Math 110A: Introduction to Partial Differential Equations
Fall 2016

Midterm 2 Review

1. Solve the wave equation $u_{tt} - c^2 u_{xx} = 0$ on the half line $x > 0$ with the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, and boundary condition at $x = 0$: (1) $u(0,t) = 0$ $(t > 0)$; (2) $u_x(0,t) = 0$ $(t > 0)$.

2. Solution formula for the wave equation $u_{tt} - c^2 u_{xx} = 0$ on a finite interval $(0, l)$ with the initial condition $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ $(0 < x < l)$ and the homogeneous Dirichlet or Neumann boundary conditions.

3. Solve the diffusion equation $u_t - ku_{xx} = f(x,t)$ $(-\infty < x < \infty, t > 0)$ with the initial condition $u(x,0) = \phi(x)$ $(-\infty < x < \infty)$. Can you verify the solution formula?

4. Solve the half-line diffusion equation $u_t - ku_{xx} = f(x,t)$ $(x > 0, t > 0)$ with the initial condition $u(x,0) = \phi(x)$ $(x > 0)$ and the boundary condition $u(0,t) = g(t)$ or $u_x(0,t) = h(t)$ $(t > 0)$.

5. Solve the wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$ on the entire line with the initial condition $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. Theorem 1 on page 71 and the inequality (4) on page 73. Verify the solution formula. (You may skip pages 76 and 77.)

6. Solve the wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$ on the half-line with the initial condition $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, and the boundary condition $u(0,t) = g(t)$ or $u_x(0,t) = h(t)$ $(t > 0)$.

7. Find the eigenvalues and eigenfunctions for $-X'' = \lambda X$ $(0 < x < l)$ with the boundary conditions: (1) $X(0) = X(l) = 0$; (2) $X'(0) = X'(l) = 0$; and (3) $X(-l) = X(l)$ and $X'(-l) = X'(l)$. What about the boundary conditions $X(0) = 0$ and $X'(l) = 0$?

8. Solve the diffusion equation $u_t - ku_{xx} = 0$ on a finite interval with $t > 0$, with the initial condition $u(x,0) = \phi(x)$ and different, homogeneous boundary conditions: Dirichlet, Neumann, or periodic, or Dirichlet at one point and Neumann at the other. Start with the separation of variables, derive and solve the eigenvalue problem, use the principle of superposition to get the solution as a series, and finally find all the coefficients.

9. Similarly, use the method of separation of variables to solve the wave equation $u_{tt} - c^2 u_{xx} = 0$ on a finite interval with the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, and different kinds of boundary conditions.

10. (1) Find the coefficients A_n $n = 1, 2, \ldots)$ in the Fourier sine series
$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$
(2) Find the coefficients A_n $n = 0, 1, \ldots)$ in the Fourier cosine series
$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$
(3) Find the coefficients A_n $n = 0, 1, \ldots)$ and B_n $(n = 1, 2, \ldots)$ in the Fourier series
$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi l}{l} + B_n \sin \frac{n\pi x}{l} \right).$$

(4) Find the coefficients c_n $(n = 0, \pm 1, \pm 2, \dots)$ in the complex Fourier series

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$

11. Why any function defined on a symmetric interval $[-l, l]$ can be uniquely decomposed into the sum of an even and an odd functions?

12. If $\phi(x)$ is even on $[-l, l]$, then all $B_n = 0$ in the Fourier series expansion of $\phi(x)$. Correct? If so, Why? What if $\phi(x)$ is odd?

13. If $\phi(x) = \sin 3x - 7\sin 5x + 2\sin 10x$, then in the Fourier sine series expansion of $\phi(x)$ on $[0, \pi]$, $A_3 = 1$, $A_5 = -7$, $A_10 = 2$, and all other A_n are 0. Correct? Why?

14. Definition of the inner product of two functions on $[a, b]$. Orthogonality.

15. Consider $-X'' + \lambda X = 0$ with symmetric boundary conditions. Then eigenfunctions corresponding to different eigenvalues are orthogonal. See Theorem 1 on page 120. Why all the eigenvalues are real-valued? See Theorem 2 on page 121. Compare with matrix eigenvalues and eigenvectors for a real, symmetric matrix.