

# **University of Chicago Political Science Math Prefresher**

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# 1 Overview

## 1.1 Introduction

The 2022 UChicago Math Prefresher for incoming Political Science graduate students will be held from September 12-14; September 19-21 and September 23rd. The course is designed as a brief review of math fundamentals – calculus, optimization, probability theory and linear algebra among other topics – as well as an introduction to programming in the R statistical computing language. The course is entirely optional and there are no grades or assignments but we encourage all incoming graduate students to attend if they are able.

## 1.2 Course Booklet

The course notes for the math and programming sections as well as all practice problems are available on this website and can be accessed by navigating the menus in the sidebar.

## 1.3 Schedule

The prefresher will run for a total of seven days September 12-14, September 19-21 and September 23rd, with breaks for the APSA conference and the new student orientation. Each day will run from around 9am to 4pm with many breaks in between.

The morning will focus on math instruction. We will have two one hour sessions from 9:30am - 10:30am and 10:45am-11:45am, with a ~15 minute break in between. These sessions will involve a combination of lectures and working through practice problems.

We will break for lunch from 12:00pm-1:00pm. On September 13th and September 19th, we will have a catered lunch with a faculty member guest. Otherwise, you are free to explore the campus for various lunch options.

The afternoon will focus on coding instruction with lecture/demonstration from 1:30pm-2:45pm. After a short break you will work together on a variety of coding exercises from 3:00-3:30pm. In the last 30 minutes we will regroup to wrap up and discuss any questions on the material.

## 1.4 Software

As the afternoons of the prefresher will involve instruction in coding, you should be sure to bring a laptop and a charging cable. In addition, prior to the start of the prefresher, please make sure to have installed the following on your computer:

- [R](#) (version 4.2.1 or higher)
- [RStudio Desktop Open Source License](#) (this is the primary IDE or integrated development environment in which we will be working)
- LaTeX: This is primarily to allow you to generate PDF documents using RMarkdown. We will use the TinyTeX LaTeX distribution which is designed to be minimalist and tailored specifically for R users. After installing R and RStudio, open up an instance of R, install the ‘tinytex’ package and run the `install_tinytex()` command

```
install.packages('tinytex')
tinytex::install_tinytex()
```

We will also spend some time discussing document preparation and typesetting using LaTeX and Markdown. For the former, we will be using the popular cloud platform [Overleaf](#), which allows for collaborative document editing and streamlines a lot of the irritating parts of typesetting in LaTeX. You should register for an account using your university e-mail as all University of Chicago students and faculty [have access](#) to an Overleaf Pro account for free.

You are also welcome to install a LaTeX editor on your local machine to work alongside the TinyTeX distribution or any other TeX distribution that you prefer such as [TexMaker](#)

## 1.5 Acknowledgments

This prefresher draws heavily on the wonderful materials that have been developed by over 20 years of instructors at the [Harvard Government Math Prefresher](#) that have been so generously distributed under the GPL 3.0 License. Special thanks to Shiro Kuriwaki, Yon Soo Park, and Connor Jerzak for their efforts in converting the original prefresher materials into the easily distributed Markdown format.

## 2 Functions and Operations

**Topics** Dimensionality; Interval Notation for  $\mathbf{R}^1$ ; Neighborhoods: Intervals, Disks, and Balls; Introduction to Functions; Domain and Range; Some General Types of Functions; log, ln, and exp; Other Useful Functions; Graphing Functions; Solving for Variables; Finding Roots; Limit of a Function; Continuity; Sets, Sets, and More Sets.

### 2.1 Summation Operators $\sum$ and $\prod$

Addition (+), Subtraction (-), multiplication and division are basic operations of arithmetic – combining numbers. In statistics and calculus, we want to add a *sequence* of numbers that can be expressed as a pattern without needing to write down all its components. For example, how would we express the sum of all numbers from 1 to 100 without writing a hundred numbers?

For this we use the summation operator  $\sum$  and the product operator  $\prod$ .

**Summation:**

$$\sum_{i=1}^{100} x_i = x_1 + x_2 + x_3 + \cdots + x_{100}$$

The bottom of the  $\sum$  symbol indicates an index (here,  $i$ ), and its start value 1. At the top is where the index ends. The notion of “addition” is part of the  $\sum$  symbol. The content to the right of the summation is the meat of what we add. While you can pick your favorite index, start, and end values, the content must also have the index.

- $\sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$
- $\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$
- $\sum_{i=1}^n c = nc$

**Product:**

$$\prod_{i=1}^n x_i = x_1 x_2 x_3 \cdots x_n$$

Properties:

- $\prod_{i=1}^n cx_i = c^n \prod_{i=1}^n x_i$
- $\prod_{i=k}^n cx_i = c^{n-k+1} \prod_{i=k}^n x_i$
- $\prod_{i=1}^n (x_i + y_i) = \text{a total mess}$
- $\prod_{i=1}^n c = c^n$

Other Useful Functions

**Factorials!:**

$$x! = x \cdot (x-1) \cdot (x-2) \cdots (1)$$

**Modulo:** Tells you the remainder when you divide the first number by the second.

- $17 \bmod 3 = 2$
- $100 \% 30 = 10$

“{example, name = “Operators”, operators}

1.  $\sum_{i=1}^5 i =$

2.  $\prod_{i=1}^5 i =$

3.  $14 \bmod 4 =$

4.  $4! =$

```{exercise, name = "Operators", operators1}

Let  $x_1 = 4, x_2 = 3, x_3 = 7, x_4 = 11, x_5 = 2$

1.  $\sum_{i=1}^3 (7)x_i$

2.  $\sum_{i=1}^5 2$

3.  $\prod_{i=3}^5 (2)x_i$

## 2.2 Introduction to Functions

A **function** (in  $\mathbf{R}^1$ ) is a mapping, or transformation, that relates members of one set to members of another set. For instance, if you have two sets: set  $A$  and set  $B$ , a function from  $A$  to  $B$  maps every value  $a$  in set  $A$  such that  $f(a) \in B$ . Functions can be “many-to-one”, where many values or combinations of values from set  $A$  produce a single output in set  $B$ , or they can be “one-to-one”, where each value in set  $A$  corresponds to a single value in set  $B$ . A function by definition has a single function value for each element of its domain. This means, there cannot be “one-to-many” mapping.

**Dimensionality:**  $\mathbf{R}^1$  is the set of all real numbers extending from  $-\infty$  to  $+\infty$  — i.e., the real number line.  $\mathbf{R}^n$  is an  $n$ -dimensional space, where each of the  $n$  axes extends from  $-\infty$  to  $+\infty$ .

- $\mathbf{R}^1$  is a one dimensional line.
- $\mathbf{R}^2$  is a two dimensional plane.
- $\mathbf{R}^3$  is a three dimensional space.

Points in  $\mathbf{R}^n$  are ordered  $n$ -tuples (just means an combination of  $n$  elements where order matters), where each element of the  $n$ -tuple represents the coordinate along that dimension.

For example:

- $\mathbf{R}^1$ : (3)
- $\mathbf{R}^2$ : (-15, 5)
- $\mathbf{R}^3$ : (86, 4, 0)

Examples of mapping notation:

Function of one variable:  $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$

- $f(x) = x + 1$ . For each  $x$  in  $\mathbf{R}^1$ ,  $f(x)$  assigns the number  $x + 1$ .

Function of two variables:  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^1$ .

- $f(x, y) = x^2 + y^2$ . For each ordered pair  $(x, y)$  in  $\mathbf{R}^2$ ,  $f(x, y)$  assigns the number  $x^2 + y^2$ .

We often use variable  $x$  as input and another  $y$  as output, e.g.  $y = x + 1$

“{example, name = “Functions”, functions}”

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For  $x \in [0, \infty]$ ,  $f : x \rightarrow x^2$  (this could also be written as  $f(x) = x^2$ ).
2. For  $x \in [-\infty, \infty]$ ,  $f : x \rightarrow x^2$ .

```
```{exercise, name = "Functions", functions1}
```

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For  $x \in [-3, \infty]$ ,  $f: x \mapsto x^2$ .

2. For  $x \in [0, \infty]$ ,  $f: x \mapsto \sqrt{x}$

Some functions are defined only on proper subsets of  $\mathbf{R}^n$ .

- **Domain:** the set of numbers in  $X$  at which  $f(x)$  is defined.
- **Range:** elements of  $Y$  assigned by  $f(x)$  to elements of  $X$ , or

$$f(X) = \{y : y = f(x), x \in X\}$$

Most often used when talking about a function  $f: \mathbf{R}^1 \rightarrow \mathbf{R}^1$ .

- **Image:** same as range, but more often used when talking about a function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^1$ .

Some General Types of Functions

**Monomials:**  $f(x) = ax^k$

$a$  is the coefficient.  $k$  is the degree.

Examples:  $y = x^2$ ,  $y = -\frac{1}{2}x^3$

**Polynomials:** sum of monomials.

Examples:  $y = -\frac{1}{2}x^3 + x^2$ ,  $y = 3x + 5$

The degree of a polynomial is the highest degree of its monomial terms. Also, it's often a good idea to write polynomials with terms in decreasing degree.

**Exponential Functions:** Example:  $y = 2^x$



## 2.3 log and exp

### Relationship of logarithmic and exponential functions:

$$y = \log_a(x) \iff a^y = x$$

The log function can be thought of as an inverse for exponential functions.  $a$  is referred to as the “base” of the logarithm.

**Common Bases:** The two most common logarithms are base 10 and base  $e$ .

1. Base 10:  $y = \log_{10}(x) \iff 10^y = x$ . The base 10 logarithm is often simply written as “ $\log(x)$ ” with no base denoted.
2. Base  $e$ :  $y = \log_e(x) \iff e^y = x$ . The base  $e$  logarithm is referred to as the “natural” logarithm and is written as “ $\ln(x)$ ”.

### Properties of exponential functions:

- $a^x a^y = a^{x+y}$
- $a^{-x} = 1/a^x$
- $a^x / a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$

### Properties of logarithmic functions (any base):

Generally, when statisticians or social scientists write  $\log(x)$  they mean  $\log_e(x)$ . In other words:  $\log_e(x) \equiv \ln(x) \equiv \log(x)$

$$\log_a(a^x) = x$$

and

$$a^{\log_a(x)} = x$$

- $\log(xy) = \log(x) + \log(y)$
- $\log(x^y) = y \log(x)$
- $\log(1/x) = \log(x^{-1}) = -\log(x)$
- $\log(x/y) = \log(x \cdot y^{-1}) = \log(x) + \log(y^{-1}) = \log(x) - \log(y)$
- $\log(1) = \log(e^0) = 0$

**Change of Base Formula:** Use the change of base formula to switch bases as necessary:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Example:

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

You can use logs to go between sum and product notation. This will be particularly important when you're learning maximum likelihood estimation.

$$\begin{aligned}\log\left(\prod_{i=1}^n x_i\right) &= \log(x_1 \cdot x_2 \cdot x_3 \cdots x_n) \\ &= \log(x_1) + \log(x_2) + \log(x_3) + \cdots + \log(x_n) \\ &= \sum_{i=1}^n \log(x_i)\end{aligned}$$

Therefore, you can see that the log of a product is equal to the sum of the logs. We can write this more generally by adding in a constant,  $c$ :

$$\begin{aligned}\log\left(\prod_{i=1}^n cx_i\right) &= \log(cx_1 \cdot cx_2 \cdots cx_n) \\ &= \log(c^n \cdot x_1 \cdot x_2 \cdots x_n) \\ &= \log(c^n) + \log(x_1) + \log(x_2) + \cdots + \log(x_n) \\ &= n \log(c) + \sum_{i=1}^n \log(x_i)\end{aligned}$$

“{example, name = “Logarithmic Functions”, log}

Evaluate each of the following logarithms

1.  $\log_4(16)$

2.  $\log_2(16)$

Simplify the following logarithm. By “simplify”, we actually really mean - use as many of the logarithmic properties as you can.

3.  $\log_4(x^3y^5)$

```
```{exercise, name = "Logarithmic Functions", log1}
```

Evaluate each of the following logarithms

1.  $\log_{\frac{3}{2}}(\frac{27}{8})$

Simplify each of the following logarithms. By "simplify", we actually really mean - use as much as possible

2.  $\log(\frac{x^9y^5}{z^3})$

3.  $\ln(\sqrt{xy})$

## 2.4 Graphing Functions

What can a graph tell you about a function?

- Is the function increasing or decreasing? Over what part of the domain?
- How “fast” does it increase or decrease?
- Are there global or local maxima and minima? Where?
- Are there inflection points?
- Is the function continuous?
- Is the function differentiable?
- Does the function tend to some limit?
- Other questions related to the substance of the problem at hand.

## 2.5 Solving for Variables and Finding Roots

Sometimes we're given a function  $y = f(x)$  and we want to find how  $x$  varies as a function of  $y$ . Use algebra to move  $x$  to the left hand side (LHS) of the equation and so that the right hand side (RHS) is only a function of  $y$ .

```
“{example, name = “Solving for Variables”, solvevar}
```

Solve for  $x$ :

1.  $y = 3x + 2$
2.  $y = e^x$

Solving for variables is especially important when we want to find the `__roots__` of an equation.

Procedure: Given  $y=f(x)$ , set  $f(x)=0$ . Solve for  $x$ .

Multiple Roots:

$$f(x)=x^2 - 9 \quad \Longleftrightarrow \quad 0=x^2 - 9 \quad \Longleftrightarrow \quad 9=x^2 \quad \Longleftrightarrow \quad x=\pm 3$$

`__Quadratic Formula:__` For quadratic equations  $ax^2+bx+c=0$ , use the quadratic formula:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

```
```{exercise, name = "Finding Roots", solvevar1}
```

Solve for  $x$ :

1.  $f(x)=3x+2 = 0$
2.  $f(x)=x^2+3x-4=0$
3.  $f(x)=e^{-x}-10 = 0$

## 2.6 Sets

**Interior Point:** The point  $\mathbf{x}$  is an interior point of the set  $S$  if  $\mathbf{x}$  is in  $S$  and if there is some  $\epsilon$ -ball around  $\mathbf{x}$  that contains only points in  $S$ . The **interior** of  $S$  is the collection of all interior points in  $S$ . The interior can also be defined as the union of all open sets in  $S$ .

- If the set  $S$  is circular, the interior points are everything inside of the circle, but not on the circle's rim.
- Example: The interior of the set  $\{(x, y) : x^2 + y^2 \leq 4\}$  is  $\{(x, y) : x^2 + y^2 < 4\}$ .

**Boundary Point:** The point  $\mathbf{x}$  is a boundary point of the set  $S$  if every  $\epsilon$ -ball around  $\mathbf{x}$  contains both points that are in  $S$  and points that are outside  $S$ . The **boundary** is the collection of all boundary points.

- If the set  $S$  is circular, the boundary points are everything on the circle's rim.
- Example: The boundary of  $\{(x, y) : x^2 + y^2 \leq 4\}$  is  $\{(x, y) : x^2 + y^2 = 4\}$ .

**Open:** A set  $S$  is open if for each point  $\mathbf{x}$  in  $S$ , there exists an open  $\epsilon$ -ball around  $\mathbf{x}$  completely contained in  $S$ .

- If the set  $S$  is circular and open, the points contained within the set get infinitely close to the circle's rim, but do not touch it.
- Example:  $\{(x, y) : x^2 + y^2 < 4\}$

**Closed:** A set  $S$  is closed if it contains all of its boundary points.

- Alternatively: A set is closed if its complement is open.
- If the set  $S$  is circular and closed, the set contains all points within the rim as well as the rim itself.
- Example:  $\{(x, y) : x^2 + y^2 \leq 4\}$
- Note: a set may be neither open nor closed. Example:  $\{(x, y) : 2 < x^2 + y^2 \leq 4\}$

**Complement:** The complement of set  $S$  is everything outside of  $S$ .

- If the set  $S$  is circular, the complement of  $S$  is everything outside of the circle.
- Example: The complement of  $\{(x, y) : x^2 + y^2 \leq 4\}$  is  $\{(x, y) : x^2 + y^2 > 4\}$ .

**Empty:** The empty (or null) set is a unique set that has no elements, denoted by  $\{\}$  or  $\emptyset$ .

- The empty set is an example of a set that is open and closed, or a “clopen” set.
- Examples: The set of squares with 5 sides; the set of countries south of the South Pole.

## Answers to Examples and Exercises

Answer to Example @ref(exm:operators):

1.  $1 + 2 + 3 + 4 + 5 = 15$
2.  $1 * 2 * 3 * 4 * 5 = 120$
3. 2

4.  $4 * 3 * 2 * 1 = 24$

Answer to Exercise @ref(exr:operators1):

1.  $7(4 + 3 + 7) = 98$

2.  $2 + 2 + 2 + 2 + 2 = 10$

3.  $2^3(7)(11)(2) = 1232$

Answer to Example @ref(exm:functions):

1. one-to-one

2. many-to-one

Answer to Exercise @ref(exr:functions1):

1. many-to-one

2. one-to-one

Answer to Example @ref(exm:log):

1. 2

2. 4

3.  $3 \log_4(x) + 5 \log_4(y)$

Answer to Exercise @ref(exr:log1):

1. 3

2.  $9 \log(x) + 5 \log(y) - 3 \log(z)$

3.  $\frac{1}{2}(\ln x + \ln y)$

Answer to Example @ref(exm:solvevar):

1.  $y = 3x + 2 \implies -3x = 2 - y \implies 3x = y - 2 \implies x = \frac{1}{3}(y - 2)$

2.  $x = \ln y$

Answer to Exercise @ref(exr:solvevar1):

1.  $\frac{-2}{3}$

2.  $x = \{1, -4\}$

3.  $x = -\ln 10$

## 3 Limits

Solving limits, i.e. finding out the value of functions as its input moves closer to some value, is important for the social scientist's mathematical toolkit for two related tasks. The first is for the study of calculus, which will be in turn useful to show where certain functions are maximized or minimized. The second is for the study of statistical inference, which is the study of inferring things about things you cannot see by using things you can see.

### Example: The Central Limit Theorem

Perhaps the most important theorem in statistics is the Central Limit Theorem,

```
For any series of independent and identically distributed random variables  $X_1, X_2, \dots$ 

$$\left[ \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \xrightarrow{d} \text{Normal}(0, 1), \right]$$

where  $\mu$  is the mean of  $X$  and  $\sigma$  is the standard deviation of  $X$ . The arrow is
That is, the limit of the distribution of the lefthand side is the distribution of the ri
```

The sign of a limit is the arrow “ $\rightarrow$ ”. Although we have not yet covered probability (in Section [@ref\(probability-theory\)](#)) so we have not described what distributions and random variables are, it is worth foreshadowing the Central Limit Theorem. The Central Limit Theorem is powerful because it gives us a *guarantee* of what would happen if  $n \rightarrow \infty$ , which in this case means we collected more data.

### Example: The Law of Large Numbers

A finding that perhaps rivals the Central Limit Theorem is the Law of Large Numbers:

```
For any draw of identically distributed independent variables with mean  $\mu$ , the sample

$$\lim_{n \rightarrow \infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$$

```

A shorthand of which is  $\bar{X}_n \rightarrow \mu$ , where the arrow is read as "converges to".

Intuitively, the more data, the more accurate is your guess. For example, the Figure @ref(fig:llnsim) shows how the sample average from many coin tosses converges to the true value : 0.5.

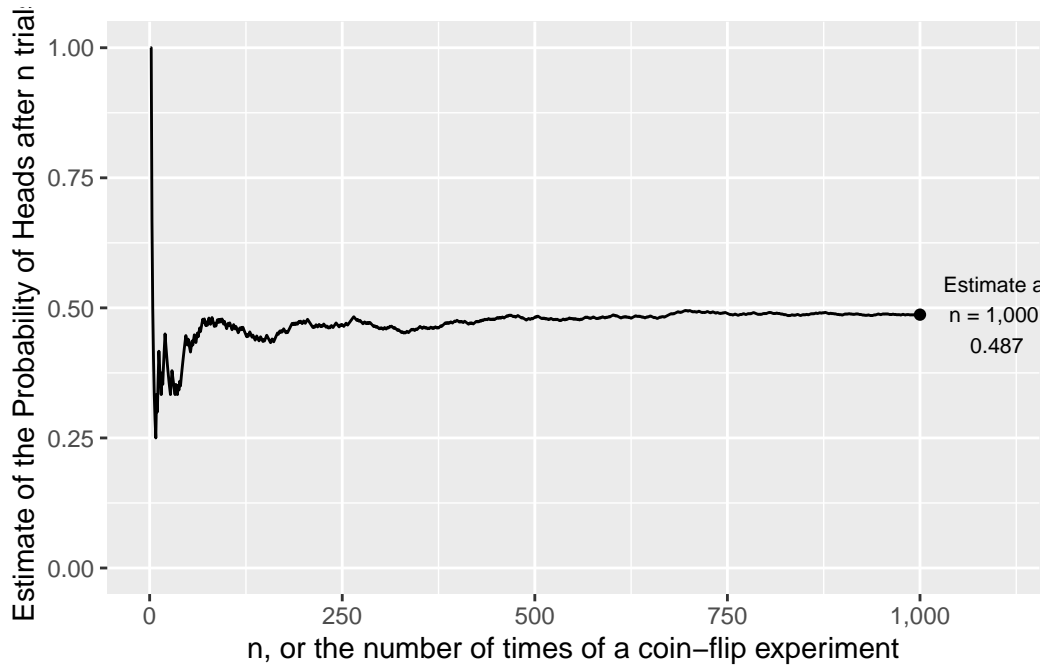


Figure 3.1: As the number of coin tosses goes to infinity, the average probability of heads converges to 0.5

### 3.1 Sequences

We need a couple of steps until we get to limit theorems in probability. First we will introduce a “sequence”, then we will think about the limit of a sequence, then we will think about the limit of a *function*.

A **sequence**  $\{x_n\} = \{x_1, x_2, x_3, \dots, x_n\}$  is an ordered set of real numbers, where  $x_1$  is the first term in the sequence and  $x_n$  is the  $n$ th term. Generally, a sequence is infinite, that is it extends to  $n = \infty$ . We can also write the sequence as  $\{x_n\}_{n=1}^{\infty}$

where the subscript and superscript are read together as “from 1 to infinity.”



How do these sequences behave?

1.  $\{A_n\} = \left\{ 2 - \frac{1}{n^2} \right\}$
2.  $\{B_n\} = \left\{ \frac{n^2+1}{n} \right\}$
3.  $\{C_n\} = \left\{ (-1)^n \left( 1 - \frac{1}{n} \right) \right\}$

We find the sequence by simply “plugging in” the integers into each  $n$ . The important thing is to get a sense of how these numbers are going to change. Example 1’s numbers seem to come closer and closer to 2, but will it ever surpass 2? Example 2’s numbers are also increasing each time, but will it hit a limit? What is the pattern in Example 3? Graphing helps you make this point more clearly. See the sequence of  $n = 1, \dots, 20$  for each of the three examples in Figure @ref(fig:seqabc).

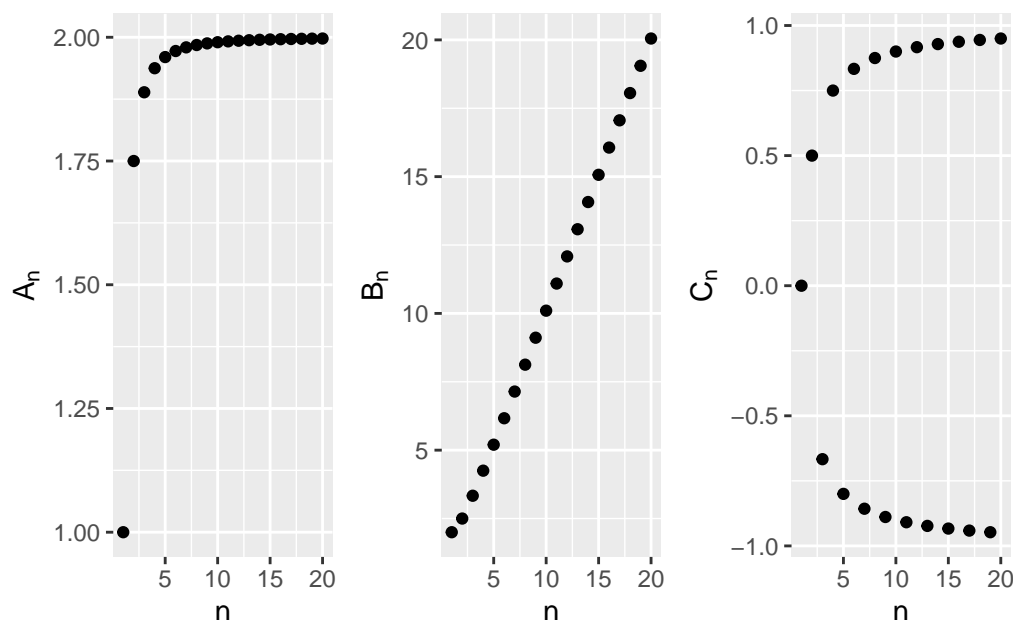


Figure 3.2: Behavior of Some Sequences

## 3.2 The Limit of a Sequence

The notion of “converging to a limit” is the behavior of the points in Example @ref(exm:seqbehav). In some sense, that’s the counterfactual we want to know. What happens as  $n \rightarrow \infty$ ?

1. Sequences like 1 above that converge to a limit.
2. Sequences like 2 above that increase without bound.
3. Sequences like 3 above that neither converge nor increase without bound — alternating over the number line.

The sequence  $\{y_n\}$  has the limit  $L$ , which we write as  $\lim_{n \rightarrow \infty} y_n = L$ .

We can also express the behavior of a sequence as bounded or not:

1. Bounded: if  $|y_n| \leq K$  for all  $n$
2. Monotonically Increasing:  $y_{n+1} > y_n$  for all  $n$
3. Monotonically Decreasing:  $y_{n+1} < y_n$  for all  $n$

A limit is *unique*: If  $\{y_n\}$  converges, then the limit  $L$  is unique.

If a sequence converges, then the sum of such sequences also converges. Let  $\lim_{n \rightarrow \infty} y_n = y$  and  $\lim_{n \rightarrow \infty} z_n = z$ . Then

1.  $\lim_{n \rightarrow \infty} [ky_n + \ell z_n] = ky + \ell z$
2.  $\lim_{n \rightarrow \infty} y_n z_n = yz$
3.  $\lim_{n \rightarrow \infty} \frac{y_n}{z_n} = \frac{y}{z}$ , provided  $z \neq 0$

This looks reasonable enough. The harder question, obviously is when the parts of the fraction *don't* converge. If  $\lim_{n \rightarrow \infty} y_n = \infty$  and  $\lim_{n \rightarrow \infty} z_n = \infty$ , What is  $\lim_{n \rightarrow \infty} y_n - z_n$ ? What is  $\lim_{n \rightarrow \infty} \frac{y_n}{z_n}$ ?

It is nice for a sequence to converge in limit. We want to know if complex-looking sequences converge or not. The name of the game here is to break that complex sequence up into sums of simple fractions where  $n$  only appears in the denominator:  $\frac{1}{n}$ ,  $\frac{1}{n^2}$ , and so on. Each of these will converge to 0, because the denominator gets larger and larger. Then, because of the properties above, we can then find the final sequence.

Find the limit of  $\lim_{n \rightarrow \infty} \frac{n+3}{n}$ .

At first glance,  $n+3$  and  $n$  both grow to  $\infty$ , so it looks like we need to divide

$$\lim_{n \rightarrow \infty} \frac{n+3}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right) =$$

so, the limit is actually 1.

After some practice, the key to intuition is whether one part of the fraction grows “faster” than another. If the denominator grows faster to infinity than the numerator, then the fraction will

converge to 0, even if the numerator will also increase to infinity. In a sense, limits show how not all infinities are the same.

Find the following limits of sequences, then explain in English the intuition for why that

1.  $\lim_{n \rightarrow \infty} \frac{2n}{n^2 + 1}$
2.  $\lim_{n \rightarrow \infty} (n^3 - 100n^2)$

### 3.3 Limits of a Function

We've now covered functions and just covered limits of sequences, so now is the time to combine the two.

A function  $f$  is a compact representation of some behavior we care about. Like for sequences, we often want to know if  $f(x)$  approaches some number  $L$  as its independent variable  $x$  moves to some number  $c$  (which is usually 0 or  $\pm\infty$ ). If it does, we say that the limit of  $f(x)$ , as  $x$  approaches  $c$ , is  $L$ :  $\lim_{x \rightarrow c} f(x) = L$ . Unlike a sequence,  $x$  is a continuous number, and we can move in decreasing order as well as increasing.

For a limit  $L$  to exist, the function  $f(x)$  must approach  $L$  from both the left (increasing) and the right (decreasing).

Let  $f(x)$  be defined at each point in some open interval containing the point  $c$ . Then

A neat, if subtle result is that  $f(x)$  does not necessarily have to be defined at  $c$  for  $\lim_{x \rightarrow c}$  to exist.

Properties: Let  $f$  and  $g$  be functions with  $\lim_{x \rightarrow c} f(x) = k$  and  $\lim_{x \rightarrow c} g(x) = \ell$ .

1.  $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$
2.  $\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x)$
3.  $\lim_{x \rightarrow c} f(x)g(x) = \left[ \lim_{x \rightarrow c} f(x) \right] \cdot \left[ \lim_{x \rightarrow c} g(x) \right]$
4.  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$ , provided  $\lim_{x \rightarrow c} g(x) \neq 0$ .

Simple limits of functions can be solved as we did limits of sequences. Just be careful which part of the function is changing.

Find the limit of the following functions.

1.  $\lim_{x \rightarrow c} k$
1.  $\lim_{x \rightarrow c} x$
1.  $\lim_{x \rightarrow 2} (2x-3)$
1.  $\lim_{x \rightarrow c} x^n$

Limits can get more complex in roughly two ways. First, the functions may become large polynomials with many moving pieces. Second, the functions may become discontinuous.

The function can be thought of as a more general or “smooth” version of sequences. For example,

Find the limit of

$$\lim_{x \rightarrow \infty} \frac{(x^4 + 3x - 99)(2 - x^5)}{(18x^7 + 9x^6 - 3x^2 - 1)(x + 1)}$$

Now, the functions will become a bit more complex:

Solve the following limits of functions

1.  $\lim_{x \rightarrow 0} |x|$
2.  $\lim_{x \rightarrow 0} \left(1 + \frac{1}{x^2}\right)$

So there are a few more alternatives about what a limit of a function could be:

1. Right-hand limit: The value approached by  $f(x)$  when you move from right to left.
2. Left-hand limit: The value approached by  $f(x)$  when you move from left to right.
3. Infinity: The value approached by  $f(x)$  as  $x$  grows infinitely large. Sometimes this may be a number; sometimes it might be  $\infty$  or  $-\infty$ .
4. Negative infinity: The value approached by  $f(x)$  as  $x$  grows infinitely negative. Sometimes this may be a number; sometimes it might be  $\infty$  or  $-\infty$ .

The distinction between left and right becomes important when the function is not determined for some values of  $x$ . What are those cases in the examples below?

## 3.4 Continuity

To repeat a finding from the limits of functions:  $f(x)$  does not necessarily have to be defined at  $c$  for  $\lim_{x \rightarrow c}$  to exist. Functions that have breaks in their lines are called discontinuous. Functions that have no breaks are called continuous. Continuity is a concept that is more fundamental to, but related to that of “differentiability”, which we will cover next in calculus.

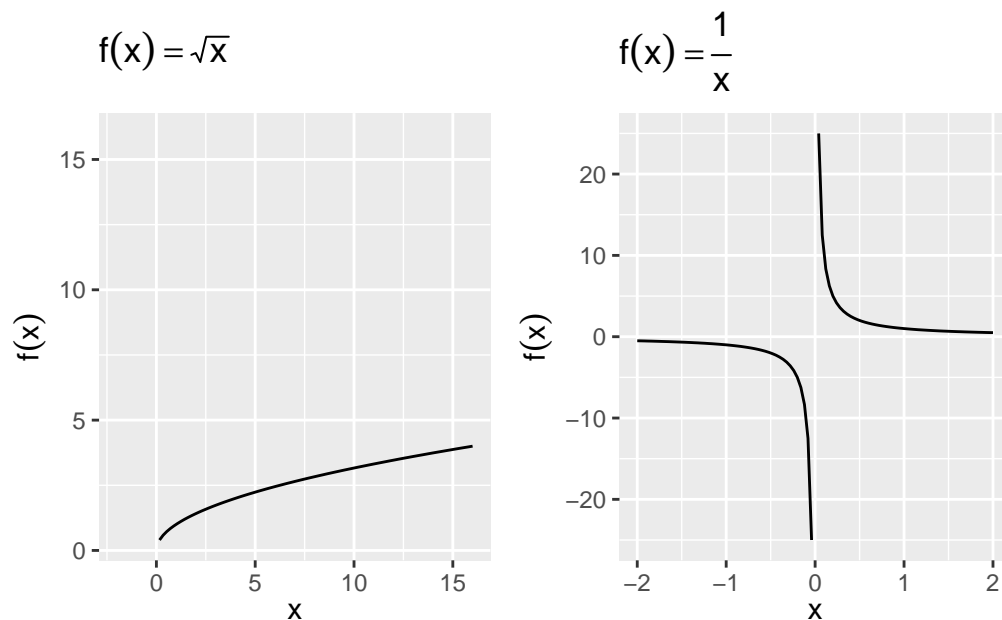


Figure 3.3: Functions which are not defined in some areas

Suppose that the domain of the function  $f$  includes an open interval containing the point

To prove that a function is continuous for all points is beyond this practical introduction to math, but the general intuition can be grasped by graphing.

For each function, determine if it is continuous or discontinuous.

1.  $f(x) = \sqrt{x}$
2.  $f(x) = e^x$
3.  $f(x) = 1 + \frac{1}{x^2}$
4.  $f(x) = \text{floor}(x)$ .

The floor is the smaller of the two integers bounding a number. So  $\text{floor}(x = 2.999)$

In Figure \@ref(fig:fig-contdiscont), we can see that the first two functions are continuous

Some properties of continuous functions:

1. If  $f$  and  $g$  are continuous at point  $c$ , then  $f + g$ ,  $f - g$ ,  $f \cdot g$ ,  $|f|$ , and  $\alpha f$  are continuous at point  $c$  also.  $f/g$  is continuous, provided  $g(c) \neq 0$ .

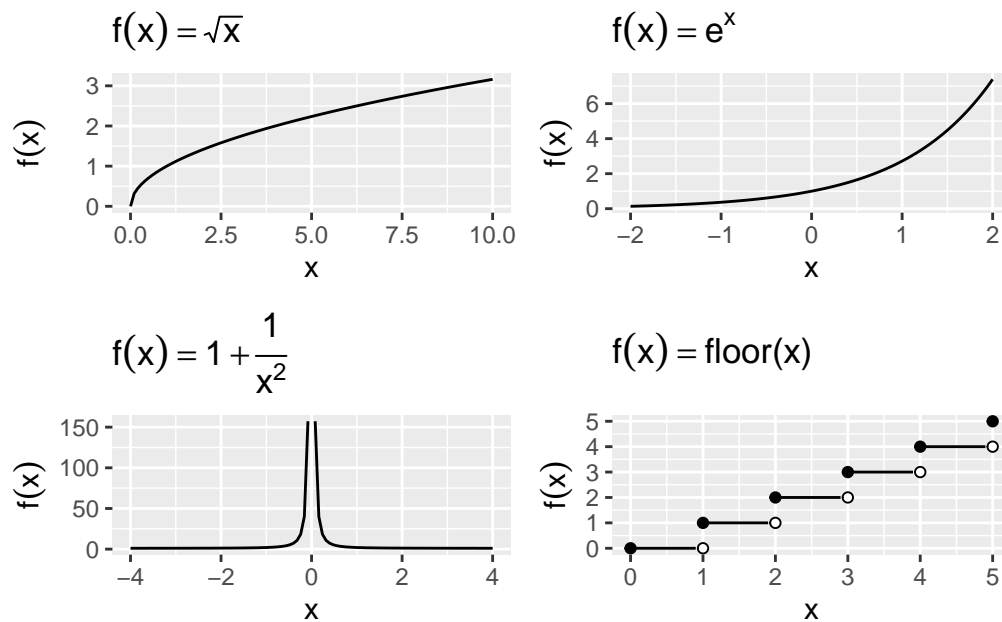


Figure 3.4: Continuous and Discontinuous Functions

2. Boundedness: If  $f$  is continuous on the closed bounded interval  $[a, b]$ , then there is a number  $K$  such that  $|f(x)| \leq K$  for each  $x$  in  $[a, b]$ .
3. Max/Min: If  $f$  is continuous on the closed bounded interval  $[a, b]$ , then  $f$  has a maximum and a minimum on  $[a, b]$ . They may be located at the end points.

Let  $f(x) = \frac{x^2 + 2x}{x}$ .

1. Graph the function. Is it defined everywhere?
2. What is the functions limit at  $x \rightarrow 0$ ?

## Answers to Examples

Example [@ref\(exm:seqbehav\)](#)

1.  $\{A_n\} = \left\{ 2 - \frac{1}{n^2} \right\} = \left\{ 1, \frac{7}{4}, \frac{17}{9}, \frac{26}{16}, \dots \right\}$
2.  $\{B_n\} = \left\{ \frac{n^2 + 1}{n} \right\} = \left\{ 2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \dots \right\}$
3.  $\{C_n\} = \left\{ (-1)^n \left( 1 - \frac{1}{n} \right) \right\} = \left\{ 0, \frac{1}{2}, -\frac{1}{3}, \frac{2}{4}, -\frac{3}{5}, \dots \right\}$

Exercise @ref(exr:limseq2)

Example @ref(exm:limfun1)

1.  $k$
2.  $c$
3.  $\lim_{x \rightarrow 2} (2x-3) = 2 \lim_{x \rightarrow 2} x - 3 \lim_{x \rightarrow 2} 1 = 1$
4.  $\lim_{x \rightarrow c} x^n = \lim_{x \rightarrow c} x \cdots [\lim_{x \rightarrow c} x] = c \cdots$

Exercise @ref(exr:limfunmax)

Although this function seems large, the thing our eyes should focus on is where the highest

```
\begin{align*}
& \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{3}{x^3} - \frac{99}{4x^4}\right) \left(-\frac{2}{x}\right)}{\times \frac{x^4}{1} \times -\frac{x^5}{1} \times \frac{1}{18x^7} \times \frac{1}{x}} \\
&= 1 \times \lim_{x \rightarrow \infty} \frac{x}{18}
\end{align*}
```

Exercise @ref(exr:discontdraw)

See Figure @ref(fig:fig-hole-0).

Divide each part by  $x$ , and we get  $x + \frac{2}{x}$  on the numerator,  $1$  on the denominator.

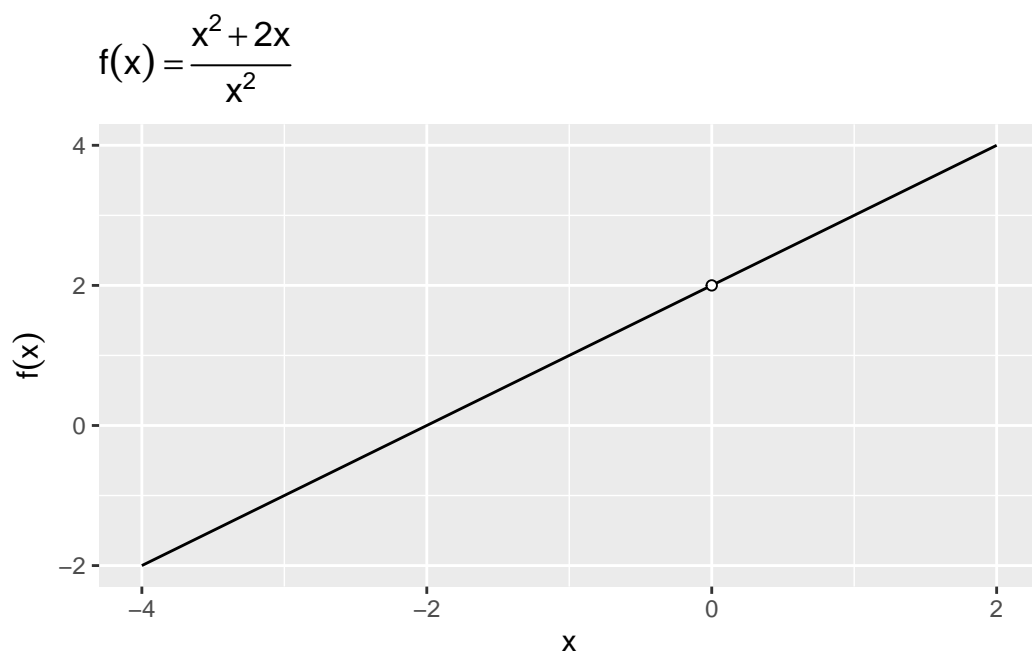


Figure 3.5: A function undefined at  $x = 0$