Practice Exam
Math 110A: Introduction to PDE

Problem 1: Indicate whether each of the statements below are true or false. **No justification is needed.**

a) If $u(x, y)$ is a solution to the transport equation: $2u_x + 3u_y = 0$, then $u(x, y)$ must be constant along the characteristic line: $3x - 2y = 1$.
b) There are infinitely many solutions to the PDE: $2u_x + 3u_y = 0$.
c) The PDE: $u_{xy} - 4x^2y^3 = 0$, is second-order, linear inhomogeneous.
d) There exists a function $u(t, x)$ which solves both the heat equation $u_t - ku_{xx} = 0$ and the wave equation $u_{tt} - u_{xx} = 0$.

Problem 2: Find the **specific** solution to the transport equation:

$$xyu_x \; + \; (1 + y^2)u_y \; = \; 0 \; ,$$

with the initial data: $u(x, 0) = x^4$.

Problem 3: Consider the wave equation IVP:

$$u_{tt} - u_{xx} = 0 \; , \qquad u(0, x) \; = \; 0 \; , \qquad u_t(0, x) \; = \; \begin{cases} 1 \; , & |x| \leqslant 1; \\ 0 \; , & |x| > 1. \end{cases}$$

(a) Calculate $u(0, t)$ explicitly for all $t > 0$ for the solution $u(x, t)$ of this problem.

(b) Show that $\lim_{t\to+\infty} u(t, x) = 1$ for all $x \in \mathbb{R}$.

Problem 4: Let $u(t, x)$ be a (smooth) solution to the wave equation: $u_{tt} - u_{xx} = 0$. Define the *Lorentz Boost* to be the differential operator: $L = t\partial_x + x\partial_t$.

(a) Show that the function: $Lu \; = \; t\partial_x u + x\partial_t u$ is also a solution of the 1-D wave equation.

(b) Define the energy for Lu to be:

$$E[Lu](t) \; = \; \frac{1}{2} \int_{-\infty}^{\infty} |\partial_t Lu(t, x)|^2 + |\partial_x Lu(t, x)|^2 \; dx \; .$$

Use part (a) above to show that for all times t:

$$E[Lu](t) \; = \; E[Lu](0) \; .$$

Problem 5: Let $u(t, x)$ be a solution to the heat equation with Neumann boundary conditions on the interval $[0, \pi]$:

$$u_t - ku_{xx} \; = \; 0 \; , \qquad u_x(0, t) \; = \; u_x(\pi, t) \; = \; 0 \; ,$$

Define the *total heat* $H[u](t)$ as follows:

$$H[u](t) \; = \; \int_0^{\pi} u(t, x) \; dx \; .$$

Show that the total heat is conserved in the sense that $H[u](t) = H[u](0)$.