University of Chicago Political Science Math Prefresher

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9/12/2022

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1 Overview

1.1 Introduction

The 2022 UChicago Math Prefresher for incoming Political Science graduate students will be held from September 12-14; September 19-21 and September 23rd. The course is designed as a brief review of math fundamentals – calculus, optimization, probability theory and linear algebra among other topics – as well as an introduction to programming in the R statistical computing language. The course is entirely optional and there are no grades or assignments but we encourage all incoming graduate students to attend if they are able.

1.2 Course Booklet

The course notes for the math and programming sections as well as all practice problems are available on this website and can be accessed by navigating the menus in the sidebar.

1.3 Schedule

The prefresher will run for a total of seven days September 12-14, September 19-21 and September 23rd, with breaks for the APSA conference and the new student orientation. Each day will run from around 9am to 4pm with many breaks in between.

The morning will focus on math instruction. We will have two one hour sessions from 9:30am - 10:30am and 10:45am-11:45am, with a ~15 minute break in between. These sessions will involve a combination of lectures and working through practice problems.

We will break for lunch from 12:00pm-1:00pm. On September 13th and Spetember 19th, we will have a catered lunch with a faculty member guest. Otherwise, you are free to explore the campus for various lunch options.

The afternoon will focus on coding instruction with lecture/demonstration from 1:30pm-2:45pm. After a short break you will work together on a variety of coding exercises from 3:00-3:30pm. In the last 30 minutes we will regroup to wrap up and discuss any questions on the material.

1.4 Software

As the afternoons of the prefresher will involve instruction in coding, you should be sure to bring a laptop and a charging cable. In addition, prior to the start of the prefresher, please make sure to have installed the following on your computer:

- R (version 4.2.1 or higher)
- RStudio Desktop Open Source License (this is the primary IDE or integrated development environment in which we will be working)
- LaTeX: This is primarily to allow you to generate PDF documents using RMarkdown. We will use the TinyTeX LaTeX distribution which is designed to be minimalist and tailored specifically for R users. After installing R and RStudio, open up an instance of R, install the 'tinytex' package and run the `install_tinytex()` command

```
install.packages('tinytex')
tinytex::install_tinytex()
```

We will also spend some time discussing document preparation and typesetting using LaTeX and Markdown. For the former, we will be using the popular cloud platform Overleaf, which allows for collaborative document editing and streamlines a lot of the irritating parts of typesetting in LaTeX. You should register for an account using your university e-mail as all University of Chicago students and faculty have access to an Overleaf Pro account for free.

You are also welcome to install a LaTeX editor on your local machine to work alongside the TinyTeX distribution or any other TeX distribution that you prefer such as TexMaker

1.5 Acknowledgments

This prefresher draws heavily on the wonderful materials that have been developed by over 20 years of instructors at the Harvard Government Math Prefresher that have been so generously distributed under the GPL 3.0 License. Special thanks to Shiro Kuriwaki, Yon Soo Park, and Connor Jerzak for their efforts in converting the original prefresher materials into the easily distributed Markdown format.

2 Sets, Operations, and Functions

2.1 Sets

Sets are the fundamental building blocks of mathematics. Events are not inherently numerical: the onset of war or the stock market crashing is not inherently a number. Sets can define such events, and we wrap math around so that we have a transparent language to communicate about those events. Combining sets with operations, relations, metrics, measures, etc… allows us to define useful mathematical structures. For example, the set of *real numbers* ($\mathbb{R}$) has a notion of *order* as well as defined *operations* of addition and multiplication.

Set : A set is any well defined collection of elements. If x is an element of S, $x \in S$.

Examples:

1. The set of choices available to a player in Rock-Paper-Scissors $\{\text{Rock}, \text{Paper}, \text{Scissors}\}$
2. The set of possible outcomes of a roll of a six-sided die $\{1, 2, 3, 4, 5, 6\}$
3. The set of all natural numbers $\mathbb{N}$
4. The set of all real numbers $\mathbb{R}$

Common mathematical notation relevant to sets:

- $\in$ = "is an element of"; $\notin$ = "is not an element of"
- $\forall$ = "for all" (univeral quantifier)
- $\exists$ = "there exists" (existential quantifier)
- $:$ = "such that"

Subset: If every element of set A is also in set B, then A is a *subset* of B. $A \subseteq B$. If, in addition to being a subset of B, A is not equal to B, A is a *proper subset* $A \subset B$.

Empty Set: a set with no elements. $S = \{\}$. It is denoted by the symbol $\emptyset$.

Cardinality: The cardinality of a set S, typically written $|S|$ is the number of members of S.

Many sets are infinite. For example, $\mathbb{N}$ the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$ - Sets with cardinality less than $|\mathbb{N}|$ are *countable* - Sets with the same cardinality as *mathbbN*| are *countably infinite* - Sets with greater cardinality than $|\mathbb{N}|$ are *uncountably infinite* (e.g. the real numbers).

Set operations:

1. **Union**: The union of two sets A and B, $A \cup B$, is the set containing all of the elements in A or B. $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$
2. **Intersection**: The intersection of sets A and B, $A \cap B$, is the set containing all of the elements in both A and B. $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$
3. **Complement**: If set A is a subset of S, then the complement of A, denoted A^C, is the set containing all of the elements in S that are not in A.

Properties of set operations:

- **Commutative**: $A \cup B = B \cup A$; $A \cap B = B \cap A$
- **Associative**: $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's laws**: $(A \cup B)^C = A^C \cap B^C$; $(A \cap B)^C = A^C \cup B^C$
- **Disjointness**: Sets are disjoint when they do not intersect, such that $A \cap B = \emptyset$. A collection of sets is pairwise disjoint (**mutually exclusive**) if, for all $i \neq j$, $A_i \cap A_j = \emptyset$. A collection of sets form a partition of set S if they are pairwise disjoint and they cover set S, such that $\bigcup_{i=1}^{k} A_i = S$.

Example 2.1.

Sets

Let set A be $\{1, 2, 3, 4\}$, B be $\{3, 4, 5, 6\}$, and C be $\{5, 6, 7, 8\}$. Sets A, B, and C are all subsets of the S which is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Write out the following sets:

1. $A \cup B$
2. $C \cap B$
3. B^c
4. $A \cap (B \cup C)$

Exercise 2.1.

Sets

Suppose you had a pair of four-sided dice. You sum the results from a single toss.

What is the set of possible outcomes?

Consider subsets $A = \{2, 8\}$ and $B = \{2, 3, 7\}$ of the sample space you found. What is

1. A^c
2. $(A \cup B)^c$

2.2 Metric spaces

A *metric space* is a set that has a notion of *distance* - called a “metric” - defined between any two elements (sometimes referred to as “points”).

The distance function $d(x, y)$ defines the distance between element x and element y

- The real numbers $\mathbb{R}$ have a single distance function: $d(x, y) = |x - y|$
- In higher-dimensional real space (e.g. $\mathbb{R}^2$), we can define multiple distance metrics between $x = (x_1, x_2)$ and $y = (y_1, y_2)$
  - “Euclidean” distance: $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
  - “Taxicab” distance: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$
  - Chebyshev distance: $d(x, y) = \max\{|x_1 - y_1| + |x_2 - y_2|\}$
- All of these generalize to $\mathbb{R}^n$

A metric is a function that satisfies the following axioms

1. A distance between a point and itself is zero $d(x, x) = 0$
2. The distance between two points is strictly positive $d(x, y) > 0 \forall x \neq y$
3. Distance from x to y is the same as the distance from y to x ($d(x, y) = d(y, x)$)
4. The “triangle inequality” holds: $d(x, z) \leq d(x, y) + d(y, z)$

Once we have a metric space, we can define some additional useful concepts

Ball: A ball of radius r centered at x_0 is a set that contains all points with a distance less than r from x_0.

Sphere: A sphere of radius r centered at x_0 is the set that contains all points with a distance exactly r from x_0.

Interior Point: The point x is an interior point of the set S if x is in S and if there is some ϵ-ball around x that contains only points in S. The **interior** of S is the collection of all interior points in S. The interior can also be defined as the union of all open sets in S.

- If the set S is circular, the interior points are everything inside of the circle, but not on the circle's rim.
- Example: The interior of the set $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 < 4\}$.

Boundary Point: The point $\mathbf{x}$ is a boundary point of the set S if every ϵ-ball around $\mathbf{x}$ contains both points that are in S and points that are outside S. The **boundary** is the collection of all boundary points.

- If the set S is circular, the boundary points are everything on the circle's rim.
- Example: The boundary of $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 = 4\}$.

Open: A set S is open if for each point $\mathbf{x}$ in S, there exists an open ϵ-ball around $\mathbf{x}$ completely contained in S.

- If the set S is circular and open, the points contained within the set get infinitely close to the circle's rim, but do not touch it.
- Example: $\{(x, y) : x^2 + y^2 < 4\}$

Closed: A set S is closed if it contains all of its boundary points.

- Alternatively: A set is closed if its complement is open.
- If the set S is circular and closed, the set contains all points within the rim as well as the rim itself.
- Example: $\{(x, y) : x^2 + y^2 \leq 4\}$
- Note: a set may be neither open nor closed. Example: $\{(x, y) : 2 < x^2 + y^2 \leq 4\}$

2.3 Operators; Sum and Product notation

Addition (+), Subtraction (-), multiplication and division are basic operations of arithmetic. In statistics or calculus, we will often want to add a *sequence* of numbers that can be expressed as a pattern without needing to write down all its components. For example, how would we express the sum of all numbers from 1 to 100 without writing a hundred numbers?

For this we use the summation operator $\sum$ and the product operator $\prod$.

Summation:

$$\sum_{i=1}^{100} x_i = x_1 + x_2 + x_3 + \cdots + x_{100}$$

The bottom of the $\sum$ symbol indicates an index (here, i), and its start value 1. At the top is where the index ends. The notion of "addition" is part of the $\sum$ symbol. The content to the right of the summation is the meat of what we add. While you can pick your favorite index, start, and end values, the content must also have the index.

A few important features of sums:

- $\sum\limits_{i=1}^{n} cx_i = c\sum\limits_{i=1}^{n} x_i$
- $\sum\limits_{i=1}^{n} (x_i + y_i) = \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} y_i$
- $\sum\limits_{i=1}^{n} c = nc$

Product:

$$\prod_{i=1}^{n} x_i = x_1 x_2 x_3 \cdots x_n$$

Properties:

- $\prod\limits_{i=1}^{n} cx_i = c^n \prod\limits_{i=1}^{n} x_i$
- $\prod\limits_{i=k}^{n} cx_i = c^{n-k+1} \prod\limits_{i=k}^{n} x_i$
- $\prod\limits_{i=1}^{n} (x_i + y_i) = \text{a total mess}$
- $\prod\limits_{i=1}^{n} c = c^n$

Other Useful Operations

Factorials!:

$$x! = x \cdot (x-1) \cdot (x-2) \cdots (1)$$

Modulo: Tells you the remainder when you divide the first number by the second.

- $17 \mod 3 = 2$
- $100 \ \% \ 30 = 10$

Example 2.2.

Operators

1. $\sum_{i=1}^{5} i =$
2. $\prod_{i=1}^{5} i =$
3. $14 \mod 4 =$
4. $4! =$

Exercise 2.2.

Operators

Let $x_1 = 4, x_2 = 3, x_3 = 7, x_4 = 11, x_5 = 2$

1. $\sum\limits_{i=1}^{3} (7)x_i$
2. $\sum\limits_{i=1}^{5} 2$
3. $\prod\limits_{i=3}^{5} (2)x_i$

2.4 Introduction to Functions

A **function** is a mapping, or transformation, that relates members of one set to members of another set. For instance, if you have two sets: set A and set B, a function from A to B maps every value a in set A such that $f(a) \in B$. Functions can be "many-to-one", where many values or combinations of values from set A produce a single output in set B, or they can be "one-to-one", where each value in set A corresponds to a single value in set B. A function by definition has a single function value for each element of its domain. This means, there cannot be "one-to-many" mapping.

Dimensionality: $\mathbf{R}^1$ is the set of all real numbers extending from $-\infty$ to $+\infty$ — i.e., the real number line. $\mathbf{R}^n$ is an n-dimensional space, where each of the n axes extends from $-\infty$ to $+\infty$.

- $\mathbf{R}^1$ is a one dimensional line.
- $\mathbf{R}^2$ is a two dimensional plane.
- $\mathbf{R}^3$ is a three dimensional space.

Points in $\mathbf{R}^n$ are ordered n-tuples (just means an combination of n elements where order matters), where each element of the n-tuple represents the coordinate along that dimension.

For example:

- $\mathbf{R}^1$: (3)
- $\mathbf{R}^2$: (-15, 5)

- $\mathbf{R}^3$: (86, 4, 0)

Examples of mapping notation:

Function of one variable: $f : \mathbf{R}^1 \to \mathbf{R}^1$

- $f(x) = x + 1$. For each x in $\mathbf{R}^1$, $f(x)$ assigns the number $x + 1$.

Function of two variables: $f : \mathbf{R}^2 \to \mathbf{R}^1$.

- $f(x, y) = x^2 + y^2$. For each ordered pair (x, y) in $\mathbf{R}^2$, $f(x, y)$ assigns the number $x^2 + y^2$.

We often use variable x as input and another y as output, e.g. $y = x + 1$

Example 2.3.

Functions

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For $x \in [0, \infty]$, $f : x \to x^2$ (this could also be written as $f(x) = x^2$).
2. For $x \in [-\infty, \infty]$, $f : x \to x^2$.

Exercise 2.3.

Functions

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For $x \in [-3, \infty]$, $f : x \to x^2$.
2. For $x \in [0, \infty]$, $f : x \to \sqrt{x}$

Some functions are defined only on proper subsets of $\mathbf{R}^n$.

- **Domain**: the set of numbers in X at which $f(x)$ is defined.
- **Range**: elements of Y assigned by $f(x)$ to elements of X, or $f(X) = \{y : y = f(x), x \in X\}$ Most often used when talking about a function $f : \mathbf{R}^1 \to \mathbf{R}^1$.
- **Image**: same as range, but more often used when talking about a function $f : \mathbf{R}^n \to \mathbf{R}^1$.

Some General Types of Functions

Monomials: $f(x) = ax^k$

a is the coefficient. k is the degree.

Examples: $y = x^2$, $y = -\frac{1}{2}x^3$

Polynomials: sum of monomials.

Examples: $y = -\frac{1}{2}x^3 + x^2$, $y = 3x + 5$

The degree of a polynomial is the highest degree of its monomial terms. Also, it's often a good idea to write polynomials with terms in decreasing degree.

2.5 Logarithms and Exponents

Exponential Functions: Example: $y = 2^x$

Relationship of logarithmic and exponential functions:

$$y = \log_a(x) \iff a^y = x$$

The log function can be thought of as an inverse for exponential functions. a is referred to as the "base" of the logarithm.

Common Bases: The two most common logarithms are base 10 and base e.

1. Base 10: $y = \log_{10}(x) \iff 10^y = x$. The base 10 logarithm is often simply written as "$\log(x)$" with no base denoted.
2. Base e: $y = \log_e(x) \iff e^y = x$. The base e logarithm is referred to as the "natural" logarithm and is written as "$\ln(x)$".

Properties of exponential functions:

- $a^x a^y = a^{x+y}$
- $a^{-x} = 1/a^x$
- $a^x/a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$

Properties of logarithmic functions (any base):

Generally, when statisticians or social scientists write $\log(x)$ they mean $\log_e(x)$. In other words: $\log_e(x) \equiv \ln(x) \equiv \log(x)$

$$\log_a(a^x) = x$$

and

$$a^{\log_a(x)} = x$$

- $\log(xy) = \log(x) + \log(y)$
- $\log(x^y) = y\log(x)$
- $\log(1/x) = \log(x^{-1}) = -\log(x)$
- $\log(x/y) = \log(x \cdot y^{-1}) = \log(x) + \log(y^{-1}) = \log(x) - \log(y)$
- $\log(1) = \log(e^0) = 0$

Change of Base Formula: Use the change of base formula to switch bases as necessary:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Example:

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

You can use logs to go between sum and product notation. This will be particularly important when you're learning how to optimize likelihood functions.

$$
\begin{aligned}
\log\left(\prod_{i=1}^{n} x_i\right) &= \log(x_1 \cdot x_2 \cdot x_3 \cdots x_n) \\
&= \log(x_1) + \log(x_2) + \log(x_3) + \cdots + \log(x_n) \\
&= \sum_{i=1}^{n} \log(x_i)
\end{aligned}
$$

Therefore, you can see that the log of a product is equal to the sum of the logs. We can write this more generally by adding in a constant, c:

$$
\begin{aligned}
\log\left(\prod_{i=1}^{n} cx_i\right) &= \log(cx_1 \cdot cx_2 \cdots cx_n) \\
&= \log(c^n \cdot x_1 \cdot x_2 \cdots x_n) \\
&= \log(c^n) + \log(x_1) + \log(x_2) + \cdots + \log(x_n) \\
&= n\log(c) + \sum_{i=1}^{n} \log(x_i)
\end{aligned}
$$

Example 2.4.

Logarithms

Evaluate each of the following logarithms

1. $\log_4(16)$
2. $\log_2(16)$

Simplify the following logarithm. By "simplify", we actually really mean - use as many of the logarithmic properties as you can.

3. $\log_4(x^3y^5)$

Exercise 2.4. Evaluate each of the following logarithms

1. $\log_{\frac{3}{2}}(\frac{27}{8})$

Simplify each of the following logarithms. By "simplify", we actually really mean - use as many of the logarithmic properties as you can.

2. $\log(\frac{x^9y^5}{z^3})$
3. $\ln\sqrt{xy}$

2.6 Graphing Functions

What can a graph tell you about a function?

- Is the function increasing or decreasing? Over what part of the domain?
- How "fast" does it increase or decrease?
- Are there global or local maxima and minima? Where?
- Are there inflection points?
- Is the function continuous?
- Is the function differentiable?
- Does the function tend to some limit?
- Other questions related to the substance of the problem at hand.

2.7 Solving for Variables and Finding Roots

Sometimes we're given a function $y = f(x)$ and we want to find how x varies as a function of y. Use algebra to move x to the left hand side (LHS) of the equation and so that the right hand side (RHS) is only a function of y.

Example 2.5.

Solving

Solve for x:

1. $y = 3x + 2$
2. $y = e^x$

Solving for variables is especially important when we want to find the **roots** of an equation: those values of variables that cause an equation to equal zero. Especially important in finding equilibria and in doing maximum likelihood estimation.

Procedure: Given $y = f(x)$, set $f(x) = 0$. Solve for x.

Multiple Roots:

$$f(x) = x^2 - 9 \quad \Longrightarrow \quad 0 = x^2 - 9 \quad \Longrightarrow \quad 9 = x^2 \quad \Longrightarrow \quad \pm\sqrt{9} = \sqrt{x^2} \quad \Longrightarrow \quad \pm 3 = x$$

Quadratic Formula: For quadratic equations $ax^2 + bx + c = 0$, use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 2.5.

Roots

Solve for x:

1. $f(x) = 3x + 2 = 0$
2. $f(x) = x^2 + 3x - 4 = 0$
3. $f(x) = e^{-x} - 10 = 0$

3 Limits

Solving limits, i.e. finding out the value of functions as its input moves closer to some value, is important for the social scientist's mathematical toolkit for two related tasks. The first is for the study of calculus, which will be in turn useful to show where certain functions are maximized or minimized. The second is for the study of statistical inference, which is the study of inferring things about things you cannot see by using things you can see.

Example: The Central Limit Theorem

Perhaps the most important theorem in statistics is the Central Limit Theorem,

Theorem 3.1 (Central Limit Theorem)**.** *For any series of independent and identically distributed random variables $X_1, X_2, \cdots$, we know the distribution of its sum even if we do not know the distribution of X. The distribution of the sum is a Normal distribution.*

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Normal(0, 1)$$

where μ is the mean of X and σ is the standard deviation of X. The arrow is read as "converges in distribution to". $Normal(0, 1)$ indicates a Normal Distribution with mean 0 and variance 1.

That is, the limit of the distribution of the lefthand side is the distribution of the righthand side.

The sign of a limit is the arrow "$\rightarrow$". Although we have not yet covered probability so we have not described what distributions and random variables are, it is worth foreshadowing the Central Limit Theorem. The Central Limit Theorem is powerful because it gives us a *guarantee* of what would happen if $n \rightarrow \infty$, which in this case means we collected more data.

Example: The Law of Large Numbers

A finding that perhaps rivals the Central Limit Theorem is the (Weak) Law of Large Numbers:

Theorem 3.2 ((Weak) Law of Large Numbers)**.** *For any draw of identically distributed independent variables with mean μ, the sample average after n draws, $\bar{X}_n$, converges in probability to the true mean as $n \to \infty$:*

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0$$

A shorthand of which is $\bar{X}_n \xrightarrow{p} \mu$, where the arrow is read as "converges in probability to".

Intuitively, the more data, the more accurate is your guess. For example, Figure 3.1 shows how the sample average from many coin tosses converges to the true value : 0.5.

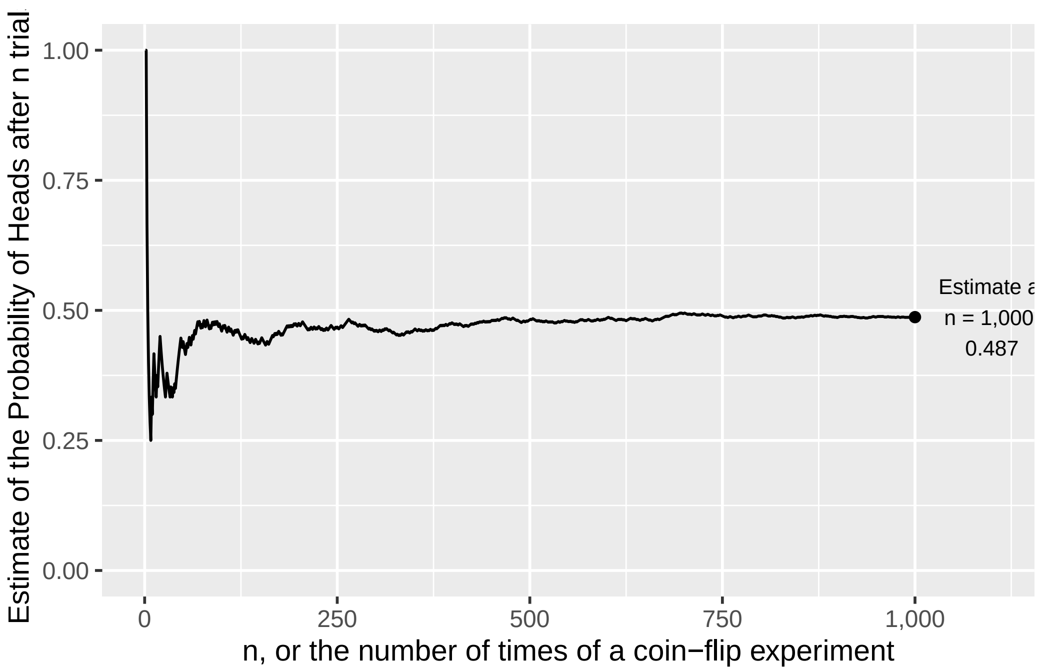

Figure 3.1: As the number of coin tosses goes to infinity, the average probabiity of heads converges to 0.5

3.1 Sequences

We need a couple of steps until we get to limit theorems in probability. First we will introduce a "sequence", then we will think about the limit of a sequence, then we will think about the limit of a *function*.

A **sequence** $\{x_n\} = \{x_1, x_2, x_3, ..., x_n\}$ is an ordered set of real numbers, where x_1 is the first term in the sequence and y_n is the nth term. Generally, a sequence is infinite, that is it extends to $n = \infty$. We can also write the sequence as $\{x_n\}_{n=1}^{\infty}$

where the subscript and superscript are read together as "from 1 to infinity."

Example 3.1.

Sequences

How do these sequences behave?

1. $\{A_n\} = \left\{2 - \frac{1}{n^2}\right\}$
2. $\{B_n\} = \left\{\frac{n^2+1}{n}\right\}$
3. $\{C_n\} = \left\{(-1)^n \left(1 - \frac{1}{n}\right)\right\}$

We find the sequence by simply "plugging in" the integers into each n. The important thing is to get a sense of how these numbers are going to change.

Graphing helps you make this point more clearly. See the sequence of $n = 1, ...20$ for each of the three examples in Figure 3.2.

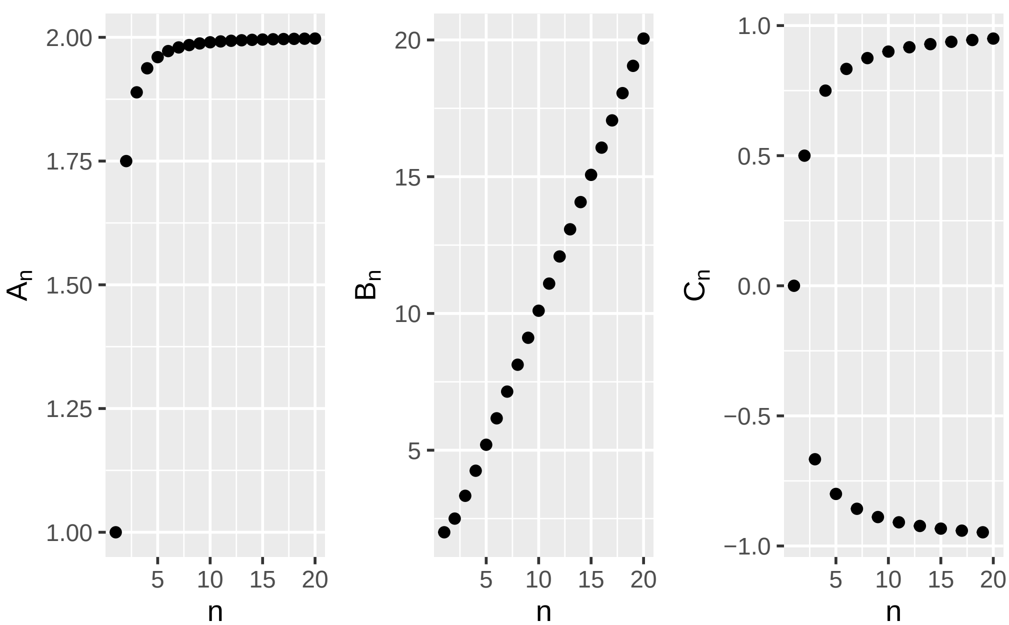

Figure 3.2: Behavior of Some Sequences

3.2 The Limit of a Sequence

The notion of "converging to a limit" is the behavior of the points in Example -@#exm-seqbehav. In some sense, that's the counterfactual we want to know. What happens as $n \to \infty$?

1. Sequences like 1 above that converge to a limit.
2. Sequences like 2 above that increase without bound.
3. Sequences like 3 above that neither converge nor increase without bound — alternating over the number line.

Definition: Limit The sequence $\{y_n\}$ has the limit L, which we write as $\lim\limits_{n\to\infty} y_n = L$, if for any $\epsilon > 0$ there is an integer N (which depends on ϵ) with the property that $|y_n - L| < \epsilon$ for each $n > N$. $\{y_n\}$ is said to converge to L. If the above does not hold, then $\{y_n\}$ diverges.

We can also express the behavior of a sequence as bounded or not:

1. Bounded: if $|y_n| \leq K$ for all n
2. Monotonically Increasing: $y_{n+1} > y_n$ for all n
3. Monotonically Decreasing: $y_{n+1} < y_n$ for all n

A limit is *unique*: If $\{y_n\}$ converges, then the limit L is unique.

If a sequence converges, then the sum of such sequences also converges. Let $\lim\limits_{n\to\infty} y_n = y$ and $\lim\limits_{n\to\infty} z_n = z$. Then

1. $\lim\limits_{n\to\infty} [ky_n + \ell z_n] = ky + \ell z$
2. $\lim\limits_{n\to\infty} y_n z_n = yz$
3. $\lim\limits_{n\to\infty} \frac{y_n}{z_n} = \frac{y}{z}$, provided $z \neq 0$

This looks reasonable enough. The harder question, obviously is when the parts of the fraction *don't* converge. If $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty} z_n = \infty$, What is $\lim_{n\to\infty} y_n - z_n$? What is $\lim_{n\to\infty} \frac{y_n}{z_n}$?

It is nice for a sequence to converge in limit. We want to know if complex-looking sequences converge or not. The name of the game here is to break that complex sequence up into sums of simple fractions where n only appears in the denominator: $\frac{1}{n}$, $\frac{1}{n^2}$, and so on. Each of these will converge to 0, because the denominator gets larger and larger. Then, because of the properties above, we can then find the final sequence.

Example 3.2.

Ratios

Find the limit of $\lim_{n\to\infty} \frac{n+3}{n}$

Solution. At first glance, $n+3$ and n both grow to ∞, so it looks like we need to divide infinity by infinity. However, we can express this fraction as a sum, then the limits apply separately:

$$\lim_{n\to\infty} \frac{n+3}{n} = \lim_{n\to\infty} \left(1 + \frac{3}{n}\right) = \underbrace{\lim_{n\to\infty} 1}_{1} + \underbrace{\lim_{n\to\infty} \left(\frac{3}{n}\right)}_{0}$$

so, the limit is actually 1.

After some practice, the key to intuition is whether one part of the fraction grows "faster" than another. If the denominator grows faster to infinity than the numerator, then the fraction will converge to 0, even if the numerator will also increase to infinity. In a sense, limits show how not all infinities are the same.

Exercise 3.1.

Limits

Find the following limits of sequences, then explain in English the intuition for why that is the case.

1. $\lim\limits_{n \to \infty} \frac{2n}{n^2+1}$
2. $\lim\limits_{n \to \infty} (n^3 - 100n^2)$

3.3 Limits of a Function

We've now covered functions and just covered limits of sequences, so now is the time to combine the two.

A function f is a compact representation of some behavior we care about. Like for sequences, we often want to know if $f(x)$ approaches some number L as its independent variable x moves to some number c (which is usually 0 or $\pm\infty$). If it does, we say that the limit of $f(x)$, as x approaches c, is L: $\lim\limits_{x \to c} f(x) = L$. Unlike a sequence, x is a continuous number, and we can move in decreasing order as well as increasing.

For a limit L to exist, the function $f(x)$ must approach L from both the left (increasing) and the right (decreasing).

Definition 3.1.

Limits of a function

Let $f(x)$ be defined at each point in some open interval containing the point c. Then L equals $\lim_{x\to c} f(x)$ if for any (small positive) number ϵ, there exists a corresponding number $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

A neat, if subtle result is that $f(x)$ does not necessarily have to be defined at c for $\lim_{x\to c}$ to exist.

Properties: Let f and g be functions with $\lim_{x\to c} f(x) = k$ and $\lim_{x\to c} g(x) = \ell$.

1. $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$
3. $\lim_{x\to c} f(x)g(x) = \left[\lim_{x\to c} f(x)\right] \cdot \left[\lim_{x\to c} g(x)\right]$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$.

Simple limits of functions can be solved as we did limits of sequences. Just be careful which part of the function is changing.

Example 3.3.

Limits of a function

Find the limit of the following functions.

1. $\lim_{x \to c} k$
2. $\lim_{x \to c} x$
3. $\lim_{x \to 2} (2x - 3)$
4. $\lim_{x \to c} x^n$

Limits can get more complex in roughly two ways. First, the functions may become large polynomials with many moving pieces. Second, the functions may become discontinuous.

The function can be thought of as a more general or "smooth" version of sequences. For example,

Example 3.4.

Limits of ratios

Find the limit of

$$\lim_{x\to\infty} \frac{(x^4+3x-99)(2-x^5)}{(18x^7+9x^6-3x^2-1)(x+1)}$$

Now, the functions will become a bit more complex:

Exercise 3.2.

Limits of a function

Solve the following limits of functions

1. $\lim_{x \to 0} |x|$
2. $\lim_{x \to 0} \left(1 + \frac{1}{x^2}\right)$

So there are a few more alternatives about what a limit of a function could be:

1. Right-hand limit: The value approached by $f(x)$ when you move from right to left.
2. Left-hand limit: The value approached by $f(x)$ when you move from left to right.
3. Infinity: The value approached by $f(x)$ as x grows infinitely large. Sometimes this may be a number; sometimes it might be ∞ or $-\infty$.
4. Negative infinity: The value approached by $f(x)$ as x grows infinitely negative. Sometimes this may be a number; sometimes it might be ∞ or $-\infty$.

The distinction between left and right becomes important when the function is not determined for some values of x. What are those cases in the examples below?

3.4 Continuity

To repeat a finding from the limits of functions: $f(x)$ does not necessarily have to be defined at c for $\lim_{x \to c}$ to exist. Functions that have breaks in their lines are called discontinuous. Functions that have no breaks are called continuous. Continuity is a concept that is more fundamental to, but related to that of "differentiability", which we will cover next in calculus.

Definition 3.2.

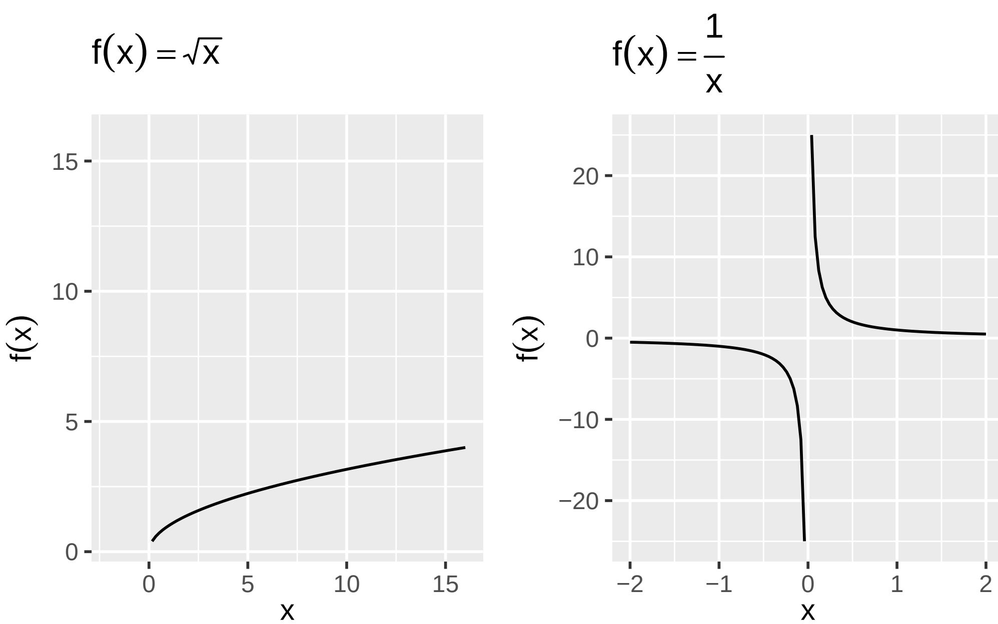

Figure 3.3: Functions which are not defined in some areas

Continuity

Suppose that the domain of the function f includes an open interval containing the point c. Then f is continuous at c if $\lim_{x \to c} f(x)$ exists and if $\lim_{x \to c} f(x) = f(c)$. Further, f is continuous on an open interval (a, b) if it is continuous at each point in the interval.

To prove that a function is continuous for all points is beyond this practical introduction to math, but the general intuition can be grasped by graphing.

Example 3.5.

Continuity

For each function, determine if it is continuous or discontinuous.

1. $f(x) = \sqrt{x}$
2. $f(x) = e^x$
3. $f(x) = 1 + \frac{1}{x^2}$
4. $f(x) = \text{floor}(x)$.

The floor is the smaller of the two integers bounding a number. So $\text{floor}(x = 2.999) = 2$, $\text{floor}(x = 2.0001) = 2$, and $\text{floor}(x = 2) = 2$.

Solution. In Figure 3.4, we can see that the first two functions are continuous, and the next two are discontinuous. $f(x) = 1 + \frac{1}{x^2}$ is discontinuous at $x = 0$, and $f(x) = \text{floor}(x)$ is discontinuous at each whole number.

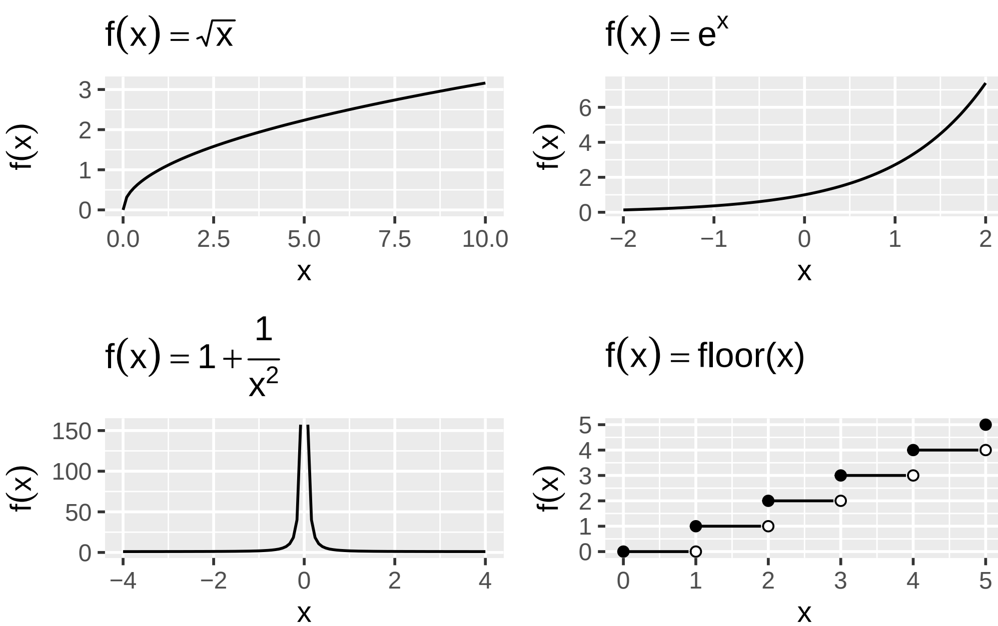

Figure 3.4: Continuous and Discontinuous Functions

Some properties of continuous functions:

1. If f and g are continuous at point c, then $f + g$, $f - g$, $f \cdot g$, $|f|$, and αf are continuous at point c also. f/g is continuous, provided $g(c) \neq 0$.
2. Boundedness: If f is continuous on the closed bounded interval $[a, b]$, then there is a number K such that $|f(x)| \leq K$ for each x in $[a, b]$.
3. Max/Min: If f is continuous on the closed bounded interval $[a, b]$, then f has a maximum and a minimum on $[a, b]$. They may be located at the end points.

Exercise

Let $f(x) = \frac{x^2+2x}{x}$.

1. Graph the function. Is it defined everywhere?
2. What is the functions limit at $x \to 0$?