Name ______________________ Student No. ______________________ Section A0_____

*No aids allowed. Answer all questions on test paper.* Total Marks: **20**

[5] 1. Under what conditions on $b$ (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Solution:** Taking the system to echelon form we obtain:

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right]$$

which means that the solvability condition is $b_3 + b_2 - 5b_1 = 0$.

[5] 2. Find a basis for $C(A)$ and a basis for $N(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

**Solution:**

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that the first two columns of $A$ span $C(A)$, and since

$$\begin{bmatrix} x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and so these two vectors span $N(A)$.

[10] 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the function $T : M_{2\times 2} \longrightarrow M_{2\times 2}$, where $M_{2\times 2}$ is the set of $2 \times 2$ matrices over $\mathbb{R}$ defined as follows:

$$T(M) = AM$$

(a) Is $M_{2\times 2}$, with matrix addition and scalar multiplication, a vector space? Why?

(b) Is $T$ a linear transformation? Why?

(c) Does $T$ have an inverse? Why?

**Solution:** (a) The quick answer is that $M_{2\times 2}$ is isomorphic to $\mathbb{R}^4$. The longer answer is to check the properties of the vector space, but they follows from the commutatitivity and associativity of matrix addition, from the zero vector being the zero matrix, and the identity matrix being the unit, and general properties of scalar multiplication.

(b) It is because

$$T(c_1M_1 + c_2M_2) = A(c_1M_1 + c_2M_2) = c_1AM_1 + c_2AM_2 = c_1T(M_1) + c_2T(M_2).$$

(c) Yes it does, $T^{-1}(M) = A^{-1}M$; note that

$$T(T^{-1}(M)) = T(A^{-1}M) = A(A^{-1}M) = (AA^{-1})M = IM = M.$$

and same for $T^{-1}(T(M))$, so $T^{-1} \circ T = T \circ T^{-1} = \mathrm{id}$. Note that $A^{-1}$ exists since $1 \cdot 4 - 3 \cdot 2 = -2 \neq 0$.