PLSC 30600

Week 2: Identification, ignorability, and propensity scores

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Recap MCAR: Missing Completely at Random

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  - $\Pr[R_i = 1] > 0$ (nonzero probability of response).
- Note: implicitly, MCAR is with respect to *all* data, observed and unobserved. So, $R_i \perp (Y_i, X_i, \ldots Z_i, \ldots)$.

Missingness wrt the full data

Definition (Full data, observed data, and response indicator)

Let U_i denote the *full-data* random vector for unit i, and let $R_i \in \{0, 1\}$ indicate whether the outcome component Y_i of U_i is observed. Write $U_i = (Y_i, X_i)$ where X_i collects all covariates measured (or conceptually present) for unit i. We observe

$$Y_i^* = Y_i R_i + (-99)(1 - R_i).$$

MCAR: Missing Completely at Random

Definition (Missing Completely at Random (MCAR))

Let $U_i = (Y_i, X_i)$ be the full-data vector and let $R_i \in \{0, 1\}$ be the response indicator. We say Y_i is *missing completely at random* if the following conditions hold:

1. $R_i \perp\!\!\!\perp U_i$ (missingness is independent of the full data),
2. $\Pr(R_i = 1) > 0$ (nonzero probability of response).

Equivalently, $R_i \perp\!\!\!\perp (Y_i, X_i)$ and $\Pr(R_i = 1) > 0$.

MAR: Missing at Random

Definition (Missing at Random (MAR))

Let Y_i and R_i be random variables and let $R_i \in \{0, 1\}$ be the response indicator. We observe X_i for all units (even when $R_i = 0$). We say Y_i is *missing at random conditional on* X_i if the following conditions hold:

1. $R_i \perp\!\!\!\perp Y_i \mid X_i$ (conditional independence of missingness and outcome),
2. $\exists \varepsilon > 0$ such that $\Pr(R_i = 1 \mid X_i) > \varepsilon$ a.s. (positivity).

Proposition (MCAR implies MAR)

If $R_i \perp\!\!\!\perp (Y_i, X_i)$, then $R_i \perp\!\!\!\perp Y_i \mid X_i$. Therefore, MCAR implies MAR (with the same positivity condition).

Recap: missing data → potential outcomes

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- Identification requires assumptions about the assignment mechanism.

Recap: Manski and Nagin (1998) Utah juvenile court data

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- How should judges sentence convicted juvenile offenders?
- Juvenile offenders in Utah may be assigned to residential or non-residential treatment programs.

  Treatment (sentencing): $D = 1$ residential, $D = 0$ non-residential

  Outcome (recidivism): $Y = 1$ recidivates, $Y = 0$ does not recidivate

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- Aside: Does this independence need to hold for the full data?

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- Under this assumption, $\mathrm{E}\,[\tau_i]$ is point identified.
- Aside: Does this independence need to hold for the full data?
- Implicitly: $R_i \perp\!\!\!\perp U_i^{pre}$ (missingness is independent of the full pre-intervention data)

ATE under random assignment

- If D_i is randomly assigned, then:

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- If D_i is randomly assigned, then:

$$\mathrm{E}\big[Y_i(1)\big] = \mathrm{E}\big[Y_i \mid D_i = 1\big], \quad \mathrm{E}\big[Y_i(0)\big] = \mathrm{E}\big[Y_i \mid D_i = 0\big].$$

$$\mathrm{E}\left[\tau_i\right] = \mathrm{E}\big[Y_i \mid D_i = 1\big] - \mathrm{E}\big[Y_i \mid D_i = 0\big].$$

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$$\widehat{\mathrm{E}}[\tau_i] = \frac{\sum_{i=1}^n Y_i D_i}{\sum_{i=1}^n D_i} - \frac{\sum_{i=1}^n Y_i(1-D_i)}{\sum_{i=1}^n (1-D_i)}.$$

Difference-in-means estimator (matrix form)

- Let

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}, \quad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

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- Let $n_1 = \mathbb{1}^\top \boldsymbol{D}$ and $n_0 = \mathbb{1}^\top(\mathbb{1} - \boldsymbol{D})$.

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- Let $n_1 = \mathbb{1}^\top \boldsymbol{D}$ and $n_0 = \mathbb{1}^\top(\mathbb{1} - \boldsymbol{D})$. Mechanically,

$$n_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix} = \sum_{i=1}^{n} 1 D_i,$$

$$n_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 - D_1 \\ 1 - D_2 \\ \vdots \\ 1 - D_n \end{bmatrix} = \sum_{i=1}^{n} 1(1 - D_i).$$

Difference-in-means estimator (matrix form)

$$\widehat{\mathrm{E}}\left[\tau_i\right] = \left(\frac{\boldsymbol{D}}{n_1} - \frac{\mathbb{1} - \boldsymbol{D}}{n_0}\right)^\top \boldsymbol{Y}$$

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$$\widehat{\mathrm{E}}[\tau_i] = \left[\frac{D_1}{n_1} - \frac{1-D_1}{n_0}, \frac{D_2}{n_1} - \frac{1-D_2}{n_0}, \cdots, \frac{D_n}{n_1} - \frac{1-D_n}{n_0}\right] \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

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$$= \sum_{i=1}^n \left(\frac{D_i}{n_1} - \frac{1-D_i}{n_0}\right) Y_i = \frac{1}{n_1}\sum_{i=1}^n D_i Y_i - \frac{1}{n_0}\sum_{i=1}^n (1-D_i) Y_i$$

$$= \frac{\sum_{i=1}^n Y_i D_i}{\sum_{i=1}^n D_i} - \frac{\sum_{i=1}^n Y_i (1-D_i)}{\sum_{i=1}^n (1-D_i)}.$$

Utah example: random assignment intuition

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- Treat D_i as randomly assigned to residential vs. non-residential.
- Then treated and control groups have the same potential outcome distributions.
- Sample means within $D = 1$ and $D = 0$ identify $\mathrm{E}[Y_i(1)]$ and $\mathrm{E}[Y_i(0)]$.

Science table: random assignment

What would the difference in recidivism rates be if all juveniles were assigned to residential instead of non-residential treatment?

Target: $\mathrm{E}[Y_i(1)] - \mathrm{E}[Y_i(0)]$

i	$Y_i(0)$	$Y_i(1)$	D_i	Y_i
1	0	?	0	0
2	?	1	1	1
3	1	?	0	1
4	?	1	1	1
5	0	?	0	0
6	?	1	1	1
7	1	?	0	1
8	?	0	1	0
9	1	?	0	0
10	?	1	1	1

$$\widehat{\mathrm{E}}[Y_i(1)] = \frac{1+1+1+0+1}{5} = 0.8, \quad \widehat{\mathrm{E}}[Y_i(0)] = \frac{0+1+0+1+1}{5} = 0.6$$

$$\widehat{\mathrm{E}}[\tau_i] = 0.8 - 0.6 = 0.2$$

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  - $\exists \varepsilon > 0$ such that $\varepsilon < \Pr[D_i = 1 \mid X_i] < 1 - \varepsilon$ (positivity).
- Random assignment implies strong ignorability (with pre-treatment X_i). Strong ignorability does *not* imply random assignment.

ATE under strong ignorability

- If D_i is strongly ignorable conditional on X_i, then:

ATE under strong ignorability

- If D_i is strongly ignorable conditional on X_i, then:

$$\begin{aligned}\mathrm{E}[\tau_i] = \sum_x \mathrm{E}\big[Y_i \mid D_i = 1, X_i = x\big] \Pr[X_i = x] - \\ \sum_x \mathrm{E}\big[Y_i \mid D_i = 0, X_i = x\big] \Pr[X_i = x].\end{aligned}$$

Conditional ATE under strong ignorability

- For any $x \in \mathrm{Supp}\,[X_i]$:

Conditional ATE under strong ignorability

- For any $x \in \mathrm{Supp}\,[X_i]$:

$$\mathrm{E}\,[\tau_i \mid X_i] = \mathrm{E}\big[Y_i \mid D_i = 1, X_i\big] - \mathrm{E}\big[Y_i \mid D_i = 0, X_i\big].$$

Post-stratification plug-in estimator under MAR

- Under MAR, $Y \perp R \mid X$, so:
$$\mathrm{E}[Y] = \sum_{x \in \mathrm{Supp}[X]} \mathrm{E}[Y \mid R = 1, X = x] \Pr[X = x].$$

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- Sample plug-in estimator:
$$\widehat{\mathrm{E}}[Y] = \sum_{x \in \mathrm{Supp}[X]} \left(\frac{\sum_{i=1}^n Y_i^* \mathbb{1}(R_i = 1)\mathbb{1}(X_i = x)}{\sum_{i=1}^n \mathbb{1}(R_i = 1)\mathbb{1}(X_i = x)} \right) \left(\frac{\sum_{i=1}^n \mathbb{1}(X_i = x)}{n} \right).$$

Science table: ignorability (conditioning on X)

i	X_i	$Y_i(0)$	$Y_i(1)$	D_i	Y_i
1	A	0	?	0	0
2	A	?	1	1	1
3	B	1	?	0	1
4	B	?	1	1	1
5	A	0	?	0	0
6	A	?	1	1	1
7	B	1	?	0	1
8	B	?	0	1	0
9	A	1	?	0	0
10	B	?	1	1	1

$$\widehat{\mathrm{E}}[Y_i \mid D_i = 1, X_i = \mathsf{A}] = 1, \quad \widehat{\mathrm{E}}[Y_i \mid D_i = 0, X_i = \mathsf{A}] = 1/3$$

$$\widehat{\mathrm{E}}[Y_i \mid D_i = 1, X_i = \mathsf{B}] = 2/3, \quad \widehat{\mathrm{E}}[Y_i \mid D_i = 0, X_i = \mathsf{B}] = 1$$

$$\widehat{\mathrm{E}}[Y_i(1)] = \frac{5}{10} \cdot 1 + \frac{5}{10} \cdot \frac{2}{3} = 0.833, \quad \widehat{\mathrm{E}}[Y_i(0)] = \frac{5}{10} \cdot \frac{1}{3} + \frac{5}{10} \cdot 1 = 0.667$$

$$\widehat{\mathrm{E}}[\tau_i] = 0.833 - 0.667 = 0.167$$

Response propensity function

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- The response propensity function is:
$$p_R(x) = \Pr[R_i = 1 \mid X_i = x], \quad \forall x \in \mathrm{Supp}[X_i].$$
- The random variable $p_R(X_i)$ is the (response) propensity score for unit i.

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- It is the conditional probability of response given covariates.

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  - $R_i \perp X_i \mid p_R(X_i)$ (balance conditional on $p_R(X_i)$).

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    - $Y_i \perp R_i \mid p_R(X_i)$ (independence of outcome and response conditional on $p_R(X_i)$).

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  - $Y_i \perp R_i \mid p_R(X_i)$ (independence of outcome and response conditional on $p_R(X_i)$).
  - $\exists \varepsilon > 0$ such that $\varepsilon < \Pr[R_i = 1 \mid p_R(X_i)] < 1 - \varepsilon$.

Science table: MAR and the propensity score

i	$X_{[1]i}$	$X_{[2]i}$	$Y_i(0)$	R_i	$Y_i^*(0)$
1	A	0	0	1	0
2	A	0	?	0	-99
3	B	0	1	1	1
4	B	0	?	0	-99
5	A	1	0	1	0
6	A	1	?	0	-99
7	B	1	1	1	1
8	B	1	?	0	-99
9	A	0	1	1	1
10	B	1	?	0	-99

Science table: add the (empirical) propensity score

i	$X_{[1]i}$	$X_{[2]i}$	$\widehat{p}_R(X_i)$	$Y_i(0)$	R_i	$Y_i^*(0)$
1	A	0	0.67	0	1	0
2	A	0	0.67	?	0	-99
3	B	0	0.50	1	1	1
4	B	0	0.50	?	0	-99
5	A	1	0.50	0	1	0
6	A	1	0.50	?	0	-99
7	B	1	0.33	1	1	1
8	B	1	0.33	?	0	-99
9	A	0	0.67	1	1	1
10	B	1	0.33	?	0	-99

$$\widehat{p}_R(X_i) = \frac{\sum_{i=1}^{n} R_i \mathbb{1}(X_{[1]i}, X_{[2]i})}{\sum_{i=1}^{n} \mathbb{1}(X_{[1]i}, X_{[2]i})}$$

$X_{[1]}$	$X_{[2]}$	n	$\widehat{p}_R$
A	0	3	0.67
A	1	2	0.50
B	0	2	0.50
B	1	3	0.33

Estimation with the propensity score

$$\widehat{\mathrm{E}}[Y_i(0)] = \frac{3}{10}\widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.67] + \frac{4}{10}\widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.50] + \frac{3}{10}\widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.33]$$

$$\widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.67] = \frac{1}{2}, \quad \widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.50] = \frac{1}{2},$$

$$\widehat{\mathrm{E}}[Y_i \mid R_i = 1, \widehat{p}_R(X_i) = 0.33] = 1$$

$$= \frac{3}{10} \cdot \frac{1}{2} + \frac{4}{10} \cdot \frac{1}{2} + \frac{3}{10} \cdot 1 = 0.65$$

Treatment propensity function

- Let $Y_i(0)$, $Y_i(1)$, and D_i be random variables with $\mathrm{Supp}\,[D_i] = \{0, 1\}$.

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- Let $Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$.

- The treatment propensity function is:
$$p_D(x) = \Pr[D_i = 1 \mid X_i = x], \quad \forall x \in \text{Supp}[X_i].$$

- The random variable $p_D(X_i)$ is the (treatment) propensity score for unit i.

Strong ignorability and the propensity score

- Let $Y_i(0)$, $Y_i(1)$, and D_i be random variables with $\mathrm{Supp}\,[D_i] = \{0, 1\}$.

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  - $(Y_i(0), Y_i(1)) \perp D_i \mid p_D(X_i)$ (conditional independence).

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- If D_i is strongly ignorable conditional on X_i, then:
  - $D_i \perp X_i \mid p_D(X_i)$ (balance conditional on $p_D(X_i)$).
  - $(Y_i(0), Y_i(1)) \perp D_i \mid p_D(X_i)$ (conditional independence).
  - $\exists \varepsilon > 0$ such that $\varepsilon < \Pr[D_i = 1 \mid p_D(X_i)] < 1 - \varepsilon$.

Science table: strong ignorability and the propensity score

i	$X_{[1]i}$	$X_{[2]i}$	$Y_i(0)$	$Y_i(1)$	D_i	Y_i
1	A	0	0	?	0	0
2	A	0	?	1	1	1
3	B	0	1	?	0	1
4	B	0	?	1	1	1
5	A	1	0	?	0	0
6	A	1	?	1	1	1
7	B	1	1	?	0	1
8	B	1	?	0	1	0
9	A	0	1	?	0	0
10	B	1	?	1	1	1

Science table: add the (empirical) propensity score

i	$X_{[1]i}$	$X_{[2]i}$	$p_D(X_i)$	$Y_i(0)$	$Y_i(1)$	D_i	Y_i
1	A	0	0.33	0	?	0	0
2	A	0	0.33	?	1	1	1
3	B	0	0.50	1	?	0	1
4	B	0	0.50	?	1	1	1
5	A	1	0.50	0	?	0	0
6	A	1	0.50	?	1	1	1
7	B	1	0.67	1	?	0	1
8	B	1	0.67	?	0	1	0
9	A	0	0.33	1	?	0	0
10	B	1	0.67	?	1	1	1

$X_{[1]}$	$X_{[2]}$	n	$p_D(X)$
A	0	3	0.33
A	1	2	0.50
B	0	2	0.50
B	1	3	0.67

Estimation with the treatment propensity score

$$
\begin{aligned}
\widehat{\mathrm{E}}\left[Y_i(1)\right] = {} & \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.33\right] + \\
& \frac{4}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.50\right] + \\
& \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.67\right]
\end{aligned}
$$

$$
= \frac{3}{10} \cdot 1 + \frac{2}{10} \cdot 1 + \frac{2}{10} \cdot 1 + \frac{3}{10} \cdot \frac{1}{2} = 0.85
$$

Estimation with the treatment propensity score

$$\widehat{\mathrm{E}}\left[Y_i(1)\right] = \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.33\right] + \frac{4}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.50\right] + \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.67\right]$$

$$= \frac{3}{10} \cdot 1 + \frac{2}{10} \cdot 1 + \frac{2}{10} \cdot 1 + \frac{3}{10} \cdot \frac{1}{2} = 0.85$$

$$\widehat{\mathrm{E}}\left[Y_i(0)\right] = \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 0, \widehat{p}_D(X_i) = 0.33\right] + \frac{4}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 0, \widehat{p}_D(X_i) = 0.50\right] + \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 0, \widehat{p}_D(X_i) = 0.67\right]$$

$$= \frac{3}{10} \cdot 0 + \frac{2}{10} \cdot 0 + \frac{2}{10} \cdot 1 + \frac{3}{10} \cdot 1 = 0.5$$

Estimation with the treatment propensity score

$$\widehat{\mathrm{E}}\left[Y_i(1)\right] = \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.33\right] + \frac{4}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.50\right] + \frac{3}{10}\widehat{\mathrm{E}}\left[Y_i \mid D_i = 1, \widehat{p}_D(X_i) = 0.67\right]$$

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$$= \frac{3}{10} \cdot 0 + \frac{2}{10} \cdot 0 + \frac{2}{10} \cdot 1 + \frac{3}{10} \cdot 1 = 0.5$$

$$\widehat{\mathrm{E}}\left[\tau_i\right] = 0.85 - 0.5 = 0.35$$

Post-treatment variables

- Conditioning on post-treatment variables can break ignorability.

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- DAG intuition: conditioning on descendants can open biasing paths.
- Preview: we will formalize this with DAGs in Weeks 3–4.

Post-treatment variables: why independence fails

- Example: $D \to M \to Y$ with M a post-treatment mediator.

Post-treatment variables: why independence fails

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- Conditioning on M forces treated/control units to have the same mediator value, which mixes causal pathways and changes the estimand.

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- Example: $D \to M \to Y$ with M a post-treatment mediator.
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- Conditioning on M forces treated/control units to have the same mediator value, which mixes causal pathways and changes the estimand.

$$\Pr[M = 1 \mid D = 1] \neq \Pr[M = 1 \mid D = 0] \quad \Rightarrow \quad D \not\perp M.$$

Generalized potential outcomes model

- Allow multi-valued treatments: $D_i \in \mathrm{Supp}\,[D_i]$.

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Generalized potential outcomes model

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- Generalized potential outcomes: $Y_i(d)$ for each $d \in \mathrm{Supp}\left[D_i\right]$.
- Switching equation: $Y_i = \sum_{d \in \mathrm{Supp}\left[D_i\right]} Y_i(d)\, \mathbb{1}(D_i = d)$.
- Preview: generalized propensity scores summarize $\Pr[D_i = d \mid X_i]$.

Average marginal causal effect

- If $Y_i(d)$ is differentiable with respect to d, the **average marginal causal effect** is:

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$$\text{AMCE} = \mathrm{E}\left[\frac{\partial \mathrm{E}\left[Y_i(D_i) \mid D_i, X_i\right]}{\partial D_i}\right].$$

- Under conditional independence, the CEF is causal, so:

Average marginal causal effect

- If $Y_i(d)$ is differentiable with respect to d, the **average marginal causal effect** is:
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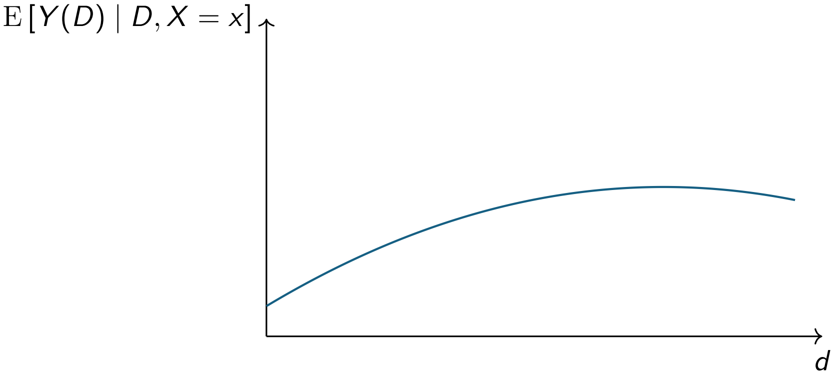
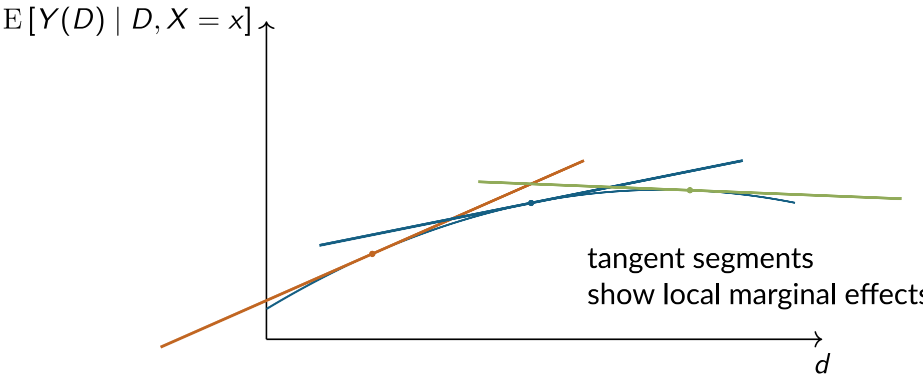
AMCE: local marginal effects

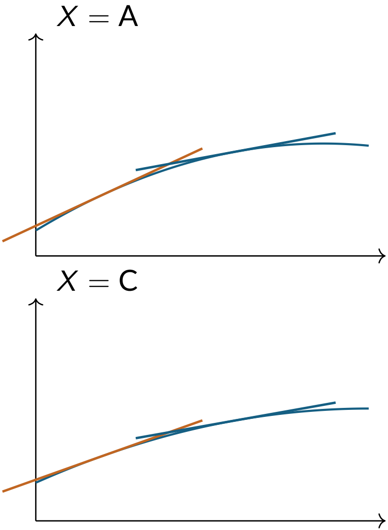


AMCE: local marginal effects



AMCE heterogeneity across covariates

Local slopes vary by X



Practice: AMCE with a non-uniform treatment

- Let $D \sim \mathrm{Beta}(2, 2)$ on $(0, 1)$ and $X \sim \mathrm{Bernoulli}(p)$, independent.

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$$m(d, x) \;=\; \alpha \;+\; \beta d \;+\; \gamma d^2 \;+\; \eta x d.$$

- Compute the average marginal causal effect:

$$\mathrm{AMCE} \;\equiv\; \mathrm{E}\left[\frac{\partial}{\partial d} m(D, X)\right].$$

AMCE

Step 1: Differentiate $m(d, x)$ with respect to d.

$$m(d, x) = \alpha + \beta d + \gamma d^2 + \eta x d.$$

Differentiate term-by-term:

$$\frac{\partial}{\partial d} m(d, x) = 0 + \beta + 2\gamma d + \eta x = \beta + 2\gamma d + \eta x.$$

AMCE

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Step 2: Use linearity of expectation.

$$\mathrm{AMCE} = \mathrm{E}\left[\beta + 2\gamma D + \eta X\right] = \beta + 2\gamma \mathrm{E}\left[D\right] + \eta \mathrm{E}\left[X\right].$$

AMCE

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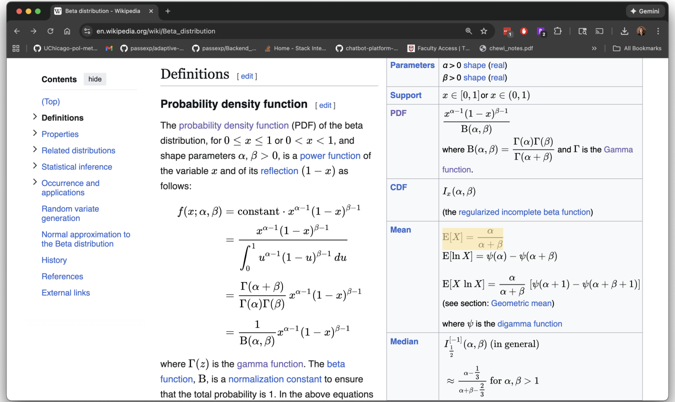
Since $X \sim \mathrm{Bernoulli}(p)$,

$$\mathrm{E}\left[X\right] = p.$$

Step 3: Compute the needed moment for $D \sim \mathrm{Beta}(2, 2)$.



Beta distribution - Wikipedia

en.wikipedia.org/wiki/Beta_distribution

Contents hide

Definitions [edit]

Probability density function [edit]

The probability density function (PDF) of the beta distribution, for $0 \le x \le 1$ or $0 < x < 1$, and shape parameters $\alpha, \beta > 0$, is a power function of the variable x and of its reflection $(1-x)$ as follows:

$$\begin{aligned} f(x;\alpha,\beta) &= \mathrm{constant}\cdot x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} \end{aligned}$$

where $\Gamma(z)$ is the gamma function. The beta function, B, is a normalization constant to ensure that the total probability is 1. In the above equations

Parameters	$\alpha > 0$ shape (real) $\beta > 0$ shape (real)
Support	$x \in [0,1]$ or $x \in (0,1)$
PDF	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$ where $\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and Γ is the Gamma function.
CDF	$I_x(\alpha,\beta)$ (the regularized incomplete beta function)
Mean	$\mathrm{E}[X] = \frac{\alpha}{\alpha+\beta}$ $\mathrm{E}[\ln X] = \psi(\alpha) - \psi(\alpha+\beta)$ $\mathrm{E}[X\ln X] = \frac{\alpha}{\alpha+\beta}[\psi(\alpha+1) - \psi(\alpha+\beta+1)]$ (see section: Geometric mean) where ψ is the digamma function
Median	$I^{[-1]}_{\frac{1}{2}}(\alpha,\beta)$ (in general) $\approx \frac{\alpha-\frac{1}{3}}{\alpha+\beta-\frac{2}{3}}$ for $\alpha,\beta > 1$

AMCE

Step 3: Compute the needed moment for $D \sim \mathrm{Beta}(2, 2)$.
Compute $\mathrm{E}[D]$. For the Beta distribution,

$$\mathrm{E}[D] = \frac{\alpha}{\alpha + \beta} = \frac{2}{2+2} = \frac{1}{2}.$$

AMCE

Step 4: Substitute back into AMCE.

$$\text{AMCE} = \beta + 2\gamma\frac{1}{2} + \eta p = \beta + \gamma + \eta p.$$

$$\boxed{\text{AMCE} = \beta + \gamma + \eta p.}$$

References I

Manski, C. F. and Nagin, D. S. (1998). Bounding disagreements about treatment effects: A case study of sentencing and recidivism. *Sociological methodology*, 28(1):99–137.