

Math 102 - Winter 2013 - Midterm I

Name: _____

Student ID: _____

Section time: _____

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 50 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		7
2		12
3		12
4		19
Total		50

Problem 1. [*7 points.*]

Find the LU -decomposition of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}.$$

Problem 2. [12 points.]

Are the following vector spaces or not? Simply write "VECTOR SPACE" or "NOT VECTOR SPACE". No justification is necessary.

(i) [2] The set of upper triangular $n \times n$ matrices.

(ii) [2] The set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3 : x_1 x_2 x_3 \geq 0 \right\}.$

(iii) [2] The set $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ in } \mathbb{R}^4 : 2x - 3y + z - 2w = 0 \text{ and } x - 2y - z + 3w = 0 \right\}.$

(iv) [2] The set of polynomials $P(x, y)$ of degree at most 3 in two variables such that

$$P(0, 0) = \frac{\partial P}{\partial x}(0, 0) = 0.$$

(v) [2] The set of $n \times n$ matrices A such that $\mathbf{rref} A = I_n$.

(vi) [2] The set of $n \times n$ matrices A that commute with a fixed permutation matrix P that is,
 $PA = AP$.

Problem 3. [12 points.]

Consider the vector space \mathcal{P} of polynomials $f(x)$ of degree less or equal to 2. Let

$$T : \mathcal{P} \rightarrow \mathcal{P}$$

be the transformation

$$T(f) = f'' + xf'.$$

(i) [4] Show that $\{1, x, x^2 - x - 1\}$ is a basis for \mathcal{P} .

(ii) [4] Briefly explain why T is a linear transformation.

(iii) [4] Find the matrix of the transformation T in the basis of part (i).

Problem 4. [19 points.]

Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}.$$

- (i) [4] Give a basis for $C(A)$. What is the rank of A ?

(ii) [4] Give a basis for the null space of A . What is the nullity of A ?

(iii) [3] Show that the columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ of A are linearly dependent by exhibiting explicit relations between them.

(iv) [2] Does A admit a left inverse, a right inverse, neither, or both?

(v) [3] What is the dimension of the left null space of A ? What is the dimension of the row space of A ?

(vi) [3] Write down the general solution to the following system of equations

$$Ax = \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}.$$

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During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most half a page, front and back.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 50 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		7
2		22
3		10
4		10
Total		50

Problem 1.

Find the LU -decomposition of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}.$$

Solution: *We start with the row operations*

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$$

yielding the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix}.$$

Next, we look at

$$R_3 \rightarrow R_3 + 5R_2$$

yielding the upper triangular matrix

$$U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

The lower triangular matrix equals

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}.$$

Problem 2.

Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}.$$

- (i) Give a basis for the column space $C(A)$.

Solution: *We begin by row reducing the matrix*

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}.$$

We have the row operations

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 + R_1$$

yielding

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 9 & 3 \end{bmatrix}.$$

We next perform the row operation

$$R_3 \rightarrow R_3 + 3R_2$$

yielding the matrix

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for $C(A)$ is given by the pivot columns of A . The pivot columns are the first and second column, hence

$$C(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix} \right\}.$$

The rank of A equals 2..

(iii) Give a basis for the null space of A . What is the nullity of A ?

Solution: We need to row reduce further in order to find the null space of A . In part (i) we arrived at the echelon form

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can further calculate

$$R_1 \rightarrow R_1 - 2R_2$$

to obtain the rref

$$\begin{bmatrix} -2 & 0 & 3 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_3 and x_4 while the pivot variables are x_1, x_2 . We have

$$-2x_1 + 3x_3 + 3x_4 = 0 \implies x_1 = \frac{3}{2}x_3 + \frac{3}{2}x_4$$

$$x_2 - 3x_3 - x_4 = 0 \implies x_2 = 3x_3 + x_4.$$

We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} \frac{3}{2} \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$N(A) = \text{span} \left\{ \begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- (iv) Show that the columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ of A are linearly dependent by exhibiting explicit relations between them.

Solution: *Vectors in the null space give relations between the columns of A . We have*

$$\begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \in N(A) \implies \frac{3}{2}\mathbf{c}_1 + 3\mathbf{c}_2 + \mathbf{c}_3 = 0$$

$$\begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in N(A) \implies \frac{3}{2}\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4 = 0.$$

- (v) Does A admit a left inverse, a right inverse, either or neither?

Solution: *Neither since the rank does not match the number of columns nor the number of rows.*

- (vi) What is the dimension of the left null space of A ? What is the dimension of the row space of A ?

Solution: *The left null space has dimension equal to number of rows minus the rank, in our case $3 - 2 = 1$. The row space has dimension equal to the rank, namely 2.*

(vii) Write down the general solution to the following system of equations

$$Ax = \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}.$$

Solution: *The particular solution is $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The general solution is of the form*

$$x = x_p + x_h = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 3.

Are the following vector spaces or not?

- (i) The set of upper triangular $n \times n$ matrices.

Solution: *Vector space: sum of upper triangular matrices is upper triangular. Multiplication of upper triangular matrices by scalars is upper triangular.*

- (ii) The set $\{(x_1, x_2, x_3) \text{ in } \mathbb{R}^3 : x_1 x_2 x_3 \geq 0\}$.

Solution: *Not a vector space. For instance $(-1, -1, 1)$ is in the set, but its scalar multiple $(1, 1, -1)$ is not.*

- (iii) The set $\{(x, y, z, w) \text{ in } \mathbb{R}^4 : 2x - 3y + z - 2w = 0 \text{ and } x - 2y - z + 3w = 0\}$.

Solution: *Vector space. The set in question is the null space of the matrix*

$$A = \begin{bmatrix} 2 & -3 & 1 & -2 \\ 1 & -2 & -1 & 3 \end{bmatrix}.$$

Null spaces are always vector spaces.

- (iv) The set of polynomials $P(x, y)$ of degree at most 3 in two variables such that

$$P(0, 0) = \frac{\partial P}{\partial x}(0, 0) = 0.$$

Solution: *Vector space. Sum of two polynomials P and Q with this property and scalar multiples still satisfy the equation in the question.*

- (v) The set of $n \times n$ matrices $\mathbf{rref} A = I$.

Solution: *Not a vector space. The zero vector is not part of the set.*

- (vi) The set of $n \times n$ matrices A that commute with a fixed permutation matrix P that is, $PA = AP$.

Solution: *Vector space. If $PA = AP$ and $PB = BP$ then $P(A + B) = (A + B)P$. Similarly, if $PA = AP$ then $P(cA) = cPA = cAP = (cA)P$.*

Problem 4.

Consider the vector space \mathcal{P} of polynomials $f(x)$ of degree less or equal to 2. Let

$$T : \mathcal{P} \rightarrow \mathcal{P}$$

be the transformation

$$f \rightarrow f'' + xf'.$$

- (i) Show that $\{1, x, x^2 - x - 1\}$ is a basis for \mathcal{P} .

Solution: We know that $\{1, x, x^2\}$ is a basis of \mathcal{P} hence $\dim \mathcal{P} = 3$. We need to show that $\{1, x, x^2 - x - 1\}$ is also a basis, hence we need to prove that the three polynomials are linearly independent. Assume otherwise. Then

$$a \cdot 1 + b \cdot x + c \cdot (x^2 - x - 1) = 0$$

and we need to show that $a = b = c = 0$. Indeed, rewrite the above equation as

$$(a - c) + (b - c)x + cx^2 = 0 \implies a - c = b - c = c = 0 \implies a = b = c = 0$$

proving linear independence, as needed.

- (ii) Explain why T is a linear transformation.

Solution: To check T is a linear transformation we need to show

$$T(f + g) = T(f) + T(g), \quad T(cf) = cT(f).$$

The first equality rewrites as

$$(f + g)'' + x(f + g)' = (f'' + xf') + (g'' + xg')$$

which is clearly true. The second equality is

$$(cf)'' + x(cf)' = c(f'' + xf')$$

which is also true.

- (iii) Find the matrix of the transformation T in the above basis.

Solution: We have

$$T(1) = 0$$

$$T(x) = x$$

$$T(x^2 - x - 1) = 2 + x(2x - 1) = 2(x^2 - x - 1) + 1 \cdot x + 4 \cdot 1.$$

The matrix of the transformation is

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$