Name ______________________ Student No. ______________________ Section A0_____

No aids allowed. Answer all questions on test paper.
Total Marks: 40 — 8 questions (plus a 9th *bonus question*), 5 points per question.
The exam has 9 pages of questions, and two extra pages at the end.

1. Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$$

(a) Under what conditions on $b = (b_1, b_2, b_3)$ does $Ax = b$ have a solution?

(b) Find a basis for the column space of A.

(c) Find a basis for the nullspace of A.

(d) What is the rank of A^T?

Solution: See worked out example on page 84.

2. Prove that $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

   Solution: First note that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. To see this note that the columns of AB are linear combinations of columns of A; hence $\operatorname{col}(AB) \subseteq \operatorname{col}(A)$, and therefore $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. Similarly the rows of AB are linear combinations of the rows of B, and so $\operatorname{row}(AB) \subseteq \operatorname{row}(B)$, and since row rank is the same as column rank, i.e., *the* rank, it follows that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. From this we can conclude that since $\operatorname{rank}(AB)$ is less than or equal to both $\operatorname{rank}(A)$ and $\operatorname{rank}(B)$, the statement in the questions is also true.

3. Use the Gauss-Jordan elimination on $\begin{bmatrix} A & I \end{bmatrix}$ to find A^{-1} where:

$$A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -a & -b+ad & -c+fb+ea-fda \\ 0 & 1 & -d & -e+fd \\ 0 & 0 & 1 & -f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summarize your answer here: $A^{-1} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$

4. Suppose that A is an $n \times n$ matrix and

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and let $p(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0$ be a polynomial of degree n in λ such that $p(\lambda) = \det(A - \lambda I)$.

Show that $p(A) = 0_{n\times n}$, where $0_{n\times n}$ is an $n \times n$ matrix of zeros.

Solution: Note that $p(\lambda) = \Pi_{i=1}^n(\lambda - \lambda_i)$, since the λ_i's are the roots of this polynomial. Also note that $p(A) = p(S\Lambda S^{-1}) = Sp(\Lambda)S^{-1}$. To see this last part remember that $(SXS^{-1})^i = SX^iS^{-1}$. So if we show that $p(\Lambda) = 0$ we will have that $p(A) = 0$. And:

$$p(\Lambda) = \Pi_{i=1}^n(\Lambda - \lambda_i I),$$

and we can visualize $\Pi_{i=1}^n(\Lambda - \lambda_i I)$ as the product of Λ n-times, where each copy of Λ is has a zero replacing an element on the diagonal. For example, if the size was 3×3, we would have:

$$\begin{bmatrix} 0 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & 0 & \\ & & \lambda_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 0 \end{bmatrix}$$

You can see that this product is zero, and an argument by induction show that it is zero for any size Λ.

5. If V is the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 0)$ find:

   (a) a basis for the orthogonal complement $V^\perp$;

   (b) the projection matrix P onto V;

   (c) the vector in V closest to the vector $b = (0, 1, 0, -1) \in V^\perp$.

   Solution: (a) Basis is $(-1, 1, 0, 0), (-1, 0, 0, 1)$. (b) The projection is:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

   (c) The closest vector is $p = \mathbb{O}$.

6. Let F_n be matrices defined as follows:

$$F_1 = \begin{bmatrix} 1 \end{bmatrix} \quad F_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad F_3 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad F_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and in general, F_n is an $n \times n$ matrix with the main diagonal of 1s, immediately above the main diagonal it has -1s, and immediately below the main diagonal it has 1s, and all other entries are zero.

Show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$. Conclude that $\det(F_n)$ computes what famous quantity?

Solution: Take the cofactor expansion of $\det(F_n)$ along the first row. It gives

$$(-1)^{1+1} \cdot 1 \cdot \det(F_{n-1}) + (-1)^{1+2} \cdot (-1) \det(F_{n-2}),$$

where it is clear that F_n with the first row and first column removed is just F_{n-1}, but F_n with first row and *second* column removed is not exactly F_{n-2}, in fact, if we remove the first row and second column of F_n, the upper-left corner of the resulting matrix looks as follows:

$$\begin{bmatrix} 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and observe that the determinant of this matrix is given by 1 times the determinant of the principal submatrix, which is $\det(F_{n-2})$.

This means that $\det(F_n)$ is the n-th Fibonacci number.

7. Suppose that a small car-rental company has three locations: San Diego (SD), Los Angeles (LA) and San Francisco (SF). Every month half of those in SD and in LA go to SF, and the other half stay where they are. The cars in SF are split equally between SD and LA. Set up the 3×3 transition matrix A, and find the steady state u_∞ corresponding to the eigenvalue $\lambda = 1$.

   Solution:

$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

   and the steady state $u_\infty = (1/3, 1/3, 1/3)$.

8. Prove **Schur's Lemma**: for any (square) matrix A, there exists a unitary matrix U such that $U^{-1}AU = T$ where T is upper triangular.

   Make sure that you explain, as part of your proof, why every square matrix *has at least one* unit eigenvector.

   Proof: Steps:

   (a) Let $Ax_1 = \lambda_1 x_1$ where x_1 is a unitary eigenvector. (Every matrix has at least one eigenvector; we divide it by its norm.) To show that there is at least one eigenvector (non zero by dfn.) note that each A has at least one eigenvalue, since $p(\lambda) = \det(A - \lambda I)$ has at least one root (it is a polynomial of degree n = size of A). Let λ_1 be this eigenvalue; note that $\det(A - \lambda_1 I) = 0$, and so $A - \lambda_1 I$ is singular, and hence has a non-trivial nullspace. Let x_1 be a non-trivial vector in $N(A - \lambda_1 I)$.

   (b) Let $U_1 = \begin{bmatrix} x_1 & q_2 & q_3 & \dots & q_n \end{bmatrix}$, where $\{q_1 = x_1, q_2, q_3, \dots, q_n\}$ are obtained by the Gram-Schmidt orthonormalization procedure from, say, the linearly independent set $\{x_1, e_{i_1}, e_{i_2}, \dots, e_{i_{n-1}}\}$.

   (c) We show that $U_1^{-1}AU_1 = \left[\begin{array}{c|c} \lambda_1 & \\ 0 & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right]$

$$
\begin{aligned}
AU_1 &= A \begin{bmatrix} x_1 & q_2 & q_3 & \dots & q_n \end{bmatrix} \\
&= \begin{bmatrix} Ax_1 & Aq_2 & Aq_3 & \dots & Aq_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 x_1 & Aq_2 & Aq_3 & \dots & Aq_n \end{bmatrix} \\
&= U_1 \begin{bmatrix} \lambda_1 & & & & \\ 0 & & & & \\ 0 & U_1^{-1}Aq_2 & U_1^{-1}Aq_3 & \dots & U_1^{-1}Aq_n \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}
\end{aligned}
$$

   (d) Repeat with the principal submatrix.

   (e) In the end $(\Pi_{i=n}^{1} U_i^{-1}) A (\Pi_{i=1}^{n} U_1)$ gives us an upper triangular matrix, and since a product of unitary matrices is unitary, and the inverse of a unitary matrix is also unitary, the lemma follows with $U = (\Pi_{i=1}^{n} U_i)$.

9. **Bonus Question:** Use Schur's Lemma (question 8) to show the following:

$$A \text{ is unitarily diagonalizable} \iff A \text{ is normal.}$$

"$\Rightarrow$" If A is unitarily diagonalizable, then there exists a unitary matrix U such that $U^{-1}AU = D$; then

$$\begin{aligned} AA^H &= (UDU^{-1})(UDU^{-1})^H = UDU^H UD^H U^H = UDD^H U^H = UD^H DU^H \\ &= UD^H U^H UDU^H = A^H A, \end{aligned}$$

and so A is normal.

"$\Leftarrow$" Suppose that A is normal. By Schur's Lemma, for any matrix A, $U^H AU = T$, where T is upper triangular. Since A is normal, so is T:

$$TT^H = (U^H AU)(U^H AU)^H = U^H AA^H U = U^H A^H AU = T^H T.$$

But an upper-triangular matrix is normal iff it is in fact diagonal. We can show this by induction on the size of T; if it is 1×1 it is always diagonal. For the induction step, notice that the principal submatrix of an upper (resp. lower) triangular matrix is also upper (resp. lower) triangular matrix, and furthermore the principal submatrix of LT (resp. TL) is product of the principal submatrix of L (resp. T) times the principal submatrix of T (resp. L). These observations make the proof of the induction step easy.

extra page

extra page