Exam 1, Mathematics 102
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Name:
Student ID:
Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (40 points)

(1) Write down a system of 3 linear equations in 3 variables whose solution set in $\mathbb{R}^3$ coincides with the position vectors of the points lying on the unique line $\mathcal{L}$ which passes through the point $P_0(1, 2, 0)$ and has direction vector

$$\vec{d} := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(2) Solve the system of linear equations you wrote down in part (i) by the method of row reduction.

(1) The equation for $\mathcal{L}$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

which gives $z = t$
$y = 2$
$x = 1 + t = 1 + z$

One of ~~many~~ many possible answers is

$$\begin{cases} x - z = 1 \\ y = 2 \\ x + y - z = 3 \end{cases}$$

$\longrightarrow$

(2)

$$\left.\begin{aligned} x - z &= 1 \\ y &= 2 \\ x + y - z &= 3 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & -1 & 3 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable

Hence, z is free variable

Set $z = t$

Row 1 gives: $x - z = 1$

so $x = z + 1 = t + 1$

Row 2 gives: $y = 2$

$$\begin{aligned} x &= 1 + t \\ y &= 2 \\ z &= t \end{aligned} \Rightarrow$$

set of solution to linear equations is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

which is the same as the line $\mathcal{L}$, as expected

II. (20 points)

(1) Write down the equation of the unique plane Π in $\mathbb{R}^3$ which contains the points $P_1(1,0,0)$, $P_2(0,1,0)$ and $P_3(0,0,1)$.
(2) Find the intersection between the plane Π and the line which is perpendicular on Π and contains the point $O(0,0,0)$.

(1) Π is set of points which satisfies

$ax + by + cz = d$ for some a, b, c, d

Plug in P_1: $a \cdot 1 + b \cdot 0 + c \cdot 0 = d \Rightarrow a = d$

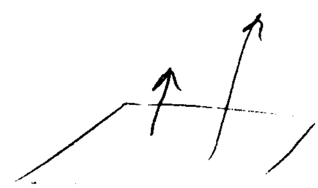
P_2: $a \cdot 0 + b \cdot 1 + c \cdot 0 = d \Rightarrow b = d$

P_3: $a \cdot 0 + b \cdot 0 + c \cdot 1 = d \Rightarrow c = d$

so eqt'n for Π is $dx + dy + dz = d$

$d \neq 0$, so can multiply by $\frac{1}{d}$,

get $\boxed{x + y + z = 1}$



(2) $\mathcal{L}$ is perpendicular on Π

so direction vector for $\mathcal{L}$ = normal vector for Π = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\mathcal{L}$ contains $(0,0,0)$ $\Rightarrow$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is the equation for $\mathcal{L}$

To find where $\mathcal{L}$ and Π intersect,

we suppose a point in $\mathcal{L}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$

satisfies $x + y + z = 1$

$\Rightarrow (t) + (t) + (t) = 1 \quad \Rightarrow 3t = 1$

$\Rightarrow t = 1/3$

Hence, $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ is a point which satisfies

$x + y + z = 1$ and is on the line $\mathcal{L}$,

so it is the intersection.

III. **(40 points)**

(1) Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times r}(\mathbb{R})$, for some natural numbers m, n, r. Prove that if the rows of A (viewed as vectors in $\mathbb{R}^n$) are linearly dependent, then the columns of $(B^t \cdot A^t)$ (viewed as vectors in $\mathbb{R}^r$) are linearly dependent.

(2) Without using determinants decide whether the rows of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

are linearly independent or not. Justify your answer.

(1) Let $r_i(A)$ denote the i^{th} row of A. If $\{r_i(A)\}$ are linearly dependent, then there is some row vector $\vec{\lambda}_0 \in \mathbb{R}^m$, $\vec{\lambda}_0 \neq \vec{0}$, with $\vec{\lambda}_0 \begin{bmatrix} r_1(A) \\ r_2(A) \\ \vdots \\ r_m(A) \end{bmatrix} = 0$; that is, $\vec{\lambda}_0 A = 0$.

Now the columns of $B^T A^T$ are the rows of $(B^T A^T)^T = (A^T)^T (B^T)^T = AB$,

so it suffices to find a nonzero $\vec{\lambda} \in \mathbb{R}^m$ with $\vec{\lambda}(AB) = 0$. (because then the ~~columns~~ rows of AB are linearly dependent)

But ~~the~~ $\vec{\lambda} = \vec{\lambda}_0$ works, as $\vec{\lambda}_0 (AB) = (\vec{\lambda}_0 A) B$
$= (0) B$
$= 0.$ //

(2) Row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3-\frac{2}{3}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Since this square matrix has echelon form with no nonzero rows, it is invertible.

$\Longleftrightarrow$ its rows are linearly dependent.