MATH 102 - PRACTICE EXAM #1

Closed book, no notes, calculators, headphones. Each problem is worth the same number of points

1) a) Use Gaussian elimination to solve the system $Ax=b$ below for $x^T=(u,v,w)$:

$u+v+w=6$, $u+2v+2w=11$, $2u+3v-4w=3$.

b) Factor the matrix A on the left hand side of the system as $A=LU$.

2) Define the following and give an example. a) Pivot position in a matrix A; b) Span of a set of vectors; c) linearly independent vectors; d) linear transformation; e) matrix of a linear transformation; f) free variable; g) elementary row operations h) basis of a vector space

3) a) Define subspace W of a vector space V.

b) Which of the following subsets W of $V=\mathbb{R}^4$ are subspaces

i) solutions x of $x_1+x_2+x_3+x_4=0$.

ii) solutions x of $x_1+x_2+x_3+x_4=1$.

4) True - False. Tell whether the following statements are true or false. Give a brief reason for your answer.

a) The following vectors are linearly independent
$a^T=(1, 1, 0, 0)$, $b^T=(1, 0, 1, 0)$, $c^T=(0, 0, 1, 1)$, $d^T=(0, 1, 0, 1)$.

b) 4 vectors can span 5 dimensional space $\mathbb{R}^5$.

c) If $Ax=0$ has more than 1 solution x, so does $Ax=b$.

d) Suppose the reduced row echelon form of A is U. Then $Col(A)=Col(U)$.

e) $AB=BA$ for any non-singular matrices A and B.

5) a) Define column space Col(A) of an mxn matrix A. Then define the null space Nul(A).

b) Compute the Column space of the matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix}$. Then compute Nul(A).

c) Obtain the consistency relations for a vector b to be in the column space of the matrix A of part b.

d) Show that these consistency relations come from vectors x which are orthogonal to all vectors b in the column space of A. What is the orthogonal complement of Col(A)?

6) a) Give 5 equivalent definitions for a matrix to be nonsingular.

b) Show that the product of 2 nonsingular matrices is also nonsingular.

7) a) State the part of the fundamental theorem of linear algebra, part I, which gives the relation between the dimensions of the Null Space(A) and Column Space (A).

b) Prove part a).

8) True - False. Tell whether the following statements are true or false. Give a brief reason for your answer.

a) The non-singular 3x3 matrices span the vector space of all 3x3 matrices.

b) Gaussian elimination on a matrix A produces a unique matrix U in echelon form.

c) Row space (A) = Column Space(A).

d) Suppose B is a subspace of V and $B^\perp$ denotes the orthogonal complement of B. Then $B^{\perp\perp}=B$.

9) a) Let V be the vector space of all polynomials of degree less than or equal to 3. Write down the 4x4 matrix of the linear transformation from V to V which takes a polynomial to its derivative using the basis $\{1, x, x^2, x^3\}$.

b) Find the 2x2 matrix of the linear transformation that rotates a vector in $\mathbb{R}^2$ through an angle of π.

Math 102 Practice Exam #1 Solutions

1. a)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array}\right) \xrightarrow{L_1^{-1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & -6 & -9 \end{array}\right) \xrightarrow{L_2^{-1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -7 & -14 \end{array}\right)$$

$$L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \qquad L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{pmatrix}$$

$$U\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -14 \end{pmatrix} \Leftrightarrow \left\{\begin{array}{r} u+v+w=6 \\ v+w=5 \\ -7w=-14 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} w=2 \\ v=3 \\ u=1 \end{array}\right\} \Rightarrow \boxed{\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}$$

b)

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Proof

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$L_2^{-1}(L_1^{-1}A) = U \Rightarrow L_1^{-1}A = L_2U \Rightarrow A = L_1L_2U$$

$$L = L_1L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = L}$$

$$\boxed{U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{pmatrix}}$$

$$\boxed{A = LU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{pmatrix} \checkmark$$

(2)

a) When matrix A is put into echelon form U by elementary row operations the pivots are the 1^{st} non-0 entries in the rows of U.

Example Problem 1a. The pivots are $1, 1, -7$

b) Span of a set $S = \{\vec{v}_1, \dots, \vec{v}_r\} = \{ \sum_{j=1}^{r} c_j \vec{v}_j \mid c_j \in \mathbb{R} \}$
i.e all linear combinations of the vectors in S.

Example. Consider $S = \{$columns of A from 1a$)\}$
$$\text{Span}(\text{Columns of } A) = \mathbb{R}^3 = \text{Col}(A)$$

c) linearly independent vectors $\{\vec{v}_1, \dots, \vec{v}_r\}$ means no vector $\vec{v}_j$ is in $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{j-1}, \vec{v}_{j+1}, \dots, \vec{v}_r\}$
i.e., $\sum_{j=1}^{r} c_j \vec{v}_j = 0 \implies$ all scalars $c_j = 0$.

Example.
$\{\binom{1}{0}, \binom{0}{1}\}$ linearly independent vectors in $\mathbb{R}^2$

d) linear transformation $T: V \longrightarrow W$ is a function mapping vectors $\vec{x}$ in vector space V to vectors $T\vec{x}$ in vector space W having the property
$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2), \text{ for all } \vec{v}_j \in V,\ c_j \in \mathbb{R}$$

Example
$V = \{$ all polynomials $\}$. The derivative $Dp = p'(x)$ is linear (for $p \in V$).

e) $T: V \longrightarrow W$ as in d)
$\mathcal{B}$ = basis V, $\quad C$ = basis W
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \quad C = \{\vec{w}_1, \dots, \vec{w}_m\}$

$\boxed{\underset{\mathcal{B}\to C}{\text{Mat}}\, T}$ has j^{th} column the scalars obtained by writing $T\vec{v}_j = \sum_{i=1}^{m} a_{ij} \vec{w}_i$, namely $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$

Example $V = \mathbb{P}_3 = \{$ polynomials of degree $\le 3\}$
$\mathcal{B}_3 = \{1, x, x^2, x^3\}$

$D: \mathbb{P}_3 \to \mathbb{P}_3 \quad Dp = \frac{dp}{dx} = p'(x)$, (the derivative) $\quad \underset{\mathcal{B}_3 \to \mathcal{B}_3}{\text{Mat}}\, D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(2 continued)

f) free variable is a variable corresponding to a non-pivot column of U = an echelon form of matrix A

Example. $\begin{array}{c} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \quad x_1 \ \ x_2 \ \ x_3 \ \ x_4 \end{array}$ free variables x_2, x_4

g) elementary row operations
(i) multiply every element of row by $c \in \mathbb{R}$, $c \neq 0$
(ii) permute or interchange 2 rows
(iii) replace row j by $(\text{row } j + c(\text{row } i))$ for $c \in \mathbb{R}$

Example.
See problem 1 a) We did 3 type (iii) operations.

h) basis of a vector space is a set of linearly independent vectors that spans the space

Example See example for part e).

3) a) V = vector space

$W \subset V$ is a subspace $\Leftrightarrow \left\{ \begin{array}{l} \text{for every } (\forall)\ \vec{x}, \vec{y} \in W \\ \qquad \forall\ c \in \mathbb{R} \\ c\vec{x} \text{ and } \vec{x}+\vec{y} \in W \end{array} \right\}$

b) i) $\left.\begin{array}{l}(1\,1\,1\,1)\vec{x} = 0 \\ (1\,1\,1\,1)\vec{y} = 0 \\ c \in \mathbb{R}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(1\,1\,1\,1)(\vec{x}+\vec{y}) = (1111)\vec{x} + (1111)\vec{y} = 0+0 = 0 \\ (1\,1\,1\,1)(c\vec{x}) = c(1111)\vec{x} = 0\end{array}\right.$

$\Rightarrow$ this is a subspace $= \mathrm{Null}(1111)$

ii) Not a subspace

$\left.\begin{array}{l}(1111)\vec{x} = 1 \\ (1111)\vec{y} = 1 \\ c \in \mathbb{R}\end{array}\right\} \Rightarrow \begin{array}{l}(1111)(\vec{x}+\vec{y}) = 2 \neq 1 \\ (1111)c\vec{x} = c \text{ may not be } 1\end{array}$

So only (i) is a subspace

4) T-F

a) F

$$\begin{pmatrix}1&1&0&0\\1&0&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix} \to \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix} \to \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&0&1&1\\0&0&1&1\end{pmatrix} \to \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix}$$

$A = (a\,b\,c\,d)$

only 3 pivots

They are dependent as $\mathrm{Nul}(A) \neq \{0\}$

b) F Since $\dim(\mathbb{R}^5) = 5$ a spanning set must have ≥ 5 vectors.

c) F If $\vec{b} \notin \mathrm{Col}(A)$, then $A\vec{x} = \vec{b}$ has no solution $\vec{x}$ e.g. $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ has ∞ # solutions $\begin{pmatrix}0\\y\end{pmatrix}$, y arbitrary; $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ has no solution

d) F $A = \begin{pmatrix}1&1&1\\1&2&3\\2&2&2\end{pmatrix} \sim \begin{pmatrix}1&1&1\\0&1&2\\0&0&0\end{pmatrix} = U$

$\mathrm{Col}(U)$ consists of the x,y-plane

$\mathrm{Col}(A)$ is a plane but not in the $x.y$-plane

4) continued

e) F

$$\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix} \neq \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&2\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&2\end{pmatrix} \neq \begin{pmatrix}1&-1\\0&2\end{pmatrix} \checkmark$$

5) a) $\boxed{\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m, \text{ if } A \text{ is } m \times n}$

$\boxed{\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \subset \mathbb{R}^n}$

b) $$\left(\begin{array}{ccc|c}1&2&3&b_1\\-1&-2&-3&b_2\\0&1&1&b_3\end{array}\right) \xrightarrow{L^{-1}} \left(\begin{array}{ccc|c}1&2&3&b_1\\0&0&0&b_2+b_1\\0&1&1&b_3\end{array}\right) = (U|c)$$

pivots are in columns 1 and 2

free variables x_3

a Basis $\mathrm{Col}(A)$ = column 1 and column 2 $\boxed{\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\-2\\1\end{pmatrix}\right\}}$

$\mathrm{Nul}(A) = \mathrm{Nul}(U)$ $\left\{\begin{array}{l}x_1 + 2x_2 + 3x_3 = 0 \\ x_2 + x_3 = 0\end{array}\right.$ x_3 free

$x_2 = -x_3$

$x_1 = -2x_2 - 3x_3 = 2x_3 - 3x_3 = -x_3$

$\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix} \Rightarrow \mathrm{Nul}\,A$ has basis $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$, $\boxed{\mathrm{Nul}(A) = \mathbb{R}\begin{pmatrix}-1\\-1\\1\end{pmatrix}}$

c) Consistency Relations

From part b)

$\boxed{b_2 + b_1 = 0 \Leftrightarrow \vec{b} \in \mathrm{Col}(A)}$

$(1,1,0)\vec{b} = 0$

d) $\boxed{\mathrm{Col}(A)^{\perp} = \mathbb{R}\begin{pmatrix}1\\1\\0\end{pmatrix}}$

check $(1,1,0)\begin{pmatrix}1\\-1\\0\end{pmatrix} = 0 \checkmark$

$(1,1,0)\begin{pmatrix}2\\-2\\1\end{pmatrix} = 0 \checkmark$

6) a) $A \in \mathbb{R}^{n\times n}$ is nonsingular $\Leftrightarrow$ $\det A \neq 0$

$\Leftrightarrow$ $A \overset{\text{Gauss}}{=} PLU \Rightarrow$ there are n pivots in U

$\Leftrightarrow$ n columns of A are linearly independent

$\Leftrightarrow$ n rows " " " "

$\Leftrightarrow$ $\text{Nul}(A) = \{\vec{0}\}$

The Favorite $\Leftrightarrow$ there is $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $B = A^{-1}$

$\Leftrightarrow$ $\text{Nul}(A^T) = \{\vec{0}\}$

$\Leftrightarrow$ $\text{Col}(A) = \mathbb{R}^n$

b) $A \cdot A^{-1} = A^{-1} A = I$ & $BB^{-1} = B^{-1}B = I$

$$\Longrightarrow (B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

So $B^{-1}A^{-1} = (AB)^{-1}$

7) a) $\dim \text{Nul}(A) + \dim \text{Col}(A) = n$ if $A \in \mathbb{R}^{m\times n}$

b) Gaussian elimination says $A = PLU$ where U is in echelon form.

$\dim \text{Col}(A) = \#(\text{pivots in } U) = r$

$+ \quad \dim \text{Nul}(A) = n - r = \#$ free variables

Sum $\dim \text{Col}(A) + \dim \text{Nul}(A) = r + (n-r) = n$ ✓

8) T-F

a) T we can write the elementary 3×3 matrices E_{ij} with 1 in i,j place & 0 elsewhere as linear combos of non-singular matrices

e.g. $E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Apply Permutation on right &/or left to get rest

The elementary matrices E_{ij}, $1 \le i,j \le 3$ form a basis for $\mathbb{R}^{3\times 3}$

They are linear combinations of non-singular matrices.

8) continued

b) F $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

very different echelon forms

are the reduced echelon forms different?

c) F Row Space $(1\,1\,1\,1) \subset \mathbb{R}^4$

Col Space $(1\,1\,1\,1) \subset \mathbb{R}$

These are different 1-dimensional vector spaces.

d) T $B \subset \mathbb{R}^n$

$(B^{\perp})^{\perp} \supset B$ as $\vec{x} \in B \Rightarrow \vec{x} \perp$ any vector in $B^{\perp}$

$\dim (B^{\perp})^{\perp} = n - (\dim B^{\perp}) = n - (n - \dim B) = \dim B$

$\Rightarrow B = B^{\perp\perp}$

A subspace having the same dimension as $B^{\perp\perp}$ must contain a basis of $B^{\perp\perp}$ and thus equals $B^{\perp\perp}$,

Note: $\dim B + \dim B^{\perp} = n$

If $\left\{ \begin{array}{l} \{\vec{v}_1, \dots, \vec{v}_r\} \text{ is a basis of } B \\ \{\vec{w}_1, \dots, \vec{w}_s\} \text{ " " " " } B^{\perp} \end{array} \right\}$ then $\{\vec{v}_1, \dots, \vec{v}_r, \vec{w}_1, \dots, \vec{w}_s\}$ is a basis of $\mathbb{R}^n$.

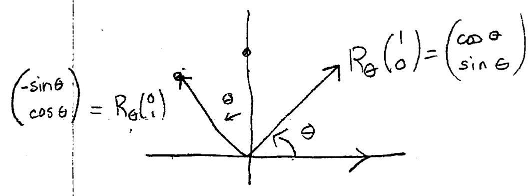
span $\vec{v} \in \mathbb{R}^n \Rightarrow \vec{p} = \text{Proj}_B \vec{v} \in B$ & $\vec{v} - \vec{p} \perp \vec{p} \in B^{\perp}$

$\Rightarrow \vec{v} = \vec{v} - \vec{p} + \vec{p}$.

linearly independent since $\perp$ vectors are

9) a) See #2e) Example

b) $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$$R_\pi = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$