Math 140A, Winter 2009. Final Exam.

Answer five questions. Full credit will be given for five correct answers. If you hand in solutions to more than five problems then only the first five solutions will be graded.

If you make use of theorems proved in class or in the book, you should give a statement, name or number of the result which you are using.

Problem 1 (20%).

(a) Let X be the collection of sequences of the form $\{x_n\}$ such that each x_n is a positive integer and $x_1 < x_2 < x_3 < \ldots$. Is the set X countable? Explain your answer.

(b) Let Y be the set of sequences of the form $\{y_n\}$ such that each y_n is a non-negative integer and for every N the sum $y_1 + y_2 + \cdots + y_N$ is less than 100. Is the set Y countable? Explain your answer.

Problem 2 (20%). An open ball in $\mathbb{R}^2$ is a set of the form $B_r = \{x : |x - p| < r\}$ for some point p in $\mathbb{R}^2$ and radius $r > 0$. A closed ball $\overline{B_r}$ is the closure of an open ball.

(a) Is it true that every non-empty open subset of $\mathbb{R}^2$ is a countable union of open balls?
(b) Is it true that every non-empty open set in $\mathbb{R}^2$ is a countable union of closed balls?

Justify your answers.

Problem 3 (20%). Suppose that $f : E \to \mathbb{R}$ is a **uniformly continuous** function defined on a subset E of a metric space X.

(a) Show that if $\{x_n\}$ is a Cauchy sequence in E then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

(b) Show that there exists a continuous function g from the closure $\bar{E}$ into $\mathbb{R}$ which extends f, in the sense that $g(x) = f(x)$ for every x in E.

Hint: If p is a limit point of E then choose a sequence $\{x_n\}$ in E converging to p and use (a) to help you define $g(p)$. Then prove that g is well defined and continuous.

Problem 4 (20%). Decide in which of the following cases there exists a continuous function f from A onto B with a continuous inverse. Either write down such a function or explain why none exists.

(a) $A = [0, 1]$, $\quad B = \mathbb{R}$.
(b) $A = (0, 1) \cup (2, 3)$, $\quad B = (0, 2)$.
(c) $A = (-1, 1)$, $\quad B = (-\infty, \infty)$.
(d) $A = \mathbb{R}$, $\quad B = [0, \infty)$.

Problem 5 (20%). Decide which of the following statements are always true and which are sometimes false, and justify your answers.

(a) The intersection of two perfect sets is perfect.

(b) Let X be a metric space. If $f : X \to [0,1]$ is a continuous function then for **every** sequence $\{x_n\}$ in X, the sequence $\{f(x_n)\}$ converges.

(c) If $\{p_n\}$ is a sequence of real numbers converging to p, then the sequence $m_n = \max\{p_n, p_{n+1}, p_{n+2}\}$ converges to p.

(d) If $\{x_n\}$, $\{y_n\}$ are real sequences such that the sum $x_n + y_n$ converges, then the difference $x_n - y_n$ must also converge.

Problem 6 (20%). Let G be the set of those points $(a, b, c, d) \in \mathbb{R}^4$ such that $ad - bc \neq 0$.

(a) Prove that the set G is dense in $\mathbb{R}^4$.

(b) Prove that the set G is open in $\mathbb{R}^4$. (Hint: use continuity.)

(c) Decide whether the complement $\mathbb{R}^4 \setminus G$ is compact and justify your answer.

Problem 7 (20%). Let X be a metric space with a metric d. For a subset B of X and a point x in X, define the distance from x to B by

$$d(x, B) \ = \ \inf\{d(x, b) : b \in B\}.$$

For two subsets A and B of X, define

$$d(A, B) \ = \ \inf\{d(a, B) : a \in A\}.$$

(a) Show that if B is a closed subset of X and $x \notin B$ then $d(x, B) > 0$.

(b) Show that if B is any subset of X then the function $x \mapsto d(x, B)$ is continuous on X.

(c) Show that if A is a compact subset of X and B is a closed subset of X and $A \cap B = \emptyset$ then $d(A, B) > 0$.

Problem 8 (20%). Give an example of

(a) A function from $(0, 1)$ to $\mathbb{R}$ which is continuous but not uniformly continuous.

(b) A function f from the set $S = [0, 1] \times [0, 1]$ to $\mathbb{R}$ such that $f(S)$ is **unbounded**, but for every fixed value $a \in [0, 1]$, the two functions $f_1(t) = f(t, a)$ and $f_2(t) = f(a, t)$ are continuous on $[0, 1]$

(c) A function from $[0, 1]$ to $\mathbb{R}$ which is discontinuous at every point.

(d) A continuous real valued function g on $\mathbb{R}$ and a closed subset V of $\mathbb{R}$ such that $g(V)$ is not closed in $\mathbb{R}$.

Problem 9 (20%). If you feel that questions 1-8 are too difficult and you are at risk of flunking the test, then write an essay summarizing what you learned from the course. (This problem requires a thoughtful answer. A list of statements of results copied from Rudin would **not** get any credit!)