Name ______ Student No. _____ Section A0_____

No aids allowed. Answer all questions on test paper. Total Marks: 15

[5] 1. Suppose that $A = [a \ b \ c]$ where a,b,c are linearly independent column vectors. Suppose that q_1,q_2,q_3 are the result of applying the Gram-Schmidt orthonormalization procedure to a,b,c.

Fill out the R matrix in the A = QR factorization of A:

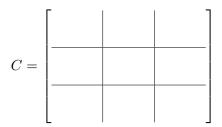
Solution:

$$\left[\begin{array}{ccc} q_{1}^{T}a & q_{1}^{T}b & q_{1}^{T}c \\ & q_{2}^{T}b & q_{2}^{T}c \\ & & q_{3}^{T}c \end{array}\right]$$

[5] 2. Consider

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right].$$

Compute the matrix of cofactors $C = (C_{ij})$ of A:



and use C to give the inverse of A.

Solution: Recall that $C_{ij} = (-1)^{i+j} \det(M_{ij})$; then:

$$C = \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array} \right]$$

and since $\det(A)=1$ (as A is upper-triangular with 1s on the main diagonal), it follows that $A^{-1}=C^T.$

$$A = \left[\begin{array}{cc} 5 & 4 \\ 4 & 5 \end{array} \right]$$

and find all its square-roots, i.e., matrices R such that $R^2=A$.

Solution:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

This matrix has 4 square-roots, all of the form:

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \pm 3 & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

[3] 4. **Bonus Question:** Let $\operatorname{trace}(A) = \sum_{i=1}^{n} a_{ii}$. That is, the trace of a matrix is the sum of the elements on the main diagonal. Show that if an $n \times n$ matrix A has n distinct eigenvalues, then $\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i$.

Solution: Since tr(AB) = tr(BA), so we have that $tr(D) = tr(S^{-1}AS) = tr(S^{-1}SA) = tr(A)$, where D is the diagonal matrix of eigenvalues.