

Social Science Inquiry II

Week 3: A brief introduction to probability

Molly Offer-Westort

Department of Political Science,
University of Chicago

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Housekeeping

- ▶ Homework assignments
 - ▶ Feedback is very helpful (thank you!)
 - ▶ If you spend > 30 minutes without making progress, go to Stack Overflow or solutions.

Loading packages for this class

```
> library(ggplot2)
> library(gridExtra)
> set.seed(60637)
```

What do we do with data

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These latter two are called *inference*, and we'll need some additional tools and assumptions to make headway on them

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 - ▶ Often, we see only part of the picture we'd like to talk about. We only see some of the data, or we can only see one version of events.
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 - ▶ It also allows us to *formalize our uncertainty* when making inference.

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- ▶ If we are describing events with counterfactuals, such as turning out to vote or testing positive for COVID, it can be useful to describe those events as probabilistic.
- ▶ These are *models* of how the world works, and they help us make sense of the fundamentally squishy nature of social science research. With limited information about the world, we operate with uncertainty. Assigning probabilities conditional on the information we *do* have helps us formalize that uncertainty.

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- ▶ These are *models* of how the world works, and they help us make sense of the fundamentally squishy nature of social science research. With limited information about the world, we operate with uncertainty. Assigning probabilities conditional on the information we *do* have helps us formalize that uncertainty. Even if we don't necessarily believe that human behavior is “random.”

Flipping a coin twice

Suppose we are flipping a coin twice, and the coin is fair. This is a random process, and we will describe the probability space associated with this process.

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- ▶ P : Probability measure. An operator that assigns probability to all events in the event space.
 - ▶ Here, the event that we get a head on the first flip, $P[A] = 1/2$.

An aside on mathematical notation

Mathematical notation is a tool that gives us a common language to express concepts with precision. There are different conventions in different communities, and no approach is “right” or “wrong”—it’s just a question of whether your notation is appropriately communicating to your audience what you want it to.

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We have just used probability notation as a way to fully describe any random generative process.

Sampling

We can simulate our double coin flip process in R, using the `sample()` function. There are four possible outcomes, and all are equally likely.

```
> Omega <- c('HH', 'HT', 'TH', 'TT')
> probs <- c(0.25, 0.25, 0.25, 0.25)
> sample(x = Omega,
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[1] "HH"
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[1] "HH"
```

We can run this simulation many times, and our results should *approximately* follow the probabilities we assigned.

```
> n <- 1000
> result_n <- sample(x = Omega,
+       size = n,
+       prob = probs,
+       replace = TRUE)
> table(result_n)

result_n
  HH  HT  TH  TT
247 241 258 254
```


Independent events

Two events are *independent* if

$$P[AB] = P[A]P[B]$$

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Notational aside: The event AB is that both A and B happen. There are other ways to write this, including $A \cap B$.

Returning to our example of flipping two fair coins. Let's say:

- ▶ Event A : we get a head on the first coin flip; $A = \{HT, HH\}$.
- ▶ Event B : we get a head on the second coin flip; $B = \{TH, HH\}$.
- ▶ We can see the event AB as the overlap in their respective sets, $AB = \{HH\}$

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$$P[A] = P[\{HT\}] + P[\{HH\}] = 0.25 + 0.25 = 0.5$$

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Then, we can calculate the product of the probabilities, and the probability of the joint event.

$$P[A]P[B] = 0.5 \times 0.5 = 0.25$$

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We see that they are the same, so we have independence.

$$P[A]P[B] = P[AB]$$

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> (PAB <- mean(result_n == 'HH'))
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> PA*PB
[1] 0.277092
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The proportions look pretty close.

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> (PAB <- mean(result_n == 'HH'))  
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> PA*PB  
[1] 0.146601
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Conditional probability

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This can also be read as, out of all of the times event B occurs, how many times does event A also occur?

For our coin flip example, we'll stick with:

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What is $P[B|A]$?

Bayes Rule

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Why is Bayes rule so useful? It basically tells us how we update probability based on observed data.

Bayes Rule

Example: Suppose everyone in the University of Chicago community is given a new rapid test for COVID.

- ▶ We are concerned with *false negatives*, when we get a negative test for a person who actually *is* infected.
- ▶ And *false positives*, when we get a positive test for a person who *is not* infected.

Bayes Rule

Example: Suppose everyone in the University of Chicago community is given a new rapid test for COVID.

A student gets a positive test back. What is the probability that they have COVID?

Bayes Rule

- ▶ Event A : person has COVID
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Should we think these events are independent?

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If you have COVID, your test will turn up positive 95% of the time : $P[B|A]$

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> PB_if_A <- 0.95
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$$P[A|B] = \frac{P[B|A]P[A]}{P[B]}$$

```
> ( PB_if_A * PA )/PB
```

```
[1] 0.57
```

Prosecutor's fallacy

$$P[A|B] \stackrel{?}{=} P[B|A]$$

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As a general rule, we cannot assume the conditional probability of A given B is the same as the probability of B given A

Prosecutor's fallacy

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- ▶ This fallacy often occurs when we observe evidence that we are very unlikely to see among innocent people, and very likely to observe among guilty people
- ▶ But the overall number of innocent people in the population is very large, so the number of false positives is much higher than the number of true positives

Why thinking about conditional probability is so important in social science research

“20% of people hospitalized with COVID-19 are vaccinated”

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Does this tell us that vaccines aren't very effective?

Why thinking about conditional probability is so important in social science research

“20% of people hospitalized with COVID-19 are vaccinated”

Does this tell us that vaccines aren't very effective?

Some things we should think about:

- ▶ What percentage of the population in areas served by hospitals are vaccinated?
- ▶ Are people who are vaccinated at higher risk for breakthrough cases? Or more likely to be hospitalized? Are they older, or have pre-existing conditions?

Why thinking about conditional probability is so important in social science research

What data do you need to answer the question: “Do white officers shoot minority citizens at a higher rate than non-white officers?”

Remember this article?



NATIONAL

New Study Says White Police Officers Are Not More Likely To Shoot Minority Suspects

July 26, 2019 · 5:21 PM ET

Heard on [All Things Considered](#)



MARTIN KASTE



Remember this article?

Sign in

More officer diversity won't cut racial disparity in US police shootings - study

Research found as percentage of black officers who fired in fatal shootings increased, the citizen shot was more likely to be black

Miranda Bryant *in New York*

Mon 22 Jul 2019 17:56 EDT

Officer characteristics and racial disparities in fatal officer-involved shootings

David J. Johnson^{a,b,1}, Trevor Tress^b, Nicole Burkel^b, Carley Taylor^b, and Joseph Cesario^b

^aDepartment of Psychology, University of Maryland at College Park, College Park, MD 20742; and ^bDepartment of Psychology, Michigan State University, East Lansing, MI 48824

Edited by Kenneth W. Wachter, University of California, Berkeley, CA, and approved June 24, 2019 (received for review March 5, 2019)

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- ▶ Their database only has data on fatal shootings—not on cases where people were NOT shot.
- ▶ Their results report “whether a person fatally shot was more likely to be Black (or Hispanic) than White” conditional on the race of the officer involved
- ▶ Does this address original claim?

“White officers are not more likely to shoot minority civilians than non-White officers”

$$P[\text{shot}|\text{White officer, minority civilian}] - \\ P[\text{shot}|\text{Minority officer, minority civilian}]$$

What data are we missing to address this question?

“White officers are not more likely to shoot minority civilians than non-White officers”

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- ▶ Dean Knox & Jonathan Mummolo wrote a letter critiquing the original article, based on the authors' failure to appropriately apply Bayes Rule;
- ▶ The article was eventually retracted
- ▶ (It's a bit more complicated than that, but know your conditional probability when thinking about difficult subjects!)

Random variables

- ▶ A random variable is a mapping X from our sample space Ω , to the Real numbers.

$$X : \Omega \rightarrow \mathbb{R}$$

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- ▶ Random variables are ways to quantify random events described by our sample space.
- ▶ We'll mostly work with random variables going forward, but it's important to remember that the random variable is built on the foundations of the sample space – and often, **you'll be the one deciding how that quantification happens.**

For example, with our two coin flips, let $X(\omega)$ be the number of heads in the sequence ω .

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Then the random variable, and its probability distribution, can be described as:

ω	$P[\{\omega\}]$	$X(\omega)$
TT	1/4	0
TH	1/4	1
HT	1/4	1
HH	1/4	2

and,

x	$P[X = x]$
0	1/4
1	1/2
2	1/4

We can simulate this in 'R' as well.

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```
> X <- c(0, 1, 2)
> probs <- c(0.25, 0.5, 0.25)
> sample(x = X,
+       size = 1,
+       prob = probs)
[1] 0
>
```

We can simulate this in 'R' as well.

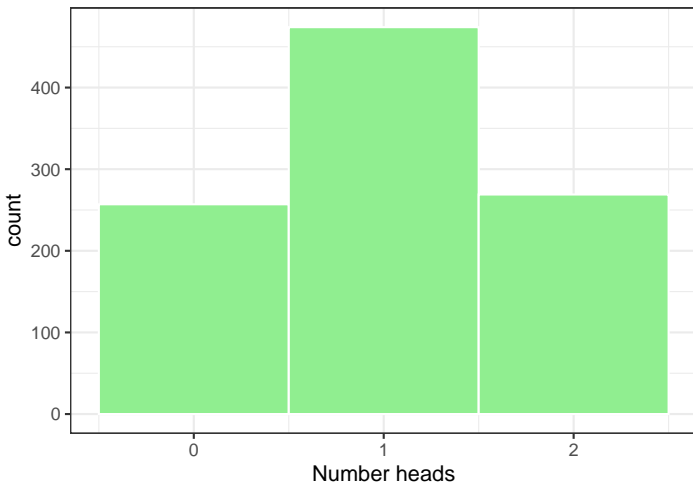
```
> X <- c(0, 1, 2)
> probs <- c(0.25, 0.5, 0.25)
> sample(x = X,
+       size = 1,
+       prob = probs)
[1] 0
>
> n <- 1000
> result_n <- sample(x = X,
+                   size = n,
+                   prob = probs,
+                   replace = TRUE)
> table(result_n)
result_n
  0    1    2
257 474 269
```

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> X <- c(0, 1, 2)
> probs <- c(0.25, 0.5, 0.25)
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+                   size = n,
+                   prob = probs,
+                   replace = TRUE)
> table(result_n)
result_n
 0    1    2
257 474 269
> prop.table(table(result_n))
result_n
 0    1    2
0.257 0.474 0.269
>
```


We can plot a histogram to look at the distribution of results.

```
> ggplot(data.frame(result_n), aes(x = result_n)) +  
+   geom_histogram(bins = 3, color = 'white', fill = 'lightgreen') +  
+   theme_bw() + xlab('Number heads')  
>
```



Probability Mass Function of a discrete random variable

- ▶ A random variable is *discrete* if it takes countably many values.

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$$f(x) = P[X = x]$$

Probability Mass Function of a discrete random variable

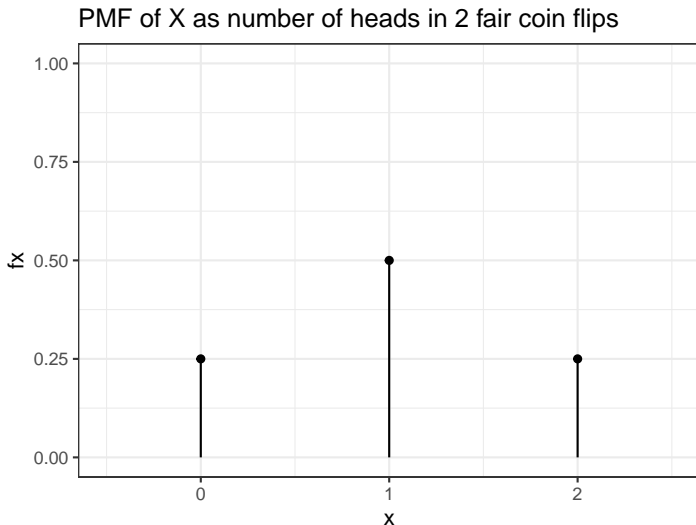
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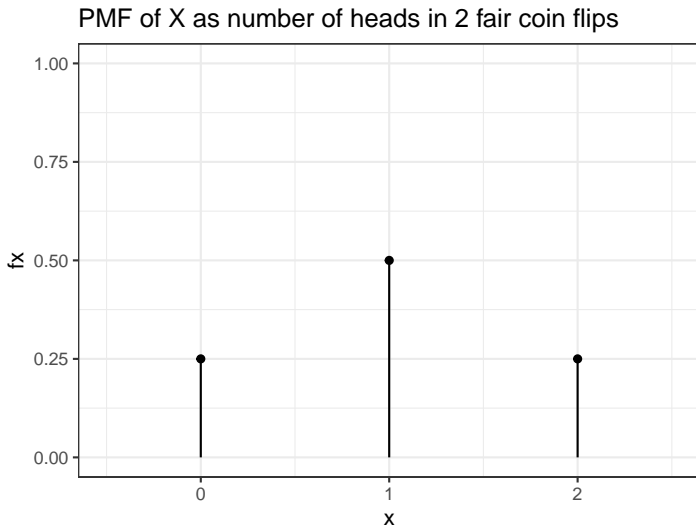
For our coin flip example,

$$f(x) = \begin{cases} 1/4 & x = 0 \\ 1/2 & x = 1 \\ 1/4 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Illustrating the PMF of a discrete RV



Illustrating the PMF of a discrete RV



Note that the probabilities sum to 1. This is one of the foundational axioms of probability.

Cumulative Distribution Functions

The cumulative distribution function of X tells us the probability we will see an outcome less than or equal to some value x .

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$$F(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

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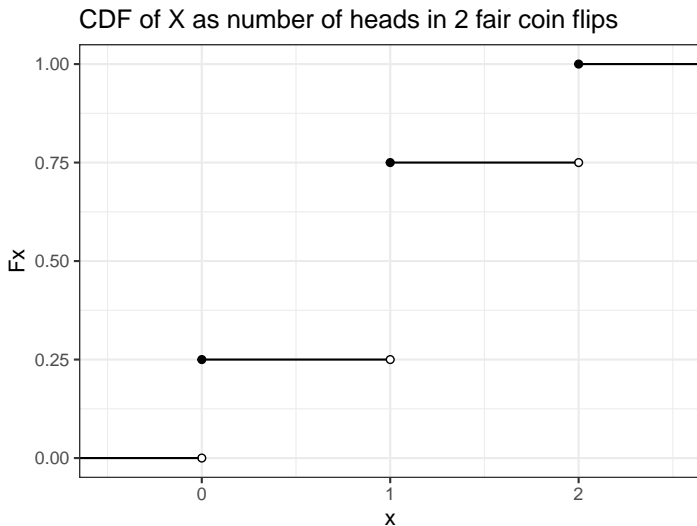
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For our coin flip example,

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CDFs are really useful, because if we know the CDF, we can fully describe the distribution of *any* random variable.

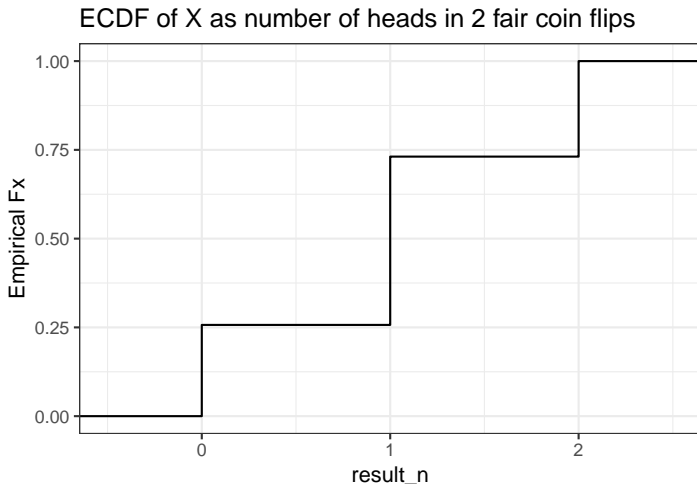
Illustrating the CDF of a RV



Illustrating the CDF of a RV

And we can use `ggplot2` to see what the *Empirical* CDF looks like

```
> ggplot(data.frame(result_n), aes(x = result_n)) +  
+   stat_ecdf() +  
+   coord_cartesian(xlim = c(-0.5, 2.5)) +  
+   ylab('Empirical Fx') +  
+   ggtitle('ECDF of X as number of heads in 2 fair coin flips') + theme_bw()  
>
```



Joint and conditional relationships

Bivariate relationships

We often care about how random variables vary with each other

- ▶ age and voter turnout
- ▶ sex and income
- ▶ education and earnings

Bivariate relationships

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- ▶ age and voter turnout
- ▶ sex and income
- ▶ education and earnings

Just like with univariate random variables, we can describe these bivariate relationships by their distributions

Joint PMF of discrete random variables

$$f(x, y) = P[X = x, Y = y]$$

Returning to our example of flipping two fair coins

- ▶ Let X be 1 if we get *at least one heads*, and 0 otherwise
- ▶ Let Y be 1 if we get *two heads* in our two coin flips, and 0 otherwise

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- ▶ Let X be 1 if we get *at least one heads*, and 0 otherwise
- ▶ Let Y be 1 if we get *two heads* in our two coin flips, and 0 otherwise

Then the joint probability distribution can be described as:

ω	$P[\{\omega\}]$	$X(\omega)$	$Y(\omega)$
TT	1/4	0	0
TH	1/4	1	0
HT	1/4	1	0
HH	1/4	1	1

or, considering the joint PMF,

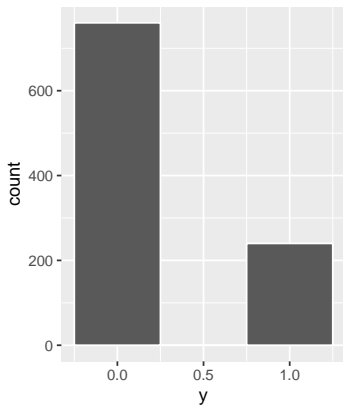
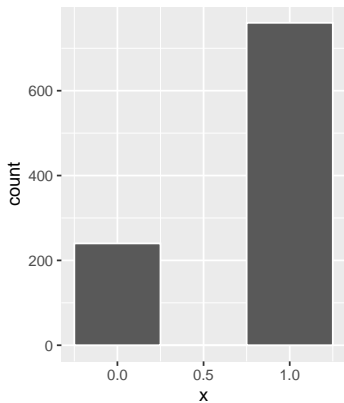
$$f(x, y) = \begin{cases} 1/4 & x = 0, y = 0 \\ 1/2 & x = 1, y = 0 \\ 1/4 & x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

```

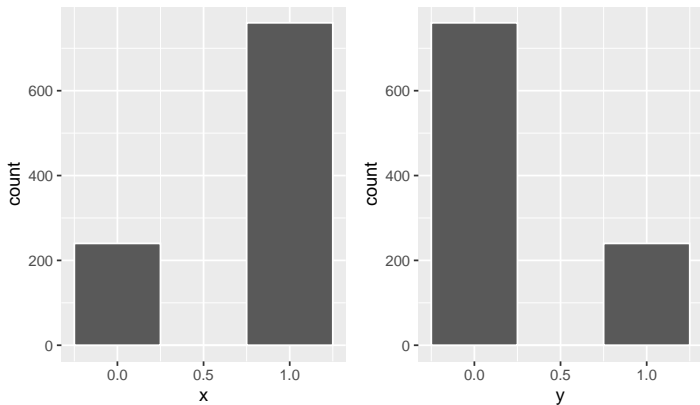
> Omega <- c('HH', 'HT', 'TH', 'TT')
> probs <- c(0.25, 0.25, 0.25, 0.25)
> result_n <- sample(x = Omega,
+                   size = n,
+                   prob = probs,
+                   replace = TRUE)
> result_mat <- data.frame(omega = result_n,
+                          x = ifelse(result_n == 'TT', 0, 1),
+                          y = ifelse(result_n == 'HH', 1, 0))
> options <- list(theme(panel.grid.minor = element_blank()),
+                 # save some style options
+                 scale_x_continuous(breaks = c(0, 1))) +
+   theme_bw()
> p1 <- ggplot(result_mat) +
+   geom_histogram(aes(x = x), bins = 3,
+                 position = 'identity',
+                 color = 'white') +
+   options
> p2 <- ggplot(result_mat) +
+   geom_histogram(aes(x = y),
+                 bins = 3,
+                 position = 'identity',
+                 color = 'white') +
+   options
>

```

```
> grid.arrange(p1, p2, ncol = 2)
```

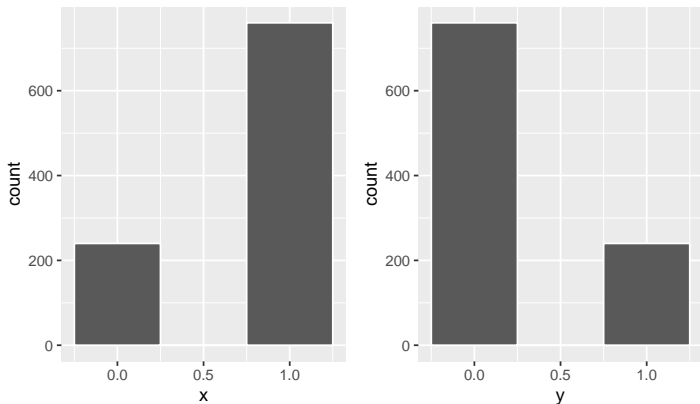


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Seeing X and Y plotted side by side doesn't really give us a full picture of their relationship.

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These are the *marginal* distributions of X and Y , i.e., their distributions where we *marginalize* or sum over the distribution of the other random variable.

Marginal distributions

$$f_X(x) = P[X = x] = \sum_y P[X = x, Y = y] = \sum_y f_{X,Y}(x, y)$$

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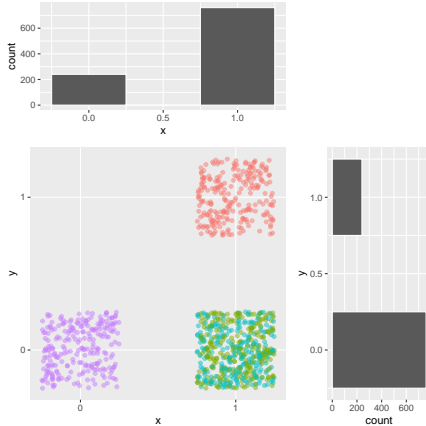
	Y = 0	Y = 1	
X = 0	1/4	0	1/4
X = 1	1/2	1/4	3/4
	3/4	1/4	

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X = 0	1/4	0	1/4
X = 1	1/2	1/4	3/4
	3/4	1/4	

Notational aside: we can subscript X in f_X to denote that it is the mass function of X specifically, as X and Y have different probability mass functions. But often we will just omit the subscript for convenience.



Plotting X and Y jointly gives us a better understanding of their joint relationship.

Conditional distributions

We are also often interested in conditional relationships.

$$f_{Y|X}(y|x) = P[Y = y|X = x] = \frac{P[X = x, Y = y]}{P[X = x]} = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

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$$f_{Y|X}(y|x) = P[Y = y|X = x] = \frac{P[X = x, Y = y]}{P[X = x]} = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

For example,

$$f_{Y|X}(y|x) = \begin{cases} 1 & x = 0, y = 0 \\ 2/3 & x = 1, y = 0 \\ 1/3 & x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

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For example,

$$f_{Y|X}(y|x) = \begin{cases} 1 & x = 0, y = 0 \\ 2/3 & x = 1, y = 0 \\ 1/3 & x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, what is the probability of observing two heads, conditional on having observed at least one heads?

Summarizing single variable distributions

Expectation

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- ▶ Expectation is an *operator* on a random variable; it maps the distribution of X to a specific number.
- ▶ Specifically, the expectation operator tells us about the mean, or average value of X across its distribution.

Notational aside: it is common to write the expectation of a distribution as μ .

Let's flip a single coin, and let X be 1 if we get a head, and 0 otherwise.

Let's flip a single coin, and let X be 1 if we get a head, and 0 otherwise.

$$f(x) = \begin{cases} 1/2 & x = 0 \\ 1/2 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

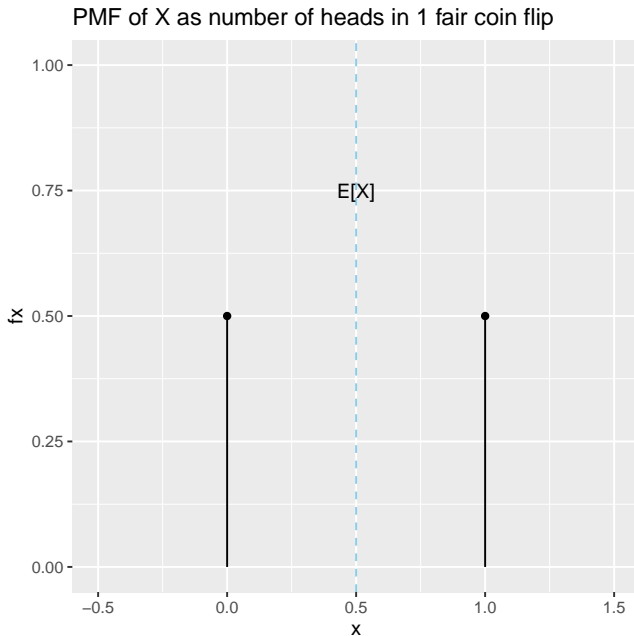
Let's flip a single coin, and let X be 1 if we get a head, and 0 otherwise.

$$f(x) = \begin{cases} 1/2 & x = 0 \\ 1/2 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Mathematically,

$$\begin{aligned} E[X] &= \sum_x xf(x) \\ &= 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Visually,



Spread of a distribution

We often describe the spread of a distribution by its variance

$$\text{Var}[X] = E[(X - E[X])^2]$$

Or equivalently,

$$= E[X^2] - E[X]^2$$

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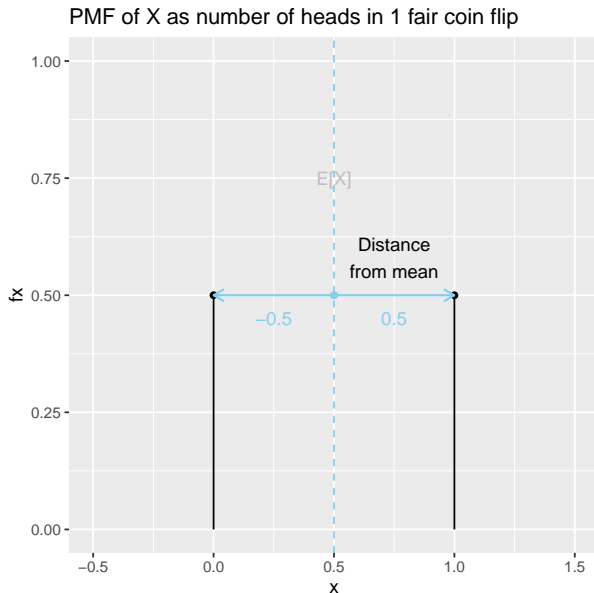
Or equivalently,

$$= E[X^2] - E[X]^2$$

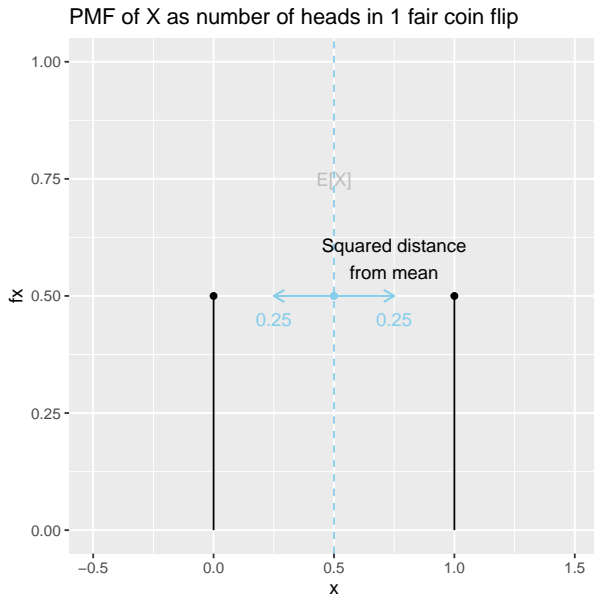
The standard deviation is the square root of the variance.

Notational aside: it is common to write the variance of a distribution as σ^2 , or the standard deviation as σ .

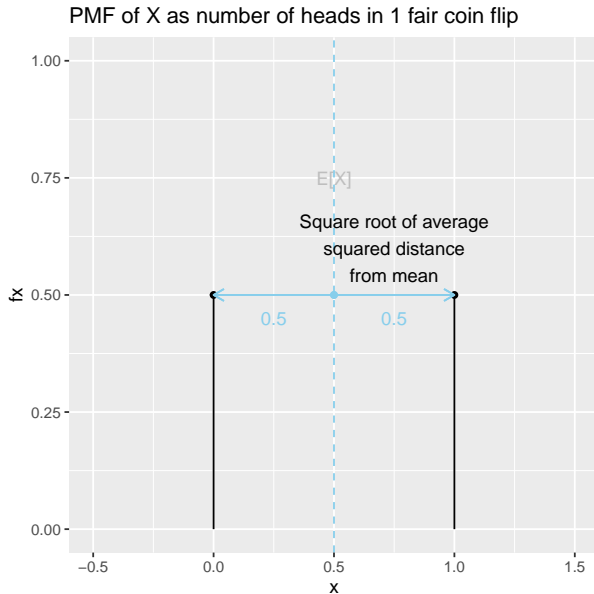
The variance is the average squared distance from the mean. The standard deviation is the square root of this.



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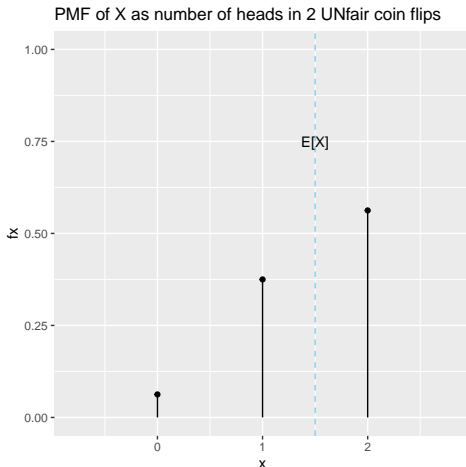
Let's take another example, where we flip a coin twice, and let X be the number of heads. However, let's say our coin is *not* fair, and the probability of getting a heads is 0.75.

Let's take another example, where we flip a coin twice, and let X be the number of heads. However, let's say our coin is *not* fair, and the probability of getting a heads is 0.75.

The random variable's probability distribution is then:

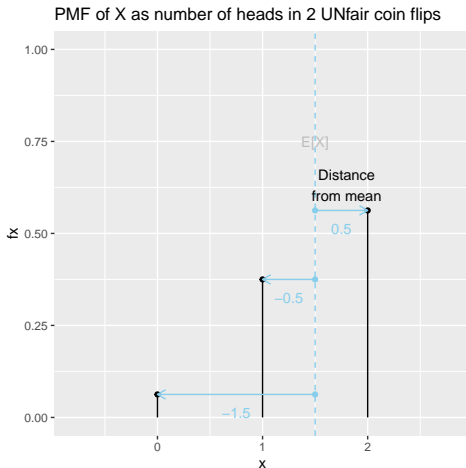
$$f(x) = \begin{cases} 1/16 & x = 0 \\ 3/8 & x = 1 \\ 9/16 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let's take a look at the mean.



$$\begin{aligned} E[X] &= \sum_x x f(x) \\ &= 0 \times \frac{1}{16} + 1 \times \frac{3}{8} + 2 \times \frac{9}{16} \\ &= \frac{24}{16} = 1.5 \end{aligned}$$

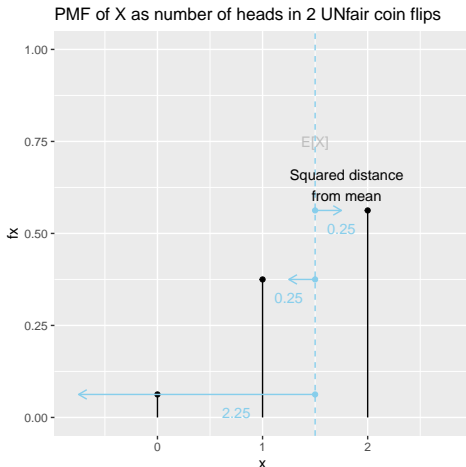
And the spread.



Variance = average squared distance from the mean

$$\text{Var}[X] = E[(X - E[X])^2]$$

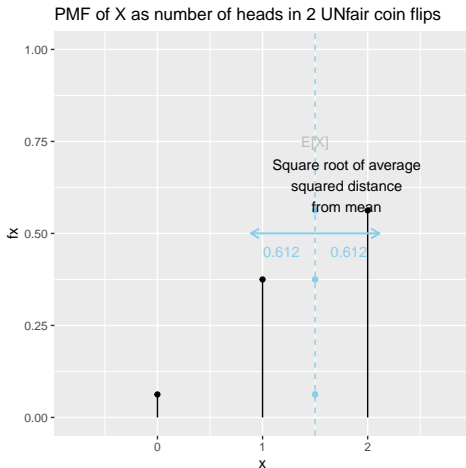
And the spread.



Variance = average squared distance from the mean

$$\begin{aligned}\text{Var}[X] &= E[(X - E[X])^2] \\ &= 2.25 \times \frac{1}{16} + 0.25 \times \frac{3}{8} + 0.25 \times \frac{9}{16} = 0.375\end{aligned}$$

And the spread.



SD = square root of variance

$$= \sqrt{0.375} = 0.612$$

Applications

- ▶ Coin flips are a pretty trivial example of a random event \rightarrow random variable.

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- ▶ Coin flips are a pretty trivial example of a random event \rightarrow random variable.
- ▶ But often, as researchers, our job is to map events that happen in the world to variables in our data sets.

Summarizing joint distributions

Covariance

$$\text{Cov}[X, Y] = E[(X - E[X])(Y - E[Y])]$$

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Covariance is how much X and Y vary together.

- ▶ If covariance is positive, when the value of X is large (relative to its mean), the value of Y will also tend to be large (relative to its mean)
- ▶ If covariance is negative, when the value of X is large (relative to its mean), the value of Y will tend to be small (relative to its mean)

Correlation

$$\rho[X, Y] = \frac{\text{Cov}[X, Y]}{\sigma[X]\sigma[Y]}$$

Rescaled version of covariance

- ▶ positive when covariance is positive
- ▶ negative when covariance is negative

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$$\rho[X, Y] = \frac{\text{Cov}[X, Y]}{\sigma[X]\sigma[Y]}$$

Rescaled version of covariance

- ▶ positive when covariance is positive
- ▶ negative when covariance is negative

$$-1 \leq \rho[X, Y] \leq 1$$

References I

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Flipping two coins event space:

$$\begin{aligned} S = & \{\emptyset, \\ & \{HH\}, \{HT\}, \{TH\}, \{TT\}, \\ & \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\} \\ & \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, \\ & \{HH, HT, TH, TT\}\} \end{aligned}$$

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