Mueller Matrix Probability Math

Joshua Johnson

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1 Introduction

Mueller matrices are transformations that turn Stokes' vectors into other Stokes' vectors. Given a deterministic system, we can find the appropriate Mueller matrix and know exactly how any inputted Stokes' vector will transform. However, when transmitting light through the atmosphere, there are random effects that change the polarization of light, so we do not know a priori which Mueller matrix to use. To model this situation, we can associate a probability to obtaining a particular Mueller matrix, allowing us to determine the probability of obtaining a particular output Stokes' vector from a given input Stokes' vector.

2 Constructing the matrix function

We desire a matrix-valued function

$$M : [0,1] \to \mathcal{M}_{4\times 4}(\mathbb{R}), \quad t \mapsto M_t,$$

which sends the real parameter t to a real-valued 4×4 matrix M_t. We want this function to satisfy a few properties:

- M_0 should be the identity matrix, corresponding to the situation where the polarization of incident light is unaltered.
- M_1 should correspond to a bit-flip, i.e. reversing the handedness of circular polarization.
- $M_{1/2}$ should be the Mueller matrix from the single pm fiber experiment (still waiting on this).
- All other values should also be valid physical Mueller matrices.

With these conditions in mind, we define

$$M_t = \begin{cases} M_0 + 2(M_{1/2} - M_0)t, & t \in [0, 1/2] \\ M_1 + 2(M_{1/2} - M_1)(1-t), & t \in [1/2, 1]. \end{cases} \tag{1}$$

In words, this function takes the prescribed values at $t = 0, 1/2, 1$ and linearly interpolates in between, giving us a continuous path in matrix space. To see that all M_t are actually physical Mueller matrices, observe that a matrix M is a physical Mueller matrix if and only if the covariance matrix $H(M)$ is Hermitian and positive semidefinite. The map $M \mapsto H(M)$ is linear, and the cone of positive semidefinite matrices is convex, so convex combinations of physical Mueller matrices remain physical.

3 Adding probability

Now, we introduce some probability on the parameter t. We sample t from a Poisson distribution. Note that this distribution is discrete, so the path defined in (1) is no longer continuous, but as the number of counts in the Poisson distribution increases, we will still have a close approximation to a continuous curve. Recall that for a Poisson distribution with mean λ, the probability of observing k events is

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (k \in \mathbb{Z}_{\geq 0}).$$

In principle, there is a nonzero probability for observing any arbitrarily large number of events k, but beyond some point, the probability is so low that we don't want to consider those values in our model. Let p^* be this cutoff probability, and let N be the integer that minimizes $|p^* - P(N)|$. Then we can restrict our range of events to $k \in [0, N]$.

It remains to map $[0, N]$ onto $[0, 1]$ in an appropriate way. By this, we mean that we want 0 to map to 0, the most likely number of events — the integer k^* which minimizes $|k^* - \lambda|$ — to map to 1/2, and N to map to 1. This is simply accomplished by

$$\phi : [0, N] \to [0, 1], \quad k \mapsto \begin{cases} \frac{k}{2k^*}, & k \in [0, k^*] \\ \frac{x-N}{2(N-k^*)} + 1, & k \in [k^*, N] \end{cases}$$

We can now say, "with probability $P(k)$, a Stokes' vector S will be transformed into $S' = M_{\phi(k)} S$."