# Social Science Inquiry II

## Week 5: Uncertainty and inference, part II

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# Loading packages for this class

```
> library(ggplot2)
> set.seed(60637)
```

# Continuing inference

- Last class, we assumed we had a finite population that we observe all of, and the source of randomness in what we observed was due to random assignment of treatment. This is called *randomization inference.*

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- Now, we'll assume that our data is produced from a random generative process, where we're sampling from some (potentially infinite) population distribution that is not fully observed. This is the type of inference we use for survey sampling.
- It's important to consider *what the source of randomness is* and *what population we're making inferences about.*

# Back to estimation

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- The random variable's probability distribution is then:

$$f(x) = \begin{cases} 1/16 & x = 0 \\ 3/8 & x = 1 \\ 9/16 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

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- For example, with respect to the Pager (2003) data, we can use a process like this to model the probability that an employer will hire a white applicant without a criminal record.
- We might say that there are different random processes, with different probabilities of success, for whites with and without criminal records, and blacks with and without criminal records.

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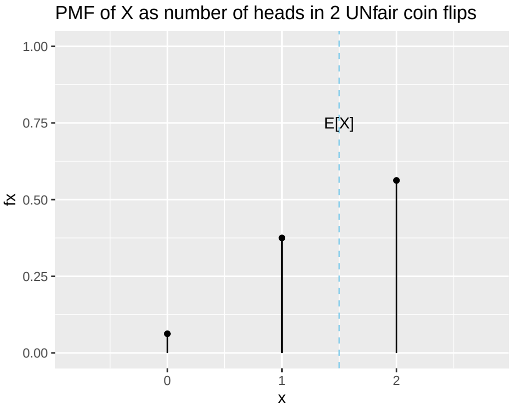
- For example, with respect to the Pager (2003) data, we can use a process like this to model the probability that an employer will hire a white applicant without a criminal record.
- We might say that there are different random processes, with different probabilities of success, for whites with and without criminal records, and blacks with and without criminal records.
- Here, where we have mutliple coin flips, we can compare that to the probability distribution of hires for two people with the same profile.

# Let's take a look at the mean.



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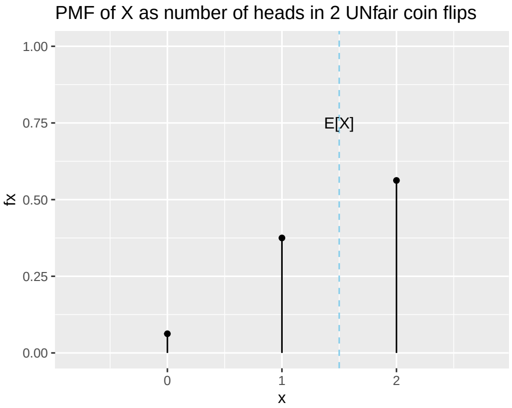
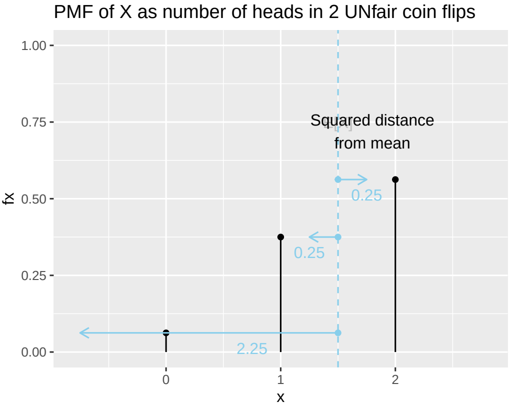



$$
\begin{aligned}
\mathrm{E}[X] &= \sum_x x f x \\
&= 0 \times \frac{1}{16} + 1 \times \frac{3}{8} + 2 \times \frac{9}{16} = \frac{24}{16} \\
&= 1.5
\end{aligned}
$$

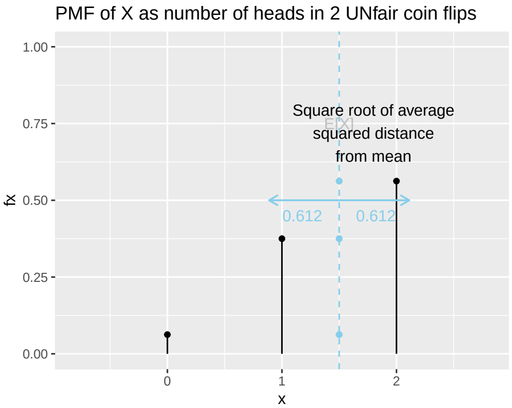
## And the spread.




Variance = average squared distance from the mean

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\ &= 2.25 \times \frac{1}{16} + 0.25 \times \frac{3}{8} + 0.25 \times \frac{9}{16} \\ &= 0.375\end{aligned}$$

# And the spread.




SD $=$ square root of variance

$$= \sqrt{0.375} = 0.612$$

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> x_observed <- sample(X, prob = probs,
+                      replace = TRUE,
+                      size = n)
> head(x_observed)
```

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[1] 1 0 1 1 2 2
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> mean(x_observed)

[1] 1.514
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[1] 1 0 1 1 2 2

> mean(x_observed)

[1] 1.514

> var(x_observed)

[1] 0.3661702

> sd(x_observed)

[1] 0.60512
```

- The process that we just did – sampling and estimation based on observed data – is a very common process in empirical research.

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- But we may notice that the mean, variance, and standard deviation are not exactly what we calculated analytically.

Let's try it again.

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```
> x_observed <- sample(X,
+                      prob = probs,
+                      replace = TRUE,
+                      size = n)
> mean(x_observed)
[1] 1.496
> var(x_observed)
[1] 0.3823664
> sd(x_observed)
[1] 0.6183578
```

- The values that we get are close, but not identical.

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- This is because what we are observing in practice is a *sample* from the data.

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- This is what we use statistics for, and why we talk about probability AND statistics.
- Probability gives us a model of the world.
- Statistics give us a way to relate the data that we see to the model.

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- Formally, that educated guess is called *estimation*.

# Sample mean

Let's repeat our random sampling from the double coin flip, but we'll consider a smaller sample, of size $n = 100$.

```
> n <- 100
> x_observed <- sample(X,
+                      prob = probs,
+                      replace = TRUE,
+                      size = n)
> head(x_observed)
[1] 1 1 1 2 2 1
```

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```
> mean(x_observed)
[1] 1.44
```

- We differentiate the *sample mean* from the *population mean* because the sample mean will vary with every new sample we draw.

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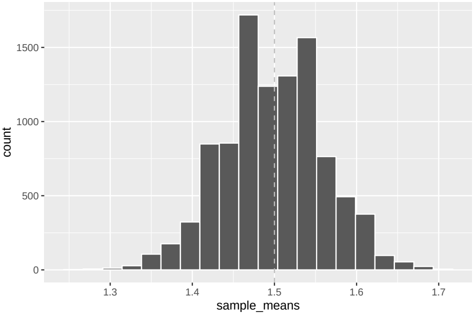
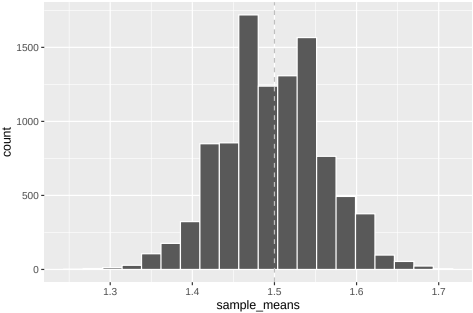
- We'll use a simulation with `replicate()` to see what would happen if we took a sample of size $n = 100$ from the population distribution many times.

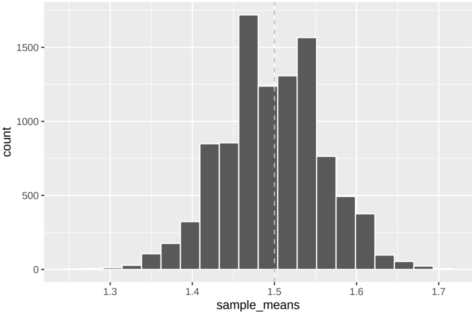
```
> n_iter <- 10000
> x_mat <- replicate(n_iter, sample(X,
+                                    prob = probs,
+                                    replace = TRUE,
+                                    size = n))
> dim(x_mat)
[1]   100 10000
> head(x_mat[,1])
[1] 0 2 2 2 2 2
> head(x_mat[,2])
[1] 2 2 1 2 1 0
```

```r
> sample_means <- apply(x_mat, 2, mean)
> length(sample_means)
[1] 10000
> head(sample_means)
[1] 1.59 1.51 1.54 1.47 1.43 1.46
```


count
1500
1000
500
0
1.3
1.4
1.5
1.6
1.7
sample_means




We see the sample means are roughly distributed around the mean of the underlying population, `Ex`.




We see the sample means are roughly distributed around the mean of the underlying population, `Ex`.

The expected value of the sample mean is the population mean.

# Sample variance

We can estimate the mean of the population using the sample mean.
What about the sample variance?

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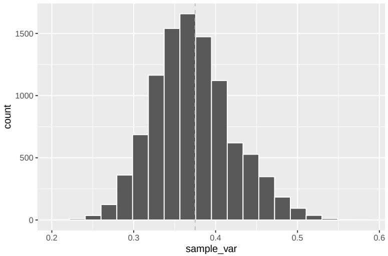

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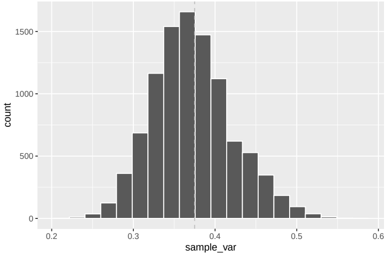


We see the sample variances are roughly distributed around the variance of the underlying population, `sdx`$^2$.

The formula for the unbiased sample variance is:

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Why do we divide by $n-1$, instead of $n$?

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- The sample mean, $\bar{X}_n$, has an expected value of $\mathrm{E}[X]$.
- However, because it is made up of the $1, \ldots, n$ $X_i$ that we actually observe, the expected difference between $(X_i - \bar{X}_n)$ is a little bit smaller than the expected difference between $(X_i - \mathrm{E}[X])$.

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```

R uses the formula for the unbiased sample variance.

# Standard error of the estimator

- The sample mean is itself a random variable, and so it has its own mean and variance. The mean of the sample mean is the population mean. The variance of the sample mean is:

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- And the standard deviation of the sample mean is:

$$\sqrt{\mathrm{Var}[\bar{X}_n]} = \sqrt{\frac{\mathrm{Var}[X]}{n}}$$

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  - The *standard error* describes the sampling variation of an **estimator**; i.e., how much our estimates will vary based on the random sample that we draw.
  - *standard deviation* describes the underlying variation in the **population distribution**.

Let's check this in our simulation. We saw that mathematically, $\mathrm{Var}[X]$ was 0.375. So

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> var(unlist(sample_means))
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It's not exactly what we calculated mathematically.

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- But *this estimate is itself a random variable*, with, again, its own sampling distribution. We will get slightly different estimates of the sampling variance of the sample mean each time we take our 10000 separate samples.
- In practice, we will estimate the standard error of the sample mean by plugging our unbiased sample variance formula into the standard error formula:

$$\hat{\mathrm{se}} = \sqrt{S_n^2/n}$$

# References I

Pager, D. (2003). The mark of a criminal record. American journal of sociology, 108(5):937–975.