Social Science Inquiry II

Week 5: Uncertainty and inference, part II

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Loading packages for this class

```
> library(ggplot2)
> set.seed(60637)
```

Continuing inference

- Last class, we assumed we had a finite population that we observe all of, and the source of randomness in what we observed was due to random assignment of treatment. The inference we used there is called *randomization inference.*

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- Now, we'll assume that our data is produced from a random generative process, where we're sampling from some (potentially infinite) population distribution that is not fully observed. The inference we will use in this setting is the type of inference we use for survey sampling.
- It's important to consider *what the source of randomness is* and *what population we're making inferences about.*

Back to estimation

- Returning to our example where we flip a coin twice, let X be the number of heads we observe. Our coin is *not* fair, and the probability of getting a heads is 0.75.

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- The random variable's probability distribution is then:

$$f(x) = \begin{cases} 1/16 & x = 0 \\ 3/8 & x = 1 \\ 9/16 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

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- For example, with respect to the Pager (2003) data, we can use a process like this to model the probability that an employer will hire a white applicant without a criminal record.
- We might say that there are different random processes, with different probabilities of success, for whites with and without criminal records, and blacks with and without criminal records.

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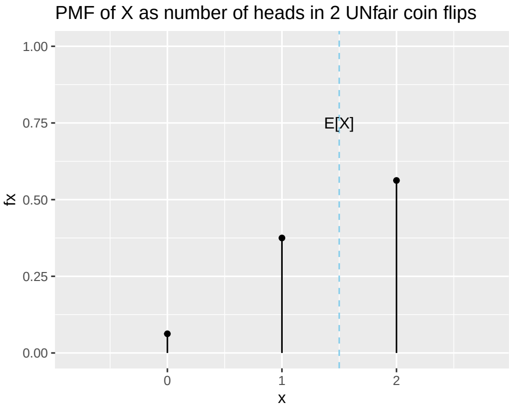
- We might say that there are different random processes, with different probabilities of success, for whites with and without criminal records, and blacks with and without criminal records.
- Here, where we have multiple coin flips, we can compare that to the probability distribution of hires for two people with the same profile.

Let's take a look at the mean.



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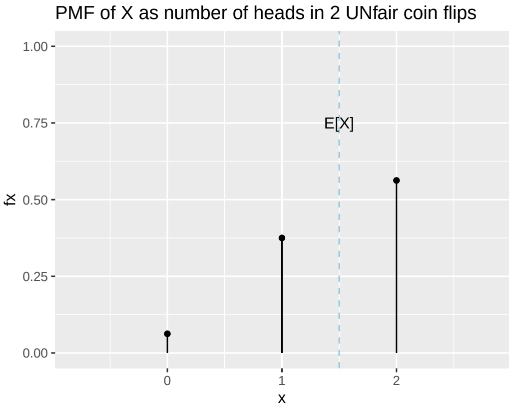
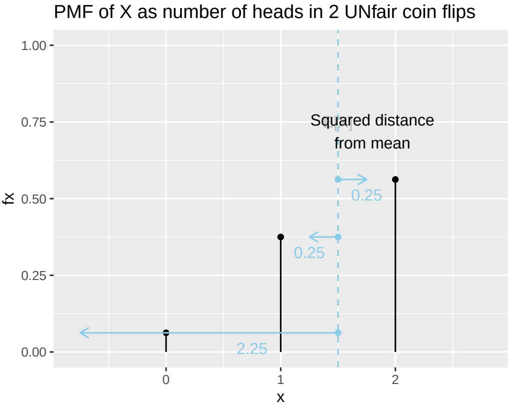


$$\mathrm{E}[X] = \sum_x x f x$$
$$= 0 \times \frac{1}{16} + 1 \times \frac{3}{8} + 2 \times \frac{9}{16} = \frac{24}{16}$$
$$= 1.5$$

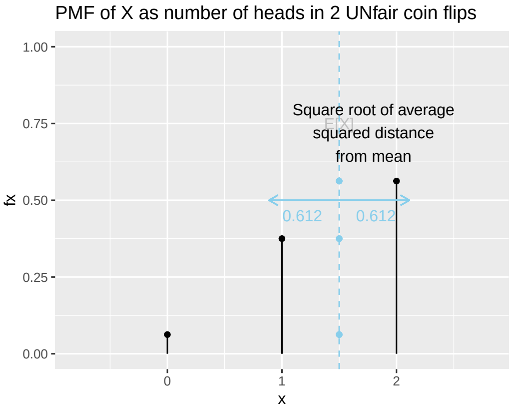
And the spread.



Variance = average squared distance from the mean

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\ &= 2.25 \times \frac{1}{16} + 0.25 \times \frac{3}{8} + 0.25 \times \frac{9}{16} \\ &= 0.375\end{aligned}$$

And the spread.



SD = square root of variance

$$= \sqrt{0.375} = 0.612$$

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> x_observed <- sample(X, prob = probs,
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```
> mean(x_observed)
```

```
[1] 1.514
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> mean(x_observed)

[1] 1.514

> var(x_observed)

[1] 0.3661702

> sd(x_observed)

[1] 0.60512
```

- The process that we just did – sampling and estimation based on observed data – is a very common process in empirical research.

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- But we may notice that the mean, variance, and standard deviation are not exactly what we calculated analytically.

Let's try it again.

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```
> x_observed <- sample(X,
+                      prob = probs,
+                      replace = TRUE,
+                      size = n)
> mean(x_observed)
[1] 1.496
> var(x_observed)
[1] 0.3823664
> sd(x_observed)
[1] 0.6183578
```

- The values that we get are close, but not identical.

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- This is because what we are observing in practice is a *sample* from the data.

Sampling and statistics

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  - (Estimators are a class of statistics that we use to approximate specific estimands. Test statistics are the specific statistics we use to test hypotheses.)

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- This is what we use statistics for, and why we talk about probability AND statistics.
- Probability gives us a model of the world.
- Statistics give us a way to relate the data that we see to the model.

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- Formally, that educated guess is called *estimation*.

Sample mean

Let's repeat our random sampling from the double coin flip, but we'll consider a smaller sample, of size $n = 100$.

```
> n <- 100
> x_observed <- sample(X,
+                      prob = probs,
+                      replace = TRUE,
+                      size = n)
> head(x_observed)
[1] 1 1 1 2 2 1
```

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```
> mean(x_observed)
[1] 1.44
```

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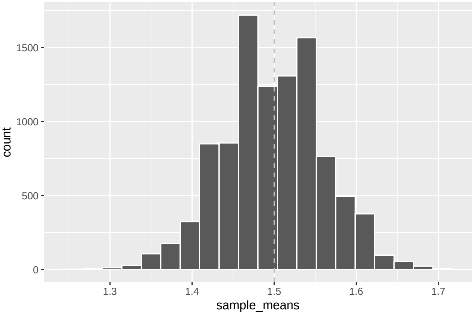
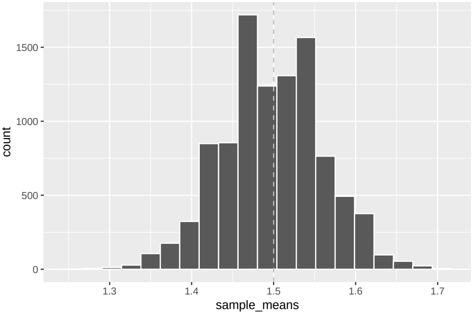
- We'll use a simulation with `replicate()` to see what would happen if we took a sample of size $n = 100$ from the population distribution many times.

```
> n_iter <- 10000
> x_mat <- replicate(n_iter, sample(X,
+                                   prob = probs,
+                                   replace = TRUE,
+                                   size = n))
> dim(x_mat)
[1]   100 10000
> head(x_mat[,1])
[1] 0 2 2 2 2 2
> head(x_mat[,2])
[1] 2 2 1 2 1 0
```

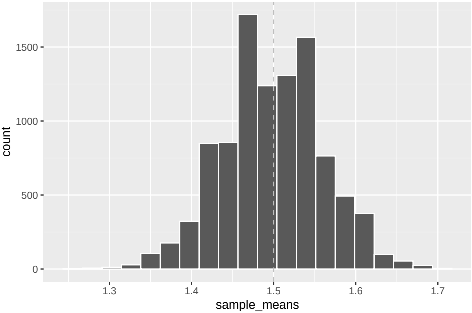
```
> sample_means <- apply(x_mat, 2, mean)
> length(sample_means)
[1] 10000
> head(sample_means)
[1] 1.59 1.51 1.54 1.47 1.43 1.46
```



count
1500
1000
500
0
1.3
1.4
1.5
1.6
1.7
sample_means



We see the sample means are roughly distributed around the mean of the underlying population, Ex.



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The expected value of the sample mean is the population mean.

Sample variance

We can estimate the mean of the population using the sample mean.
What about the sample variance?

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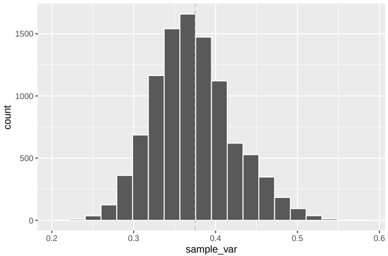

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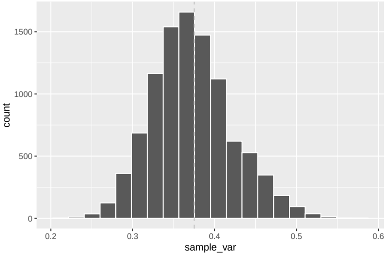

We see the sample variances are roughly distributed around the variance of the underlying population, $\texttt{sdx}^2$.

The formula for the unbiased sample variance is:

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$$\mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}\left[X\right])^2\right]$$

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Why do we divide by $n-1$, instead of n?

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- However, because it is made up of the $1, \ldots, n$ X_i that we actually observe, the expected difference between $(X_i - \bar{X}_n)$ is a little bit smaller than the expected difference between $(X_i - \mathrm{E}[X])$.

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R uses the formula for the unbiased sample variance.

Standard error of the estimator

- The sample mean is itself a random variable, and so it has its own mean and variance. The mean of the sample mean is the population mean. The variance of the sample mean is:

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- And the standard deviation of the sample mean is:

$$\sqrt{\mathrm{Var}[\bar{X}_n]} = \sqrt{\frac{\mathrm{Var}[X]}{n}}$$

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    - *standard deviation* describes the underlying variation in the **population distribution**.

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- But *this estimate is itself a random variable*, with, again, its own sampling distribution. We will get slightly different estimates of the sampling variance of the sample mean each time we take our 10000 separate samples.
- In practice, we will estimate the standard error of the sample mean by plugging our unbiased sample variance formula into the standard error formula:

$$\hat{\text{se}} = \sqrt{S_n^2/n}$$

Weak Law of Large Numbers

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- Convergence in probability, $\xrightarrow{p}$, here means that the probability that we measure a value of $\bar{X}_n$ that is any arbitrary distance away from $\mathrm{E}[X]$ is decreasing with our sample size.

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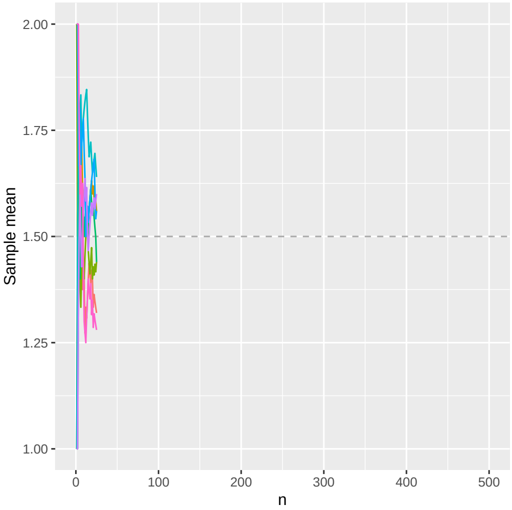

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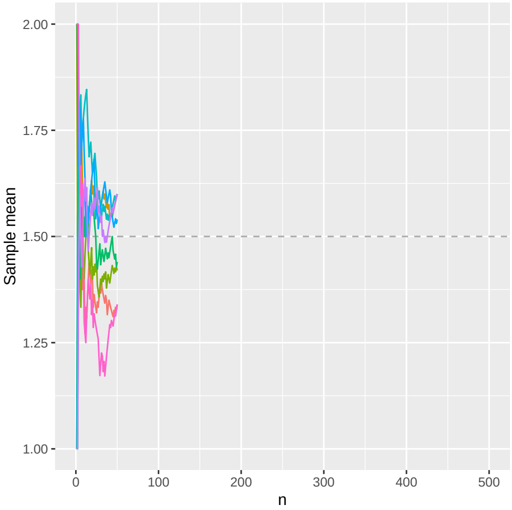

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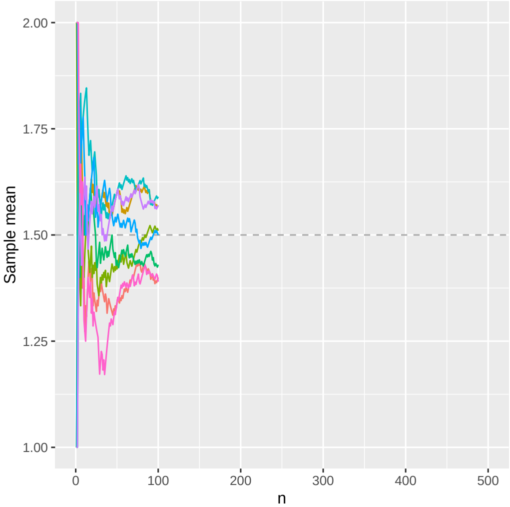

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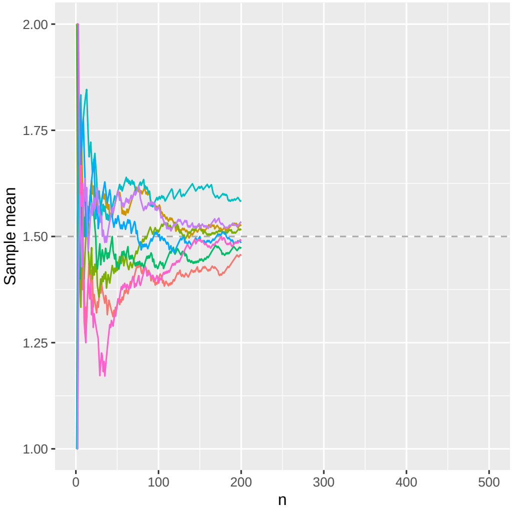

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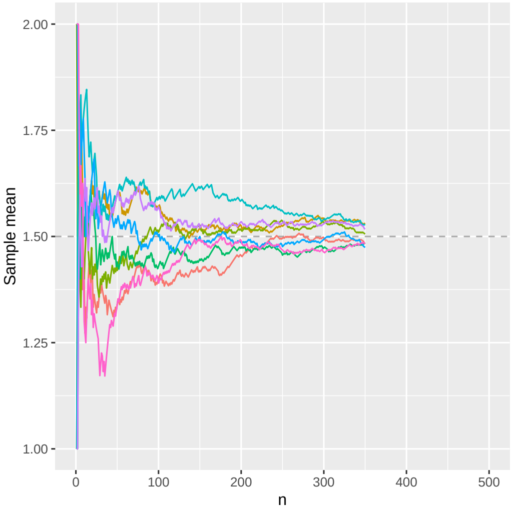

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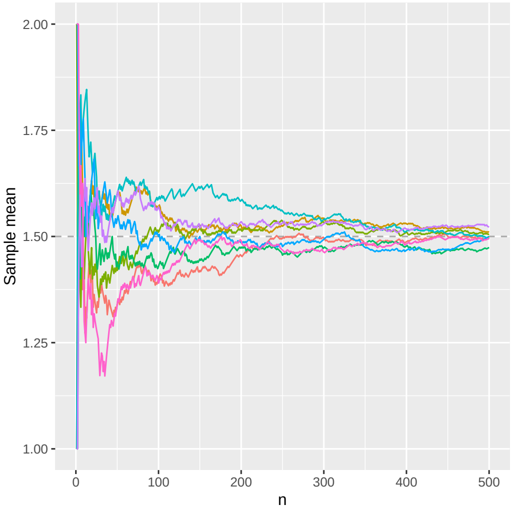

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- Why is the WLLN so helpful to us?
- Given a sufficient sample from a population, we can estimate features of a random variable to arbitrary precision
- This is why we can use sample analogs of population features, like the sample mean, as plug-in estimators to estimate the population quantities.

Reading papers

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  - Research design/strategy

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How would you answer these questions with the Pager (2003) paper?

References I

Pager, D. (2003). The mark of a criminal record. American journal of sociology, 108(5):937–975.