e^{tA}.

13. Solve the following system using Cramer's rule:

$$x + 2y = a, \quad -3x + 4y = b.$$

14. Consider the three vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} p \\ 0 \\ p \end{bmatrix}.$$

(i) Find the volume of the tetrahedron spanned by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in terms of the parameter p.

(ii) For what value of p are the three vectors in the same plane?

15. The Laguerre polynomials are important in quantum mechanics, in writing down the solution of the Schrödinger equation for the hydrogen atom.

(i) Show that
$$(f, g) = \int_0^\infty f(x)g(x)\, e^{-x}\, dx$$
is an inner product on the space $\mathcal{P}$ of polynomials of degree at most 2.

(ii) Starting with the basis $\{1, x, x^2\}$ obtain an orthogonal basis for $\mathcal{P}$ using Gramm-Schmidt method. The resulting poiynomials are the Laguerre polynomials.

For this problem you may use the values of the integrals (called the gamma function):
$$\int_0^\infty x^n e^{-x}\, dx = n!.$$

MATH 102 - SOLUTIONS TO PRACTICE PROBLEMS - MIDTERM II

Several correct answers are possible for some of these questions.

1.

(i) Let $A = \begin{bmatrix} -2 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. We have $V = C(A)$ hence

$$V^{\perp} = N(A^T).$$

Row reducing A^T we find the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

whose null space is spanned by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(ii) We apply Gram-Schmid to the basis we found above. We nornalize the first basis vector

$$u_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and compute an orthogonal vector to it

$$y_2 = v_2 - (v_2 \cdot u_1)u_1 = v_2 + \sqrt{3}u_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Normalizing the answer again, we obtain

$$u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

(iii) We let

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

whose columns are the vectors u_1, u_2 we found in (ii). The projection has matrix

$$AA^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$

(iv) The projection of the vector onto $V^\perp$ equals

$$AA^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \\ 4/3 \\ 1/3 \end{bmatrix}.$$

The projection onto V and that onto $V^\perp$ add up to the original vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The projection onto V becomes

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2/3 \\ 4/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

2.

(i) We have $A^+ = (A^T A)^{-1} A^T$. We find

$$A^T A = \begin{bmatrix} 4 & -6 \\ -6 & 34 \end{bmatrix} \implies (A^T A)^{-1} = \frac{1}{100}\begin{bmatrix} 34 & 6 \\ 6 & 4 \end{bmatrix}.$$

This yields

$$A^+ = \frac{1}{10}\begin{bmatrix} 4 & 1 & 4 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

(ii) The matrix of the orthogonal projection is

$$AA^+ = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

(iii) The least squares solution is

$$x^\star = A^+ b = \frac{1}{10}\begin{bmatrix} 4 & 1 & 4 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 29/10 \\ -9/10 \end{bmatrix}$$

(iv) We run the Gramm-Schmid process for the vectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ -4 \end{bmatrix}.$$

For the first step we have

$$q_1 = \frac{u_1}{||u_1||}$$

where $||u_1|| = 2$ hence

$$q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

This step yields already the first row of Q, namely the vector q_1.

The second step yields the second rows of Q and R. We first orthogonalize

$$y_2 = u_2 - (u_2 \cdot q_1)q_1.$$

We have

$$u_2 \cdot q_1 = -3$$

yielding

$$y_2 = u_2 + 3q_1 = \begin{bmatrix} 5/2 \\ -5/2 \\ 5/2 \\ -5/2 \end{bmatrix}.$$

Next, we have

$$||y_2|| = 5$$

yielding the second normalized vector

$$q_2 = \frac{y_2}{||y_2||} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

We create the matrix Q from the vectors q_1, q_2 and the matrix R from the dot products we computed during Gramm-Schmidt. We have

$$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \quad R = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

(v) We explained in class that the least squares solution is found by solving the system

$$Rx^\star = Q^T b$$

which in our case becomes

$$\begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} x^\star = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/2 \\ -9/2 \end{bmatrix}.$$

This can be solved by back substitution yielding

$$5x_2^\star = -9/2 \implies x_2^\star = -9/10$$

$$2x_1^\star - 3x_2^\star = 17/2 \implies x_1^\star = 29/10.$$

3. The determinant is equal to 5. This question is quite straightforward and there are many ways in which we can carry out the row reduction, so i will not write out the full solution. Similarly, for the cofactor method, you can expand along the rows or columns of your choice.

4. We have

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

where C is the matrix of cofactors. We have $\det A = -1$ and the following list of cofactors

$$C_{11} = 8, C_{21} = 2, C_{31} = -1, C_{12} = -3, C_{22} = -1, C_{32} = 0, C_{13} = -1,\ C_{23} = 0, C_{33} = 0.$$

We obtain

$$A^{-1} = \begin{bmatrix} -8 & -2 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

5.

(i) Assume

$$A\vec{v} = \lambda v.$$

Then

$$A^2\vec{v} = AA\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}.$$

Similarly,

$$A^3\vec{v} = AA^2\vec{v} = \lambda^2 A\vec{v} = \lambda^3\vec{v}.$$

Continuing, we obtain

$$A^n\vec{v} = \lambda^n\vec{v}.$$

This proves λ^n is an eigenvalue for A^n with eigenvector $\vec{v}$.

(ii) We have

$$\begin{aligned} p(A)\vec{v} &= (c_n A^n + c_{n-1}A^{n-1} + \ldots + c_0 I)\vec{v} \\ &= c_n A^n\vec{v} + c_{n-1}A^{n-1}\vec{v} + \ldots + c_0\vec{v} \\ &= c_n\lambda^n\vec{v} + c_{n-1}\lambda^{n-1}\vec{v} + \ldots + c_0\vec{v} \\ &= (c_n\lambda^n + \ldots + c_0)\vec{v} = p(\lambda)\vec{v}. \end{aligned}$$

This proves $p(\lambda)$ is an eigenvalue for $p(A)$ with eigenvector $\vec{v}$.

6.

(i) No, the matrix is not diagonalizable. The characteristic polynomial of A is

$$\det(\lambda I - A) = (1-\lambda)(\lambda-2)^2.$$

The eigenspace E_2 is obtained from the null space of

$$2I - A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{bmatrix}.$$

We don't need to row reduce the matrix to find its nullity. The following shortcut can be applied: the row space of this matrix is visibly 2 dimensional (the second and third row are multiples), so the rank is 2. By the rank nullity theorem, the nullity must be 1. But $\lambda = 2$ is an eigenvalue with multiplicity 2, so we do not get enough eigenvectors for $\lambda = 2$ to form an eigenbasis.

(ii) The matrix is symmetric hence diagonalizable.

(iii) The characteristic polynomial of this triangular matrix is

$$(\lambda + 1)(\lambda - 1)\lambda(\lambda - 2)(\lambda - 3)(\lambda - 4)(\lambda - 10)$$

which has 7 distinct roots/eiegnvalues. Since the eigenvalues are distinct, the matrix is diagonalizable.

7.

(i) We have

$$A = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

Note that

$$v_k = Av_{k-1} = AAv_{k-2} = \ldots = A \ldots Av_0 = A^k v_0.$$

(ii) We have $\mathrm{Tr} A = \frac{6}{5}, \det A = \frac{1}{5}$. The characteristic polynomial is

$$\lambda^2 - \frac{6}{5}\lambda + \frac{1}{5} = 0.$$

The eigenvalues are

$$\lambda_1 = 1, \text{ and } \lambda_2 = \frac{1}{5}.$$

The eigenvalue $\lambda_1 = 1$ has eigenspace

$$E_1 = N(I - A) = N\left(\begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}.$$

The eigenspace of $\lambda_2 = \frac{1}{5}$ is

$$E_{\frac{1}{5}} = N(\frac{1}{5}I - A) = N\left(\begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{3}{5} \end{bmatrix}\right) = \text{span }\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

(iii) We have

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = w_1 + 2w_2,$$

where w_1 and w_2 are the two eiegnvectors found in (ii) corresponding to $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{5}$.

(iv) We have

$$v_{2013} = A^{2013}\begin{bmatrix}3\\1\end{bmatrix} = A^{2013}(w_1 + 2w_2) = A^{2013}w_1 + 2A^{2013}w_2 = w_1 + 2\left(\frac{1}{5}\right)^{2013} w_2,$$

where we used that w_1 and w_2 are eigenvectors for A^{2013} with eigenvalues 1^{2013} and $\left(\frac{1}{5}\right)^{2013}$. The second term clearly is negligible. Thus the dominant term is

$$v_{2013} \mapsto w_1 = \begin{bmatrix}1\\3\end{bmatrix}.$$

Therefore, the customers of XYZ will spend approximately 1 million dollars on apples and 3 million on oranges.

8.

(i) For $v \in V$, we have $R(v) = v$ so 1 is an eigenvalue. For $v \in V^{\perp}$, we have $R(v) = -v$ hence -1 is an eigenvalue. There are no other eigenvalues since V and $V^{\perp}$ together span $\mathbb{R}^n$.

(ii) If

$$Tv = \lambda v \implies T^5 v = \lambda^5 v.$$

Since $T^5 = T$ we have

$$T^5 v = Tv \implies \lambda^5 v = \lambda v \implies \lambda^5 = \lambda$$

which gives $\lambda \in \{0, 1, -1\}$.

9. The matrix is invertible. We expand the determinant along the first column. All entries contribute rational numbers with one exception: the irrational number π. To find the minor of π we calculate the determinant of

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{23} & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 1 & 9 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{3} & 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{3} & 0 & 1 & 0 & 3 \end{bmatrix}.$$

We expand along rows and columns. We see that this minor equals 6. Thus2

$$\det A = -6\pi + \text{rational number}.$$

The determinant is thus an irrational number, hence nonzero. Therefore, A is invertible.

10. The matrices AB and BA are similar. Indeed,

$$AB = B^{-1}(BA)B.$$

Therefore AB and BA have the same characteristic polynomial.

11. If A is symmetric, then

$$A = PDP^{-1}$$

for some diagonal matrix D. Then

$$\exp(A) = P\exp(D)P^{-1}.$$

But

$$\exp(A) = I \implies P\exp(D)P^{-1} = I \implies \exp(D) = I,$$

after multiplying by P^{-1} and P to the left and right respectively.

Now if D has diagonal entries $\lambda_1, \ldots, \lambda_n$ then $\exp(D)$ has diagonal entries $e^{\lambda_1}, \ldots, e^{\lambda_n}$. But the diagonal entries must be 1 hence

$$e^{\lambda_1} = \ldots = e^{\lambda_n} = 1 \implies \lambda_1 = \ldots = \lambda_n = 0 \implies D = 0.$$

Then

$$A = PDP^{-1} = 0.$$

12.

(i) We compute

$$\det(\lambda I - A) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

so the eigenvalues are $\lambda = 1$ and $\lambda = 2$. We find the eigenvectors. For $\lambda = 1$ we need to compute

$$A - I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}.$$

Its null space is spanned by the vector

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Similarly, $\lambda = 2$ yields the eigenvector

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

obtained by finding the nullspace of $N(A - 2I)$. Thus

$$C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

We have

$$A = CDC^{-1}.$$

(ii) We compute

$$A^k = CD^kC^{-1}$$

where

$$D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix}.$$

This yields

$$A^k = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2^k - 2 & -3 \cdot 2^k + 3 \\ 2^{k+1} - 2 & -2^{k+1} + 3 \end{bmatrix}.$$

(iii) We have

$$e^{tA} = Ce^{tD}C^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3e^{2t} - 2e^t & -3e^{2t} + 3e^t \\ 2e^{2t} - 2e^t & -2e^{2t} + 3e^t \end{bmatrix}.$$

13. We have

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}.$$

Thus $\det A = 10$. Using

$$B_1 = \begin{bmatrix} a & 2 \\ b & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & a \\ -3 & b \end{bmatrix} \implies \det B_1 = 4a - 2b, \det B_2 = b + 3a.$$

Cramer's rule yields

$$x = \frac{\det B_1}{\det A} = \frac{4a - 2b}{10} = \frac{2a - b}{5}$$

and similarly

$$y = \frac{\det B_2}{\det A} = \frac{b + 3a}{10}.$$

14.

(i) The tetrahedron has volume one sixth of the volume of the box spanned by the vectors i.e.

$$\frac{1}{6}|\det A|$$

where

$$A = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 3 & 0 & p \end{bmatrix}.$$

We have $\det A = -2p$ hence the volume equals $\frac{|p|}{3}$.

(ii) The three vectors lie in the same plane when the volume they span is zero. Hence we need $p = 0$, using (i).

15.

(i) We need to verify the axioms of inner products. There are 4 such axioms:

- $(f, g + h) = (f, g) + (f, h)$;
- $(f, g) = (g, f)$;
- $c(f, g) = (f, cg) = (cf, g)$;
- $(f, f) \geq 0$ with equality if and only if $f = 0$.

The first three axioms follow from definitions. Indeed, the first axiom reads

$$(f, g+h) = \int_0^\infty f(x)(g(x)+h(x))e^{-x}\,dx = \int_0^\infty f(x)g(x)e^{-x}\,dx + \int_0^\infty f(x)h(x)e^{-x}\,dx = (f, g)+(f, h)$$

which is clearly satisfied. The second is verified the same way:

$$(f, g) = \int_0^\infty f(x)g(x)e^{-x}\,dx = \int_0 g(x)f(x)e^{-x}\,dx = (g, f).$$

and the third is entirely similar (and left to the reader). For the last axiom, we calculate

$$(f,f)=\int_0^\infty f(x)^2e^{-x}\,dx\geq 0$$

since we are integrating a nonnegative function $(f(x))^2e^{-x}\geq 0$. Equality happens if and only if $(f(x))^2e^{-x}=0 \implies f=0$.

(ii) Using the orthogonalization procedure for the polynomials

$$P_1=1, P_2=x, P_3=x^2$$

we find:

Step 1: $Q_1=P_1=1$;

Step 2:

$$Q_2=P_2-\frac{(P_2,Q_1)}{(Q_1,Q_1)}Q_1.$$

We have

$$(P_2,Q_1)=\int_0^\infty x\cdot 1\cdot e^{-x}\,dx=1$$

and

$$(Q_1,Q_1)=\int_0^\infty 1\cdot 1\cdot e^{-x}\,dx=1.$$

Here we used the values of the integral supplied by the problem. This yields

$$Q_2=x-1.$$

Step 3:

$$Q_3=P_3-\frac{(P_3,Q_1)}{(Q_1,Q_1)}Q_1-\frac{(P_3,Q_2)}{(Q_2,Q_2)}Q_2.$$

We have

$$(P_3,Q_1)=\int_0^\infty x^2\cdot 1\cdot e^{-x}\,dx=2!=2$$

and

$$(Q_1,Q_1)=\int_0^\infty 1\cdot 1\cdot e^{-x}\,dx=1.$$

Again we used the integrals provided by the text of the problem. Similarly,

$$(P_3,Q_2)=\int_0^\infty x^2\cdot(x-1)\cdot e^{-x}\,dx=\int_0^\infty x^3e^{-x}\,dx-\int_0^\infty x^2e^{-x}\,dx=3!-2!=4.$$

Also

$$(Q_2,Q_2)=\int_0^\infty (x-1)\cdot(x-1)\cdot e^{-x}\,dx=\int_0^\infty (x^2-2x+1)e^{-x}\,dx$$

$$=\int_0^\infty x^2e^{-x}\,dx-2\int_0^\infty xe^{-x}\,dx+\int_0^\infty e^{-x}\,dx=2!-2\cdot 1+1=1.$$

This yields

$$Q_3=x^2-\frac{2}{1}\cdot 1-\frac{4}{1}(x-1)=x^2-4x+2.$$

The basis of Laguerre polynomials for $\mathcal{P}$ is $\{1,x-1,x^2-4x+2\}$.