Today: $\boxed{\text{linear functions}}$ + $\boxed{\text{matrices}}$

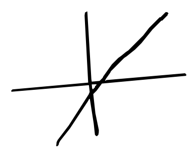
linear: if

additivity: $F(x+y) = F(x) + F(y)$

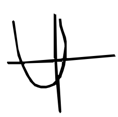
homogeneity: $F(ax) = aF(x)$

exs:

- line:



- expectation functions are linear

non ex:  parabola

↳ called higher order polynomials, where degree > 1

x^2, x^3 etc

we will approximate w/ linear equations

linear equations:

$$b = a_1 x_1 + a_2 x_2 + \dots a_n x_n$$

($a_1 \dots a_n$ are #s - parameters)

($x_1 \dots x_n$ variables)

- group of linear equations = linear system

ex: tanks purchased from NATO allies, by Ukraine
$1 \dots i \dots n$

- sps y_i constant

$$y_i = 20$$

- sps it depends on # of senior leadership visits x_i

$$y_i = 20 + 2x_i$$

$$\boxed{y_i = \beta_0 + \beta_1 x_i}$$ → we could graph this as a line!

- tells us that each additional unit of x (leadership visits) will produce a 2 unit increase in y

in real life, don't observe what β_0 and β_1 are - we observe x and y. task: figure out β_0 & β_1!

sps: france has 1 visit. france sells 1 tank
germany has 2. germany sells 2 tanks

$$\boxed{\begin{aligned} 4 &= \beta_0 + \beta_1 \cdot 2 \\ 3 &= \beta_0 + \beta_1 \cdot 1 \end{aligned}}$$

solving for β_1, β_2:

$$4 - 3 = \beta_0 - \beta_0 + 2\beta_1 - 1\beta_1$$

$$1 = \beta_1$$

$$2 = \beta_0$$

matrix

when we input data into excel or R, stored as a matrix.

looks like this:

final grade	hw #1	hw #2	hw #3	final ex
95	5	10	100	45
30	0	10	13	40
100	100	0	100	100

in matrix form, actually...

$$\begin{bmatrix} 95 \\ 30 \\ 100 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 100 & 45 \\ 0 & 10 & 13 & 40 \\ 100 & 0 & 100 & 100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from our germany example...

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$\Longrightarrow$

matrix dimensionality

$m \times n$ matrices have m rows
n columns

Rows x Columns
(Roman catholic, Rum and coke)

matrices are column of row vectors or row of column vectors

$$\begin{array}{cc} \text{rows: } 1 \quad 2 \quad \cdots \quad n & \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} & \begin{matrix} 1 \\ \vdots \\ m \end{matrix} \end{array}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{C}_1 & \vec{C}_2 & \cdots & \vec{C}_n \\ | & | & & | \end{bmatrix} \quad \text{where } \vec{C}_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

OR

$$= \begin{bmatrix} — & \vec{R}_1^T & — \\ — & \vec{R}_2^T & — \\ & \vdots & \\ — & \vec{R}_m^T & — \end{bmatrix} \quad \text{where } \vec{R}_i^T = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}$$

we've already seen this . . . diff in means estimator

$$\sum \mathbb{1}[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

matrix transpose:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad X^T \text{ or } X' = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

(more on this later!)

matrices r how computers store large datasets. this is essentially what is going on when we run regressions (more on that later)

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\beta_0 + 2\beta_1 \\ 1\beta_0 + 1\beta_1 \end{bmatrix}$$

called matrix multiplication!

examples:

$$XT = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1+1\cdot3 & 1\cdot2+4 \\ 1\cdot1+2\cdot3 & 1\cdot2+2\cdot4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}$$

more generally,

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a_{11} = [1st row of A][1st column of B]

a_{21} = [2nd row of A][1st column of B]

a_{12} = [1st row of A][2nd column of B]

a_{22} = [2nd row of A][2nd column of B]

$\boxed{\text{Laws:}}$

$$A + B = B + A$$

BUT ... $AB \neq BA$

associative laws:

$$(A + B) + C = A + (B + C)$$

$$(AB)C = A(BC)$$

distributive laws:

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

Identity matrix: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AI = A$$

$$IA = A$$

$AB \neq BA$ example

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 57 & 24 \\ 89 & 40 \end{bmatrix}$$

$$BA = \begin{bmatrix} 61 & 76 \\ 27 & 36 \end{bmatrix}$$

(can also click "show solution" on matrix calculator for steps)

Can't multiply everything!

$$A = \begin{array}{c} \text{rows: } 1 \quad 2 \quad \cdots \quad n \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \vdots \\ m \end{array}$$

Call this an $m \times n$ matrix

m rows
n columns
RC
Roman catholic

if A $m \times n$ and $B = r \times p$

AB exists iff $n = r$

$\boxed{\text{why?}}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ q \end{bmatrix} = B$$

when we multiply this out, q doesn't have any parameters.

ex: in survey experiments, q may be some 4th variable that we think affects the outcome (say, voting choice). But if we didn't ask about q, we don't know what it is for each person. Thus we can't say anything about the relationship btwn q and our outcome

$\boxed{\text{Practice}}$

a) $\begin{bmatrix} 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 7 \\ 8 & 12 & 10 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 6 & 15 \\ 10 & 23 \\ 14 & 31 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \therefore \nexists$

how do we use matrices to solve systems of linear equations?

- some systems have infinite solutions –
- some systems have no solutions

→ both mean that given the data we have, we cannot make good inferences!
(more on that later)

★ maybe show in R what's going on here?

here, we have written matrices as follows:

$$[y] = [X][\beta]$$

where y is our outcome.

sometimes, we write matrices like this:

$$\left[\begin{array}{c|cc} | & | & | \\ \vec{y} & \vec{X}_1 & \vec{X}_2 \\ | & | & | \end{array}\right]$$

called an augmented matrix, solved by a process called gauss-jordan elimination

notice that $\vec{\beta}$ is always an $n \times 1$ vector! - in the augmented ex, we will solve for βs - see Bretscher pg 12