Math 142a **Name**:

December 13, 1997

Final Exam

Instructions: You may cite any theorem from the text, either by name, or by describing the result. You must re-do any problem which appeared as a homework assignment. HAPPY HOLIDAYS!

1. (15 pts.) Find $\lim_{x\to 6} \frac{\sqrt{x-2}-2}{x-6}$ by any method. Show work (but not necessary to give a formal proof).

2. (20 pts.) Let S be the set in the plane given by

$$S = \{(x,y) : 1 < x^2 + y^2 < 4\} \cup \{(0,0)\}.$$

(a) Find the following (reasons not necessary):

interior of S:

boundary of S:

(b) Is S open? Give brief reason why or why not.

3. (20 pts.) (a) Let $\{a_n\}$ be a sequence of real numbers, and $A \in \mathbb{R}$. Give the precise meaning of the equation

$$\lim_{n\to\infty} a_n = A$$

(using ϵ).

(b) Let $\{P_n\}$ be a sequence of vectors in some Euclidean space, E, and let $P \in E$. Give the precise meaning of the equation

$$\lim_{n\to\infty} P_n = P.$$

(You may want to use Part (a) for this)

4. (15 pts.) Use the Mean Value Theorem to show

$$|\sin x - \sin y| \leq |x - y|.$$

5. (15 pts.) Prove CAREFULLY, using just the definition of limit, that

$$\lim_{n \to \infty} \frac{n^3}{n^3 + n^2 + 1} = 1.$$

6. (20 pts.) Find $\lim_{x \to 0} \frac{1}{x} - \cot x + 2$. (Show work; not necessary to prove rigorously.)

7. (20 pts.) Let $\{a_n\}$ be the sequence defined inductively by

$$a_1 = \sqrt{3},\ a_2 = \sqrt{6+\sqrt{3}},\ \ldots,\ a_{n+1} = \sqrt{6+a_n}.$$

(a) Show by induction that $a_n < 3$ for all $n \geq 1$.

(b) Show that the sequence $\{a_n\}$ is monotonically increasing.

(c) Find $\lim a_n$.

8. (20 pts.) Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin \frac{k}{n}}{n}$.

9. (20 pts.) Consider a plane ℓ given in space by $P \cdot \mathbf{a} = d$.

(a) Find the point on ℓ closest to $(0, 0, 0)$.

(b) Show that $|d|/|\mathbf{a}|$ is the distance from the plane ℓ to $(0, 0, 0)$.

10. (30 pts) For each of the following statements, determine if it is always true. If so, give a reason. If not, give an example for which the statement does not hold. (Sorry, no credit for just writing True or False.)

(a) If f is continuous on $\{-1 \leq x \leq 1\}$, then $f'(0)$ exists.

(b) If $\{a_n\}$ is a sequence of real numbers satisfying $0 \leq a_{n+1} \leq \frac{n}{n+1} a_n$, for all $n \geq 1$ then $\{a_n\}$ converges.

(c) If a sequence of real numbers $\{b_n\}$ satisfies $|b_{n+1}| \leq |b_n|$, for all $n \geq 1$, then $\{b_n\}$ has a convergent subsequence.

(d) If $f(x)$ is a continuous function on $\{0 \leq x \leq 1\}$ and satisfies

$$\frac{d}{dx} \int_0^{x^2} f(t)dt = f(x^2)$$

for all $x,\ 0 \leq x \leq 1$, then $f(x) = 0$ for all x.