ECE276B-HW3

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1 Q1

1.1 Q1.1

Below is my hand-wrriten solution:

$t = 0$

$$Q^{\pi}(x_0, u_0) = \sin^2(x_0) + u_0^2 + r \cdot Q^{\pi}\left[(x_0+u_0), -\frac{x_0+u_0}{2}\right]$$
$$= \sin^2(x_0) + u_0^2 + r \cdot V^{\pi}(x_0+u_0)$$

$t > 0$

$$Q^{\pi}(x_t, u_t) = x_t^2 + u_t^2 + r\, Q^{\pi}\left(x_t+u_t, -\frac{x_t+u_t}{2}\right)$$
$$= x_t^2 + u_t^2 + r \cdot V^{\pi}(x_t+u_t)$$

$$V^{\pi}(x_t) = l\left(x_t, -\frac{x_t}{2}\right) + r V^{\pi}\left(x_t - \frac{x}{2}\right)$$
$$= x_t^2 + \frac{x_t^2}{4} + r V^{\pi}\left(\frac{x_t}{2}\right) \quad \because t > 0,\ \sin x \approx x$$

$\because V^{\pi}(x_t)$ is a quadratic form

let $V^{\pi}(x_t) = a \cdot x_t^2$

$$\therefore a \cdot x_t^2 = x_t^2 + \frac{x_t^2}{4} + \frac{1}{2} \cdot a \cdot \frac{x_t^2}{4}$$
$$a = 1 + \frac{1}{4} + \frac{1}{2} \cdot a \cdot \frac{1}{4}$$
$$a = \frac{1}{8} a + \frac{5}{4}$$
$$\frac{7}{8} a = \frac{5}{4}$$
$$a = \frac{10}{7}$$

$$\therefore V^{\pi}(x_t) = \frac{10}{7} x_t^2$$

$\therefore$ For $t = 0$

$$Q^{\pi}(x_0, u_0) = \sin^2(x_0) + u_0^2 + \frac{5}{7}(x_0+u_0)^2$$

For $t > 0$

$$Q^{\pi}(x_t, u_t) = \underline{x_t^2 + u_t^2 + \frac{5}{7}(x_t} + u_t)^2$$

Figure 1: Q1.1

1.2 Q1.2

Below is my hand-wrriten solution:

2. For $t=0$

$$Q^\pi(x_0, u_0) = \sin^2(x_0) + u_0^2 + \frac{5}{7}(x_0+u_0)^2$$

$$\frac{dQ^\pi}{du} = 2u + \frac{10}{7}(x+u)$$

$$0 = 2u + \frac{10}{7}x + \frac{10}{7}u$$

$$24u = -10x$$

$$u = -\frac{5}{12}x$$

For $t>0$

$$Q^\pi(x_t, u_t) = x_t^2 + u_t^2 + \frac{5}{7}(x_t+u_t)^2$$

$$\frac{dQ^\pi}{du} = 2u + \frac{10}{7}x + \frac{10}{7}u$$

$$u = -\frac{5}{12}x$$

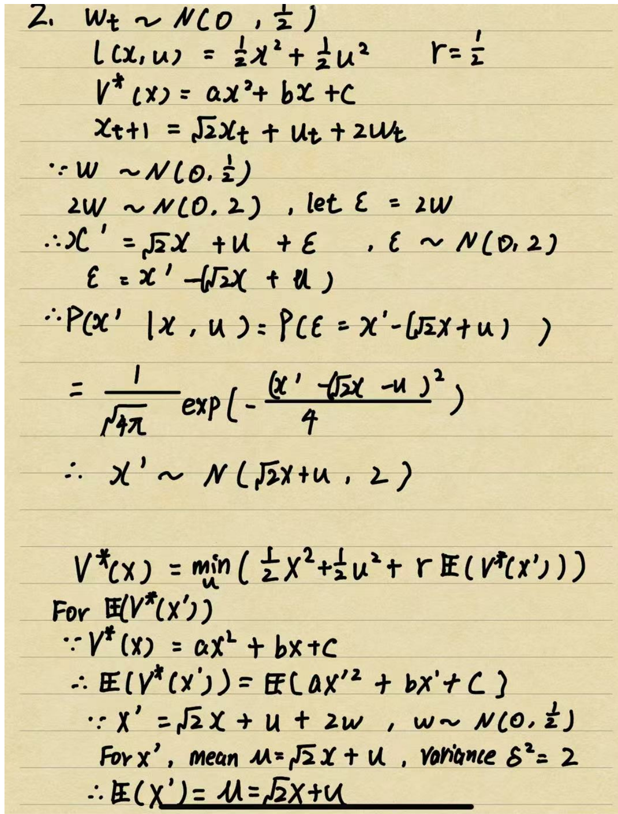
$$\therefore \pi'(x) = -\frac{5}{12}x$$

Figure 2: Q1.2

2 Q2

2.1 Q2.1

Below is my hand-wrriten solution:



2. $w_t \sim N(0, \frac{1}{2})$

$l(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2 \qquad r = \frac{1}{2}$

$V^*(x) = ax^2 + bx + c$

$x_{t+1} = \sqrt{2}x_t + u_t + 2w_t$

$\because w \sim N(0, \frac{1}{2})$

$2w \sim N(0, 2)$, let $\varepsilon = 2w$

$\therefore x' = \sqrt{2}x + u + \varepsilon$, $\varepsilon \sim N(0, 2)$

$\varepsilon = x' - (\sqrt{2}x + u)$

$\therefore P(x' \mid x, u) = P(\varepsilon = x' - (\sqrt{2}x + u))$

$$= \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(x' - \sqrt{2}x - u)^2}{4}\right)$$

$\therefore x' \sim N(\sqrt{2}x + u, 2)$

$$V^*(x) = \min_u \left(\frac{1}{2}x^2 + \frac{1}{2}u^2 + r\,\mathbb{E}(V^*(x'))\right)$$

For $\mathbb{E}(V^*(x'))$

$\because V^*(x) = ax^2 + bx + c$

$\therefore \mathbb{E}(V^*(x')) = \mathbb{E}(ax'^2 + bx' + c)$

$\because x' = \sqrt{2}x + u + 2w$, $w \sim N(0, \frac{1}{2})$

For x', mean $\mu = \sqrt{2}x + u$, variance $\delta^2 = 2$

$\therefore \mathbb{E}(x') = \underline{\mu = \sqrt{2}x + u}$

Figure 3: Q2.1

$$\mathbb{E}(X'^2) = \delta^2 + \mathbb{E}(X')^2 = 2 + u^2$$

$$= 2 + (\sqrt{2}X + u)^2$$

$$\therefore \mathbb{E}(V^*(X')) = a \cdot [2 + (\sqrt{2}X + u)^2] + b \cdot (\sqrt{2}X + u) + C$$

$$= a \cdot (2 + 2X^2 + 2\sqrt{2}Xu + u^2) + \sqrt{2}bX + bu + C$$

$$= 2a + 2aX^2 + 2\sqrt{2}aXu + au^2 + \sqrt{2}bX + bu + C$$

$$\therefore r \cdot \mathbb{E}(V^*(X')) = a + aX^2 + \sqrt{2}aXu + \frac{1}{2}au^2 + \frac{\sqrt{2}}{2}bX + \frac{1}{2}bu + \frac{1}{2}C$$

$$\therefore V^*(X) = \min_u \left[\frac{1}{2}X^2 + \frac{1}{2}u^2 + a + aX^2 + \sqrt{2}aXu + \frac{1}{2}au^2 + \frac{\sqrt{2}}{2}bX + \frac{1}{2}bu + \frac{1}{2}C \right]$$

$$\frac{dV^*}{du} = u + \sqrt{2}aX + au + \frac{1}{2}b = 0$$

$$(1+a)u = -\sqrt{2}aX - \frac{1}{2}b$$

$$u = \frac{-\sqrt{2}aX - \frac{1}{2}b}{1+a} = \frac{-\sqrt{2}a}{1+a}X + \frac{-\frac{1}{2}b}{1+a}$$

Figure 4: Q2.1

$$\therefore V^*(x) = \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{\sqrt{2}ax+\frac{1}{2}b}{1+a}\right)^2 + a + ax^2 + \sqrt{2}ax\cdot\left(-\frac{\sqrt{2}ax+\frac{1}{2}b}{1+a}\right) + \frac{1}{2}a\cdot\left(-\frac{\sqrt{2}ax+\frac{1}{2}b}{1+a}\right)^2 + \frac{\sqrt{2}}{2}bx + \frac{1}{2}b\left(\frac{-\sqrt{2}ax-\frac{1}{2}b}{1+a}\right) + \frac{1}{2}c$$

Then I use sympy to simplify the equation After getting coefficients of x^2, x and c, we have following equations:

$$a = \frac{3a}{2a+2} + \frac{1}{2a+2}$$

$$b = \frac{\sqrt{2}b}{2a+2} \qquad c = \frac{a^2}{a+1} + \frac{ac}{2a+2} + \frac{a}{a+1} + \frac{b^2}{8a+8} + \frac{c}{2a+2}$$

Lastly I use Sympy to solve once agin, have following answers

$a=1$ $\quad$ $a=-\frac{1}{2}$

$b=0$ $\quad$ $b=0$

$c=2$ $\quad$ $c=-1$

$\because$ we should have positive expected cost

$\therefore a=1$, $b=0$, $c=2$

$V^*(x) = x^2+2$

Figure 5: Q2.1

2.2 Q2.2

Below is my hand-wrriten solution:

$$2.\ Q^*(x,u) = l(x,u) + r\,\mathbb{E}[\min_{u'} Q^*(x',u')]$$

$$\because V^*(x') = \min_{u'} Q^*(x',u')$$

$$\therefore Q^*(x,u) = l(x,u) + r\,\mathbb{E}[V^*(x')]$$

$$\therefore Q^*(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2 + a + ax^2 + \sqrt{2}axu + \frac{1}{2}au^2 + \frac{\sqrt{2}}{2}bx + \frac{1}{2}bu + \frac{1}{2}c$$

$$\because a=1,\ b=0,\ c=2$$

$$\therefore Q^*(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2 + 1 + x^2 + \sqrt{2}xu + \frac{1}{2}u^2 + 1$$

$$= \frac{3}{2}x^2 + \sqrt{2}xu + u^2 + 2$$

Figure 6: Q2.2

3 Q3

3.1 Q3.1

In the given Discounted problem, we have state space $\mathcal{X} = \{1, 2\}$, control space $\mathcal{U} = \{a, b\}$. From the given transition matrix P^a and P^b, we define the transition model as follow:

$$p_f(j|i, a) = P^a_{i,j}$$

$$p_f(j|i, b) = P^b_{i,j}$$

where i, j entries in P^a specifies the transition probability from state i to j under action a; i, j entries in P^b specifies the transition probability from state i to j under action b.

Firstly, we need to add virtual terminal state to the state space, $\mathcal{X} = \mathcal{X} \cup \{\tau\}$ and $\mathcal{T} = \{\tau\}$. Therefore, the state space is $\mathcal{X} = \{1, 2, \tau\}$ We will use the same control space. Then, for the motion model, we will use $1 - \gamma$ as the probability of terminate. Therefore, we have

$$p_f(j|i, a) = \gamma p_f(j|i, a) = \gamma P^a_{i,j}, \quad p_f(j|i, b) = \gamma p_f(j|i, b) = \gamma P^b_{i,j}, \quad \text{for j} \neq \tau$$

$$p_f(j|i, a) = p_f(j|i, b) = 1 - \gamma, \quad \text{for j} = \tau$$

$$p_f(j|\tau, a) = p_f(j|\tau, b) = 0, \quad \text{for j} \neq \tau$$

$$p_f(\tau|\tau, a) = p_f(\tau|\tau, b) = 1$$

Put together, we have following transition matrices (we consider $\tau = 3$):

$$\mathbf{P}^a := \begin{bmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^b := \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

For the stage cost, we have

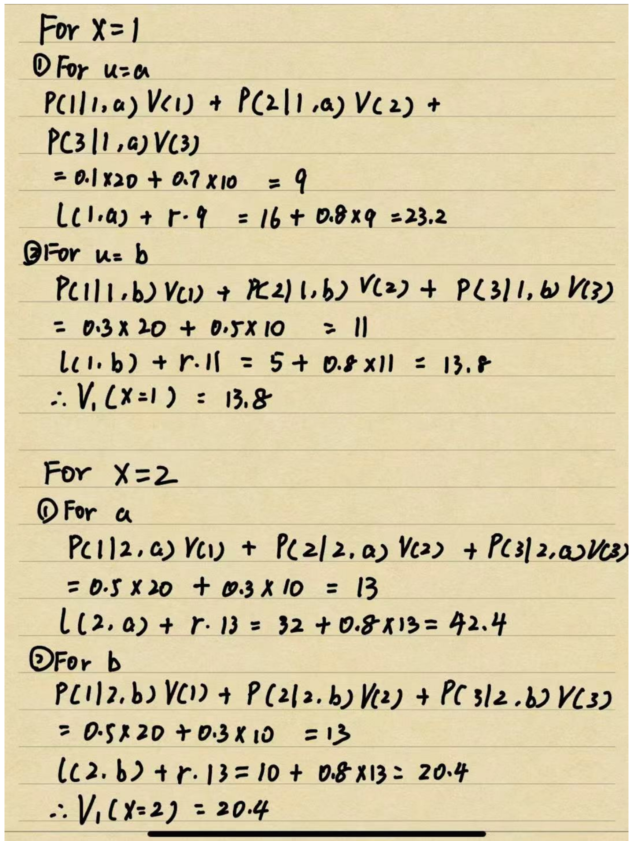
$$\ell(x, u) := \begin{cases} 16x & \text{if } u = a \text{ and } x \neq \tau \\ 5x & \text{if } u = b \text{ and } x \neq \tau. \\ 0 & \text{if } x = \tau \end{cases}$$

Lastly, for the terminal cost, we have $q(x_\tau) = 0$. The problem terminates at $T := inf\{t \geq 0 | x_t = 3\}$. Our objective is

$$\begin{aligned} V^*(\mathbf{x}) = \min_\pi \; & V^\pi(\mathbf{x}) := \mathbb{E}\left[q(x_\tau) + \sum_{t=0}^{T-1} \ell(x_t, \pi(x_t)) \,\middle|\, x_0 = \mathbf{x} \right] \\ \text{s.t.} \quad & x_{t+1} \sim p_f(\cdot \mid x_t, \pi(x_t)), \\ & x_t \in \mathcal{X}, \quad \pi(x_t) \in \mathcal{U} \end{aligned}$$

3.2 Q3.2

Below is my hand-written solution:



For $X=1$
① For $u=a$
$P(1|1,a)V(1) + P(2|1,a)V(2) +$
$P(3|1,a)V(3)$
$= 0.1\times20 + 0.7\times10 = 9$
$L(1,a) + r\cdot 9 = 16 + 0.8\times9 = 23.2$
② For $u=b$
$P(1|1,b)V(1) + P(2|1,b)V(2) + P(3|1,b)V(3)$
$= 0.3\times20 + 0.5\times10 = 11$
$L(1,b) + r\cdot 11 = 5 + 0.8\times11 = 13.8$
$\therefore V_1(X=1) = 13.8$

For $X=2$
① For a
$P(1|2,a)V(1) + P(2|2,a)V(2) + P(3|2,a)V(3)$
$= 0.5\times20 + 0.3\times10 = 13$
$L(2,a) + r\cdot 13 = 32 + 0.8\times13 = 42.4$
② For b
$P(1|2,b)V(1) + P(2|2,b)V(2) + P(3|2,b)V(3)$
$= 0.5\times20 + 0.3\times10 = 13$
$L(2,b) + r\cdot 13 = 10 + 0.8\times13 = 20.4$
$\therefore V_1(X=2) = 20.4$

Figure 7: Q3.2

3.3 Q3.3

Let ℓ^a denotes the stage cost if $u = a$ and ℓ^b denotes the stage cost if $u = b$, and we have

$$\ell^a = [16 \times 1, 16 \times 2]^T = [16, 32]^T$$

$$\ell^b = [5 \times 1, 5 \times 2]^T = [5, 10]^T$$

since $x \in \{1, 2\}$. Then, based on the lecture slides, we can formulate the given discounted problem as LP problem as follow:

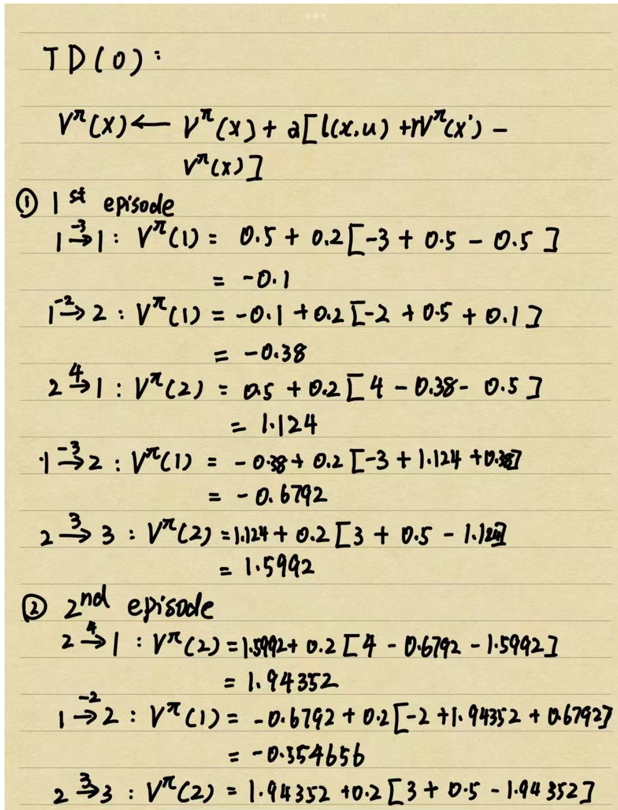
$$\max_V w^T V$$
$$\text{s.t.} \quad (I - \gamma P^u)V \leq \ell^u, \forall u \in \mathcal{U}$$

where P^u is the 2×2 transition matrix, $V \in R^2$ is the value vector, $w \in R^2$ is the weight vector and $w_i > 0 \quad \forall i = \{1, 2\}$.

In the implementation, I initialize w as random vector where each entry is uniformly sampled from $(0, 1]$. The result I have is $V^* = [35.4166667439.58333341]$, and $V^*(1) = 35.41666674$, $V^*(2) = 39.58333341$

4 Q4

Below is my hand-written process:



TD(0):

$V^\pi(x) \leftarrow V^\pi(x) + \alpha[l(x,u) + \gamma V^\pi(x') - V^\pi(x)]$

① 1st episode

$1 \xrightarrow{-3} 1 : V^\pi(1) = 0.5 + 0.2[-3 + 0.5 - 0.5]$
$= -0.1$

$1 \xrightarrow{-2} 2 : V^\pi(1) = -0.1 + 0.2[-2 + 0.5 + 0.1]$
$= -0.38$

$2 \xrightarrow{4} 1 : V^\pi(2) = 0.5 + 0.2[4 - 0.38 - 0.5]$
$= 1.124$

$1 \xrightarrow{-3} 2 : V^\pi(1) = -0.38 + 0.2[-3 + 1.124 + 0.38]$
$= -0.6792$

$2 \xrightarrow{3} 3 : V^\pi(2) = 1.124 + 0.2[3 + 0.5 - 1.124]$
$= 1.5992$

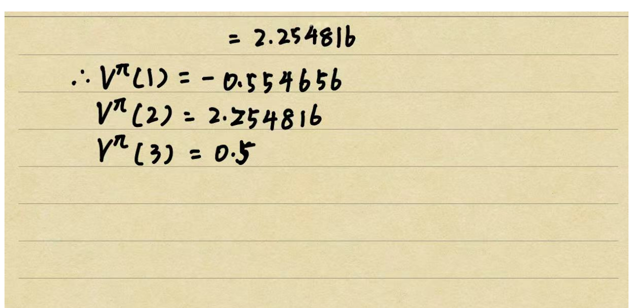
② 2nd episode

$2 \xrightarrow{4} 1 : V^\pi(2) = 1.5992 + 0.2[4 - 0.6792 - 1.5992]$
$= 1.94352$

$1 \xrightarrow{-2} 2 : V^\pi(1) = -0.6792 + 0.2[-2 + 1.94352 + 0.6792]$
$= -0.354656$

$2 \xrightarrow{3} 3 : V^\pi(2) = 1.94352 + 0.2[3 + 0.5 - 1.94352]$

Figure 8: Q4



$= 2.254816$

$\therefore V^\pi(1) = -0.554656$

$V^\pi(2) = 2.254816$

$V^\pi(3) = 0.5$

Figure 9: Q4

The result I have is $V(1) = -0554656$, $V(2) = 2.254816$, $V(3) = 0.5$

References