Week 6: Mixture Models

PLSC 40502 - Statistical Models

Review

Previously

- **Survey weighting**
  - Learning about **population** parameters from non-representative samples.
  - Two key components: **selection/non-response model** and a **measurement model** for the population targets.
  - **Calibration** weighting - find weights that satisfy a set of moment condition subject to a loss function.
- **Multilevel Regression and Post-stratification**
  - Can we estimate population proportions for small areas with a national survey?
  - **Yes** - if we pool information from other units.

This week

- **"Exploratory" modeling**
  - Unsupervised "clustering" algorithms
  - K-means clustering
  - Bayesian mixture models
  - Obtaining MLEs or posterior modes via Expectation-Maximization
- **Topic modeling**
  - A mixture model for text:
  - Documents have a topic **distribution**
  - Define a DGP for each word as a function of the document topic distribution and a topic-specific word distribution.
  - Flexible way of representing documents via a lower-dimensional summary.

Finite Mixture Models

Finite Mixture Models

- In conventional **regression** models, we assume a known distribution for the outcome given a set of covariates
  - The covariates could be something like a discrete "group" label (e.g. $X_i = x$)
- In this setting, conditional on X_i, we would model each Y_i with a known distribution (e.g. Normal)

$$Y_i | X_i = x \sim \text{Normal}(\mu_x, \sigma_x^2)$$

μ_x and σ_x^2 are the mean and variance parameters associated with the group $X_i = x$.

- Now imagine that the group indicators are **not observed** but instead rather **latent parameters**
  - We can do inference on these parameters conditional on the observed data and the model.

Finite Mixture Models

- In a **finite** mixture model, we assume that the outcome distribution for each observation i is governed by some discrete latent indicator $z_i \in \{1, 2, \ldots, K\}$
  - $z_1, z_2, \ldots, z_N$ are i.i.d. $\sim \text{Categorical}(\pi)$
- Conditional on the latent variable $z_i = k$, the outcome distribution is known.
- For example, in a **gaussian** finite mixture model, we assume

$$Y_i | z_i = k \sim \text{Normal}(\mu_k, \sigma_k^2)$$

- But unconditionally, Y_i has a **mixture** distribution in that its density is a weighted average of the component densities.
- In a **gaussian** finite mixture model, we have:

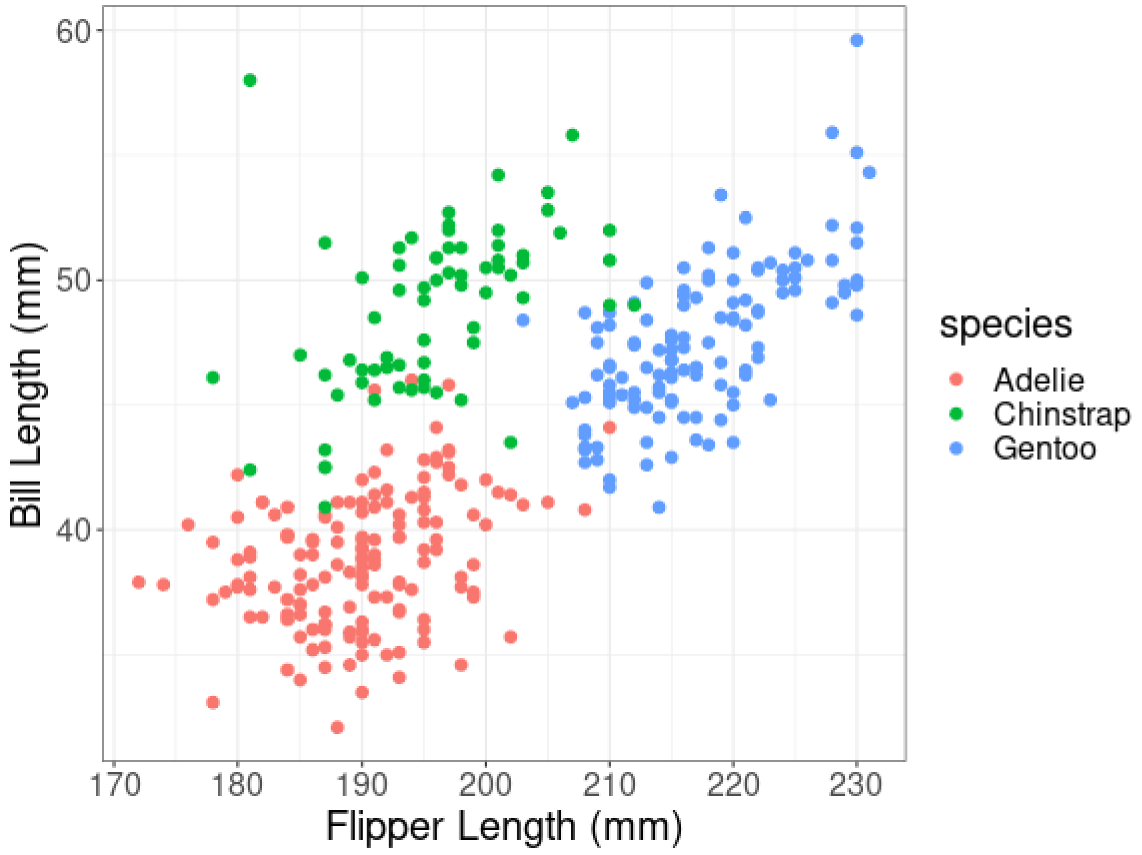
$$Y_i \sim \sum_{k=1}^{K} \pi_k \times \text{Normal}(\mu_k, \sigma_k^2)$$

where $\pi_k = Pr(z_i = k)$

- Our goal is to estimate the cluster means/variances for each cluster and the mixing proportions using likelihood inference.
  - Put priors on the mean/variance parameters and π to make it fully bayesian

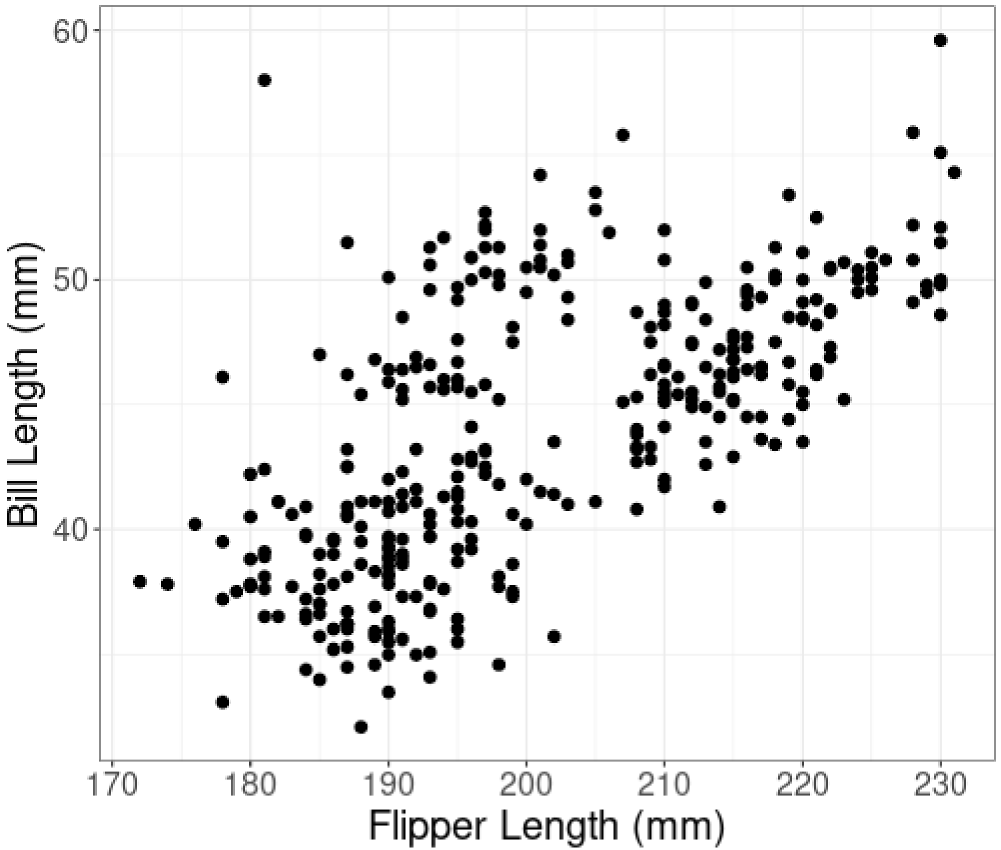
Penguins!

- For our running example, we'll look at the `palmerpenguins` dataset which contains measurements on three species of penguins in the Palmer Archipelago in Antarctica



Clustering

- Suppose we didn't observe the labels, could we still recover the latent "clusters" in the data?

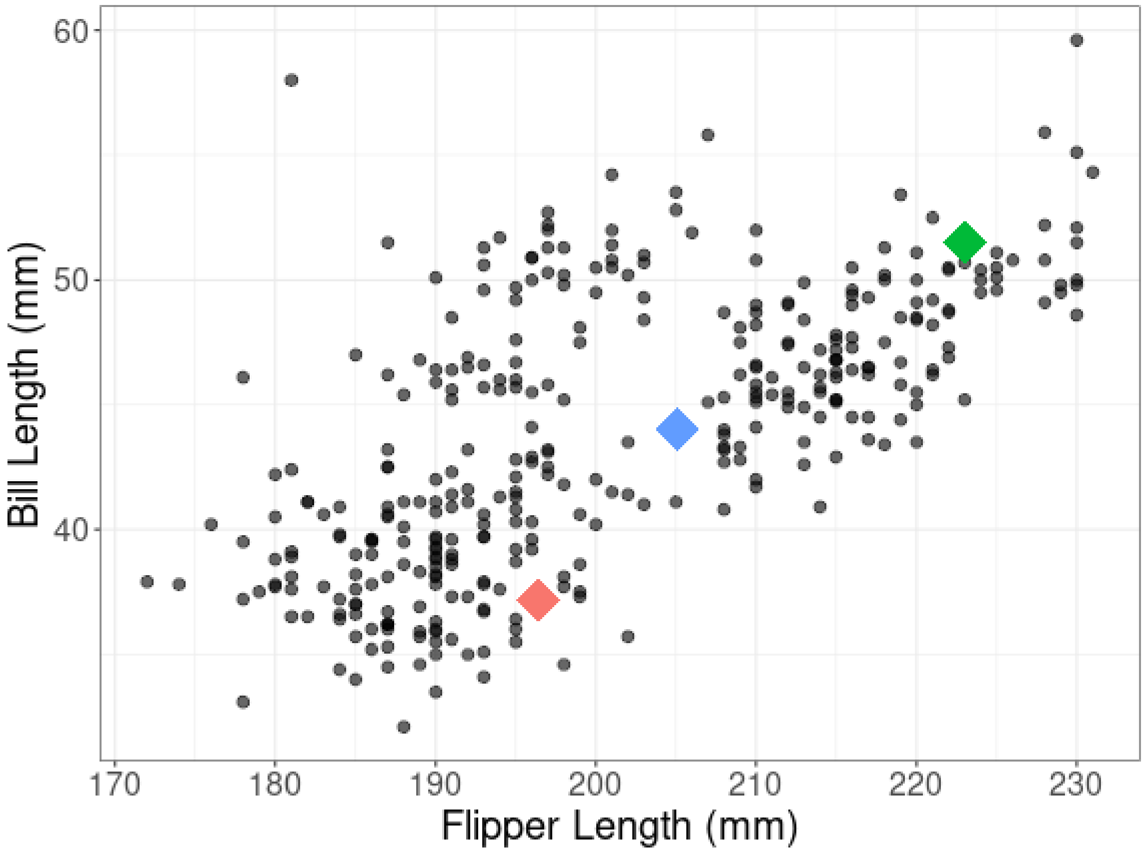


K-means

- A simple algorithm for generating clusters that has no underlying probabilistic model is the **K-means algorithm**
    - Straightforward to implement and still surprisingly popular and effective for a "first cut" at the data.
- **K-means algorithm**
    1. Initialize K distinct clusters by randomly assigning points to one of the k groups
    2. Calculate μ_k as the mean of Y_i in observations in cluster k.
    3. Reassign each point to the cluster k that has the smallest distance between Y_i and μ_k.
    4. Repeat 2-3 until no points change their assignments.

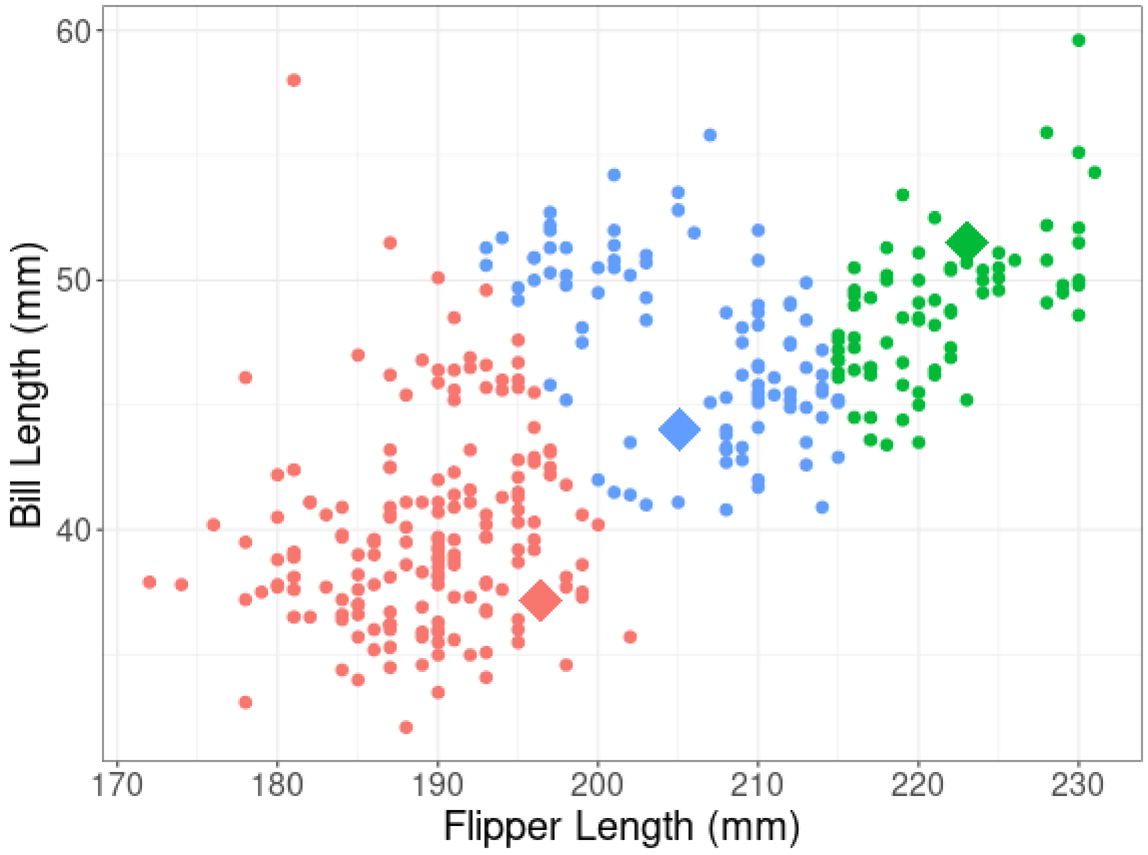
K-means example

- Let $K = 3$ and randomly pick three initial μ_k



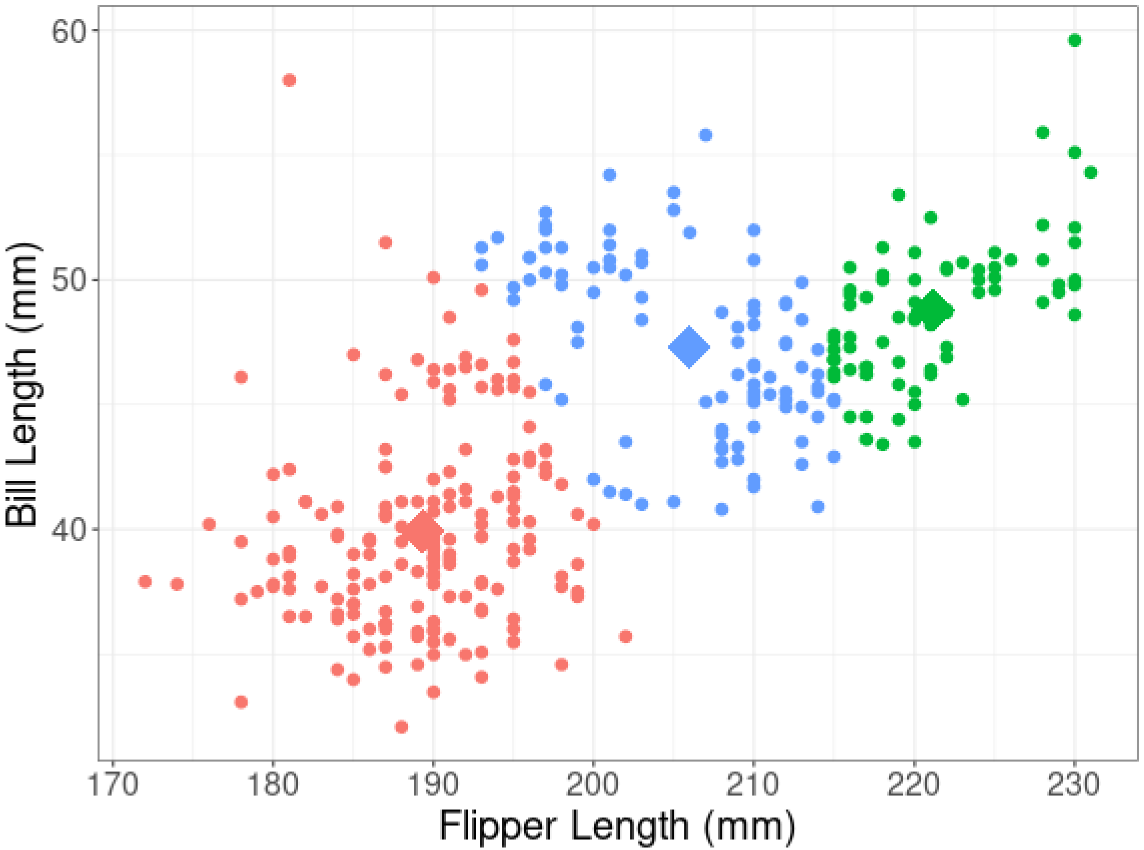
K-means example

- Assign each penguin to the cluster with the nearest cluster centroid.



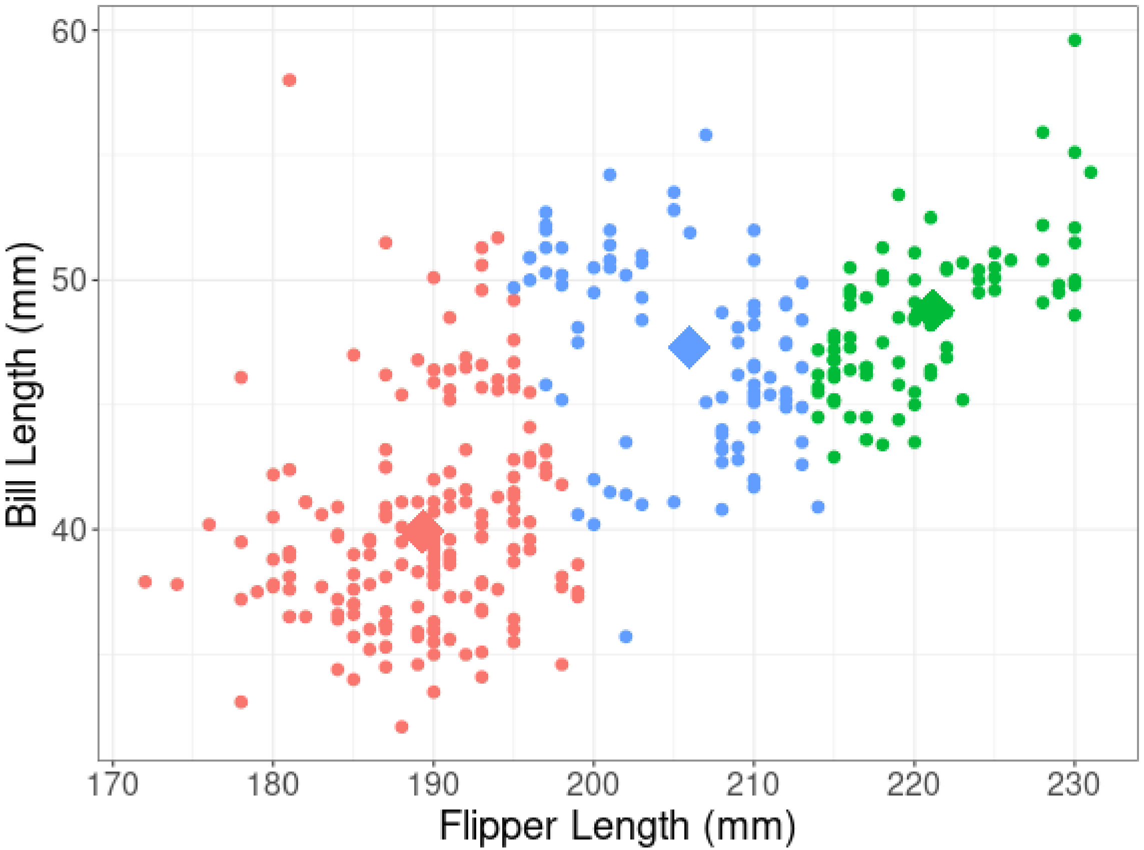
K-means example

- Recompute the cluster means



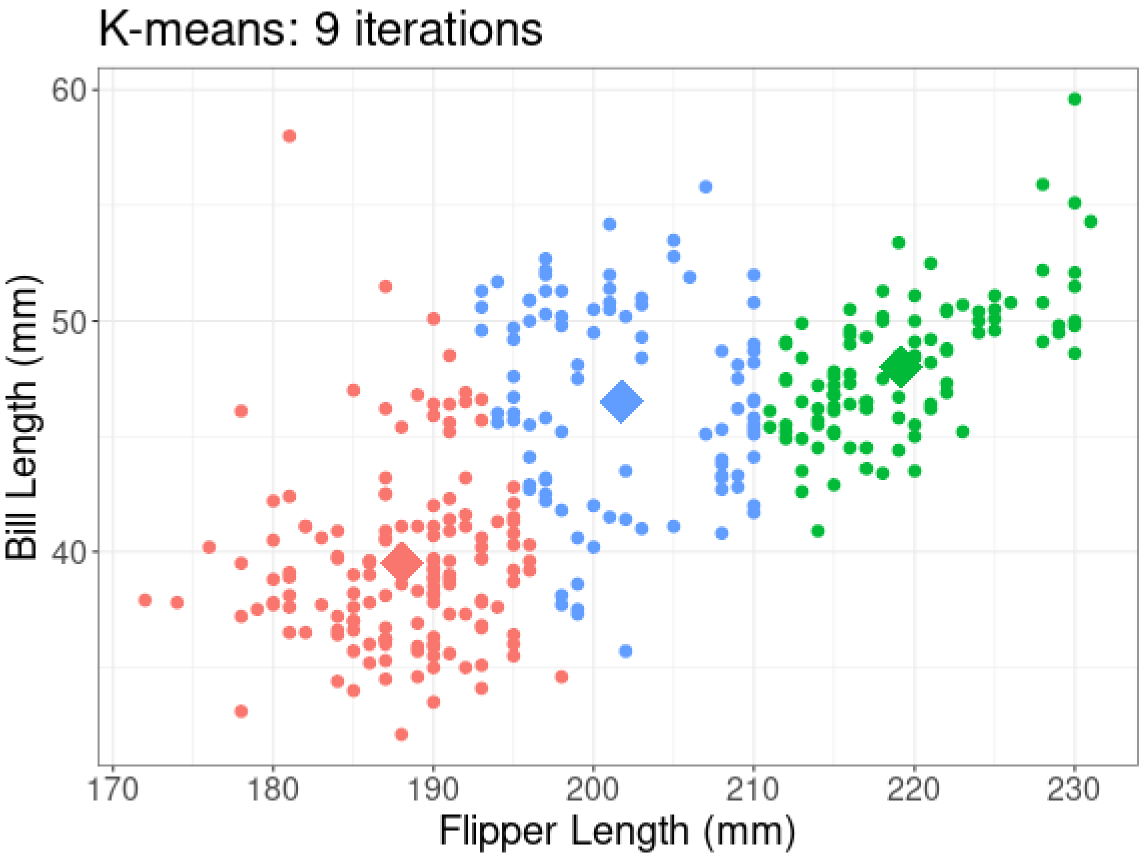
K-means example

- Re-label the points



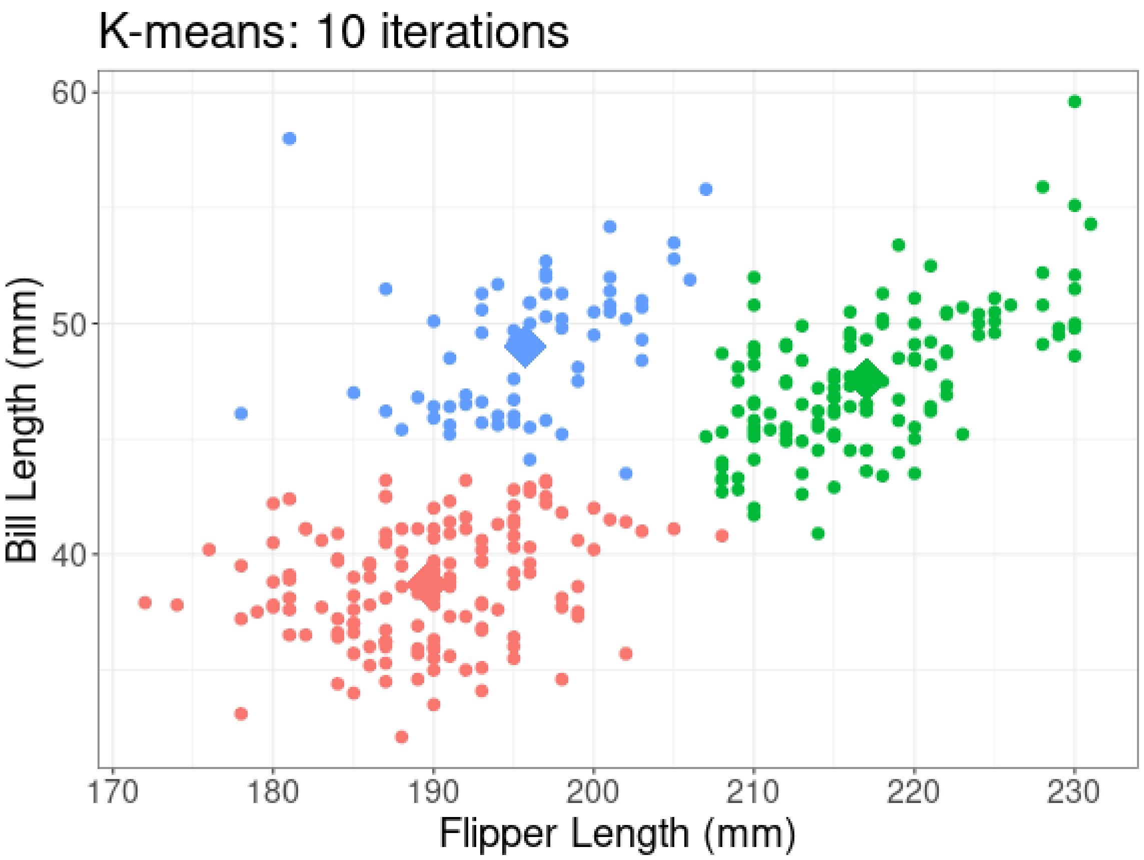
K-means example

- Iterate until convergence



K-means example

- Note some sensitivity to the choice of distance metric. Here's what it looks like if we standardize each Y by it standard deviation.



K-means example

- How do our clusters compare to the **actual** categories? Surprisingly well!

```
table(penguins_complete$cluster2, penguins_complete$species)
```

```
##
##      Adelie Chinstrap Gentoo
##   1     146         5      0
##   2       1         4    122
##   3       4        59      1
```

- But the clusters themselves don't have any underlying interpretation or meaning
  - We **assign** the interpretation to them through inspection!

K-means

- Advantages
  - Easy to implement, fast updating steps and quick convergence.
- Disadvantages
  - Multimodality (we can't really solve this one!)
  - Choice of distance metric matters.
  - Hard to make sense of outcomes that aren't in R^d (e.g. binary/discrete outcomes)
- With conventional K-means, we have no underlying probability model -- each unit is assigned to a single cluster.
  - Sometimes called **"hard"** K-means.
- We can instead implement an approach sometimes referred to as a "soft" K-means algorithm
  - Our target of inference is the **probability** that a unit belongs to each cluster.

Gaussian Mixture Models

- Assume **latent variables** $z = \{z_1, z_2, \ldots, z_N\}$

$$z_i \sim \text{i.i.d.}\text{Categorical}(\pi)$$

- Then, the outcome vector Y_i has a multivariate gaussian distribution conditional on $Z_i = k$

$$Y_i | z_i = k \sim \text{Normal}(\mu_k, \Sigma_k)$$

- Our goal is to infer μ_k and Σ_k for each cluster
    - And conditional on these estimates, we can obtain the cluster probabilities $Pr(Z_i = k | Y_i)$

Expectation-Maximization

- Let's write down the marginal likelihood, marginalizing out the latent parameters:

$$\mathcal{L}(\mu, \Sigma, \pi; \mathbf{Y}) = \prod_{i=1}^{N} \sum_{k=1}^{K} f(Y_i \mid \mu_k, \Sigma_k, Z_i = k) \times p(Z_i = k \mid \pi)$$

- This is a tricky likelihood to maximize - if we take the log, we get a log of a sum (which doesn't simplify as neatly as a log of a product)

$$\ell(\mu, \Sigma, \pi; \mathbf{Y}) = \sum_{i=1}^{N} \log\left(\sum_{k=1}^{K} \pi_k \times \mathcal{N}(Y_i \mid \mu_k, \Sigma_k) \right)$$

- Suppose we knew the Z_i as well (they were like "data"), then the "complete" log-likelihood would be easier to maximize!

$$\ell(\mu, \Sigma, \pi; \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{N} \sum_{k=1}^{K} \log\left(\pi_k^{I(Z_i=k)} \times \mathcal{N}(Y_i \mid \mu_k, \Sigma_k)^{I(Z_i=k)} \right)$$

Expectation-Maximization

- The **Expectation-Maximization** algorithm is another iterative method for obtaining a maximum likelihood estimate (or maximum a posteriori (MAP) in a Bayesian setting) when the likelihood consists of a sum/integral over the distribution of some **latent variables**
  - It uses the idea of iteratively optimizing a **lower bound** on the likelihood until convergence is reached.
  - The key trick is **Jensen's inequality**. For a **concave** function like the logarithm:

$$\log(E[X]) \geq E[\log(X)]$$

- Consider the general setting where we have a parameter θ, data $\mathbf{X}$ and discrete latent variables Z
- We want to optimize the log-likelihood:

$$\ell(\theta; \mathbf{X}) = \log \sum_{\mathbf{Z}} f(\mathbf{X}, \mathbf{Z} \mid \theta) = \log \sum_{\mathbf{Z}} f(\mathbf{X} \mid \mathbf{Z}, \theta) p(\mathbf{Z} \mid \theta)$$

Expectation-Maximization

- Instead of optimizing the log-likelihood, let's try to come up with a lower-bound. Define an arbitrary distribution on the latent variables $q(\mathbf{Z})$.
- Multiply by 1:

$$\ell(\theta; \mathbf{X}) = \log \sum_{\mathbf{Z}} f(\mathbf{X} \mid \mathbf{Z}, \theta) p(\mathbf{Z} \mid \theta) \frac{q(\mathbf{Z})}{q(\mathbf{Z})}$$

- Rearranging terms, we can see that this can be written as an expectation over the distribution q

$$\ell(\theta; \mathbf{X}) = \log E_q \left[\frac{f(\mathbf{X} \mid \mathbf{Z}, \theta) p(\mathbf{Z} \mid \theta)}{q(\mathbf{Z})} \right]$$

- By **Jensen's inequality**, we have a lower bound

$$\ell(\theta; \mathbf{X}) \geq E_q \left[\log \left(\frac{f(\mathbf{X} \mid \mathbf{Z}, \theta) p(\mathbf{Z} \mid \theta)}{q(\mathbf{Z})} \right) \right]$$

Expectation-Maximization

- And by properties of logs, our lower-bound is:

$$\ell(\theta; \mathbf{X}) \geq E_q[\log(f(\mathbf{X} \mid \mathbf{Z}, \theta))] + E_q[\log(f\mathbf{Z} \mid \theta))] - E_q[\log(q(\mathbf{Z})]$$

- We can optimize this **iteratively** by switching between finding an optimal distribution $q^{(t+1)}$ given parameter values $\theta^{(t)}$ and finding parameter values $\theta^{(t+1)}$ given an existing choice of $q^{(t)}$.

1. **Expectation** step: Find $q^{(t+1)}$ (what distribution are we taking the expectation over)
2. **Maximization** step: Find $\theta^{(t+1)}$ (given our q distribution, what is the value of the parameter values that maximizes the lower bound).

E-step

- Can we find a closed-form "optimal" solution for our update $q^{(t+1)}$ given $\theta^{(t)}$?
  - Yes, find the q that makes the inequality an equality!

$$\ell(\theta^{(t)}; \mathbf{X}) \geq E_q\left[\log\left(\frac{f(\mathbf{X} \mid \mathbf{Z}, \theta^{(t)})p(\mathbf{Z} \mid \theta^{(t)})}{q(\mathbf{Z})}\right)\right]$$

$$\ell(\theta^{(t)}; \mathbf{X}) \geq \sum_{\mathbf{Z}} \log\left(\frac{f(\mathbf{X} \mid \mathbf{Z}, \theta^{(t)})p(\mathbf{Z} \mid \theta^{(t)})}{q(\mathbf{Z})}\right)q(\mathbf{Z})$$

E-step

- Let's try to get the $\mathbf{Z}$ out of the log. Start by factoring the joint distribution in the numerator conditioning on $\mathbf{X}$ instead of $\mathbf{Z}$

$$\ell(\theta^{(t)}; \mathbf{X}) \geq \sum_{\mathbf{Z}} \log\left(\frac{f(\mathbf{Z} \mid \mathbf{X}, \theta^{(t)}) p(\mathbf{X} \mid \theta^{(t)})}{q(\mathbf{Z})}\right) q(\mathbf{Z})$$

- Now we can see the optimal choice for q^{t+1} revealed to us. Suppose we plug in $q = f(\mathbf{Z} \mid \mathbf{X}, \theta^{(t)})$, the conditional distribution of Z given X and $\theta^{(t)}$:

$$\ell(\theta^{(t)}; \mathbf{X}) \geq \sum_{\mathbf{Z}} \log(p(\mathbf{X} \mid \theta^{(t)}) \times f(\mathbf{Z} \mid \mathbf{X}, \theta^{(t)})$$

- The logged term no longer depends on $\mathbf{Z}$, so pull it out of the sum

$$\ell(\theta^{(t)}; \mathbf{X}) \geq \log(p(\mathbf{X} \mid \theta^{(t)}) \times \sum_{\mathbf{Z}} f(\mathbf{Z} \mid \mathbf{X}, \theta^{(t)})$$

E-step

- The sum is equal to 1 (sum over a PMF/PDF) and we're left with the definition of the marginal log-likelihood, so this expression holds with *equality*

$$\ell(\theta^{(t)}; \mathbf{X}) = \log(p(\mathbf{X} \mid \theta^{(t)})$$

- So our choice of distribution over which to take the expectation of the joint likelihood is $q^{(t+1)} = f(\mathbf{Z} \mid \mathbf{X}, \theta^{(t)})$

M-step

- Let's go back to our lower bound - for a given value of $q^{(t)}$, we want to find the $\theta^{(t+1)}$ that maximize the "complete data" log-likelihood

$$\ell(\theta; \mathbf{X}) \geq E_{q^{(t)}}[\log(f(\mathbf{X} | \mathbf{Z}, \theta))] + E_{q^{(t)}}[\log(f(\mathbf{Z} | \theta))] - E_{q^{(t)}}[\log(q^{(t)}(\mathbf{Z})]$$

- Since the third term doesn't depend on θ, we only need to worry about the first two.
  - This is sometimes called the "Q-Function" $Q(\theta | \theta^{(t)})$ or the expected "complete data" log-likelihood

$$Q(\theta | \theta^{(t)}) = E_{\mathbf{Z}|\mathbf{X},\theta^{(t)}}[\log(f(\mathbf{X}, \mathbf{Z} | \theta))]$$

- The M-step sets $\theta^{(t+1)}$ to the value of θ that maximizes this Q-function

$$\theta^{(t+1)} = \arg\max_{\theta} Q(\theta | \theta^{(t)})$$

Deriving EM for the GMM

- For current values of $\mu^{(t)}$, $\Sigma^{(t)}$, $\pi^{(t)}$, let's derive the Q function.
- We'll start by deriving the conditional distribution $z_i \mid Y_i, \mu^{(t)}, \Sigma^{(t)}, \pi^{(t)}$ using Bayes' rule

$$p(z_i = k \mid Y_i, \mu^{(t)}, \Sigma^{(t)}, \pi^{(t)}) = \frac{p(z_i = k \mid \pi_k^{(t)}) f(Y_i \mid \mu^{(t)}, \Sigma^{(t)}, z_i = k)}{f(Y_i \mid \mu^{(t)}, \Sigma^{(t)})}$$

- Given $z_i = k$, we know the distribution is normal at mean $\mu_k^{(t)}$ and variance $\Sigma_k^{(t)}$. And the denominator is just the marginal

$$\gamma_i^k = p(z_i = k \mid Y_i, \mu^{(t)}, \Sigma^{(t)}, \pi^{(t)}) = \frac{\pi_k^{(t)} \times \mathcal{N}(Y_i \mid \mu_k^{(t)}, \Sigma_k^{(t)})}{\sum_{j=1}^{K} \pi_j^{(t)} \times \mathcal{N}(Y_i \mid \mu_j^{(t)}, \Sigma_j^{(t)})}$$

- These weights, γ_i^k are sometimes called the "responsibility" parameters as they denote the extent to which each cluster is "responsible" for an observation.

Deriving EM for the GMM

- Recall the "complete likelihood"

$$\ell(\mu, \Sigma, \pi; \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{N}\sum_{k=1}^{K} \log\left(\pi_k^{I(Z_i=k)} \times \mathcal{N}(Y_i \mid \mu_k, \Sigma_k)^{I(Z_i=k)}\right)$$

- Simplify it a bit

$$\ell(\mu, \Sigma, \pi; \mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{N}\sum_{k=1}^{K} I(Z_i = k)\log(\pi_k) + \sum_{i=1}^{N}\sum_{k=1}^{K} I(Z_i = k)\log\mathcal{N}(Y_i \mid \mu_k, \Sigma_k)$$

- Now, taking the expectation over Z_i, the only component that is not a constant is $I(Z_i = k)$ and $E[I(Z_i = k)] = p(Z_i = k)$ (fundamental bridge).
  - And we got that (conditional) expectation in the previous section: γ_i^k
- So our Q function is

$$Q(\theta \mid \theta^{(t)}) = \sum_{i=1}^{N}\sum_{k=1}^{K} \gamma_i^k \log(\pi_k) + \sum_{i=1}^{N}\sum_{k=1}^{K} \gamma_i^k \log\mathcal{N}(Y_i \mid \mu_k, \Sigma_k)$$

Deriving EM for the GMM

- Closed form solutions are straightforward to obtain for the M-step (and follow from weighted regression)

$$n_k = \sum_{i=1}^{N} \gamma_i^k$$

$$\pi_k = \frac{n_k}{N}$$

$$\mu_k = \frac{1}{n_k} \sum_{i=1}^{N} \gamma_i^k Y_i$$

$$\Sigma_k = \frac{1}{n_k} \sum_{i=1}^{N} \gamma_i^k (Y_i - \mu_k)(Y_i - \mu_k)'$$

Implementing EM

```
gmm_loglik <- function(Y, K, mu, sigma, pi){
  # Log-likelihood of each Y
  lik_normal <- matrix(nrow=nrow(Y), ncol=K)
  for(k in 1:K){
    lik_normal[,k] <- mvtnorm::dmvnorm(Y, mu[k,], sigma[[k]])
  }
  # Log of the sums
  log_likelihood <- sum(apply(lik_normal, 1, function(x) log(sum(x*pi))))
  return(log_likelihood)
}
```

Implementing EM

```r
gmm_estep_gamma <- function(Y, K, mu, sigma, pi){
  # Calculate unnormalized gamma_k for each k
  gamma <- matrix(nrow=nrow(Y), ncol=K)
  for (k in 1:K){
    gamma[,k] <- pi[k]*mvtnorm::dmvnorm(Y, mu[k,], sigma[[k]])
  }
  # Normalize the gammas (denominator)
  gamma <- gamma/rowSums(gamma)
  return(gamma)
}
```

Implementing EM

```r
sigma_update <- function(Y, K, mu_k, gamma, n_k){ # Super inefficient but works
  # For each k
  sigma <- list()
  for (k in 1:K){
    sigma[[k]] <- matrix(data=0, nrow=ncol(Y), ncol=ncol(Y)) # blank matrix
    for (n in 1:nrow(Y)){
      sigma[[k]] <- sigma[[k]] + gamma[n,k]*outer(Y[n,] - mu_k[k,], Y[n,] - mu_k[k,])
    }
    sigma[[k]] <- sigma[[k]]/n_k[k]
  }
  return(sigma)
}
```

Implementing EM

```r
gmm_em <- function(Y, K, maxit=5000, tol=1e-8, verbose=F){
  D <- ncol(Y) # Number of dimensions
  ## Initialize the parameters (pick some reasonable starting points)
  mu_k <- MASS::mvrnorm(K, colMeans(Y), Sigma = var(Y)) # Matrix of means
  sigma_k <- list() # list of covariances
  for (k in 1:K){
    sigma_k[[k]] <- var(Y)
  }
  pi <- rep(1/K, K)
  curr_lik <- gmm_loglik(Y, K, mu_k, sigma_k, pi) #Evaluate current likelihood
  if(verbose) print(str_c("Log-Likelihood: ", curr_lik))
  for(iter in 2:maxit){
    # E-step
    gamma <- gmm_estep_gamma(Y, K, mu_k, sigma_k, pi)
    # M-step
    n_k <- colSums(gamma)
    pi <- n_k/sum(n_k)
    for(k in 1:K){
        mu_k[k,] <- colSums(gamma[,k]*Y)/n_k[k]
    }
    sigma_k <- sigma_update(Y, K, mu_k, gamma, n_k)
    # Check convergence
    new_lik <- gmm_loglik(Y, K, mu_k, sigma_k, pi)
    if (abs(new_lik - curr_lik) < tol){
      gamma <- gmm_estep_gamma(Y, K, mu_k, sigma_k, pi)
      if(verbose)print(str_c("Log-Likelihood: ", curr_lik))
      if(verbose) print(str_c("Log-Likelihood: ", new_lik))
```

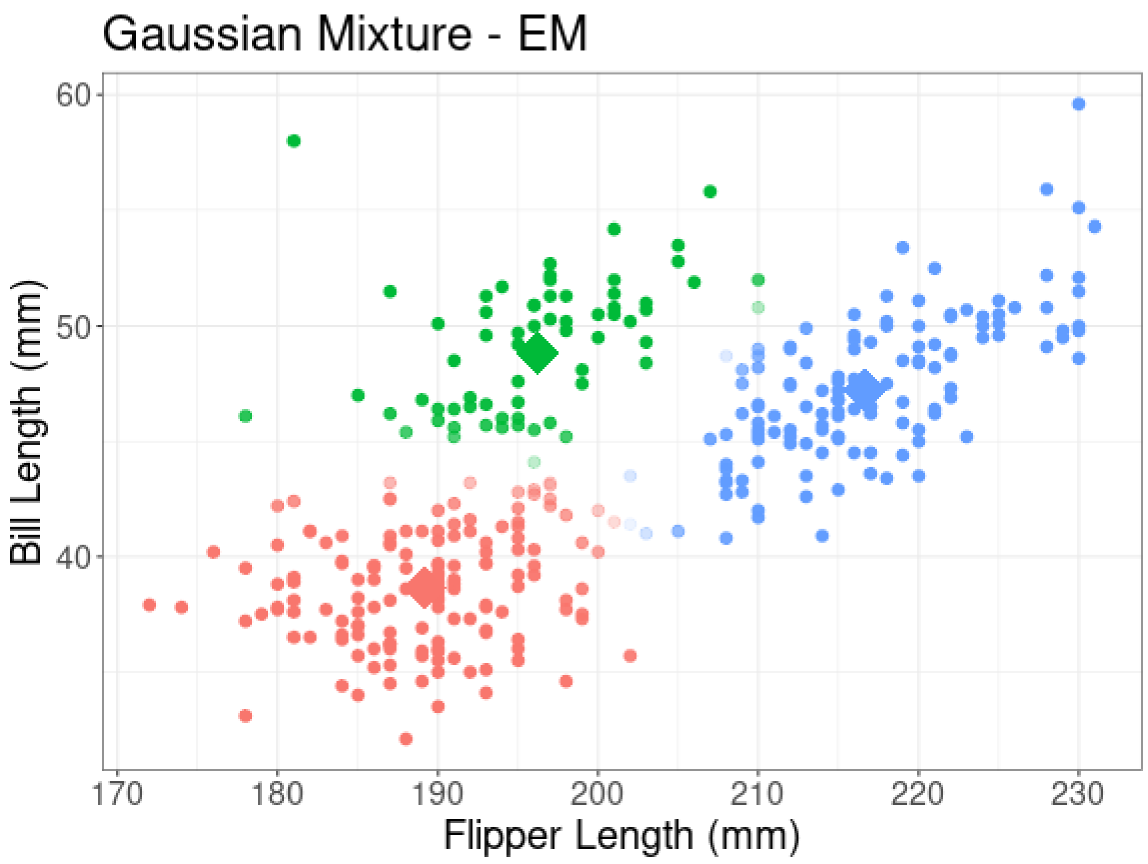
Implementing EM

```
set.seed(60639)
penguins_em_3 <- gmm_em(Y=as.matrix(penguins_complete %>% select(flipper_length_mm, bill_length
```

```
## [1] "Log-Likelihood: -2464.23892416674"
## [1] "Log-Likelihood: -2286.49407632345"
## [1] "Log-Likelihood: -2265.16441563185"
## [1] "Log-Likelihood: -2244.45549391833"
## [1] "Log-Likelihood: -2244.22093530689"
## [1] "Log-Likelihood: -2244.21933979789"
## [1] "Log-Likelihood: -2244.21927829982"
## [1] "Log-Likelihood: -2244.21927596998"
## [1] "Log-Likelihood: -2244.21927591305"
## [1] "Log-Likelihood: -2244.21927590356"
## [1] "Complete!"
```

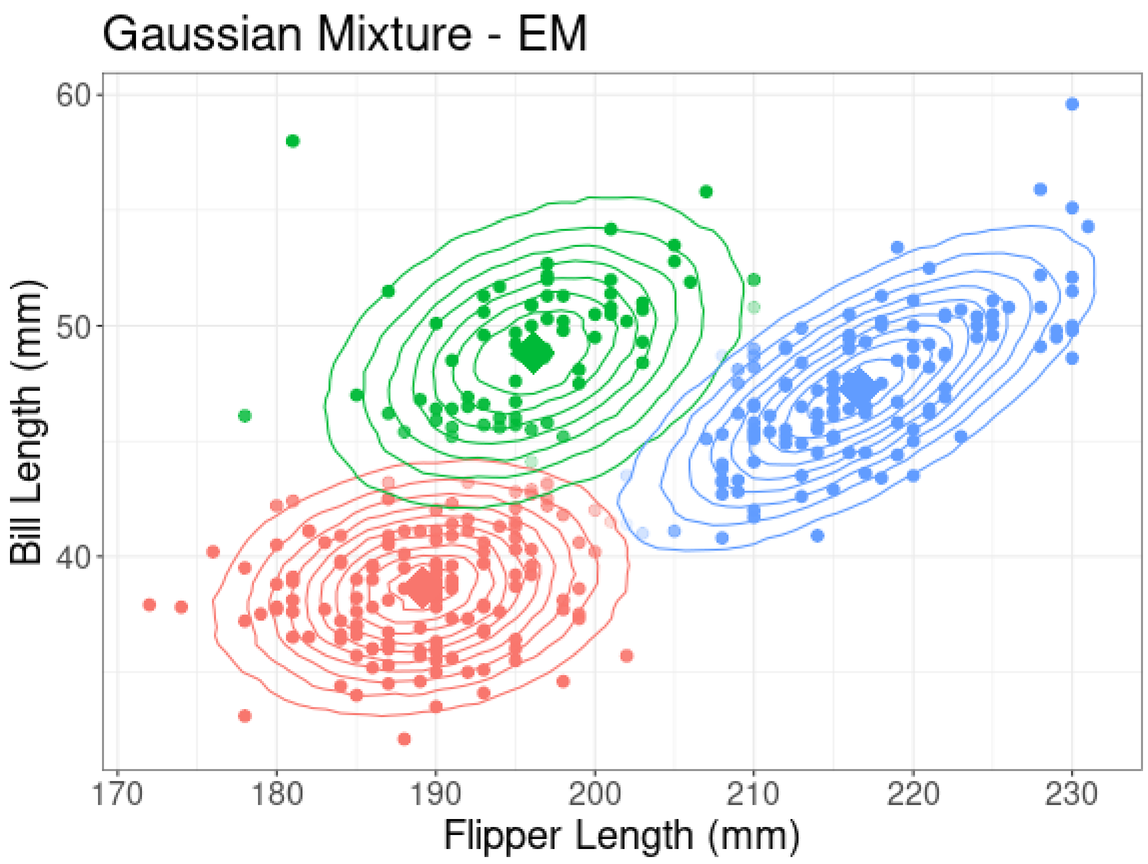
Visualizing EM

- Now our point labels have **probabilities** attached



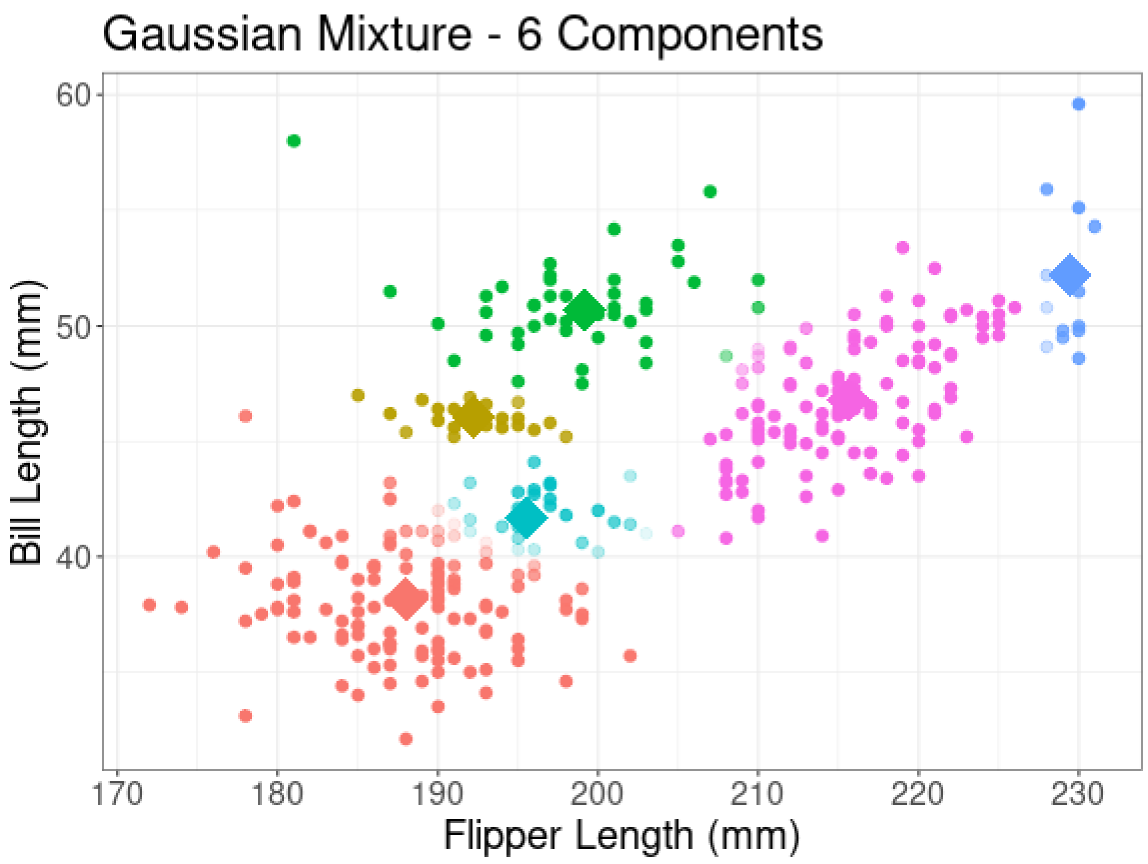
Visualizing EM

- We can also visualize the component distributions



Alternate K

- What does the 6-component mixture look like?



Challenges with mixture models

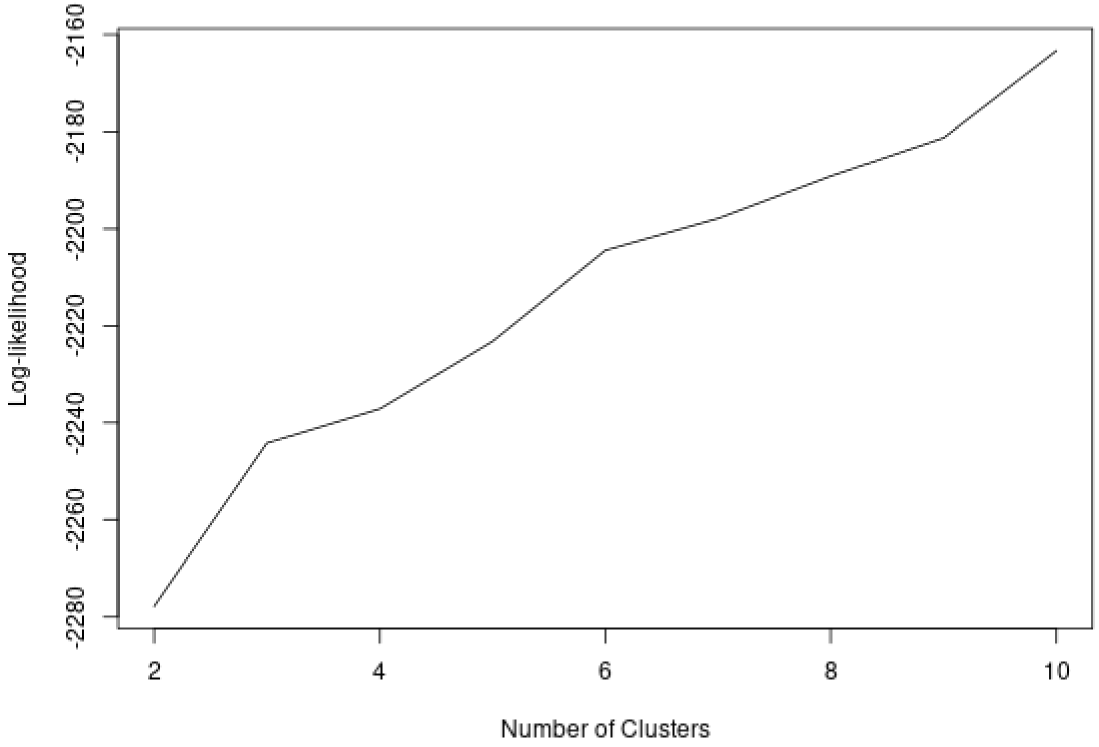
- **Highly model dependent** - our identification of the cluster centroids + variances depends heavily on our selection of the appropriate distribution for Y_i
  - Normal does okay for outcomes where it's plausible that CLT kicks in, but not true of all Y_i
- **Multi-modality** - Mixture model likelihoods often have multiple modes and EM is only guaranteed to converge to a **local** optimum
  - Can get stuck in a "bad" EM run (**Solution**: Run multiple EM chains with different starting values and pick the one with the best log-likelihood)
  - "Label-switching" problem: Permuting the labels doesn't change the likelihood
- **Challenges with Bayes** - Sampling-based inference can be tricky with GMMs due to the label-switching problem.
  - Can implement many models via MCMC/Gibbs but need tricks to avoid having the chain jump between permutations.
  - Common to use an approximation to the likelihood around the posterior mode obtained via EM.
  - Stan doesn't like sampling discrete latent variables.

Choosing K

- Choosing K is a problem of model comparison and selection.
  - Just as in regression, more parameters = better **in-sample** fit.
  - But we want to avoid **over-fitting**- Just as before, two approaches
  - **Information criteria** - Evaluate the in-sample log-likelihood at the maximum, penalized by some factor for the number of parameters
  - **Cross-validation** - Estimate the model on a **training set**. Compute the log-likelihood on a **held-out** test set.

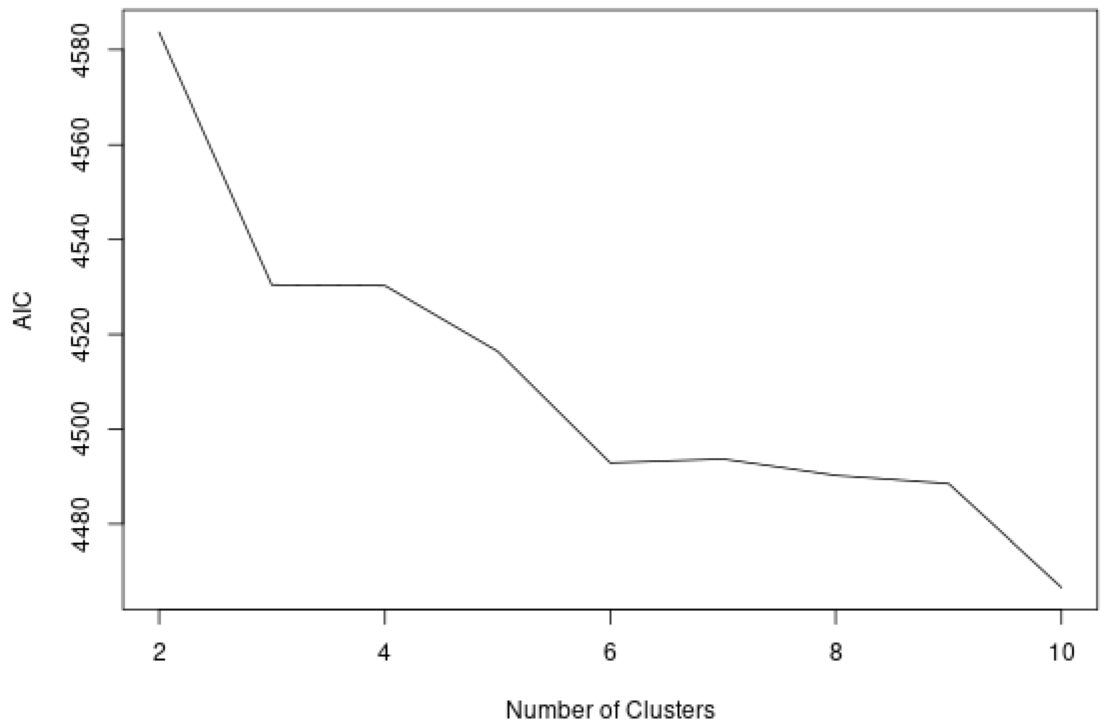
Choosing K

- Log-likelihood doesn't decrease as K increases, but common to look for an "elbow" where the increases become marginal



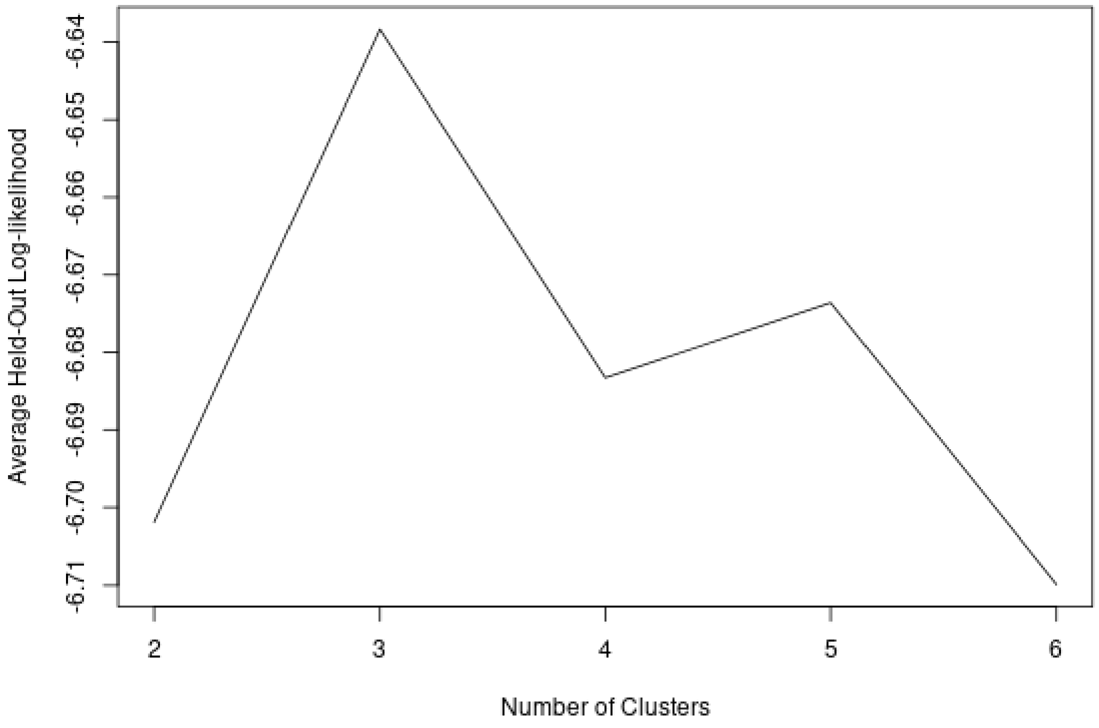
Choosing K

- Information criteria results depend on how much you penalize the additional parameters
  - $\text{AIC} = 2k - 2\ell(\hat{\theta})$ shows minimal improvement beyond 6 clusters (but still potentially some!)



Choosing K

- 10-fold cross-validation gives best performance at $K = 3$



Topic Modeling

Modeling Text

- **Topic models** are a tool for **unsupervised** analysis of text
    - Don't need to label the texts ex-ante.
- Texts are **high-dimensional** - Topic models provide a **lower-dimensional** summary of the common themes in a document
    - Model document content as a **mixture** of a finite number of K latent **topics**
    - Topics are described as a **distribution** on words.
- History of topic models
    - **Latent Dirichlet Allocation** $\rightsquigarrow$ **Correlated Topic Model** $\rightsquigarrow$ **Structural Topic Model**

Example: Political Blogs

- Our running example is the CMU 2008 Political Blog corpus (Eisenstein and Xing, 2010)
    - 13,246 posts from 8 political blogs rated liberal/conservative

```r
library(stm)
poliblog <- read_csv("data/poliblogs2008.csv")
poliblog$documents[2:10] # Example document
```

```
## [1] "I honestly don't know how either party's caucus results will play out tonight. Usually, you can
## [2] "While we stand in awe of the willingness of our troops in Iraq to sacrifice themselves for the M
## [3] "These pages recently said goodbye to global warming.  Ironically, the current spell of
## [4] "A US report shows how the enemy controlled the information on the battlefield in Fallujah and us
## [5] "Mike Huckabee is pretty slick. He's the one Republican who's been boosted by the big media. Thos
## [6] "In the aftermath of the \"upset\" victories in yesterday's Iowa caucus by Republican Mike Huckab
## [7] "FrontPage.Com has an editorial up this morning pointing out the deficiencies of both of the Iowa
## [8] "An influential policy outfit, the International Crisis Group, has called on President Musharraf
## [9] "Goodbye Iowa for 4 more years. But the Hawkeye State may have crowned the new President this tim
```

Pre-processing the text

- We will need to convert the raw documents into a **document-term matrix**
  - **Rows**: Documents
  - **Columns**: "Tokens" (words)
- The `stm` package in R, which implements the **Structural Topic Model** has a host of helper functions for parsing texts and applying conventional pre-processing techniques (stemming, stop word removal, etc...)
  - `textProcessor` wraps the popular `tm` package for text mining

```
processed <- textProcessor(poliblog$documents, metadata=poliblog,
                           lowercase=T, removestopwords=T,
                           removenumbers = T, removepunctuation=T,
                           stem = T)
```

```
out <- prepDocuments(processed$documents, processed$vocab, processed$meta,
                     lower.thresh=round(.01*nrow(poliblog)), upper.thresh = round(.99*nrow(poli
```

```
## Removing 121334 of 123990 terms (531226 of 2298953 tokens) due to frequency
## Your corpus now has 13246 documents, 2656 terms and 1767727 tokens.
```

"Bag of Words"

- We summarize each document using the counts of each token
  - These constitute our **document-term matrix** (which software packages typically store in a "sparse" format)

```
head(data.frame(token=out$vocab[out$documents$`2`[1,]], count=out$documents$`2`[2,]))
```

```
##      token count
## 1      ago     1
## 2    ahead     1
## 3    along     1
## 4 although     1
## 5    among     1
## 6    anoth     2
```

- The ordering of words is ignored - the "**bag of words**" assumption
  - Can preserve some ordering by tokenizing bigrams or trigrams rather than just unigrams (e.g. "republican party" or "daily tracking poll")
  - Interestingly, throwing out word order still typically preserves the underlying **topics** or **concepts** in a document!

Latent Dirichlet Allocation

- **Latent Dirichlet Allocation** (Blei, Ng, Jordan, 2003)
  - A generative model for **words** w - defined as unit-vectors from a discrete vocabulary $\{1, 2, \ldots, V\}$
  - **Documents** are collections of words $\mathbf{w} = \{w_1, w_2, \ldots w_N\}$
  - A **corpus** is a collection of documents $\{\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_D\}$

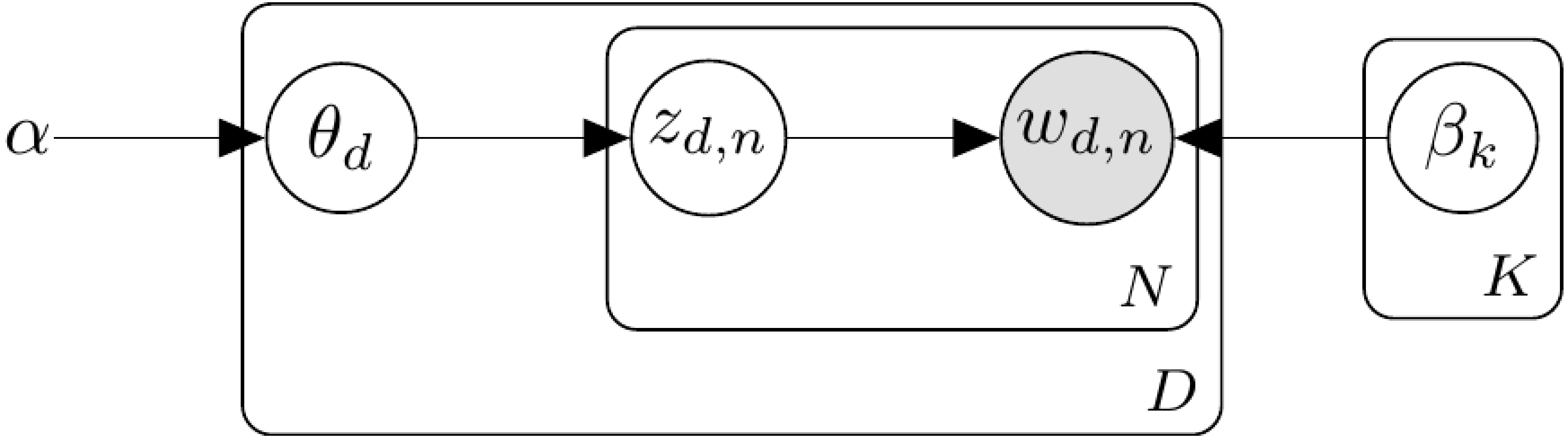
- *LDA* assumes a particular generative process for each document $\mathbf{w}_d$:

  1. Choose a **topic proportion** vector $\theta_d \sim \text{Dirichlet}(\alpha)$
  2. For each of the N words in document d $w_{d,n}$
     - Choose topic $z_{d,n} \sim \text{Multinomial}(\theta_d)$
     - Choose word $w_{d,n} \sim \text{Mutinomial}(\beta_{z_{d,n}})$

- Two main targets of inference:

  - The **document topic proportions**: $\theta = \{\theta_1, \theta_2, \ldots \theta_D\}$
  - The **topic-word distributions**: $\beta = \{\beta_1, \beta_2, \ldots \beta_k\}$

Latent Dirichlet Allocation

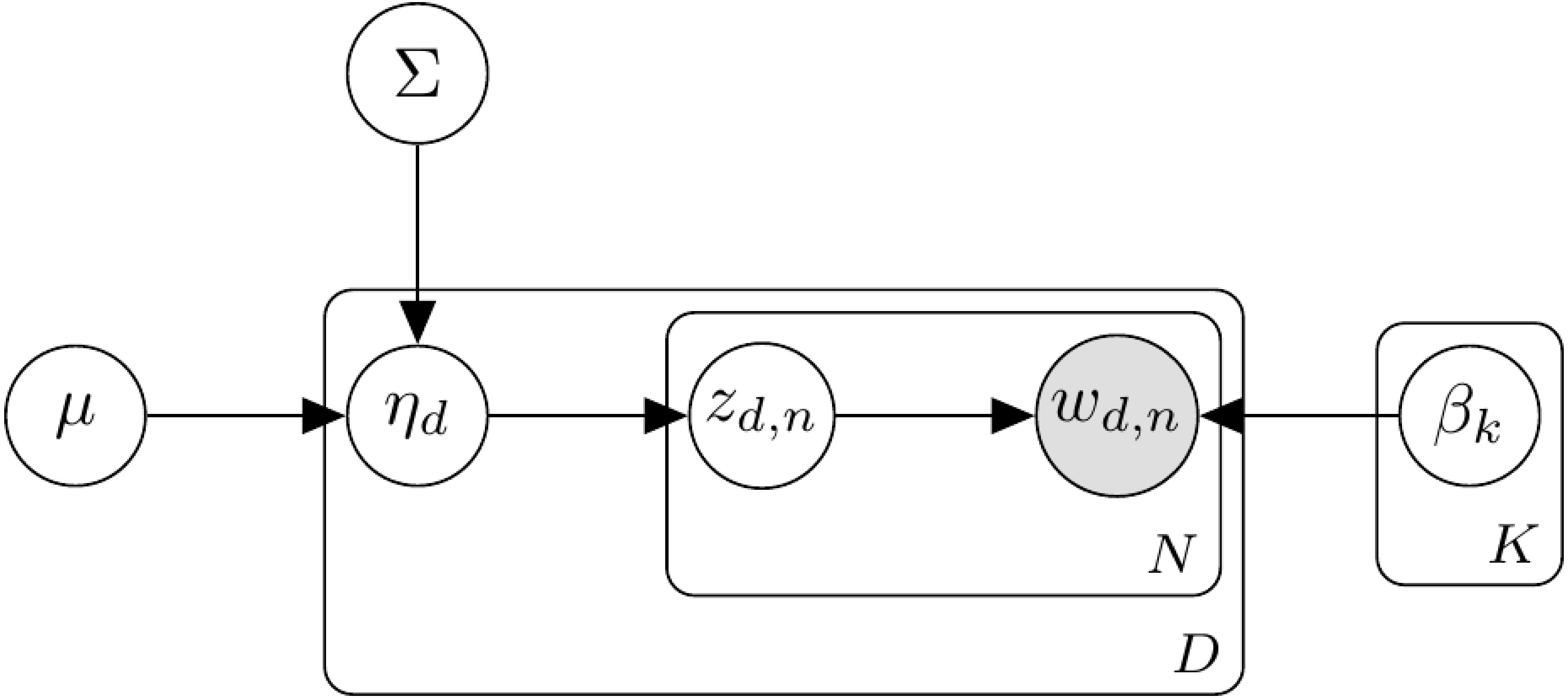


Correlated Topic Model

- LDA makes a somewhat restrictive assumption on the distribution from where the topics are drawn (the Dirichlet)
  - Topic proportions are assumed to be **independent** (a feature of the dirichlet)
  - But in reality, some topics are more likely to appear with others!
- The **Correlated Topic Model** (Blei and Raftery, 2006) addresses this
  - Replaces the dirichlet with a logistic normal distribution.
- The **Correlated Topic Model** assumes that each document $\mathbf{w}_d$ is generated by:

  1. Draw η_d from a **normal distribution** with mean μ, Σ
  2. For each of the N words $w_{d,n}$
     - Choose topic $z_{d,n} \sim \text{Multinomial}(f(\eta_d))$ where $f(\eta_{d,k}) = \exp(\eta_{d,k}) / \sum_j \exp(\eta_{d,j})$
     - Choose word $w_{d,n} \sim \text{Mutinomial}(\beta_{z_{d,n}})$

Correlated Topic Model



Structural Topic Model

- Often researchers want to incorporate **covariates** in modeling the topic proportions
  - e.g. - Do Liberal blogs talk about different topics compared to Conservative blogs?
  - Ad-hoc regressions of topic proportions on covariates will fail to incorporate uncertainty in the estimation of the topics.
- Additionally, researchers may want to have different word distributions for a topic depending on covariates
  - e.g. - Do Liberal blogs talk *about immigration* differently than Conservative blogs?
- The **Structural Topic Model** (Roberts, Stewart, Tingley, Airoldi, 2013) extends the correlated topic model to incorporate covariates:
  - Allow covariates X_d to enter into the logistic-normal distribution governing the topic proportions $\eta_d \sim \text{Normal}(\mu_d, \Sigma)$ where $\mu_{d,k} = X_d^{'}\gamma_k$
  - Allow covariates Y_d to enter into the β parameter governing topic content $\beta_{d,k} \propto \exp(m + \kappa_k^{(t)} + \kappa_{y_d}^{(c)} + \kappa_{y_d,k}^{(i)})$

Structural Topic Model

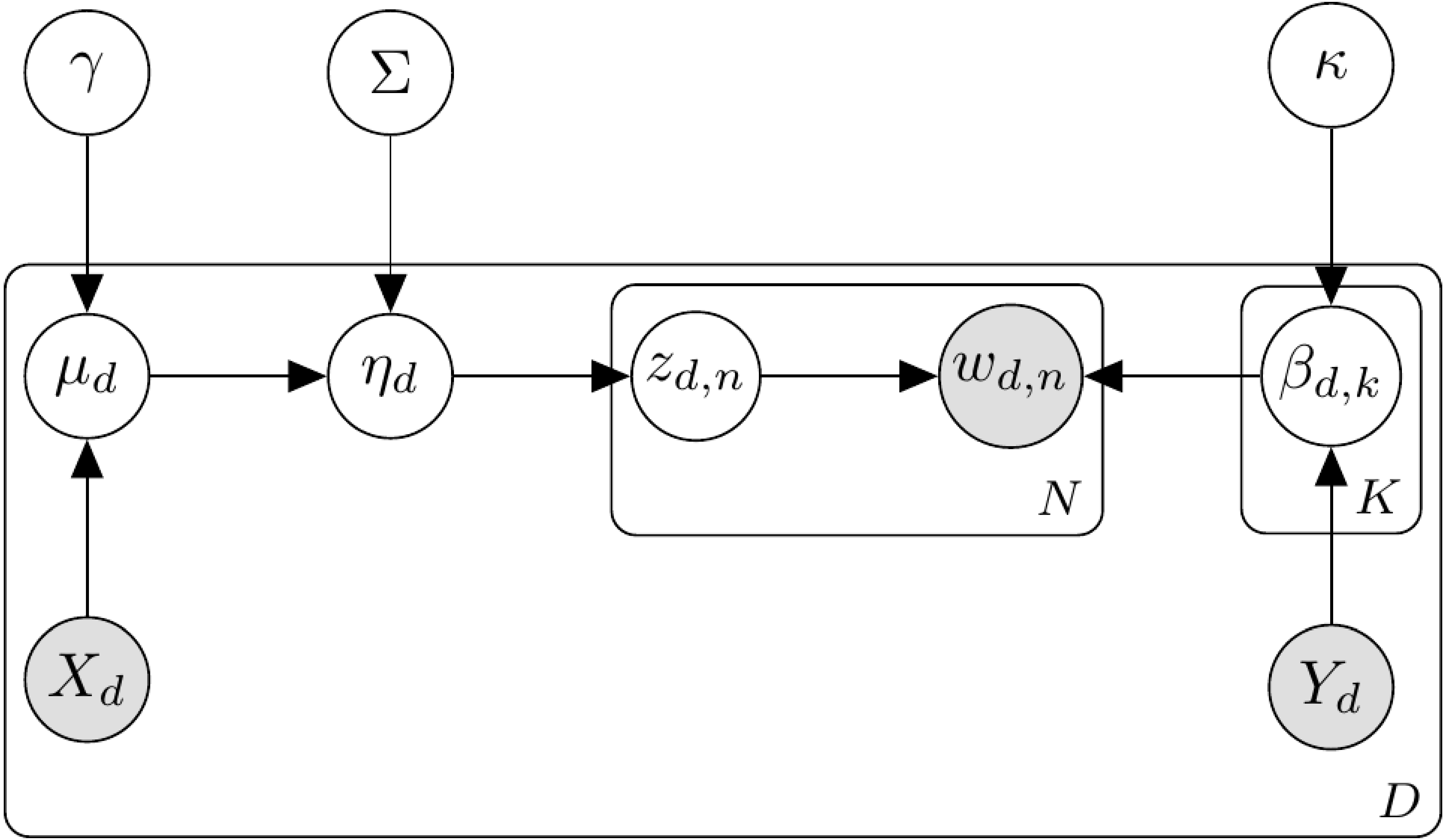


Illustration: Political Texts

- Without any covariates, `stm()` is a fast implementation of the **correlated topic model**

```
poliBlog_fit <- stm(documents = out$documents,
                    vocab = out$vocab,
                    K = 20, verbose=F,
                    max.em.its = 75, data = out$meta,
                    init.type = "Spectral", seed=60639)
```

- `stm()` implements a **Variational EM** algorithm to provide a tractable, analytical approximation to the posterior distribution
  - **Benefits**: Faster, deterministic
  - **Drawbacks**: Can understate the variance in the posterior
  - For more, see: Grimmer, 2010 "An Introduction to Bayesian Inference via Variational Approximations" *Political Analysis*

Illustration: Political Texts

- Typically, we're interested in understanding the content of the topics discovered
  - We'll inspect the **topic-word** distributions to attempt to infer the underlying topic meaning from a set of "top" words

```
labelTopics(poliBlog_fit, topics=20)
```

```
## Topic 20 Top Words:
##       Highest Prob: financi, crisi, market, govern, money, bank, bailout
##       FREX: financi, bailout, loan, mortgag, bank, fanni, market
##       Lift: mae, fanni, paulson, freddi, subprim, treasuri, bailout
##       Score: mae, bailout, mortgag, fanni, financi, loan, market
```

- Common criteria for ranking the "top" words
  - **Highest probability** - Which words have the highest probability in the topic-word distribution $\beta_k = p(w \mid z = k)$
  - **Frequency-Exclusivity** (FREX) - Harmonic mean of rank by within-topic probability and rank by topic distribution given word $p(z \mid w = v)$
  - **Lift** - Topic-word distribution divided by the empirical distribution - how **more common** are these words in this topic relative to their baseline prevalence)
  - **Score** - Similar to Lift (but in terms of log probabilities)

Illustration: Political Texts

- We can find "prototypical" documents that contain a high proportion of a particular topic

```
plotQuote(unlist(findThoughts(poliBlog_fit, texts = out$meta$documents, n=3, topics=20)$docs),
```

It would be the mother of all bailouts if it came to pass - hundreds of billions of taxpayer dollars used to shore up the two largest mortgage companies in the United States - Fannie Mae and Freddie Mac. The Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage

---

Over the weekend, the Federal Reserve opened a $30 billion line of credit for the purchase of troubled investment bank Bear Sterns and promised an open ended lending program for the biggest investment firms on Wall Street:In a third move aimed at helping banks and thrifts, the Fed also lowered the

---

When the New York Times refers to the bailout plan for Citigroup as "radical," it must be somewhere out near Mars: Federal regulators approved a radical plan to stabilize Citigroup in an arrangement in which the government could soak up billions of dollars in losses at the struggling bank, the

Illustration: Political Texts

- What about this topic?

```
labelTopics(poliBlog_fit, topics=16)
```

```
## Topic 16 Top Words:
##       Highest Prob: will, american, care, health, america, peopl, can
##       FREX: care, health, job, america, famili, educ, that
##       Lift: coal, health, prosper, poverti, care, insur, afford
##       Score: coal, health, tax, care, economi, american, insur
```

Illustration: Political Texts

- Not all topics make sense

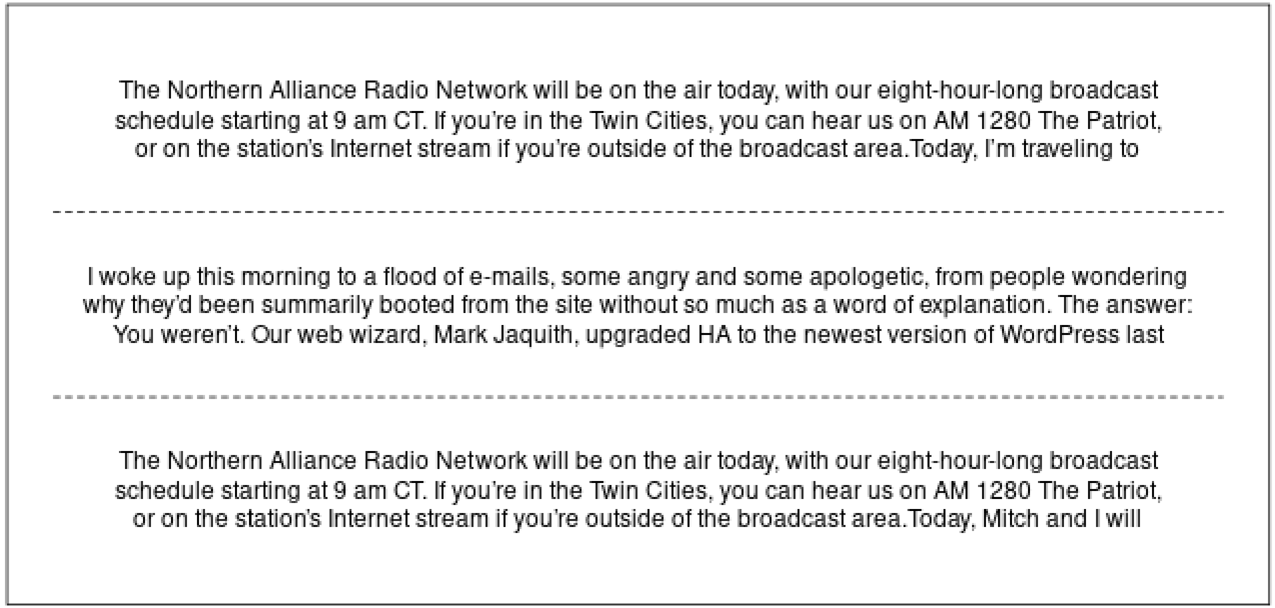
```
labelTopics(poliBlog_fit, topics=4)
```

```
## Topic 4 Top Words:
##       Highest Prob: get, one, ’re, don’t, like, want, doesn’t
##       FREX: doesn’t, ’ll, ’re, don’t, didn’t, can’t, ’ve
##       Lift: chat, see-dubya, ’ll, one’, can’t, isn’t, doesn’t
##       Score: chat, ’re, ’ll, don’t, doesn’t, ’ve, didn’t
```

Illustration: Political Texts

- Topic 4 seems to be picking up some odd stylistic feature of Minnesota conservative talk radio?

```
plotQuote(unlist(findThoughts(poliBlog_fit, texts = out$meta$documents, n=3, topics=4)$docs),
```



Choosing Models

- How do we evaluate the quality of a model?
  - ...for multiple runs of the same topic count (multiple modes)
  - ...for different numbers of topics?
- Conventional method: **held-out log-likelihood**. For an "out-of-sample" document $\mathbf{w}_{\text{out}}$:

$$\text{Perplexity} = \exp(-\log p(\mathbf{w}_{\text{out}} | \mu, \Sigma, \beta))$$

- But "highly predictive" topic models may be difficult for humans to interpret (Chang et. al., 2009).

Coherence and Exclusivity

- **Alternative** - Pick models where the topics have nice features
  - **Semantic coherence** - The most common words in a topic tend to co-occur in a document
  - **Exclusivity** - Words that have high probability in a topic have low probability in **other** topics (FREX).
- **Semantic coherence**
  - Let $D(v, v')$ define the number of times words v and v' appear together in a document.
  - Semantic coherence is defined as a sum across the W most probable words

$$C_k = \sum_{i=2}^{W} \sum_{j=1}^{i-1} \log\left(\frac{D(v_i, v_j) + 1}{D(v_j)}\right)$$

Illustration: Political Texts

- Let's compare fits across 10, 20, 30, and 40 topics

```r
modelEval <- searchK(out$documents,
                     vocab = out$vocab,
                     K = c(10,20,30,40), verbose=F,
                     max.em.its = 75, data = out$meta,
                     init.type = "Spectral", seed=60639, cores=4)
```

```
## Using multiple-cores.  Progress will not be shown.
```

Illustration: Political Texts

- Average topic exclusivity

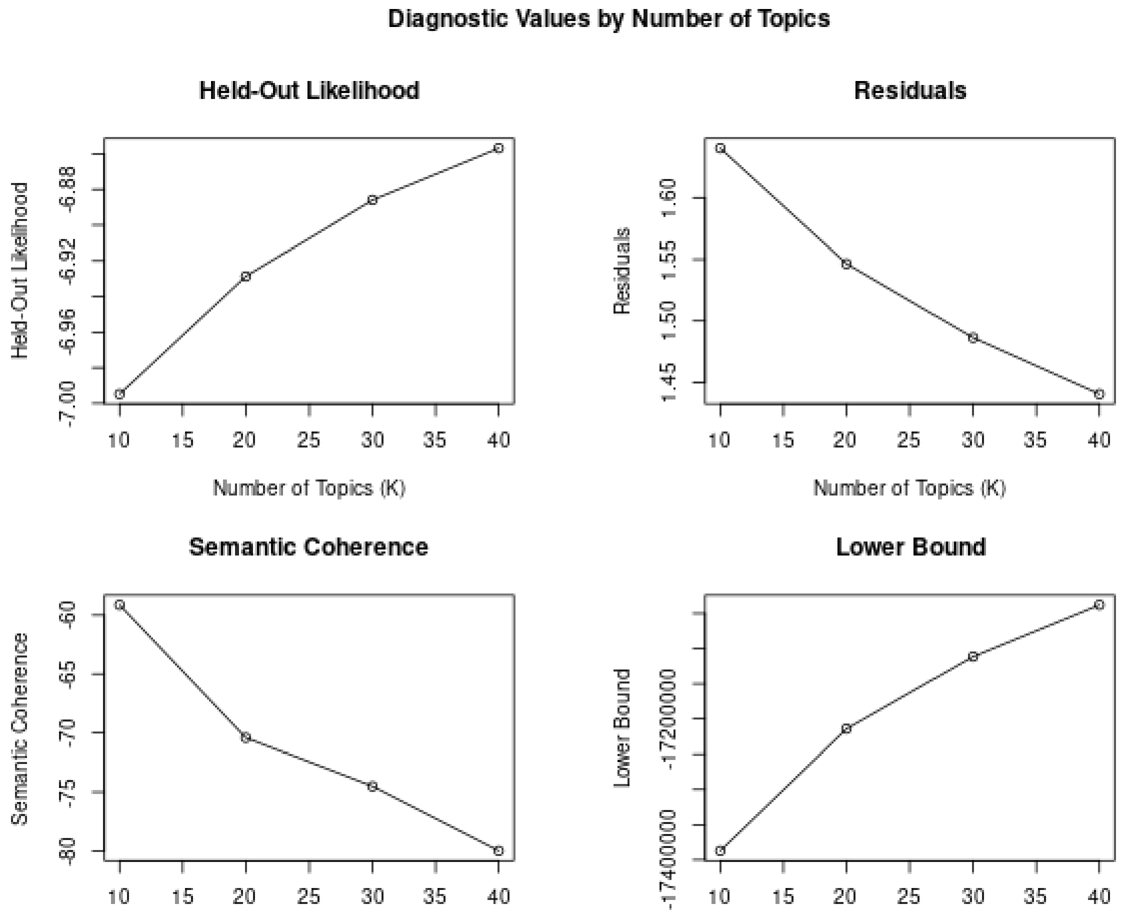
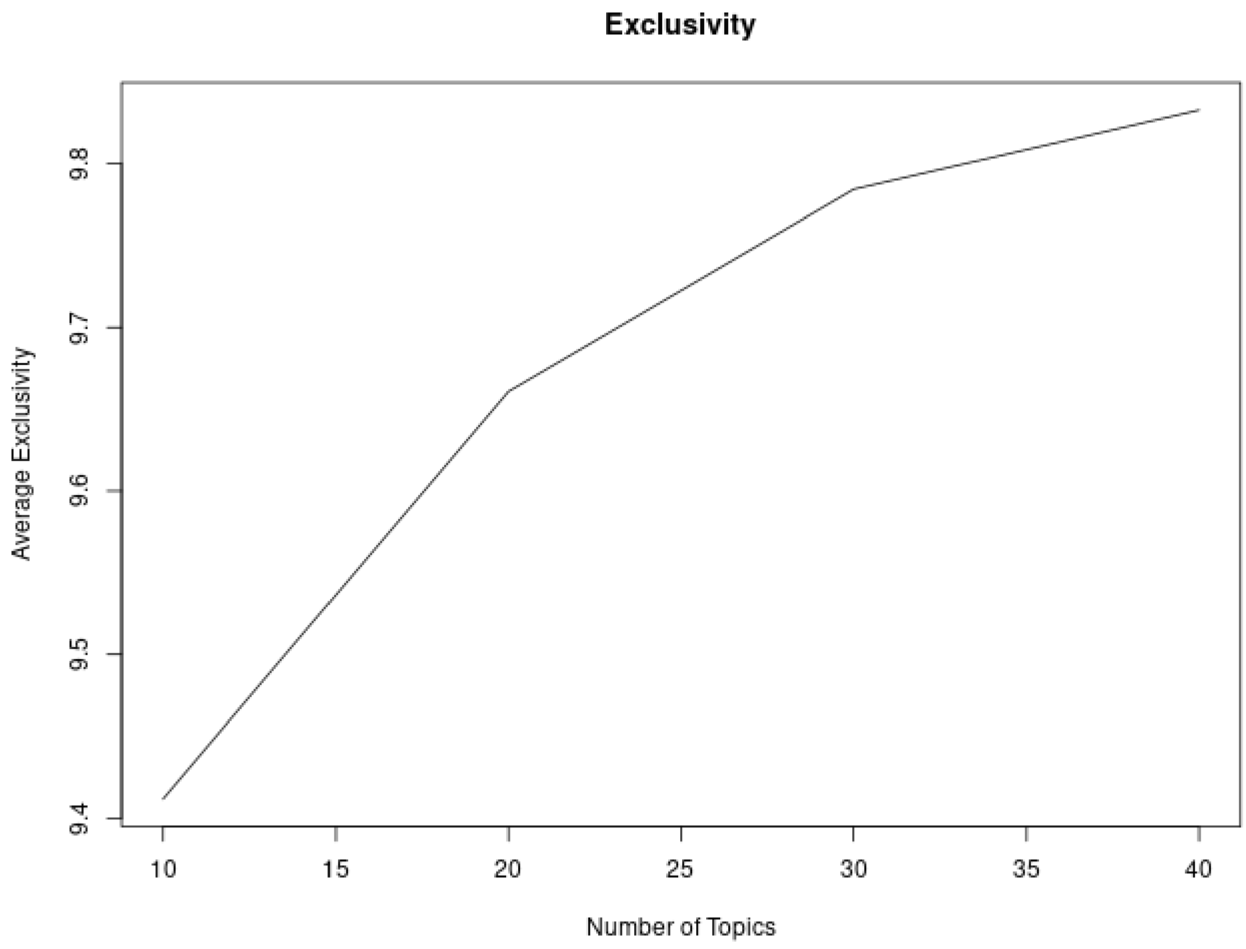


Illustration: Political Texts



Estimating coefficients

- With `stm()`, we can incorporate covariates into the topic proportions using standard formula syntax

```
poliBlog_fit_cov <- stm(documents = out$documents,
                        vocab = out$vocab,
                        prevalence = ~ rating,
                        K = 20, verbose=F,
                        max.em.its = 75, data = out$meta,
                        init.type = "Spectral", seed=60639)
```

Estimating coefficients

- Do liberal blogs talk about health care more than conservative blogs?

```
estim_hc <- estimateEffect(c(16) ~ rating, stmobj=poliBlog_fit_cov, metadata=out$meta)
summary(estim_hc)
```

```
##
## Call:
## estimateEffect(formula = c(16) ~ rating, stmobj = poliBlog_fit_cov,
##     metadata = out$meta)
##
##
## Topic 16:
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)    0.03106    0.00117    26.6   <2e-16 ***
## ratingLiberal  0.02166    0.00182    11.9   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Estimating coefficients

- Do conservative blogs talk about the financial crisis more than liberal blogs?

```
estim_fc <- estimateEffect(c(20) ~ rating, stmobj=poliBlog_fit_cov, metadata=out$meta)
summary(estim_fc)
```

```
##
## Call:
## estimateEffect(formula = c(20) ~ rating, stmobj = poliBlog_fit_cov,
##     metadata = out$meta)
##
##
## Topic 20:
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)    0.04007    0.00130   30.79   <2e-16 ***
## ratingLiberal -0.00547    0.00210   -2.61   0.0091 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```