Finite-n unbiasedness of the Horvitz-Thompson estimator with adaptively collected data.

Under the potential outcomes framework:

$$\mathrm{E}\left[\hat{\theta}_w^{HT}\right] = \frac{1}{N}\sum_{i=1}^{N} Y_i(w).$$

We require only independence of potential outcomes and treatment conditional on history, $Y_i(w) \perp\!\!\!\perp W_i | S_i$, which is given by the experimental design.

$$\mathrm{E}\left[\hat{\theta}_w^{HT}\right] = \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N} Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)}\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N} \mathrm{E}\left[Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)}\right].$$

Considering the i^{th} unit, by the Law of Iterated Expectations,

$$\mathrm{E}\left[Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)}\right] = \mathrm{E}\left[\mathrm{E}\left[Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)} \middle| S_i\right]\right]$$

Taking the interior term, $\mathrm{E}\left[Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)} \middle| S_i\right]$, by definition,

$$= \mathrm{E}\left[\frac{Y_i \mathbb{1}\{W_i = w\}}{\Pr[W_i = w | S_i]} \middle| S_i\right]$$

By the potential outcomes model,

$$= \mathrm{E}\left[Y_i(w) \times \frac{\mathbb{1}\{W_i = w\}}{\Pr[W_i = w | S_i]} \middle| S_i\right]$$

And because $Y_i \perp\!\!\!\perp W_i | S_i$,

$$= \mathrm{E}\left[Y_i(w) | S_i\right] \times \mathrm{E}\left[\frac{\mathbb{1}\{W_i = w\}}{\Pr[W_i = w | S_i]} \middle| S_i\right]$$

$$= \mathrm{E}\left[Y_i(w) | S_i\right] \times \frac{\mathrm{E}\left[\mathbb{1}\{W_i = w\} | S_i\right]}{\Pr[W_i = w | S_i]}$$

$$= \mathrm{E}\left[Y_i(w) | S_i\right] \times \frac{\Pr[W_i = w | S_i]}{\Pr[W_i = w | S_i]}$$

$$= \mathrm{E}\left[Y_i(w) | S_i\right].$$

Then returning to the Law of Iterated Expectations from above,

$$\mathrm{E}\left[Y_i \frac{\mathbb{1}\{W_i = w\}}{e_i(w; S_i)}\right] = \mathrm{E}\left[\mathrm{E}\left[Y_i(w) | S_i\right]\right] = \mathrm{E}\left[Y_i(w)\right].$$

HT estimator | HT toy example