GRE Prep: Precalculus

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1 Introduction

These are the notes for the Precalculus section for the GRE Prep session held at UCSD in August 2011.

These notes are in no way intended to *teach* you precalculus. Rather, they are here to *remind* you of the many facts that you may have forgotten. If you find *any* of the material presented here foreign or grossly unfamiliar, I *strongly* encourage you to relearn and practice that material as soon as possible.

The Mathematics Subject GRE test is a 2-hour 50-minute multiple choice test covering many sections of the undergraduate mathematics curriculum. That being said, I should emphasize that precalculus, in itself, will not have a major presence on the exam. However, *it will most certainly appear in the context of more advanced subjects.*

It is also worth noting that the Mathematics Subject GRE, at least in this author's opinion, **is primarily a test of thinking on your feet**. Research mathematics is certainly not done in the context of a timed exam. Hence, in preparing for this exam, there are a few things to keep in mind:

- For the only time in your life, memorize formulas. Unlike some of your college mathematics courses, *you will not get a cheat sheet.* Hence, the only way you will remember an obscure fact if to bring with you (in your head). I will make an attempt to point out useful formulas to memorize.
- If you know it, and you know you know it, don't practice it.
- Practice for speed. The test is only 170 minutes which only gives roughly 3 minutes per question. With high probability, you are more then capable of doing and completing most of the questions. Hence the biggest dilemma is time. Spending 20 minutes on one question is often not an effective use of time.
- That being said, pick your questions carefully. Every question has equal weight, and you need not answer all of the questions to get a very high

score. For example, 58 or 59/68 is often good enough for a score in the 90th percentile.

- The test is multiple choice ... sometimes it is better to work backwards! That is, you can just check all the proposed answer(s) and see which one works.
- Take a practice test and hopefully more than one. You *need to adapt* to the time constraint.
- Do a *quick* check on your answers. Most of the actual GRE questions are intentionally tricky to some extent. Before answering questions, see if you can figure out how they are trying to trick you. Common trick techniques they use include omitting hypotheses from a definition (e.g. not requiring function be continuous, etc.) or by considering often overlooked subtle facts (e.g., the "0" case, dividing by zero, or considering negative/complex numbers.)
- When in doubt, guess. The guessing penalty is small compared to the reward. If you can accurately eliminate a choice or two, you are worthwhile guessing for the answer.

2 Functions

A *function*, f, is an operation takes inputs from the *domain* and outputs from the *codomain*. Often such a function is written $f : X \to Y$ where f is the function, X is the domain and Y is the codomain.

Sometimes the term *codomain* and range are used interchangeably. However, they are in fact different terms. The codomain is the set where the the function is *allowed* to output where as the range is the set of all possible outputs. They are not necessarily the same.

Example 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{x}$.*
In this case, the domain and codomain are both $\mathbb{R}$ whereas the range is $\mathbb{R}\backslash\{0\}$.

Sometimes you will need to determine when a function is defined. Here are important facts (regarding functions over $\mathbb{R}$) to keep in mind:

- A fraction is defined only when its denominator is nonzero.
- Square roots (and even roots) are only defined when its contents are ≥ 0.
- Cube roots (and odd roots) are defined over all of $\mathbb{R}$.
- (Over $\mathbb{R}$), logarithms are defined when its contents are > 0

- $sin(x)$ and $cos(x)$ are defined over all of $\mathbb{R}$
- $tan(x)$ is defined everywhere except at odd multiples of $\frac{\pi}{2}$

Example 2 *Let $f : X \rightarrow \mathbb{R}$ be defined as $f(x) = \frac{1}{\sqrt{log(x)}}$.*
The largest possible $X \subset \mathbb{R}$ is $(1, \infty)$.

A function is *one-to-one* or *injective* if every output in the codomain corresponds to **at most** one element in the domain.

A function is *onto* or *surjective* if every output in the codomain corresponds to at to **least** one element in the domain.

A function is *in one-to-one correspondence*, *invertible* or *bijective* if it is both onto and one-to-one.

Example 3 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x$.*
In this case, f is both onto and one-to-one, and hence invertible.

We denote a *composition of functions*, for example $f \circ g$ as the function constructed by first applying g, then applying that output as the input into f. Simply put: $(f \circ g)(x) = f(g(x))$. Notice that the order in which the functions are applied is opposite from how they are written.

Composition of functions *is* **not commutative**. That is, $f \circ g \neq g \circ f$. However, composition of functions are associative. That is, $(f \circ g) \circ h = f \circ (g \circ h)$.

IF a function $f : X \rightarrow Y$ is invertible, then its inverse, denoted $f^{-1} : Y \rightarrow X$ is the unique function such that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(y) = y$ for all $x \in X$ and all $y \in Y$.

In the case that a function, $f(x)$ has a domain and codomain both as $\mathbb{R}$, then its inverse function can be found by replacing all cases of $f(x)$ with x and all cases of x with f^{-1} within the definition of f, then solving for f^{-1}

Example 4 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x$.*
Then the inverse can be found as follows:
$f(x) = 2x$.
$x = 2f^{-1}(x)$.
$f^{-1}(x) = \frac{x}{2}$

3 Polynomials and Conic Sections

A *polynomial* is a function which is a sum of nonnegative powers of the input. Specifically, every polynomial has the form: $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

For a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, *the degree of* $p(x)$ is the power of the largest i such that $a_i \neq 0$.

A polynomial is *quadratic* if it has degree 2.

The roots to a quadratic equation: $ax^2+bx+c = 0$ are given by the *quadratic formula* (You should have the memorized):

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

The *discriminant* of a quadratic equation is $\Delta = b^2 - 4ac$ the power of the discriminant is that is tells you information about the roots without actually computing them. $\Delta > 0$ implies there are two discount real roots. $\Delta = 0$; there is exactly one root. And $\Delta < 0$; there are two distinct non-real (complex) roots.

The *axis of symmetry* is the line that reflects the graph $y = p(x)$. For a quadratic polynomial $p(x) = ax^2 + bx + c$ is given by $x = \frac{-b}{2a}$.

The quadratic polynomial can be generalized to a *conic section* of the form:

$\pm\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1$

A *circle* with *center* (h, k) and *radius* r is more specifically by

$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$

Geometrically, a circle centered at (h, k) is the set of points which are exactly distance r from (h, k).

An *ellipse* with *center* (h, k) and *axes* a, b is more specifically by

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Geometrically, an ellipse centered at (h, k) is the set of points whose total distance from the two *foci* is $\sqrt{|a^2 - b^2|}$.

The foci are the points: $(h, k \pm \sqrt{|a^2 - b^2|})$ if $a < b$, and $(h \pm \sqrt{|a^2 - b^2|}, k)$ if $b < a$

A *hyperbola* with *center* (h, k) and *axes* a, b is more specifically by

$\pm\frac{(x-h)^2}{a^2} \mp \frac{(y-k)^2}{b^2} = 1$

Geometrically, an ellipse centered at (h, k) is the set of points whose distance to the two *foci* has difference $\sqrt{a^2 + b^2}$.

The foci are the points: $(h \pm \sqrt{a^2 + b^2}), k)$ in the +/- case, and $(h, k \pm \sqrt{a^2 - b^2}))$ in the -/+ case.

Theorem 1 *(The Fundamental Theorem of Algebra) Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *be an* $(n > 0)$ *n-th degree polynomial with coefficients in* $\mathbb{R}$*, then* $p(x)$ *has at most* n *roots (e.g. solutions to* $p(x) = 0$*). Further, upon counting for multiplicities* $p(x)$ *has exactly* n *roots in* $\mathbb{C}$*.*

Theorem 2 *(The Fundamental Theorem of Algebra, the fundamental exception) Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *be a polynomial with coefficients in* $\mathbb{R}$*, more than* n *roots (e.g. solutions to* $p(x) = 0$*), then* $p(x) = 0$*.*

It should be remarked that the Fundamental Theorem can the topic of some tricky questions.

Example 5 *Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *be an polynomial with coefficients in* $\mathbb{R}$*. What is the maximal number of roots that* $p(x)$ *can have?*
A) 1 *B)* $n - 1$ *C)* n *D)* $n + 1$ *E)* ∞
In this case, the answer is E, for is all of the a_i *are 0, there are infinite roots.*

Further, you are by no means expected to be able to prove the fundamental theorem of algebra (or no any of the techniques). Rather you only need to know how to apply it.

Fact 1 *Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *be a polynomial with at least one* $a_i \neq 0$*, then the sum of the roots (counting multiplicity) is* $\frac{-a_{n-1}}{a_n}$ *and the product of the roots (counting multiplicity) is* $\frac{(-1)^n a_0}{a_n}$*.*
This is corollary to the Fundamental Theorem of Algebra.

4 Logarithms

The logarithm written $log_b(x)$ is the induced inverse of $f(x) = b^x$. Specifically, for $b > 0$,$log_b(b^x) = x$ for all x, and $b^{log_b(x)} = x$ for all $x > 0$.

Other facts regarding $log_b(x)$:

- $log_b(x) + log_b(y) = log_b(xy)$
- $log_b(x) - log_b(y) = log_b(x/y)$
- $log_b(x^a) = a log_b(x)$
- $log_c(x) = log_b(x)/log_b(c)$
- $log_b(x)$ is only defined for $x > 0$

5 Trigonometry

All of the points of the unit circle can be given by $(\cos\theta, \sin\theta)$ Where θ is the angle measured counterclockwise from the positive x-axis. This is definition we will give here for cos and sin, far it is most applicable. Further we can define all of the remaining trigonomatric functions as follows: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$.

While it is too cumbersome to list here, it is advised you know your *unit circle*. That is you know the values for $\cos\theta$ and $\sin\theta$ for all $\theta = \frac{k\pi}{6}$ and $\frac{\ell\pi}{4}$ for $k = -12, \ldots 12$, and $\ell = -8 \ldots 8$.

You should definitely know many of the trigonometric formulae. Including, the *trigonometric Pythagorean identities*, *sine and cosine addition and subtraction forumlae*, and *double angle formulae*. They can be find in the Princeton Review's Cracking the Math GRE book on page 21 (the addition/subtraction formula for $\cos a \pm b$ are incorrect) , or on the inside front cover of your calculus text book.

The inverse trigonometric functions are written with "arc-" or "a-" or "$-^{-1}$". However, observe that since the trigonometric functions are periodic, they are not injective. For example $acos(1)$ returns *an* angle whose cosine is 1; however, there are infinitely many angles whose cosine is 1. Specifically, it is useful to know the domain and range of each of the inverse trigonometric functions:

- $arccos : [-1, 1] \to [0, \pi]$
- $arcsin : [-1, 1] \to [-\pi/2, \pi/2]$
- $arctan : (-\infty, \infty) \to (-\pi/2, \pi/2)$

To get the *general solution* to a trigonometric function apply the following rules:

- For $\cos\theta = \alpha$, then $\theta = 2\pi k \pm arccos(\alpha)$
- For $\sin\theta = \alpha$, then $\theta = 2\pi k + arcsin(\alpha)$ or $\theta = 2\pi k + \pi - arcsin(\alpha)$
- For $\tan\theta = \alpha$, then $\theta = \pi k + arctan(\alpha)$