# Social Science Inquiry II

## Week 8: Inference for multivariate regression, part II

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# Loading packages for this class

```
> set.seed(60637)
> # For plotting:
> library(ggplot2)
> # library(devtools)
> # devtools::install_github("wilkelab/ungeviz")
> library(ungeviz)
> library(ggridges)
```

- Housekeeping

# P-hacking

# P-values

Suppose $\hat{\theta}$ is the general form for an estimate produced by our estimator, and $\hat{\theta}^*$ is the value we have actually observed.

# P-values

- A two-tailed p-value under the null hypothesis is

$$p = \mathrm{P}_0[|\hat{\theta}| \geq |\hat{\theta}^*|]$$

i.e., the probability *under the null distribution* that we would see an estimate of $\hat{\theta}$ as or more extreme as what we saw from the data.

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- Suppose we have some data, $(Y, X_1, X_2, \ldots X_K)$.
- Suppose the null distribution represents the truth.
- If we test one hypothesis, what is the probability that we will find something that is statistically significant at $p \leq 0.05$?
- If we test two unrelated hypotheses, what is the probability that we will find something that is statistically significant at $p \leq 0.05$?

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- $\mathrm{P}[A] = 0.05$
- $\mathrm{P}[B] = 0.05$
- $\mathrm{P}[AB]$? The probability we see event A OR B?

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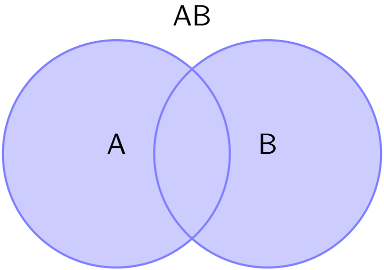

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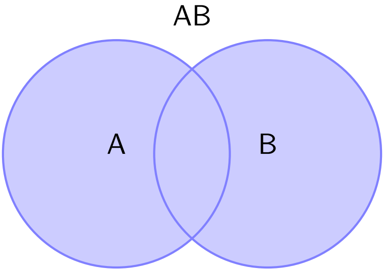

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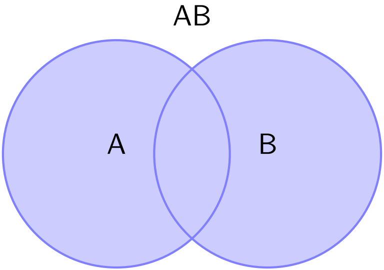

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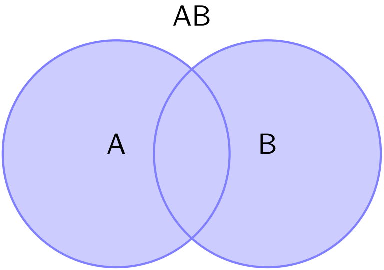

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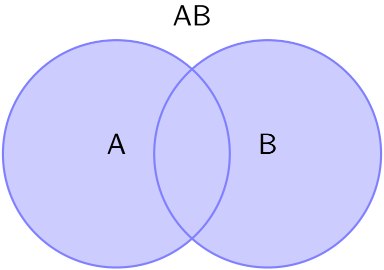

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This becomes a real problem when researchers run many tests in their papers!

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- **Type I error:** (false positive) we see an effect, where one doesn't really exist
- **Type II error:** (false negative) we didn't see an effect, but one really does exist

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These tests aren't all fully independent, but the more tests we do, the more likely we are to uncover a false positive.

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  - For $\alpha = 0.05$, $1 - 0.95^4 = 0.1855$
  - With Bonferroni correction: $1-(1-\alpha/4)^4 = 0.0491$
  - Ten independent tests: $1-(1-\alpha/10)^1 0 = 0.0489$

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    - Reject all p-values greater than or equal to $p_k$, accept all p-values less $p_k$

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# p-value adjustment

- In either case, for more complex settings, try simulation.

# Multiple testing

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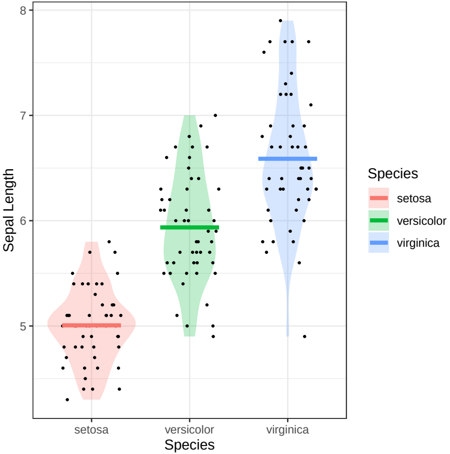
- When can you consider tests as unrelated?
- Exploratory vs. confirmatory hypotheses?

# Some alternatives to confidence intervals (via `ungeviz`)

Show the underlying data.

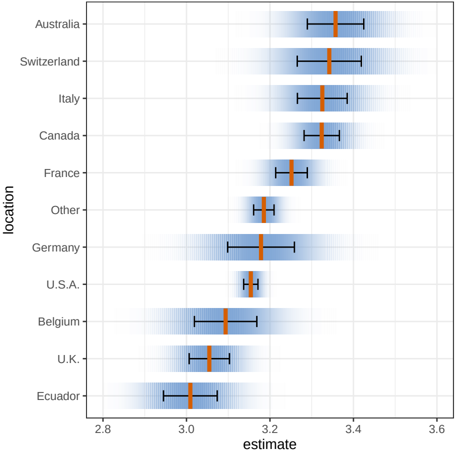
```
> ggplot(iris, aes(Species, Sepal.Length,fill = Species)) +
+     geom_violin(alpha = 0.25, color = NA) +
+     geom_point(position = position_jitter(width = 0.3, height = 0), size = 0.5) +
+     geom_hpline(aes(colour = Species), stat = "summary", width = 0.6, size = 1.5, fun = 'mean')
```



# Some alternatives to confidence intervals (via `ungeviz`)

## Shaded confidence strips.

```
> ggplot(cacao_means, aes(x = estimate, y = location)) +
+     stat_confidence_density(aes(moe = std.error), confidence = 0.68, fill = "#81A7D6", height = 0.7) +
+     geom_errorbarh(aes(xmin = estimate - std.error, xmax = estimate + std.error), height = 0.3) +
+     geom_vpline(aes(x = estimate), size = 1.5, height = 0.7, color = "#D55E00")
```



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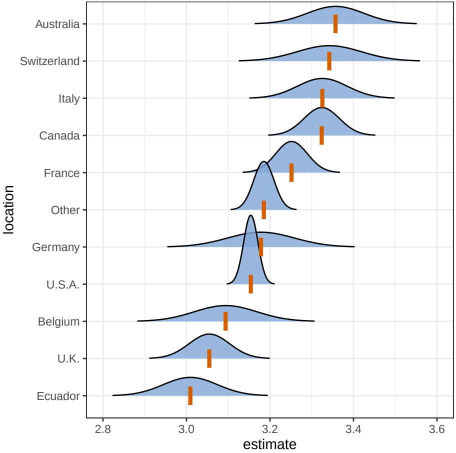
## Confidence densities.

```
> ggplot(cacao_means, aes(x = estimate, y = location)) +
+     stat_confidence_density(
+         aes(moe = std.error, height = stat(density)), geom = "ridgeline",
+         confidence = 0.68, fill = "#81A7D6", alpha = 0.8, scale = 0.08, min_height = 0.1) +
+     geom_vpline(aes(x = estimate), size = 1.5, height = 0.5, color = "#D55E00")
```



# References I

Clause Wilke: `https://wilkelab.org/ungeviz/index.html`