University of Chicago Political Science Math Prefresher

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1 Overview

1.1 Introduction

The 2022 UChicago Math Prefresher for incoming Political Science graduate students will be held from September 12-14; September 19-21 and September 23rd. The course is designed as a brief review of math fundamentals – calculus, optimization, probability theory and linear algebra among other topics – as well as an introduction to programming in the R statistical computing language. The course is entirely optional and there are no grades or assignments but we encourage all incoming graduate students to attend if they are able.

1.2 Course Booklet

The course notes for the math and programming sections as well as all practice problems are available on this website and can be accessed by navigating the menus in the sidebar.

1.3 Schedule

The prefresher will run for a total of seven days September 12-14, September 19-21 and September 23rd, with breaks for the APSA conference and the new student orientation. Each day will run from around 9am to 4pm with many breaks in between.

The morning will focus on math instruction. We will have two one hour sessions from 9:30am - 10:30am and 10:45am-11:45am, with a ~15 minute break in between. These sessions will involve a combination of lectures and working through practice problems.

We will break for lunch from 12:00pm-1:00pm. On September 13th and Spetember 19th, we will have a catered lunch with a faculty member guest. Otherwise, you are free to explore the campus for various lunch options.

The afternoon will focus on coding instruction with lecture/demonstration from 1:30pm-2:45pm. After a short break you will work together on a variety of coding exercises from 3:00-3:30pm. In the last 30 minutes we will regroup to wrap up and discuss any questions on the material.

1.4 Software

As the afternoons of the prefresher will involve instruction in coding, you should be sure to bring a laptop and a charging cable. In addition, prior to the start of the prefresher, please make sure to have installed the following on your computer:

- R (version 4.2.1 or higher)
- RStudio Desktop Open Source License (this is the primary IDE or integrated development environment in which we will be working)
- LaTeX: This is primarily to allow you to generate PDF documents using RMarkdown. We will use the TinyTeX LaTeX distribution which is designed to be minimalist and tailored specifically for R users. After installing R and RStudio, open up an instance of R, install the 'tinytex' package and run the `install_tinytex()` command

```
install.packages('tinytex')
tinytex::install_tinytex()
```

We will also spend some time discussing document preparation and typesetting using LaTeX and Markdown. For the former, we will be using the popular cloud platform Overleaf, which allows for collaborative document editing and streamlines a lot of the irritating parts of typesetting in LaTeX. You should register for an account using your university e-mail as all University of Chicago students and faculty have access to an Overleaf Pro account for free.

You are also welcome to install a LaTeX editor on your local machine to work alongside the TinyTeX distribution or any other TeX distribution that you prefer such as TexMaker

1.5 Acknowledgments

This prefresher draws heavily on the wonderful materials that have been developed by over 20 years of instructors at the Harvard Government Math Prefresher that have been so generously distributed under the GPL 3.0 License. Special thanks to Shiro Kuriwaki, Yon Soo Park, and Connor Jerzak for their efforts in converting the original prefresher materials into the easily distributed Markdown format.

2 Functions and Operations

Topics Dimensionality; Interval Notation for $\mathbf{R}^1$; Neighborhoods: Intervals, Disks, and Balls; Introduction to Functions; Domain and Range; Some General Types of Functions; log, ln, and exp; Other Useful Functions; Graphing Functions; Solving for Variables; Finding Roots; Limit of a Function; Continuity; Sets, Sets, and More Sets.

2.1 Summation Operators $\sum$ and $\prod$

Addition (+), Subtraction (-), multiplication and division are basic operations of arithmetic – combining numbers. In statistics and calculus, we want to add a *sequence* of numbers that can be expressed as a pattern without needing to write down all its components. For example, how would we express the sum of all numbers from 1 to 100 without writing a hundred numbers?

For this we use the summation operator $\sum$ and the product operator $\prod$.

Summation:

$\sum\limits_{i=1}^{100} x_i = x_1 + x_2 + x_3 + \cdots + x_{100}$

The bottom of the $\sum$ symbol indicates an index (here, i), and its start value 1. At the top is where the index ends. The notion of "addition" is part of the $\sum$ symbol. The content to the right of the summation is the meat of what we add. While you can pick your favorite index, start, and end values, the content must also have the index.

- $\sum\limits_{i=1}^{n} cx_i = c\sum\limits_{i=1}^{n} x_i$
- $\sum\limits_{i=1}^{n} (x_i + y_i) = \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} y_i$
- $\sum\limits_{i=1}^{n} c = nc$

Product:

$$\prod_{i=1}^{n} x_i = x_1 x_2 x_3 \cdots x_n$$

Properties:

- $\prod\limits_{i=1}^{n} cx_i = c^n \prod\limits_{i=1}^{n} x_i$
- $\prod\limits_{i=k}^{n} cx_i = c^{n-k+1} \prod\limits_{i=k}^{n} x_i$
- $\prod\limits_{i=1}^{n} (x_i + y_i) =$ a total mess
- $\prod\limits_{i=1}^{n} c = c^n$

Other Useful Functions

Factorials!:

$$x! = x \cdot (x-1) \cdot (x-2) \cdots (1)$$

Modulo: Tells you the remainder when you divide the first number by the second.

- $17 \mod 3 = 2$
- $100 \ \% \ 30 = 10$

"'{example, name = "Operators", operators}

1. $\sum\limits_{i=1}^{5} i =$
2. $\prod\limits_{i=1}^{5} i =$
3. $14 \mod 4 =$
4. $4! =$

```
```{exercise, name = "Operators", operators1}
Let $x_1 = 4, x_2 = 3, x_3 = 7, x_4 = 11, x_5 = 2$

1. $\sum\limits_{i=1}^{3} (7)x_i$

2. $\sum\limits_{i=1}^{5} 2$

3. $\prod\limits_{i=3}^{5} (2)x_i$
```
```

2.2 Introduction to Functions

A **function** (in $\mathbf{R}^1$) is a mapping, or transformation, that relates members of one set to members of another set. For instance, if you have two sets: set A and set B, a function from A to B maps every value a in set A such that $f(a) \in B$. Functions can be "many-to-one", where many values or combinations of values from set A produce a single output in set B, or they can be "one-to-one", where each value in set A corresponds to a single value in set B. A function by definition has a single function value for each element of its domain. This means, there cannot be "one-to-many" mapping.

Dimensionality: $\mathbf{R}^1$ is the set of all real numbers extending from $-\infty$ to $+\infty$ — i.e., the real number line. $\mathbf{R}^n$ is an n-dimensional space, where each of the n axes extends from $-\infty$ to $+\infty$.

- $\mathbf{R}^1$ is a one dimensional line.
- $\mathbf{R}^2$ is a two dimensional plane.
- $\mathbf{R}^3$ is a three dimensional space.

Points in $\mathbf{R}^n$ are ordered n-tuples (just means an combination of n elements where order matters), where each element of the n-tuple represents the coordinate along that dimension.

For example:

- $\mathbf{R}^1$: (3)
- $\mathbf{R}^2$: (-15, 5)
- $\mathbf{R}^3$: (86, 4, 0)

Examples of mapping notation:

Function of one variable: $f : \mathbf{R}^1 \to \mathbf{R}^1$

- $f(x) = x + 1$. For each x in $\mathbf{R}^1$, $f(x)$ assigns the number $x + 1$.

Function of two variables: $f : \mathbf{R}^2 \to \mathbf{R}^1$.

- $f(x, y) = x^2 + y^2$. For each ordered pair (x, y) in $\mathbf{R}^2$, $f(x, y)$ assigns the number $x^2 + y^2$.

We often use variable x as input and another y as output, e.g. $y = x + 1$

"'{example, name = "Functions", functions}

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For $x \in [0, \infty]$, $f : x \to x^2$ (this could also be written as $f(x) = x^2$).
2. For $x \in [-\infty, \infty]$, $f : x \to x^2$.

```
```{exercise, name = "Functions", functions1}

For each of the following, state whether they are one-to-one or many-to-one functions.


1. For $x \in [-3, \infty]$, $f: x \rightarrow x^2$.


2. For $x \in [0, \infty]$, $f: x \rightarrow \sqrt{x}$
```

Some functions are defined only on proper subsets of $\mathbf{R}^n$.

- **Domain**: the set of numbers in X at which $f(x)$ is defined.
- **Range**: elements of Y assigned by $f(x)$ to elements of X, or

  $$f(X) = \{y : y = f(x), x \in X\}$$

  Most often used when talking about a function $f : \mathbf{R}^1 \to \mathbf{R}^1$.
- **Image**: same as range, but more often used when talking about a function $f : \mathbf{R}^n \to \mathbf{R}^1$.

Some General Types of Functions

Monomials: $f(x) = ax^k$

a is the coefficient. k is the degree.

Examples: $y = x^2$, $y = -\frac{1}{2}x^3$

Polynomials: sum of monomials.

Examples: $y = -\frac{1}{2}x^3 + x^2$, $y = 3x + 5$

The degree of a polynomial is the highest degree of its monomial terms. Also, it's often a good idea to write polynomials with terms in decreasing degree.

Exponential Functions: Example: $y = 2^x$
```

2.3 log and exp

Relationship of logarithmic and exponential functions:

$$y = \log_a(x) \iff a^y = x$$

The log function can be thought of as an inverse for exponential functions. a is referred to as the "base" of the logarithm.

Common Bases: The two most common logarithms are base 10 and base e.

1. Base 10: $y = \log_{10}(x) \iff 10^y = x$. The base 10 logarithm is often simply written as "$\log(x)$" with no base denoted.
2. Base e: $y = \log_e(x) \iff e^y = x$. The base e logarithm is referred to as the "natural" logarithm and is written as "$\ln(x)$".

Properties of exponential functions:

- $a^x a^y = a^{x+y}$
- $a^{-x} = 1/a^x$
- $a^x/a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$

Properties of logarithmic functions (any base):

Generally, when statisticians or social scientists write $\log(x)$ they mean $\log_e(x)$. In other words: $\log_e(x) \equiv \ln(x) \equiv \log(x)$

$$\log_a(a^x) = x$$

and

$$a^{\log_a(x)} = x$$

- $\log(xy) = \log(x) + \log(y)$
- $\log(x^y) = y \log(x)$
- $\log(1/x) = \log(x^{-1}) = -\log(x)$
- $\log(x/y) = \log(x \cdot y^{-1}) = \log(x) + \log(y^{-1}) = \log(x) - \log(y)$
- $\log(1) = \log(e^0) = 0$

Change of Base Formula: Use the change of base formula to switch bases as necessary:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Example:

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

You can use logs to go between sum and product notation. This will be particularly important when you're learning maximum likelihood estimation.

$$\begin{aligned}
\log\left(\prod_{i=1}^{n} x_i\right) &= \log(x_1 \cdot x_2 \cdot x_3 \cdots x_n) \\
&= \log(x_1) + \log(x_2) + \log(x_3) + \cdots + \log(x_n) \\
&= \sum_{i=1}^{n} \log(x_i)
\end{aligned}$$

Therefore, you can see that the log of a product is equal to the sum of the logs. We can write this more generally by adding in a constant, c:

$$\begin{aligned}
\log\left(\prod_{i=1}^{n} cx_i\right) &= \log(cx_1 \cdot cx_2 \cdots cx_n) \\
&= \log(c^n \cdot x_1 \cdot x_2 \cdots x_n) \\
&= \log(c^n) + \log(x_1) + \log(x_2) + \cdots + \log(x_n) \\
&= n\log(c) + \sum_{i=1}^{n} \log(x_i)
\end{aligned}$$

"'{example, name = "Logarithmic Functions", log}

Evaluate each of the following logarithms

1. $\log_4(16)$
2. $\log_2(16)$

Simplify the following logarithm. By "simplify", we actually really mean - use as many of the logarithmic properties as you can.

3. $\log_4(x^3y^5)$

````
```{exercise, name = "Logarithmic Functions", log1}


Evaluate each of the following logarithms

1. $\log_\frac{3}{2}(\frac{27}{8})$


Simplify each of the following logarithms. By "simplify", we actually really mean - use as ma

2. $\log(\frac{x^9y^5}{z^3})$


3. $\ln{\sqrt{xy}}$
````

2.4 Graphing Functions

What can a graph tell you about a function?

- Is the function increasing or decreasing? Over what part of the domain?
- How "fast" does it increase or decrease?
- Are there global or local maxima and minima? Where?
- Are there inflection points?
- Is the function continuous?
- Is the function differentiable?
- Does the function tend to some limit?
- Other questions related to the substance of the problem at hand.

2.5 Solving for Variables and Finding Roots

Sometimes we're given a function $y = f(x)$ and we want to find how x varies as a function of y. Use algebra to move x to the left hand side (LHS) of the equation and so that the right hand side (RHS) is only a function of y.

“‘{example, name = “Solving for Variables”, solvevar}

Solve for x:
````

1. $y = 3x + 2$
2. $y = e^x$

```
Solving for variables is especially important when we want to find the __roots__ of an equati

Procedure: Given $y=f(x)$, set $f(x)=0$.  Solve for $x$.

Multiple Roots:
        $$f(x)=x^2 - 9 \quad\Longrightarrow\quad 0=x^2 - 9 \quad\Longrightarrow\quad 9=x^2 \

__Quadratic Formula:__ For quadratic equations $ax^2+bx+c=0$, use the quadratic formula: $$x=
```

```
```{exercise, name = "Finding Roots", solvevar1}

Solve for x:

1. $f(x)=3x+2 = 0$

2. $f(x)=x^2+3x-4=0$

3. $f(x)=e^{-x}-10 = 0$
```

2.6 Sets

Interior Point: The point $\mathbf{x}$ is an interior point of the set S if $\mathbf{x}$ is in S and if there is some ϵ-ball around $\mathbf{x}$ that contains only points in S. The **interior** of S is the collection of all interior points in S. The interior can also be defined as the union of all open sets in S.
```

- If the set S is circular, the interior points are everything inside of the circle, but not on the circle's rim.
- Example: The interior of the set $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 < 4\}$.

Boundary Point: The point $\mathbf{x}$ is a boundary point of the set S if every ϵ-ball around $\mathbf{x}$ contains both points that are in S and points that are outside S. The **boundary** is the collection of all boundary points.

- If the set S is circular, the boundary points are everything on the circle's rim.
- Example: The boundary of $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 = 4\}$.

Open: A set S is open if for each point $\mathbf{x}$ in S, there exists an open ϵ-ball around $\mathbf{x}$ completely contained in S.

- If the set S is circular and open, the points contained within the set get infinitely close to the circle's rim, but do not touch it.
- Example: $\{(x, y) : x^2 + y^2 < 4\}$

Closed: A set S is closed if it contains all of its boundary points.

- Alternatively: A set is closed if its complement is open.
- If the set S is circular and closed, the set contains all points within the rim as well as the rim itself.
- Example: $\{(x, y) : x^2 + y^2 \leq 4\}$
- Note: a set may be neither open nor closed. Example: $\{(x, y) : 2 < x^2 + y^2 \leq 4\}$

Complement: The complement of set S is everything outside of S.

- If the set S is circular, the complement of S is everything outside of the circle.
- Example: The complement of $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 > 4\}$.

Empty: The empty (or null) set is a unique set that has no elements, denoted by $\{\}$ or $\emptyset$.

- The empty set is an example of a set that is open and closed, or a "clopen" set.
- Examples: The set of squares with 5 sides; the set of countries south of the South Pole.

Answers to Examples and Exercises

Answer to Example @ref(exm:operators):

1. $1 + 2 + 3 + 4 + 5 = 15$
2. $1 * 2 * 3 * 4 * 5 = 120$
3. 2

4. 4 * 3 * 2 * 1 = 24

Answer to Exercise @ref(exr:operators1):

1. 7(4 + 3 + 7) = 98
2. 2 + 2 + 2 + 2 + 2 = 10
3. 2^3(7)(11)(2) = 1232

Answer to Example @ref(exm:functions):

1. one-to-one
2. many-to-one

Answer to Exercise @ref(exr:functions1):

1. many-to-one
2. one-to-one

Answer to Example @ref(exm:log):

1. 2
2. 4
3. $3\log_4(x) + 5\log_4(y)$

Answer to Exercise @ref(exr:log1):

1. 3
2. $9\log(x) + 5\log(y) - 3\log(z)$
3. $\frac{1}{2}(\ln x + \ln y)$

Answer to Example @ref(exm:solvevar):

1. $y = 3x + 2 \quad \implies \quad -3x = 2 - y \quad \implies \quad 3x = y - 2 \quad \implies \quad x = \frac{1}{3}(y - 2)$
2. $x = \ln y$

Answer to Exercise @ref(exr:solvevar1):

1. $\frac{-2}{3}$
2. x = {1, -4}
3. x = - $\ln 10$