**Linear Algebra - Exam #2 - A. Terras, May 25, 2007**

**The exam is closed book, no calculators, no computers, no notes, no headphones .... Each problem is worth the same number of points.**

1) **Define** the following and give an example.
   a) eigenvalue of a matrix
   b) orthogonal matrix

2) a) Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 \\ -5 & 0 & 6 \\ 0 & 8 & 9 \end{bmatrix}$.

   b) Find the area of the triangle with vertices at points (1,1), (2,3), (-1,5).

3) a) Use the Gram-Schmidt process to find 2 orthonormal vectors forming a basis for the column space of the matrix $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

   b) Suppose vectors $\vec{u}_1, \ldots, \vec{u}_n$, form an orthogonal basis of vector space V. If

$$\vec{y} \in V, \text{ then } \vec{y} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n, \text{ with } c_j = \frac{\vec{y}^T \cdot \vec{u}_j}{\vec{u}_j^T \cdot \vec{u}_j}.$$

   Hint. Take the inner product of vector $\vec{u}_j$ with $\vec{y} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$.

4) **True-False. Tell whether the following statements are true or false. If true, give a brief explanation and if false, give a counterexample.**
   a) Every matrix A is diagonalizable (i.e., A is of the form $A = PDP^{-1}$ with D diagonal).
   b) Det(AB) = Det(BA)
   c) Use elementary row operations (Gaussian elimination) to put a matrix A in row echelon form U. The eigenvalues of A are the same as the eigenvalues of U.

5) a) Show that an nxn matrix A with n linearly independent eigenvectors is diagonalizable (i.e., $A = PDP^{-1}$ with D diagonal).

   b) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then find the corresponding eigenvectors. Write $A = PDP^{-1}$ with D diagonal. Compute $A^9$.

Exam #2 Solutions

1) a) $\lambda \in \mathbb{C}$ is an eigenvalue of A if $A\vec{v} = \lambda\vec{v}$ for some vector $\vec{v} \in \mathbb{C}^n$ with $\vec{v} \neq \vec{0}$.

<u>Example</u>

2 is an eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

b) An nxn real matrix A is orthogonal if $AA^T = I$ = identity

<u>Example</u>. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

2) a) $\det \begin{bmatrix} 2 & 3 & 0 \\ -5 & 0 & 6 \\ 0 & 8 & 9 \end{bmatrix} = -6 \cdot 8 \cdot 2 - 9 \cdot 3(-5) = -3\{32 - 45\}$
$= 3 \cdot 13 = 39$

b) area of $\triangle$ is : $\frac{1}{2}\left|\det\begin{bmatrix} -2 & 1 \\ 4 & 2 \end{bmatrix}\right| = \frac{1}{2}|-4-4| = \frac{8}{2} = 4$

$\vec{a} = \begin{bmatrix} -1-1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 2-1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3) $\vec{v}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{[1\ 1\ 0\ 0]\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}{[1\ -4\ 0\ 1]\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\cancel{3}}{\cancel{18}\,6} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 1/6 \\ 1 - 4/6 \\ 0 \\ 1/6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 7 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{q}_1 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$\vec{q}_2 = \frac{1}{\sqrt{54}} \begin{bmatrix} 7 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

3 b) If $\vec{y} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$, then

$$\vec{u}_j^T \cdot \vec{y} = \vec{u}_j^T (c_1\vec{u}_1 + \cdots + c_n\vec{u}_n)$$

$$= c_1 \vec{u}_j^T \vec{u}_1 + \cdots + c_j \vec{u}_j^T \vec{u}_j + \cdots + c_n \vec{u}_j^T \vec{u}_n$$

(All terms $=0$ but $j^{th}$, since $\vec{u}_j^T \cdot \vec{u}_i = 0 \quad i \neq j$)

So $\vec{u}_j^T \vec{y} = c_j \vec{u}_j^T \vec{u}_j$

as $\vec{u}_j$ form a basis

Divide by $\vec{u}_j^T \vec{u}_j$ (a non-0 scalar since $\vec{u}_j \neq \vec{0}$)

Then $c_j = \dfrac{\vec{u}_j^T \vec{y}}{\vec{u}_j^T \vec{u}_j} = \dfrac{\vec{y}^T \cdot \vec{u}_j}{\vec{u}_j^T \cdot \vec{u}_j}$. (by symmetry of inner product)

4 a) False $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ not diagonalizable

b) True $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

assuming $A$ & $B$ square

c) False $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

eigenvalues are $3, -1$ — eigenvalues $1, 1$

5 a) $A\vec{v}_i = \lambda_i \vec{v}_i$, $i = 1, \ldots, n$, $\lambda_i \in \mathbb{C}$

$v_i$ linearly independent in $\mathbb{C}^n$, $i = 1, \ldots, n$

$P = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$ $n \times n$ matrix

$$AP = [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n]$$
$$= [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_n\vec{v}_n]$$
$$= [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$



Set $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

So $AP = P\Lambda$.

$P$ non-singular as columns are basis $\mathbb{C}^n$

So $P^{-1}$ exists, So

$A = APP^{-1} = P\Lambda P^{-1}$.

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

$\lambda = 3, -1$

$Nul(A - 3I)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Nul(A + I)$ " " $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^9 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^9 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$