Last week: solutions to systems of lin eq
matrix form
matrix multiplication

this week:
- inverses
- dot product
- when can a matrix be inverted?

INVERSES

- Remember from last week, we introduced the identity matrix I_n:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{aligned} AI &= A \\ IA &= A \end{aligned}$$

essentially, I is a matrix that does <u>nothing</u> to A

we also have another concept called the <u>inverse</u>

$$A^{-1}A = I$$
$$AA^{-1} = I$$

(this is similar to thinking of A^{-1} as acting as a divisor on A: for a # $A=5$, $A^{-1} = \frac{1}{5}$ — the # that turns it back into identity matrix)

from last week:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$
$$\vec{y} = X\vec{\beta}$$

we really want to know what values of β make this system work - now, need in the form $\vec{\beta} = Z\vec{y}$

this $\mathbf{Z}$ is the inverse of $\mathbf{X}$!

$$\vec{y} = \mathbf{X}\vec{\beta}$$

$$\mathbf{Z}\vec{y} = \underbrace{\mathbf{Z}\mathbf{X}}_{= I \text{ by inverse..}}\vec{\beta}$$

(remember, CANNOT add $\mathbf{Z}$ to either side of $\mathbf{X}\vec{\beta}$ - needs to go on the same "side" as it did to $\vec{y}$)

$$\mathbf{Z}\vec{y} = \mathbf{I}\vec{\beta}$$

$$\mathbf{Z}\vec{y} = \vec{\beta}$$

Theorem: Inverse for A, a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$A^{-1} = \frac{1}{\underbrace{ad - bc}_{\text{called the } \boxed{\text{determinant}}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

note: not all matrices are invertible - when $\det(A) = 0$!

$\boxed{\text{ex!}}$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\det(A) = -1$$

Practice

ex 1

$$4 = \beta_0 + 2\beta_1$$
$$2 = \beta_0 + 3\beta_1$$
$$y = X \beta_1 \qquad X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$X^{-1} = \frac{1}{1\cdot 3 - 1\cdot 2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad X^{-1} y = \begin{bmatrix} 3\cdot 4 - 2\cdot 2 \\ -4\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\boxed{\text{ex 2}}$ $0 = \beta_0 + 2\beta_1$, $X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$

$-1 = 4\beta_1$

$$X^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/4 \end{bmatrix}$$

$$X^{-1}Y = \begin{bmatrix} 1/2 \\ -1/4 \end{bmatrix}$$

$\boxed{\text{ex 3}}$ $12 = \beta_0 + 2\beta_1$, $X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$6 = \beta_0 + 1\beta_1$

$$X^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \quad X^{-1}Y = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

→ when does inverse not exist?

sps we have

$89 = \beta_0 + 2\beta_1$

$178 = 2\beta_0 + 4\beta_1$

$$\begin{bmatrix} 89 \\ 178 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

the solution to this system is

$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 89 - 2t \\ t \end{bmatrix}$ for <u>any</u> t - so no inverse

$\boxed{\text{RANK}}$

def: the <u>rank</u> of a $k \times r$ matrix is the number of linearly <u>independent</u> columns.

- def a square ($k \times k$) matrix is non singular if rank $= k$. also called a matrix w/ full rank

- non-singular matrixes have an inverse!

from above ex:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$$

notice that $\vec{a}_2 = 2\vec{a}_1$!

(this is linear dependence)

so not nonsingular! and not invertible

DOT PRODUCT

→ quant measure of similarity

→ if $\vec{a}, \vec{b}$ $k \times 1$:

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$$

then the dot product is

$$a^T b = a_1 b_1 + a_2 b_2 + \dots + a_k b_k = \sum_k a_j b_j$$

matrix multiplication gives a series of dot products!

for A, B a 3×3 matrix,

$$A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}, \quad B = \begin{bmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{bmatrix}$$

$$A^T = \begin{bmatrix} \vec{a}_1^{\,T} \\ a_2^T \\ a_3^T \end{bmatrix}$$

$$AB = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & a_1^T b_3 \\ a_2^T b_1 & a_2^T b_2 & a_2^T b_3 \\ a_3^T b_1 & a_3^T b_2 & a_3^T b_3 \end{bmatrix}$$

does this look familiar?

$$\text{cov}(x,y) = \frac{1}{n-1} \sum\left[(x_i - \bar{x})(y_i - \bar{y})\right] = \frac{1}{n-1} \sum x_i y_i$$

you will metimes see something called a variance covariance matrix:

$$A = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A \cdot A = \begin{bmatrix} \text{var}(x) & \text{cov}(x,y) \\ \text{cov}(x,y) & \text{var}(y) \end{bmatrix}$$

note: we call 2 vectors orthogonal if $\vec{a}^T \vec{b} = 0$
(more on orthogonality next week!)

MATRIX INVERSE LAWS

* $(A^{-1})^T = (A^T)^{-1}$

$$(AC)^{-1} = C^{-1}A^{-1}$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

also have these 2 identities:

① $A^{-1} + C^{-1} = A^{-1}(A+C)C^{-1}$

$$(A+C)^{-1} = A^{-1}(A^{-1}+C^{-1})^{-1}C^{-1}$$

Lets prove ①:

$$X^{-1} + Y^{-1} = X^{-1}YY^{-1} + X^{-1}XY^{-1}$$

↳ multiply by $\underbrace{YY^{-1}}_{=I}$ and $\underbrace{X^{-1}X}_{=I}$

$$= X^{-1}(YY^{-1} + XY^{-1})$$

↳ distributive law (pull out X^{-1})

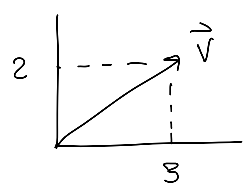
$$= X^{-1}(Y + X)Y^{-1}$$

↳ distributive law (pull out Y^{-1})

$\boxed{\text{BRIEF}}$ introduction to vectors (if time!)

what is a vector?

$\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$, but can also be thought of as an arrow from the origin to a point in space.



$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(this is $\mathbb{R}^2$)

Length of a vector: $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

(dot product!)

$\|\vec{v}\| = \sqrt{3^2 + 2^2}$

↳ pythagorean theorem

matrices and vectors live in <u>vector spaces</u>

(next week!)