Name: Time section meets:

Mathematics 20F Professor Linda Rothschild
June 13, 1996

Final Exam-Version A

Instructions: Answer all questions. Use calculators for computation whenever it is quicker. There is not enough time to do all problems by hand. Indicate what the calculator has shown. Show some work or reason for each answer. For justification, you can mention any fact cited in the text, but you cannot cite something shown in a homework problem. There are two versions of this exam; anyone who gives some answers corresponding to the other version will receive a 0 *for the final.*

1. & 2.	20
3. & 4.	23
5.	30
4.	18
6.	12
7. & 8.	17
9.	30
10.	7
11.	12
12.	25
13.	12
14.	12
Total	200

Have a great summer!

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. (10 pts.) Find the general solution of the system

$$\begin{aligned} x_1 + 5x_3 + 6x_4 &= 6 \\ x_1 + x_2 + 2x_4 &= 1 \\ 3x_1 + 2x_2 + 6x_3 + 11x_4 &= 11, \end{aligned}$$

2. (10 pts.) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_2 - x_3 \\ x_1 + 3x_2 \\ x_3 + x_4 \end{bmatrix}.$$

(a) Find the standard matrix for T.

(b) Find a basis for the kernel of T.

(15 pts.) For $A = \begin{bmatrix} 2 & 2 & 4 & 5 & 0 \\ 0 & 2 & 2 & 1 & 3 \\ 1 & 1 & 3 & 4 & 2 \\ 5 & 5 & 11 & 14 & 2 \end{bmatrix}$, find the following.

(a) rank of A

(b) a basis for the row space of A

(c) a basis for all $b \in \mathbb{R}^4$ for which $Ax = b$ has a solution.

4. (8 pts.) If the eigenvalues of a 3×3 matrix A are 1, 2, and 3, what are the eigenvalues of A^{-1}? Give a brief reason for your answer.

5. (30 pts.) For each statement, mark it True or False. If true, give a brief reason. If false, explain why or give a "counterexample". No credit if reason is wrong.

(a) If A and B are 2×2 matrices, with A invertible, and if $AB = 0$, then $B = 0$.

(b) If A is a 2×2 matrix which is diagonalizable, then A is symmetric.

(c) If the eigenvalues of a 3×3 matrix A are 0, 1, and 2, then A is diagonalizable.

(d) If B is diagonalizable and B^{-1} exists, then B^{-1} is also diagonalizable.

(e) Suppose A and B are square matrices, and B is obtained from A by row operations. Then every eigenvalue of A is an eigenvalue of B.

6. (12 pts.) Let V be the plane in $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$.

Find the vector in V closest to $y = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$.

7. (9 pts.) Determine if the set of vectors in $\mathbb{R}^4$,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is linearly independent.

8. (8 pts.) Suppose you know that the determinant of the matrix

$$A = \begin{bmatrix} 1 & a & 2 \\ 3 & b & 5 \\ -1 & c & -3 \end{bmatrix}$$

is 3, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector for which $A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Find x_2. (Hint: There is an easy way to do this one! Answer should not depend on the numbers a, b, c; x_2 is the only component of x which can be determined from the information given.)

9. (30 pts.) For each of the following, read carefully and determine if it is true or false. GIVE ANSWER ONLY, True or False. NO REASONS NEED BE GIVEN, NOR WILL REASONS BE GRADED. EACH PART IS WORTH ± 3 POINTS. IF CORRECT, IT WILL BE GRADED $+3$. IF INCORRECT, IT WILL BE GRADED -3. (However, the minimum score for this problem will be 0.)

(a) If A is any matrix, the system $A\mathbf{x} = 0$ must have at least one solution.

(b) If A is an $n \times n$ orthogonal matrix, then $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

(c) If a square matrix A is diagonalizable, then its rows must be linearly independent.

(d) If V is a vector space and there is no set of n vectors which spans V, then $\dim(V) > n$.

(e) If there is a linearly DEPENDENT set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of vectors in V, then $\dim(V) < 4$.

(f) The mapping $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 + 1 \end{bmatrix}$ with $a, b \in \mathbb{R}$ is a linear mapping of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

(g) If A and B are square matrices which are similar to each other, then they must have the same eigenvalues.

(h) If A and B are square matrices which are similar to each other, then they must have the same eigenvectors.

(i) If A is a square matrix, and 0 is an eigenvalue of the matrix $A - 2I$, then 2 must be an eigenvalue of A.

(j) There exists a linear mapping of $\mathbb{R}^3$ onto $\mathbb{R}^4$.

10. (7 pts.) Give an example of 2×2 matrices A and B such that A and B have the same characteristic poynomial, but A is diagonalizable while B is not diagonalizable. (Not necessary to show why your example works.)

11. (12 pts.) For the matrix $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, find vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and a diagonal matrix D so that

$$P^{-1}AP = D,$$

where $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. (OK to use a calculator for everything here.)

12. (25 pts.) In this problem A is a square matrix. Give a very brief reason for each answer.

(a) If A is not invertible, what number must be an eigenvalue of A?

(b) If $\dim \mathrm{Nul}(A) = 1$, what is the rank of A?

(c) If A is invertible, what is the (row) reduced echelon form (rref) of A?

(d) If A is not invertible, find $\det A$.

(e) If $A^2 = 0$, show that A is not invertible.

13. (12 pts.) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in a vector space V and $\mathbf{v}_4$ is a vector in V which is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, show (carefully) that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

14. (12 pts.) Describe the set of all polynomials $p(t) = a + bt + ct^2 + dt^3$ whose graph passes through the points $(0, 1)$, $(1, -2)$, $(2, 0)$.