PROBLEMS IN ELEMENTARY NUMBER THEORY

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God does arithmetic. **Gauss**

Contents

1. Preface

The heart of Mathematics is its problems. Paul Halmos

1.1. **Introduction.** The purpose of this book is to present a collection of interesting questions in *Number Theory*. Many of the problems are mathematical competition problems all over the the world including IMO, APMO, APMC, Putnam, etc. I have given sources of the problems at the end of the book. This book is available at

http://my.netian.com/∼ideahitme/eng.html

1.2. **How You Can Help.** This book is an unfinished manuscript. I would greatly appreciate hearing about any errors in the book, even minor ones. I also would like to hear about other interesting problems in *Number Theory*. You can send all comments to the author at

ideahitme@hotmail.com

2. Notations and Abbreviations

Notations

$\mathbf{Z}$ the set of integers
$\mathbf{N}$ the set of positive integers
$\mathbf{N}_0$ the set of nonnegative integers
$\sum_{d|n} f(d) = \sum_{d\in D(n)} f(d)$ $(D(n) = \{d \in \mathbf{N} : d|n\})$
$[x]$ the greatest integer less than or equal to x
$\{x\}$ the fractional part of x ($\{x\} = x - [x]$)
$\phi(n)$ the number of positive integers less than n that are relatively prime to n
$m|n$ n is a multiple of m.

Abbreviations

IMO International Mathematical Olympiads
APMO Asian Pacific Mathemats Olympiads

3. Divisibility Theory I

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. Paul Erdös

1. *The integers a and b have the property that for every nonnegative integer n the number of $2^n a + b$ is the square of an integer. Show that $a = 0$.*

2. *Let n be a positive integer such that $2 + 2\sqrt{28n^2 + 1}$ is an integer. Show that $2 + 2\sqrt{28n^2 + 1}$ is the square of an integer.*

3. *Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that*

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

4. *Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Show that*

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

5. *Let n be a positive integer with $n \geq 3$. Show that*

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

6. *Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.*

7. *Determine if there exists a positive integer n such that n has exactly* 2000 *prime divisors and $2^n + 1$ is divisible by n.*

8. *Let m and n be natural numbers such that*

$$A = \frac{(m + 3)^n + 1}{3m}.$$

is an integer. Prove that A is odd.

9. *Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1}.*

10. *Determine all positive integers n for which there exists an integer m so that $2^n - 1$ divides $m^2 + 9$.*

11. *Let n be a positive integer. Show that the product of n consecutive integers is divisible by $n!$*

12. *Prove that the number*

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 *for any integer* $n \geq 0$.

13. *(Wolstenholme's Theorem) Prove that if*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ *is a prime, then* p^2 *divides the numerator.*

14. *If* p *is a prime number greater than* 3 *and* $k = [\frac{2p}{3}]$. *Prove that the sum*

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

15. *Show that* $\binom{2n}{n} | LCM[1, 2, \cdots, 2n]$ *for all positive integers* n.

16. *Let* m *and* n *be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1)$.

17. *Show that the coefficients of a binomial expansion* $(a+b)^n$ *where* n *is a positive integer, are odd, if and only if* n *is of the form* $2^k - 1$ *for some positive integer* k.

18. *Prove that the expression*

$$\frac{gcd(m,n)}{n} \binom{n}{m}$$

is an integer for all pairs of positive integers $n \geq m \geq 1$.

19. *For which positive integers* k, *is it true that there are infinitely many pairs of positive integers* (m, n) *such that*

$$\frac{(m+n-k)!}{m!n!}$$

is an integer ?

20. *Show that if* $n \geq 6$ *is composite, then* n *divides* $(n-1)!$.

21. *Show that there exist infinitely many positive integers* n *such that* $n^2 + 1$ *divides* $n!$.

22. *Let p and q be natural numbers such that*
$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$
Prove that p is divisible by 1979.

23. *Let $b > 1$, a and n be positive integers such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.*

24. *Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than* 3*. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.*

25. *Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that*
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

26. *Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a+1)^n$.*

27. *Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.*

28. *An integer $n > 1$ and a prime p are such that n divides $p - 1$, and p divides $n^3 - 1$. Show that $4p + 3$ is the square of an integer.*

29. *Let n and q be integers, $n \geq 5$, $2 \leq q \leq n$. Prove that $q - 1$ divides $\left[\frac{(n-1)!}{q}\right]$. (Note : $[x]$ is the largest integer not exceeding x.)*

30. *If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.*

31. *Let p be a prime with $p > 5$, and let*
$$S = \{p - n^2 | n \in \mathbf{N}, n^2 < p\}.$$
Prove that S contains two elements a, b such that $1 < a < b$ and a divides b.

32. *Let n be a positive integer. Prove that the following two statements are equivalent.*

- ∘ *n is not divisible by* 4
- ∘ *There exist $a, b \in \mathbf{Z}$ such that $a^2 + b^2 + 1$ is divisible by n.*

33. *Determine the greatest common divisor of the elements of the set*
$$\{n^{13} - n | n \in \mathbf{Z}\}.$$

34. *Show that there are infinitely many composite n such that $3^{n-1} - 2^{n-1}$ is divisible by n*

35. *Suppose that $2^n + 1$ is an odd prime for some positive integer n. Show that n must be a power of 2.*

36. *Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.*

37. *Suppose that $4^n + 2^n + 1$ is prime for some positive integer n. Show that n must be a power of 3.*

38. *Let b, m, n be positive integers $b > 1$ and m and n are different. Suppose that $b^m - 1$ and $b^n - 1$ have the same prime divisors. Show that $b + 1$ must be a power of 2.*

39. *Show that a and b have the same parity if and only if there exist integers c and d such that $a^2 + b^2 + c^2 + 1 = d^2$.*

40. *Let n be a positive integer with $n > 1$. Prove that*

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

41. *Let n be a positive integer. Prove that*

$$\frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2n+1}$$

is not an integer.

42. *Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n + k)!$ equals k.*

43. *Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.*

44. *Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.*

45. *Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.*

46. *Suppose that x, y, z are positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$.*

4. Divisibility Theory II

Number theorists are like lotus-eaters - having tasted this food they can never give it up. Leopold Kronecker

1. *Determine all integers $n > 1$ such that*

$$\frac{2^n + 1}{n^2}$$

is an integer.

2. *Determine all pairs (n, p) of nonnegative integers such that*

- ○ *p is a prime,*
- ○ *$n < 2p$, and*
- ○ *$(p-1)^n + 1$ is divisible by n^{p-1}.*

3. *Determine all pairs (n, p) of positive integers such that*

- ○ *p is a prime, $n > 1$,*
- ○ *$(p-1)^n + 1$ is divisible by n^{p-1}.* [1]

4. *Find an integer n, where $100 \le n \le 1997$, such that*

$$\frac{2^n + 2}{n}$$

is also an integer.

5. *Find all triples (a, b, c) such that $2^c - 1$ divides $2^a + 2^b + 1$.*

6. *Find all $n \in \mathbf{N}$ such that $2^{n-1} | n!$.*

7. *Find all integers a, b, c with $1 < a < b < c$ such that*

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc - 1.$$

8. *Find all positive integers, representable uniquely as*

$$\frac{x^2 + y}{xy + 1},$$

where x, y are positive integers.

9. *Determine all ordered pairs (m, n) of positive integers such that*

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

10. *Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.*

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[1] The anwser is $(n, p) = (2, 2), (3, 3)$. Note that this problem is a very *nice* generalization of the above two IMO problems !

11. *Find all positive integers n such that $3^n - 1$ is divisible by 2^n.*

12. *Find all positive integers n such that $9^n - 1$ is divisible by 7^n.*

13. *Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab.*

14. *Determine all pairs (x, y) of positive integers with $y|x^2+1$ and $x|y^3+1$.*

15. *Determine all pairs (a, b) of positive integers such that ab^2+b+7 divides $a^2b + a + b$.*

16. *Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.*

17. *Find the largest positive integer n such that n is divisible by all the positive integers less than $n^{1/3}$.*

18. *Find all $n \in \mathbf{N}$ such that $3^n - n$ is divisible by 17.*

19. *Suppose that a, b are natural numbers such that*

$$p = \frac{4}{b}\sqrt{\frac{2a-b}{2a+b}}$$

is a prime number. What is the maximum possible value of p?

20. *Find all positive integer N which have the following properties*

- ∘ *N has exactly 16 positive divisors $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = N$,*
- ∘ *The divisor with d_2 is equal to $(d_2 + d_4)d_6$.*

21. *Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2 \cdots, d_{16}$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.*

22. *Suppose that n is a positive integer and let*

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2$$

23. *Let $n \geq 2$ be a positive integer, with divisors*

$$1 = d_1 \leq d_2 \leq \cdots \leq d_k = n\ .$$

Prove that

$$d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$$

is always less than n^2, and determine when it is a divisor of n^2.

24. *Find all positive integers n such that*

(a) n *has exactly* 6 *positive divisors* $1 < d_1 < d_2 < d_3 < d_4 < n$,
(b) $1 + n = 5(d_1 + d_2 + d_3 + d_4)$.

25. *Determine all three-digit numbers N having the property that N is divisible by* 11, *and* $\frac{N}{11}$ *is equal to the sum of the squares of the digits of N.*

26. *When* 4444^{4444} *is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)*

27. *A wobbly number is a positive integer whose digits in base* 10 *are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.*

28. *Determine all pairs of integers (a, b) such that*

$$\frac{a^2}{2a^2b - b^3 + 1}$$

is a positive integer.

5. Congruences

Mathematics is the queen of the sciences and number theory is the queen of mathematics. Johann Carl Friedrich Gauss

1. *The number* 21982145917308330487013369 *is the thirteenth power of a positive integer. Which positive integer?*

2. *Determine all positive integers $n \geq 2$ that satisfy the following condition ; For all integers a, b relatively prime to n,*

$$a \equiv b (mod\ n) \Longleftrightarrow ab \equiv 1 (mod\ n).$$

3. *Determine all positive integers n such that $xy + 1 \equiv 0 (mod\ n)$ implies that $x + y \equiv 0 (mod\ n)$.*

4. *Let p be a prime number. Determine the maximal degree of a polynomial $T(x)$ whose coefficients belong to $\{0, 1, \cdots, p-1\}$ whose degree is less than p, and which satisfies*

$$T(n) = T(m) (mod\ p) \Longrightarrow n = m (mod\ p)$$

for all integers n, m.

5. *Let n be a positive integer. Prove that n is prime if and only if*

$$\binom{n-1}{k} \equiv (-1)^k (mod\ n)$$

for all $k \in \{0, 1, \cdots, n-1\}$.

6. *(Morley) Show that*

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} (mod\ p^3)$$

for all prime numbers p with $p \geq 5$.

7. *Show that there exists a composite number n such that $a^n \equiv a\ (mod\ n)$ for all $a \in \mathbf{Z}$.*

8. *Let p be a prime number of the form $4k+1$. Suppose that $2p+1$ is prime. Show that there is no $k \in \mathbf{N}$ with $k < 2p$ and $2^k \equiv 1\ (mod\ 2p+1)$*

9. *Let n be a positive integer. Show that there are infinitely many primes p such that the smallest positive primitive root of p is greater than n,*

10. *The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?*

11. *During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.*

12. *Let p be an odd prime number. Show that the smallest positive quadratic nonresidue of p is smaller than $\sqrt{p}+1$.*

13. *Show that for each odd prime p, there is an integer g such that $1 < g < p$ and g is a primitive root modulo p^n for every positive integer n.*

6. Primes and Composite Numbers

Wherever there is number, there is beauty. Proclus Diadochus

1. *Show that there are infinitely many primes.*

2. *Show that there exist two consecutive integer squares such that there are at least* 1000 *primes between them.*

3. *Let* a, b, c, d *be integers with* $a > b > c > d > 0$*. Suppose that* $ac + bd = (b + d + a - c)(b + d - a + c)$*. Prove that* $ab + cd$ *is not prime.*

4. *Prove that there is no noncontant polynomial* $f(x)$ *with integral coefficients such that* $f(n)$ *is prime for all* $n \in \mathbf{N}$*.*

5. *A prime* p *has decimal digits* $p_n p_{n-1} \cdots p_0$ *with* $p_n > 1$*. Show that the polynomial* $p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$ *cannot be represented as a product of two nonconstant polynomials with integer coefficients*

6. *Let* $n \geq 2$ *be an integer. Prove that if* $k^2 + k + n$ *is prime for all integers* k *such that* $0 \leq k \leq \sqrt{\frac{n}{3}}$*, then* $k^2 + k + n$ *is prime for all integers* k *such that* $0 \leq k \leq n - 2$*.*

7. *Prove that for any prime* p *in the interval* $\left(n, \frac{4n}{3}\right]$*,* p *divides*

$$\sum_{j=0}^{n} \binom{n}{j}^4$$

8. *Let* a*,* b*, and* n *be positive integers with* $gcd(a, b) = 1$*. Without using the Dirichlet's theorem*[2]*, show that there are infinitely many* $k \in \mathbf{N}$ *such that* $gcd(ak + b, n) = 1$*.*

9. *Without using the Dirichlet's theorem, show that there are infintely many primes ending in the digit* 9*.*

10. *Let* p *be an odd prime. Without using the Dirichlet's theorem, show that there are infintely many primes of the form* $2pk + 1$*.*

11. *Show that, for each* $r \geq 1$*, there are infintely many primes* $p \equiv 1 \pmod{2^r}$*.*

12. *Prove that if* p *is a prime, then* $p^p - 1$ *has a prime factor that is congruent to* 1 *modulo* p*.*

13. *Let* p *be a prime number. Prove that there exists a prime number* q *such that for every integer* n*,* $n^p - p$ *is not divisible by* q*.*

[2] For any $a, b \in \mathbf{N}$ with $gcd(a, b) = 1$, there are infinitely many primes of the form $ak + b$.

14. *Let $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_n$ be the first n prime numbers, where $n \geq 3$. Prove that*

$$\frac{1}{{p_1}^2} + \frac{1}{{p_2}^2} \cdots + \frac{1}{{p_n}^2} + \frac{1}{p_1 p_2 \cdots p_n} < \frac{1}{2}.$$

15. *Let p_n be the nth prime : $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$. Show that the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

16. *Prove that log $n \geq k$ log 2, where n is a natural number and k be the number of distinct primes that divide n.*

17. *Find the smallest prime which is not the differnce (in some order) of a power of* 2 *and a power of* 3.

18. *Find the sum of all distinct positive divisors of the number* 104060401.

19. *Prove that* 1280000401 *is composite.*

20. *Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.*

21. *Find the factor of $2^{33} - 2^{19} - 2^{17} - 1$ that lies between* 1000 *and* 5000.

22. *Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.*

23. *Show that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbf{N}_0$.*

24. *Show that $n^{\pi(2n)-\pi(n)} < 4^n$ for all positive integer n.*

25. *Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after* 1996 *such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes ?*

7. Rational and Irrational Numbers

God made the integers, all else is the work of man. Leopold Kronecker

1. *Find all polynomials W with real coefficients possessing the following property : if $x+y$ is a rational number, then $W(x)+W(y)$ is rational as well.*

2. *Show that any positive rational number can be represented as the sum of three positive rational cubes.*

3. *Prove that every positive rational number can be represented under the form*

$$\frac{a^3+b^3}{c^3+d^3}$$

for some positive integers a, b, c, d.

4. *The set S is a finite subset of $[0,1]$ with the following property : for all $s \in S$, there exist a, $b \in S \bigcup \{0,1\}$ with $a, b \neq x$ such that $x = \frac{a+b}{2}$. Prove that all the numbers in S are rational.*

5. *Let $S = \{x_0, x_1, \cdots, x_n\} \subset [0,1]$ be a finite set of real numbers with $x_0 = 0$ and $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, exept for the distance 1. Prove that all of the x_i are rational.*

6. *Find the smallest positive integer n such that*

$$0 < n^{\frac{1}{4}} - [n^{\frac{1}{4}}] < 0.00001.$$

7. *Prove that for any positive integers a and b*

$$\left|a\sqrt{2}-b\right| > \frac{1}{2(a+b)}.$$

8. *Prove that there exist positive integers m and n such that*

$$\left|\frac{m^2}{n^3}-\sqrt{2001}\right| < \frac{1}{10^8}.$$

9. *Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that*

$$\left|a+b\sqrt{2}+c\sqrt{3}\right| > \frac{1}{10^{21}}.$$

10. *(Hurwitz) Prove that for any irrational number ξ, there are infinitely many rational numbers $\frac{m}{n}$ $((m,n) \in \mathbf{Z} \times \mathbf{N})$ such that*

$$\left|\xi - \frac{n}{m}\right| < \frac{1}{\sqrt{5}m^2}.$$

11. *You are given three lists A, B, and C. List A contains the numbers of the form* 10^k *in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:*

A	*B*	*C*
10	*1010*	*20*
100	*1100100*	*400*
1000	*1111101000*	*13000*
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$*, there is exactly one number in exactly one of the lists B or C that has exactly n digits.*

12. *(Beatty) Prove that if* α *and* β *are positive irrational numbers satisfying* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$*, then the sequences*

$$[\alpha], [2\alpha], [3\alpha], \cdots$$

and

$$[\beta], [2\beta], [3\beta], \cdots$$

together include every positive integer exactly once.

13. *For a positive real number* α*, define*

$$S(\alpha) = \{[n\alpha] | n = 1, 2, 3, \cdots\}.$$

Prove that $\mathbf{N}$ *cannot be expressed as the disjoint union of three sets* $S(\alpha)$*,* $S(\beta)$*, and* $S(\gamma)$*.*

14. *Show that* π *is irrational.*

15. *Show that* $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ *is irrational.*

16. *Show that* $\cos \frac{\pi}{7}$ *is irrational.*

17. *Show that* $\frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{2003}}\right)$ *is irrational.*

18. *Show that* $\cos 1^\circ$ *is irrational.*

19. *Prove that there cannot exist a positive rational number x such that*

$$x^{[x]} = \frac{9}{2}$$

holds. (Note that $[x]$ *is the largest less than, or equal to, x.)*

20. *Let* x, y, z *non-zero real numbers such that* xy*,* yz*,* zx *are rational.*

(a) Show that the number $x^2 + y^2 + z^2$ *is rational.*
(b) If the number $x^3 + y^3 + z^3$ *is also rational, show that x, y, z are rational.*

21. *Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression.*

22. *Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.*

8. Diophantine Equations I

In the margin of his copy of Diophantus' Arithmetica, Pierre de Fermat wrote : *To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.*

1. *Does there exist a solution to the equation*

$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$

in integers x, y, z, u, v greater than 1998?

2. *Find all pairs (x, y) of positive rational numbers such that $x^2 + 3y^2 = 1$.*

3. *Find all pairs (x, y) of rational numbers such that $y^2 = x^3 - 3x + 2$.*

4. *Show that there are infinitely many pairs (x, y) of rational numbers such that $x^3 + y^3 = 9$.*

5. *Show that the equation*

$$a^2 = b^3 + b^2 + b + 1$$

has infinitely many integral solutions.

6. *Determine all pairs (x, y) of positive integers satisfying the equation*

$$(x + y)^2 - 2(xy)^2 = 1.$$

7. *Show that the equation*

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.

8. *Determine with proof all those integers a for which the equation*

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct integer solutions x, y.

9. *Prove that there are unique positive integers a and n such that*

$$a^{n+1} - (a + 1)^n = 2001.$$

10. *Find all $(x, y, n) \in \mathbf{N}^3$ such that $gcd(x, n + 1) = 1 and$ $x^n + 1 = y^{n+1}$.*

11. *Find all $(x, y, z) \in \mathbf{N}^3$ such that $x^4 - y^4 = z^2$.*

12. *Find all pairs (x, y) of positive integers that satisfy the equation*

$$y^2 = x^3 + 16.$$

13. *Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y.*

14. *Prove that there are no integers x, y satisfying $x^2 = y^5 - 4$.*

15. *The polynomial $W(x) = x^4 - 3x^3 + 5x^2 - 9x$ is given. Determine all pairs of different integers a and b satisfying the equation $W(a) = W(b)$.*

16. *Find all positive integers n for which the equation*

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

17. *Determine all positive integer solutions (x, y, z, t) of the equation*

$$(x + y)(y + z)(z + x) = xyzt$$

for which $gcd(x, y) = gcd(y, z) = gcd(z, x) = 1$.

18. *Find all $(x, y, z, n) \in \mathbf{N}^4$ such that $x^3 + y^3 + z^3 = nx^2y^2z^2$.*

19. *Determine all positive integers n for which the equation*

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

has an integer as a solution.

20. *Prove that the equation*

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

21. *Find all integers (a, b, c, x, y, z) such that*

$$a + b + c = xyz, x + y + z = abc, a \geq b \geq c \geq 1, x \geq y \geq z \geq 1.$$

22. *Find all $(x, y, z) \in \mathbf{N}^3$ such that $x^3 + y^3 + z^3 = x + y + z = 3$.*

23. *Prove that if n is a positive integer such that the equation*

$$x^3 - 3xy^2 + y^3 = n.$$

has a solution in integers (x, y), then it has at least three such solutions.
Show that the equation has no solutions in integers when $n = 2891$.

24. *What is the smallest positive integer t such that there exist integers $x_1, x_2, \cdots, x_t$ with*

$$x_1{}^3 + x_2{}^3 + \cdots + x_t{}^3 = 2002^{2002} \quad ?$$

25. *Solve in integers the following equation*

$$n^{2002} = m(m + n)(m + 2n) \cdots (m + 2001n).$$

26. *Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.*

27. *Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a and b. Prove that there exist infinitely many triples (x, y, z) of distinct positive integers x, y, z such that*

$$x^a + y^b = z^c.$$

28. *Find all pairs of integers (x, y) satisfying the equality*

$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0$$

9. Diophantine Equations II

The positive integers stand there, a continual and inevitable challenge to the curiousity of every healthy mind. Godfrey Harold Hardy

1. *(Erdös) Show that the equation* $\binom{n}{k} = m^l$ *has no integral solution with* $l \geq 2$ *and* $4 \leq k \leq n-4$.

2. *Find all positive integers* x, y *such that* $7^x - 3^y = 4$.

3. *Show that* $|12^m - 5^n| \geq 7$ *for all* $m, n \in \mathbf{N}$.

4. *Show that there is no positive integer* k *for which the equation*

$$(n-1)! + 1 = n^k$$

is true when n *is greater than* 5.

5. *Determine all integers* a *and* b *such that*

$$(19a+b)^{18} + (a+b)^{18} + (19b+a)^{18}$$

is a positive square.

6. *Let* b *be a positive integer. Determine all* 200*-tuple integers of non-negative integers* $(a_1, a_2, \cdots, a_{2002})$ *satisfying*

$$\sum_{j=1}^{n} {a_j}^{a_j} = 2002b^b.$$

7. *Is there a positive integers* m *such that the equation*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infintely many solutions in positive integers a, b, c *?*

8. *Consider the system*

$$x + y = z + u$$
$$2xy = zu$$

Find the greatest value of the real constant m *such that* $m \leq \frac{x}{y}$ *for any positive integer solution* (x, y, z, u) *of the system, with* $x \geq y$.

9. *Determine all positive rational number* $r \neq 1$ *such that* $r^{\frac{1}{r-1}}$ *is rational.*

10. *Show that the equation* $\{x^3\} + \{y^3\} = \{z^3\}$ *has infinitely many rational non-integer solutions.*

11. *Let* n *be a positive integer. Prove that the equation*

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

12. *Find all pairs* (x, y) *of positive rational numbers such that* $x^y = y^x$

13. *Find all pairs* (a, b) *of positive integers that satisfy the equation*

$$a^{b^2} = b^a.$$

14. *Find all pairs* (a, b) *of positive integers that satisfy the equation*

$$a^{a^a} = b^b.$$

15. *Let* x, a, b *be positive integers such that* $x^{a+b} = a^b b$. *Prove that* $a = x$ *and* $b = x^x$.

16. *Find all pairs* (m, n) *of integers that satisfy the equation*

$$(m-n)^2 = \frac{4mn}{m+n-1}$$

17. *Find all pairwise relatively prime positive integers* l, m, n *such that*

$$(l+m+n)\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)$$

is an integer.

18. *Let* x, y, z *be integers with* $z > 1$. *Show that*

$$(x+1)^2 + (x+2)^2 + \cdots + (x+99)^2 \neq y^z.$$

10. Functions in Number Theory

Gauss once said "Mathematics is the queen of the sciences and number theory is the queen of mathematics." If this be true we may add that the Disauistiones is the Magna Charta of number theory. M. Cantor

1. *Let $\sigma(n)$ denote the sum of the positive divisors of the positive integer n. and $\phi(n)$ the Euler phi-function. Show that $\phi(n)+\sigma(n) \geq 2n$ for all positive integers n.*

2. *Let n be an integer with $n \geq 2$. Show that $\phi(2^n - 1)$ is divisible by n.*

3. *Show that if the equation $\phi(x) = n$ has one solution it always has a second solution, n being given and x being the unknown.*

4. *Find the total number of different integer values the function*

$$f(x) = [x] + [2x] + \left[\frac{5x}{3}\right] + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

5. *Show that $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}]$ for all positive integer n.*

6. *Let $d(n)$ denote the number of positive divisors of the natural number n. Prove that $d(n^2+1)^2$ does not become monotonic from any given point onwards.*

7. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers n such that $n = d(n)^2$.*

8. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers k such that*

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbf{N}$.

9. *Show that for all positive integers m and n,*

$$gcd(m,n) = m + n - mn + 2\sum_{k=0}^{m-1} \left[\frac{kn}{m}\right].$$

10. *Show that for all primes p,*

$$\sum_{k=1}^{p-1} \left[\frac{k^3}{p}\right] = \frac{(p+1)(p-1)(p-2)}{4}$$

11. *Let p be a prime number of the form $4k+1$. Show that*

$$\sum_{k=1}^{p-1}\left(\left[\frac{2k^2}{p}\right]-2\left[\frac{k^2}{p}\right]\right)=\frac{p-1}{2}$$

12. *Let p be a prime number of the form $4k+1$. Show that*

$$\sum_{i=1}^{k}\left[\sqrt{ip}\right]=\frac{p^2-1}{12}$$

13. *Let a, b, n be positive integers with $gcd(a,b)=1$. Prove that*

$$\sum_{k}\left\{\frac{ak+b}{n}\right\}=\frac{n-1}{2},$$

where k runs through a complete system of residues modulo m.

14. *The function $\mu : \mathbf{N} \longrightarrow \mathbf{C}$ is defined by*

$$\mu(n)=\sum_{k\in R_n}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right),$$

where $R_n=\{k\in\mathbf{N}|1\le k\le n, gcd(k,n)=1\}$. Show that for all positive integer n, $\mu(n)$ is an integer.

15. *(Gauss) Show that for all $n \in \mathbf{N}$,*

$$n=\sum_{d|n}\phi(d).$$

16. *Let m, n be positive integers. Prove that, for some positive integer a, each of $\phi(a)$, $\phi(a+1)$, $\cdots$, $\phi(a+n)$ is a multiple of m.*

17. *For a positive integer n, let $d(n)$ be the number of all positive divisord of n. Find all positive integers n such that $d(n)^3=4n$.*

11. Sequences of Integers

A peculiarity of the higher arithmetic is the great difficulty which has often been experienced in proving simple general theorems which had been suggested quite naturally by numerical evidence. Harold Davenport

1. *Let m be a positive integer. Define the sequence $\{a_n\}_{n\geq 0}$ by*

$$a_0 = 0,\ a_1 = m,\ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

2. *Let $P(x)$ be a nonzero polynomial with integral coefficients. Let $a_0 = 0$ and for $i \geq 0$ define $a_{i+1} = P(a_i)$. Show that $gcd(a_m, a_n) = a_{gcd(m,n)}$ for all $m, n \in \mathbf{N}$*

3. *An integer sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_0 = 0,\ a_1 = 1,\ a_{n+2} = 2a_{n+1} + a_n$$

Show that 2^k divides a_n if and only if 2^k divides n.

4. *An integer sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 1,\ a_{n+1} = a_n + [\sqrt{a_n}]$$

Show that a_n is a square if and only if $n = 2^k + k - 2$ for some $k \in \mathbf{N}$.

5. *Let $f(n) = n + [\sqrt{n}]$. Prove that, for every positive integer m, the sequence*

$$m, f(m), f(f(m)), f(f(f(m))), \cdots$$

contains at least one square of an integer.

6. *An integer sequence $\{a_n\}_{n\geq 1}$ is given such that*

$$2^n = \sum_{d|n} a_d$$

for all $n \in \mathbf{N}$. Show that a_n is divisible by n.

7. *If $a_0, a_1, \cdots a_{n-1}$ are integers, show that*

$$\prod_{0\leq i<j\leq n-1} \frac{a_i - a_j}{i - j}$$

is an integer.

8. *Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that the product $(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$ is divisible by the product $c_1 c_2 \cdots c_n$.*

9. *Show that for all prime numbers p*

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

10. *The sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 1,\ a_2 = 2,\ a_3 = 24,\ a_{n+2} = \frac{6{a_{n-1}}^2 a_{n-3} - 8a_{n-1}{a_{n-2}}^2}{a_{n-2}a_{n-3}} (n \geq 4)$$

Show that for all n, a_n is an integer.

11. *Show that there is a unique sequence of integers $\{a_n\}_{n\geq 1}$ with*

$$a_1 = 1,\ a_2 = 2,\ a_4 = 12,\ a_{n+1}a_{n-1} = {a_n}^2 + 1 \ \ (n \geq 2).$$

12. *The sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 1,\ a_{n+1} = 2a_n + \sqrt{3{a_n}^2 + 1} \ \ (n \geq 1)$$

Show that a_n is an integer for every n.

13. *Prove that the sequence $\{y_n\}_{n\geq 1}$ defined by*

$$y_0 = 1,\ y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5{a_n}^2 - 4}\right) \ \ (n \geq 0)$$

consists only of integers.

14. *(C. von Staudt) The Bernoulli sequence*[3] *$\{B_n\}_{n\geq 0}$ is defined by*

$$B_0 = 1,\ B_n = -\frac{1}{n+1}\sum_{k=0}^{n} \binom{n+1}{k} B_k \ \ (n \geq 1)$$

Show that for all $n \in \mathbf{N}$,

$$(-1)^n B_n - \sum \frac{1}{p},$$

is an integer where the summation being extended over the primes p such that $p|2k-1$.

15. *Let n be a positive integer. Show that*

$$\sum_{i=1}^{n} \tan^2 \frac{i\pi}{2n+1}$$

is an odd integer.

16. *An integer sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 2,\ a_{n+1} = \left[\frac{3}{2}a_n\right]$$

Show that there are infinitely many even and infinitely many odd integers.

[3] $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \cdots$

17. *Prove or disprove that there exists a positive real number* u *such that* $[u^n] - n$ *is an even integer for all positive integer* n.

18. *Let* $\{a_n\}$ *be a strictly increasing positive integers sequece such that* $gcd(a_i, a_j) = 1$ *and* $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. *Show that the infinite series*

$$\sum_{i=1}^{\infty} \frac{1}{a_i}$$

converges.

19. *Let* $\{n_k\}_{k\geq 1}$ *be a sequence of natural numbers such that for* $i < j$, *the decimal representation of* n_i *does not occur as the leftmost digits of the decimal representation of* n_j. *Prove that*

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}.$$

20. *An integer sequence satisfies* $a_{n+1} = {a_n}^3 + 1999$. *Show that it contains at most one square.*

21. *Let* $n > 6$ *be an integer and* $a_1, a_2, \ldots, a_k$ *be all the natural numbers less than* n *and relatively prime to* n. *If*

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n *must be either a prime number or a power of* 2.

22. *Show that if an infinite arithmetic progression of positive integers contains a square and a cube, it must contain a sixth power.*

23. *Let* $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, *and*

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4.$$

Determine $a_{14^{14}}$.

24. *Prove that there exists two strictly increasing sequences* a_n *and* b_n *such that* $a_n(a_n + 1)$ *divides* ${b_n}^2 + 1$ *for every natural* n.

25. *Let* k *be a fixed positive integer. The infinite sequence* a_n *is defined by the formulae*

$$a_1 = k + 1, a_{n+1} = {a_n}^2 - ka_n + k \ \ (n \geq 1).$$

Show that if $m \neq n$, *then the numbers* a_m *and* a_n *are relatively prime.*

26. *The Fibonacci sequence* $\{F_n\}$ *is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $gcd(F_m, F_n) = F_{gcd(m,n)}$ *for all* $m, n \in \mathbf{N}$.

27. *The Fibonacci sequence* $\{F_n\}$ *is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn-1} - {F_{n-1}}^m$ *is divisible by* ${F_n}^2$ *for all* $m \geq 1$ *and* $n > 1$.

28. *The Fibonacci sequence* $\{F_n\}$ *is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn} - {F_{n+1}}^m + {F_{n-1}}^m$ *is divisible by* ${F_n}^3$ *for all* $m \geq 1$ *and* $n > 1$.

29. *Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.*

30. *The sequence* $\{x_n\}$ *is defined by*

$$x_0 \in [0,1],\ x_{n+1} = 1 - |1 - 2x_n|.$$

Prove that the sequence is periodic if and only if x_0 *is irrational.*

31. *Let* x_1 *and* x_2 *be relatively prime positive integers. For* $n \geq 2$, *define* $x_{n+1} = x_n x_{n-1} + 1$.

(a) Prove that for every $i > 1$, *there exists* $j > i$ *such that* ${x_i}^i$ *divides* ${x_j}^j$.

(b) Is it true that x_1 *must divide* ${x_j}^j$ *for some* $j > 1$ *?*

12. Combinatorial Number Theory

In great mathematics there is a very high degree of unexpectedness, combined with inevitability and economy. Godfrey Harold Hardy

1. *Prove that the set of integers of the form $2^k - 3 (k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.*

2. *The set of positive integers is parititioned into finitely many subsets. Show that some subset S has the following property : for every positive integer n, S contains infinitely many multiples of n.*

3. *Let M be a positive integer and consider the set*

$$S = \{n \in \mathbf{N} | M^2 \le n < (M+1)^2\}.$$

Prove that the products of the form ab with $a, b \in S$ are distinct.

4. *Let S be a set of integers such that*

- *there exist $a, b \in S$ with $gcd(a, b) = gcd(a-2, b-2) = 1$.*
- *if x and y are elements of S, then $x^2 - y$ also belongs to S. Prove that S is the set of all integers.*

5. *Show that for each $n \ge 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$*

6. *Let a and b be positive integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, \cdots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i \le k)$.*

7. *Prove that $n \ge 3$ be a prime number and $a_1 < a_2 < \cdots < a_n$ be integers. Prove that $a_1, cdots, a_n$ is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \cdots\}$ into classes $A_1, A_2, \cdots, A_n$ such that*

$$a_1 + A_1 = a_2 + A_2 = \cdots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a | a \in A\}$.

8. *Let n be an integer, and let X be a set of $n+2$ integers each of absolute value at most n. Show that there exist three distinct numbers $a, b, c \in X$ such that $c = a + b$.*

9. *Let $m \ge 2$ be an integer. Find the smallest integer $n > m$ such that for any partition of the set $\{m, m+1, \cdots, n\}$ into two subsets, at least one subset contains three numbers a, b, c such that $c = a^b$.*

10. *Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.*

11. *Let a and b be non-negative integers such that $ab \geq c^2$ where c is an integer. Prove that there is a positive integer n and integers $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ such that*

$$x_1{}^2 + \cdots + x_n{}^2 = a, y_1{}^2 + \cdots + y_n{}^2 = b, x_1y_1 + \cdots + x_ny_n = c$$

12. *Let m and n be positive integers. If $x_1, x_2, \cdots, x_m$ are postive integers whose average is less than $n+1$ and if $y_1, y_2, \cdots, y_n$ are postive integers whose average is less than $m+1$, prove that some sum of one or more x's equals some sum of one or more y's.*

13. *For every natural number n, $Q(n)$ denote the sum of the digits in the decimal representation of n. Prove that there are infinitely many natural number k with $Q(3^k) > Q(3^{k+1})$.*

14. *Let n, k be positive integers such that n is not divisible by 3 and k is greater or equal to n. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in the decimal representation is k.*

15. *Prove that for every real number M there exists an infinite arithmetical progression such that*

- *each term is a positive integer and the common difference is not divisible by 10.*
- *the sum of digits of each term exceeds M.*

16. *Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is colored either blue or white. It is given that*

- *for each $i \in M$, both i and $n - i$ have the same color;*
- *for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color.*

Prove that all numbers in M must have the same color.

17. *Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.*

18. *Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?*

19. *Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different ?*

20. *Find the maximum number of pairwise disjoint sets of the form*

$$S_{a,b} = \{n^2 + an + b | n \in \mathbf{Z}\},$$

with $a, b \in \mathbf{Z}$.

21. *Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \dots\ 2p\}$ are there, the sum of whose elements is divisible by p?*

22. *Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|e_i| < m$ for all i, such that $e_1 a_1 + e_2 a_2 + \cdots + e_n a_n$ is a multiple of m^n.*

23. *Determine the smallest integer $n \geq 4$ for which one can choose four different numbers $a, b, c,$ and d from any n distinct integers such that $a + b - c - d$ is divisible by 20*

24. *A sequence of integers $a_1, a_2, a_3, \cdots$ is defined as follows : $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any $i, j,$ and k in $\{1, 2, 3, \cdots, n+1\}$, not necessarily distinct. Determine a_{1998}.*

25. *Prove that for each positive integer n, there exists a positive integer with the following properties :*

- *It has exactly n digits.*
- *None of the digits is 0.*
- *It is divisible by the sum of its digits.*

26. *Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, \cdots, k\}$ such that no n distinct elements of S add up to m.*

27. *Find the number of subsets of $\{1, 2, \cdots, 2000\}$, the sum of whose elements is divisible by 5.*

28. *Let A be a non-empty set of positive integers. Suppose that there are positive integers $b_1, \cdots, b_n$ and $c_1, \cdots, c_n$ such that*

(i) for each i the set $b_i A + c_i = \{b_i a + c_i | a \in A\}$ is a subset of A, and
(ii) the sets $b_i A + c_i$ and $b_j A + c_j$ are disjoint whenever $i \neq j$.

Prove that
$$\frac{1}{b_1} + \cdots + \frac{1}{b_n} \leq 1.$$

29. *A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the unions of pairwise disjoint historic sets.*

30. *Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2, \cdots\}$ is called ideal if $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2, \cdots\}$.*

31. *Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.*

32. *Let A be a set of N residues (mod N^2). Prove that there exists a set B of N residues (mod N^2) such that the set $A+B=\{a+b|a\in A, b\in B\}$ contains at least half of all the residues (mod N^2).*

33. *Determine the largest positive integer n for which there exists a set S with exactly n numbers such that*

(i) each member in S is a positive integer not exceeding 2002,
(ii) if a and b are two (not necessarily different) numbers in S, then there product ab does not belong to S.

34. *Prove that, for any integer $a_1 > 1$ there exist an increasing sequence of positive integers $a_1, a_2, a_3, \cdots$ such that*

$$a_1 + a_2 + \cdots + a_n | {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$$

for all $k \in \mathbf{N}$.

13. Additive Number Theory

On Ramanujan, G. H. Hardy Said : I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied,

"it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

1. *Show that any integer can be expressed as a sum of two squares and a cube.*

2. *Prove that any positive integer can be represented as an aggregate of different powers of* 3*, the terms in the aggregate being combined by the signs* $+$ *and* $-$ *appropriately chosen.*

3. *For each positive integer* n, $S(n)$ *is defined to be the greatest integer such that, for every positive integer* $k \le S(n)$, n^2 *can be written as the sum of* k *positive squares.*

(a) *Prove that* $S(n) \le n^2 - 14$ *for each* $n \ge 4$.
(b) *Find an integer* n *such that* $S(n) = n^2 - 14$.
(c) *Prove that there are infintely many integers* n *such that* $S(n) = n^2 - 14$.

4. *For each positive integer* n *, let* $f(n)$ *denote the number of ways of representing* n *as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance,* $f(4) = 4$*, because the number* 4 *can be represented in the following four ways:*

$$4; 2+2; 2+1+1; 1+1+1+1.$$

Prove that, for any integer $n \ge 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

5. *The positive function* $p(n)$ *is defined as the number of ways that the positive integer* n *can be written as a sum of positive integers.* [4] *Show that, for all* $n > 1$,

$$2^{[\sqrt{n}]} < p(n) < n^{3[\sqrt{n}]}.$$

[4] For example, $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, and so $p(5) = 7$.

6. *Let $a_1 = 1$, $a_2 = 2$, be the sequence of positive integers of the form $2^\alpha 3^\beta$, where α and β are nonnegative integers. Prove that every positive integer is exprssible in the form*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_n},$$

where no summand is a multiple of any other.

7. *Let n be a non-negative integer. Find the non-negative integers a, b, c, d such that*

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

8. *Find all integers $m > 1$ such that m^3 is a sum of m squares of consecutive integers.*

9. *A positive integer n is a square-free integer if there is no prime p such that $p^2|n$. Show that every integer greater than 1 can be written as a sum of two square-free integers.*

10. *Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.*

11. *(Jacobsthal) Let p be a prime number of the form $4k+1$. Suppose that r is a quadratic residue of p and that s is a quadratic nonresidue of p. Show that $p = a^2 + b^2$, where*

$$a = \frac{1}{2}\sum_{i=1}^{p-1}\left(\frac{i(i^2-r)}{p}\right), b = \frac{1}{2}\sum_{i=1}^{p-1}\left(\frac{i(i^2-s)}{p}\right).$$

Here, $\left(\frac{k}{p}\right)$ denotes the Legendre Symbol.

12. *Let p be a prime with $p \equiv 1 (mod\ 4)$. Let a be the unique integer such that*

$$p = a^2 + b^2, a \equiv -1 (mod\ 4), b \equiv 0 (mod\ 2)$$

Prove that

$$\sum_{i=0}^{p-1}\left(\frac{i^3 + 6i^2 + i}{p}\right) = 2\left(\frac{2}{p}\right)a.$$

13. *Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \le \sqrt{n}$, then p divides ab. Determine all such n.*

14. *If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.*

15. *Let A be the set of positive integers represented by the form $a^2 + 2b^2$, where a, b are integers and $b \neq 0$. Show that p is a prime number and $p^2 \in A$, then $p \in A$.*

16. *Show that an integer can be expressed as the difference of two squares if and only if it is not of the form $4k + 2 (k \in \mathbf{Z})$.*

17. *Show that there are infinitely many positive integers which cannot be expressed as the sum of squares.*

18. *Show that any integer can be expressed as the form* $a^2 + b^2 - c^2$, *where* $a, b, c \in \mathbf{Z}$.

19. *Let* a *and* b *be positive integers with* $\gcd(a, b) = 1$. *Show that every integer greater than* $ab - a - b$ *can be expressed in the form* $ax + by$, *where* $x, y \in \mathbf{N}_0$.

20. *Let* a, b *and* c *be positive integers, no two of which have a common divisor greater than* 1. *Show that* $2abc - ab - bc - ca$ *is the largest integer which cannot be expressed in the form* $xbc + yca + zab$, *where* $x, y, z \in \mathbf{N}_0$

21. *Determine, with proof, the largest number which is the product of positive integers whose sum is* 1976.

22. *(Zeckendorf) Any positive integer can be represented as a sum of Fibonacci numbers, no two of which are consecutive.*

23. *Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.*

24. *Let* $a_1, a_2, a_3, \cdots$ *be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form* $a_i + 2a_j + 4a_l$, *where* i, j, *and* k *are not necessarily distinct. Determine* a_{1998}.

25. *A finite sequence of integers* $a_0, a_1, \cdots, a_n$ *is called quadratic if for each* $i \in \{1, 2, \cdots, n\}$ *we have the equality* $|a_i - a_{i-1}| = i^2$.

(a) Prove that for any two integers b *and* c, *there exists a natural number* n *and a quadratic sequence with* $a_0 = b$ *and* $a_n = c$.

(b) Find the smallest natural number n *for which there exists a quadratic sequence with* $a_0 = 0$ *and* $a_n = 1996$.

14. The Geometry of Numbers

Srinivasa Aiyangar Ramanujan said *"An equation means nothing to me unless it expresses a thought of God."*

1. *Prove no three lattice points in the plane form an equilateral triangle.*

2. *The sides of a polygon with* 1994 *sides are* $a_i = \sqrt{i^2+4}$ $(i = 1, 2, \cdots, 1994)$. *Prove that its vertices are not all on lattice points.*

3. *A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is* $\frac{1}{2}$.

4. *Let* R *be a convex region*[5] *symmetrical about the origin with area greater than* 4. *Then* R *must contain a lattice point*[6] *different form the origin.*

5. *Show that the number* $r(n)$ *of representations of* n *as a sum of two squares has average value* π, *that is*

$$\frac{1}{n}\sum_{m=1}^{n} r(m) \to \pi \text{ as } n \to \infty.$$

[5]For any two points of R, their midpoint also lies in R.
[6]A point with integral coordinates

15. Miscellaneous Problems

Mathematics is not yet ready for such problems. Paul Erdös

1. *Let p be an odd prime. Determine positive integers x and y for which $x \le y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is nonnegative and as small as possible.*

2. *Let $\alpha(n)$ be the number of digits equal to one in the dyadic representation of a positive integer n. Prove that*

(a) the inequality $\alpha(n^2) \le \frac{1}{2}\alpha(n)(1+\alpha(n))$ holds,
(b) the above inequality is equality for infinitely many positive integers, and
(c) there exists a sequence $\{n_i\}$ such that $\frac{\alpha(n_i{}^2)}{\alpha(n_i)} \to 0$ as $i \to \infty$.

3. *Show that if a and b are positive integers, then*

$$\left(a+\frac{1}{2}\right)^n + \left(b+\frac{1}{2}\right)^n$$

is an integer for only finitely many positive integer n.

4. *If x is a real number such that $x^2 - x$ is an integer, and for some $n \ge 3$, $x^n - x$ is also an integer, prove that x is an integer.*

5. *Suppose that x and y are complex numbers such that*

$$\frac{x^n - y^n}{x - y}$$

is an integer for some four consecutive positive integers n. Prove that it is an integer for all positive integers n.

6. *Determine the maximum value of $m^2 + n^2$,where m and n are integers satisfying $m, n \in \{1, 2, ..., 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.*

7. *Denote by S the set of all primes p such that the decimal representation of $\frac{1}{p}$ has the fundamental period of divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write*

$$\frac{1}{p} = 0.a_1a_2\cdots a_{3r}a_1a_2\cdots a_{3r}\cdots,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \ge 1$ define $f(k,p)$ by

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

a) Prove that S is finite.
b) Find the hightst value of $f(k,p)$ for $k \ge 1$ and $p \in S$.

8. *Determine all pairs (a,b) of real numbers such that $a[bn] = b[an]$ for all positive integer n. (Note that $[x]$ denotes the greatest integer less than or equal to x.)*

9. *Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by*

$$a = n^{\frac{1}{3}}, b = \frac{1}{a-[a]}, c = \frac{1}{b-[b]},$$

where $[x]$ denotes the integer part of x. Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra + sb + tc = 0$.

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The only way to learn Mathematics is to do Mathematics. Paul Halmos

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18. Appendix

How Many Problems Are In This Book ?

`Divisibility Theory I` : 46 problems
`Divisibility Theory II` : 28 problems
`Congruences` : 13 problems
`Primes and Composite Numbers` : 25 problems
`Rational and Irrational Numbers` : 22 problems
`Diophantine Equations I` : 28 problems
`Diophantine Equations II` : 18 problems
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