Name ________________________ Student No. ________________________ Section A0_____

No aids allowed. Answer all questions on test paper. Total Marks: **15**

[5] 1. Suppose that $A = [\ a \quad b \quad c\ ]$ where a, b, c are linearly independent column vectors. Suppose that q_1, q_2, q_3 are the result of applying the Gram-Schmidt orthonormalization procedure to a, b, c.

Fill out the R matrix in the $A = QR$ factorization of A:

$$A = [\ a \quad b \quad c\ ] = [\ q_1 \quad q_2 \quad q_3\ ] \left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \end{array} \right]$$

Solution:

$$\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

[5] 2. Consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute the matrix of cofactors $C = (C_{ij})$ of A:

$$C = \left[\begin{array}{c|c|c} \quad & \quad & \quad \\ \hline \quad & \quad & \quad \\ \hline \quad & \quad & \quad \end{array}\right]$$

and use C to give the inverse of A.

Solution: Recall that $C_{ij} = (-1)^{i+j} \det(M_{ij})$; then:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and since $\det(A) = 1$ (as A is upper-triangular with 1s on the main diagonal), it follows that $A^{-1} = C^T$.

[5] 3. Diagonalize the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

and find all its square-roots, i.e., matrices R such that $R^2 = A$.

Solution:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

This matrix has 4 square-roots, all of the form:

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \pm 3 & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

[3] 4. **Bonus Question:** Let $\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$. That is, the trace of a matrix is the sum of the elements on the main diagonal. Show that if an $n \times n$ matrix A has n distinct eigenvalues, then $\text{trace}(A) = \sum_{i=1}^{n} \lambda_i$.

Solution: Since $\text{tr}(AB) = \text{tr}(BA)$, so we have that $\text{tr}(D) = \text{tr}(S^{-1}AS) = \text{tr}(S^{-1}SA) = \text{tr}(A)$, where D is the diagonal matrix of eigenvalues.