MATH 20E FINAL EXAM Fall 98, Lindblad.

1. Consider the function $f(x, y) = 3e^{x^2y-2}$.
a) In which direction does the function increase the fastest at the point $(x, y) = (1, 2)$?
b) Give the equation for the tangent plane to the surface $z = f(x, y)$ at the point $(1, 2, 3)$.

2. Let $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$.
a) Find $\nabla \times \mathbf{F}$.
b) Find $\int_C \mathbf{F} \cdot d\mathbf{R}$ where C is a circle $x^2 + y^2 = r^2$ going counterclockwise.
c) Is $\mathbf{F}$ conservative in the plane minus the origin?
If $\mathbf{F}$ is conservative find a potential ϕ so $\mathbf{F} = \nabla\phi$, otherwise explain why not.
d) Find the flow lines for $\mathbf{F}$.

3. Let $\mathbf{F} = \dfrac{y\mathbf{i} - x\mathbf{j}}{x^2 + y^2}$.
a) Find $\nabla \times \mathbf{F}$.
b) Find $\int_C \mathbf{F} \cdot d\mathbf{R}$ where C is a circle $x^2 + y^2 = r^2$ going counterclockwise.
c) Is $\mathbf{F}$ conservative in the plane minus the origin?
If $\mathbf{F}$ is conservative find a potential ϕ so $\mathbf{F} = \nabla\phi$, otherwise explain why not.
d) Find the flow lines for $\mathbf{F}$.

4. Let $F(u, v) = (v\cos u,\ 2v\sin u)$ be a mapping from the uv-plane to the xy-plane. Let R be the rectangle in the uv-plane defined by $0 \leq u \leq \pi/2$ and $2 \leq v \leq 4$ and let S be the image of R in the xy-plane.
a) Calculate $F(u, v)$ for the four vertices of R. In the xy-plane sketch the image under F of the four sides of R, and shade the region of S.
b) Compute the Jacobian of F.
c) Compute the area of S.

5. a) Find the area of the part of the sphere
$S = \{(x, y, z);\ x^2 + y^2 + (z-1)^2 = 4,\ 0 \leq z \leq 1\}$, centered at $(0, 0, 1)$.

b) Find $\iiint_B f(x, y, z)\, dxdydz$ where $f(x, y, z) = x^2$ and
$B = \{(x, y.z);\ x^2 + y^2 + (z-1)^2 = 4\}$ is the whole ball.

6. Let S be as in problem (5a): $S = \{(x,y,z);\, x^2+y^2+(z-1)^2 = 4,\, 0 \le z \le 1\}$ and let n be the exterior unit normal to S.

a) Find $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F} = (z+x-y)\mathbf{i} + (x+y-z)\mathbf{j} + (y+z-x)\mathbf{k}$.

b) Find $\displaystyle\iint_M \mathbf{G} \cdot \mathbf{n} dS$, where $M = \{(x,y.z);\, x^2+y^2+(z-1)^2 = 4\}$ is the whole surface of the sphere and $\mathbf{G} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$.
(Hint: $\nabla \cdot G = 0$ when $(x,y,z) \neq (0,0,0)$.)

7. a) State Greens formula for a domain D in the plane bounded by a curve C.
b) Explain how one can use Greens formula to find the area of the domain D.
c) Consider the cycloid given parametrically by $\begin{cases} x = 2(t - \sin t), \\ y = 2(1-\cos t) \end{cases}$.
Compute the area under the arch of the cycloid and above the x-axis from $(0,0)$ to $(4\pi, 0)$.

8. Let S be the surface given parametrically by $(x,y,z) = (\cos u,\, 2\sin u + v,\, v)$, where $0 \le u \le 2\pi$ and $0 \le v \le 4$.
a) Find the normal to S.
b) Find the area of S.

9. Compute $\int_C \mathbf{F} \cdot d\mathbf{R}$ where $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j} + e^{z \sin z}\mathbf{k}$ and C is the closed curve $(x,y,z) = (\cos t, \sin t, \sin^2 t)$, $0 \le t \le 2\pi$.