Problem set 3: Summarizing distributions

Due October 21, 2024, at 10am

(Your name here)

Note: Start with the file `ps3_2024_summarizing_distributions.qmd` *(available from the github repository at https://github.com/UChicago-pol-methods/IntroQSS-F24/tree/main/assignments). Modify that file to include your answers. Make sure you can "render" the file (e.g. in RStudio by clicking on the* `Render` *button). Submit both the qmd file and the PDF via Canvas.*

Question 1: Expected value

Consider the random variable X characterized by the following PMF.

x	$f(x)$
0	.3
1	.5
2	.15
3	.05

(1a) Compute $\mathrm{E}[X]$. Show your work.

Answer:

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_x x f(x) \\
&= 0 \times .3 + 1 \times .5 + 2 \times .15 + 3 \times .05 \\
&= .95
\end{aligned}
$$

(1b) Write an `R` function to compute the expectation of any discrete random variable. The arguments to your function should include the values the random variable can take on (`x`) and the probability it takes on each value (`probs`). Use your function to confirm your answer from question (1a).

Answer:

```
exp_func <- function(x, probs){
  sum(x * probs)
}
x_vals <- c(0, 1, 2, 3)
x_probs <- c(.3, .5, .15, .05)
exp_func(x = x_vals, probs = x_probs)
```

```
[1] 0.95
```

(1c) Compute the MSE ($\mathrm{E}\left[(X-c)^2\right]$) for $c=0$ and $c=1$. Show your work.

Answer:

$$\begin{aligned}
\mathrm{E}\left[(X-0)^2\right] &= \sum_x (x-0)^2 f(x) \\
&= (0-0)^2 \times .3 + (1-0)^2 \times .5 + (2-0)^2 \times .15 + (3-0)^2 \times .05 \\
&= 0 + .5 + .6 + .45 \\
&= 1.55
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[(X-1)^2\right] &= \sum_x (x-1)^2 f(x) \\
&= (0-1)^2 \times .3 + (1-1)^2 \times .5 + (2-1)^2 \times .15 + (3-1)^2 \times .05 \\
&= .3 + 0 + .15 + .2 \\
&= .65
\end{aligned}$$

(1d) Write a function to compute the MSE for any discrete random variable at a value `c`. The arguments to your function should include the values the random variable can take on (`x`), the probability it takes on each value (`probs`), and the value `c` being considered. Use your function to confirm your answers from (1c).

Answer:

```
mse_func <- function(x, probs, c){
  sum(probs*(x - c)^2)
}
mse_func(x = x_vals, probs = x_probs, c = 0)
```

```
[1] 1.55
```

```r
mse_func(x = x_vals, probs = x_probs, c = 1)
```

```
[1] 0.65
```

(1e) Create a vector `cs` that contains numbers in the sequence `0.5, 0.55, 0.6, 0.65, 0.7, . . ., 2`. Using a for-loop and your function from (1d), compute the MSE for the random variable X whose PMF was given above at each value of `cs` and store the result in a vector called `mses`.

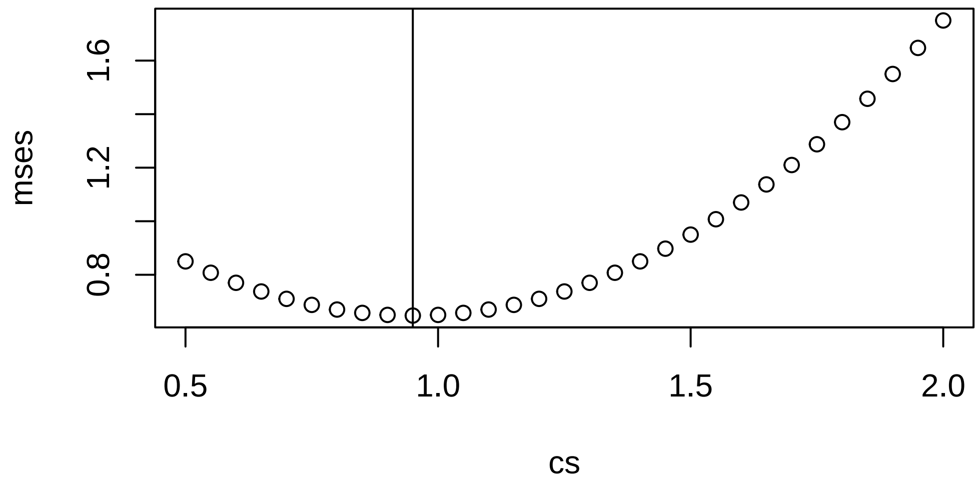
Answer:

```r
cs <- seq(from = .5, to = 2, by = .05)
mses <- rep(NA, times = length(cs))
for(i in 1:length(mses)){
  mses[i] <- mse_func(x = x_vals, probs = x_probs, c = cs[i])
}
```

(1f) Using the `plot()` command, make a scatterplot showing the MSE of X (vertical axis) at each value of $c \in \{0.5, 0.55, 0.6, \ldots, 2\}$ (horizontal axis). Use `abline()` to add a vertical line at $E[X]$.

Answer:

```r
plot(cs, mses)
abline(v = .95)
```



Question 2: Variance

Consider the random variable X characterized by the following PMF:

x	$f(x)$
1	.2
2	.7
3	.1

(2a) Confirm that the variance of X is the same whether we compute it by the formula in Definition 2.1.12 or the Alternative Formula in Theorem 2.1.13. (We want the two variance computations, not the proof.)

Answer: First we calculate $\mathrm{E}[X]$:

$$\begin{aligned}\mathrm{E}[X] &= \sum_x x f(x) \\ &= 1 \times .2 + 2 \times .7 + 3 \times .1 \\ &= 1.9\end{aligned}$$

Using Definition 2.1.12:

$$\begin{aligned}\mathrm{V}[X] &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\ &= \mathrm{E}\left[(X - 1.9)^2\right] \\ &= (-.9)^2 \times .2 + .1^2 \times .7 + 1.1^2 \times .1 \\ &= .29\end{aligned}$$

To use Theorem 2.1.13, we calculate $\mathrm{E}[X^2]$:

$$\begin{aligned}\mathrm{E}[X^2] &= \sum_x x^2 f(x) \\ &= 1 \times .2 + 4 \times .7 + 9 \times .1 \\ &= 3.9\end{aligned}$$

Using Theorem 2.1.13:

$$\begin{aligned}\mathrm{V}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 \\ &= 3.9 - 3.61 \\ &= .29\end{aligned}$$

(2b) Write an `R` function to compute the variance of any discrete random variable. The arguments to your function should include the values the random variable can take on (`x`) and the probability it takes on each value (`probs`). Use your function to confirm your answer from question (2a).

Answer:

```r
# two versions, both of which use the expectation function I wrote above

# this version uses the E[(X - E[X])^2] formulation
var_func2 <- function(x, probs){
  ex <- exp_func(x, probs)
  exp_func((x - ex)^2, probs)
}

# this version uses the E[X^2] - E[X]^2 formulation
var_func <- function(x, probs){
  exp_func(x^2, probs) - exp_func(x, probs)^2
}

# confirming equality
var_func(x = c(1, 2, 3), probs = c(.2, .7, .1))
```

```
[1] 0.29
```

```r
var_func2(x = c(1, 2, 3), probs = c(.2, .7, .1))
```

```
[1] 0.29
```

(2c) Explain what definition/property/mathematical operation is being used in each step of the following proof.

For random variable X and $a \in \mathbb{R}$, $\mathrm{V}[aX] = a^2\mathrm{V}[X]$,

$$
\begin{aligned}
\mathrm{V}[aX] &= \mathrm{E}\left[(aX - \mathrm{E}[aX])^2\right] && \text{(Step 1)} \\
&= \mathrm{E}\left[(aX - a\mathrm{E}[X])^2\right] && \text{(Step 2)} \\
&= \mathrm{E}\left[a^2(X - \mathrm{E}[X])^2\right] && \text{(Step 3)} \\
&= a^2\mathrm{E}\left[(X - \mathrm{E}[X])^2\right] && \text{(Step 4)} \\
&= a^2\mathrm{V}[X] && \text{(Step 5)}
\end{aligned}
$$

Answer:

In Step 1, . . .

In Step 2, . . .

In Step 3, . . .

In Step 4, . . .

In Step 5, . . .

Answer:

In Step 1, we use the **definition of variance** applied to the random variable aX.

In Step 2, we move a constant outside of the expectations operator using **the property $E[aX] = aE[X]$ that is stated in Theorem 2.1.6**, and that also follows from Linearity of Expectations. This property follows from the fact that (with discrete X) an expectation is a (weighted) sum, and in this case each element of that sum is multiplied by a, so it can be brought outside the summation.

In Step 3, we do a mathematical operation to **move a outside of the parentheses**. In more detail, we first express $aX - a\mathrm{E}[X]$ as $a(X - \mathrm{E}[X])$, then we apply the square to both of them, i.e. $(a(X - \mathrm{E}[X]))^2 = a^2(X - \mathrm{E}[X])^2$.

In Step 4, again move a constant outside of the expectations operator using **the property $E[aX] = aE[X]$** (see Step 2).

In Step 5, we use the **definition of variance**.

(2d) Use your variance function to confirm that $\mathrm{V}[aX] = a^2\mathrm{V}[X] \forall a \in \mathbb{R}$. (You can just show this for one value of a.)

Answer

```
a <- 5
var_func(x = a*c(0, 1, 2), probs = c(.6, .1, .3))
```

```
[1] 20.25
```

```
var_func(x = c(0, 1, 2), probs = c(.6, .1, .3))*a^2
```

```
[1] 20.25
```

(2e) Use your function to confirm that $V[X + c] = V[X] \, \forall \, c \in \mathbb{R}$. (You can just show this for one value of c.)

Answer

```r
c <- 6
var_func(x = c(0, 1, 2), probs = c(.6, .1, .3))
```

```
[1] 0.81
```

```r
var_func(x = c(0, 1, 2) + 6, probs = c(.6, .1, .3))
```

```
[1] 0.81
```