Fourier Transforms!

PHYS 250 (Autumn 2025) – Lecture 11

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November 5, 2024

Outline

Moving towards physics data analysis

As we discussed last time, I would like to take the direction of this quarter more towards practical physics data analysis and algorithms. We will start with **Fourier Transforms** and **Neural Networks** and then analyze data from the **CMB, LIGO, and/or the Large Hadron Collider**.

Fourier Analysis and Neural Networks

- **Fourier Series and Analysis:**
  - Discussing the basics of Fourier Series
  - Evaluate, computationally, the coefficients of a simple series for both a square and sawtooth wave
  - Extend discussion to the **Fast Fourier Transform**
- **Neural Networks:**
  - Training computers to discover, identify, and analyze patterns in data
  - Modeling perspective on what a neural network achieves
  - Structure and function of a neuron
  - Mathematical properties of a neural network

This will be a mixture of Python Notebooks and Lecture Slides

Outline

Square Wave

Let's start with the Fourier Series for the **square wave** function.

As you may have heard in a variety of contexts (e.g. PHYS 133!) we can decompose any periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or, equivalently, complex exponentials).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{1}$$

This is possible because the trigonometric functions for a **set of complete, orthogonal basis vectors** that span the space.

Now open up the `Fourier-Series.ipynb` jupyter notebook and we will discuss this more deeply!

Fourier series for a square wave

We may determine the coefficients of a sine and cosine expansion to be:

$$a_n = \frac{2}{n\pi} \sin\left(n\omega_0 \frac{\pi}{2}\right) \tag{2}$$

which yields a discrete function:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{3}$$

Now open up the `Fourier-Transforms-Analysis.ipynb` jupyter notebook so that we can look at this in more detail!

Outline

1. *Plan going forward*
   - Data analysis tools
2. *The Square Wave*
   - Square Wave
3. *Extension to the Fourier Transform*
   - Euler's Formula
   - Fourier Transforms

Euler's Formula

We can generalize this by making use of Euler's Formula.

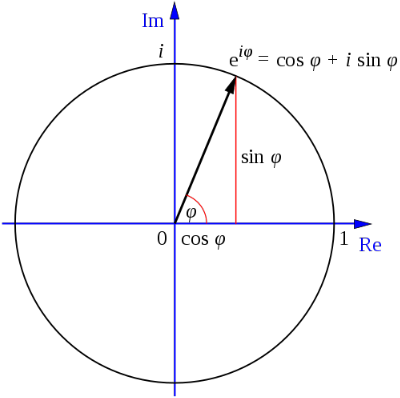
Euler's formula states that for any real number ϕ:

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \qquad (4)$$

When $\phi = \pi$, Euler's formula evaluates to

$$e^{i\pi} + 1 = 0, \qquad (5)$$

which is known as Euler's identity.



The implication is that it is possible to recover the amplitude of each wave in a Fourier series using an integral, which has many useful properties (in particular, that it's then continuous).

Fourier Transforms (I)

I will use the following definitions for the Fourier transform $\hat{f}(\xi)$ of a function $f(x)$, where x typically represents either a **spatial or time domain**, and ξ typically represents a corresponding inverse notion of **spatial or time frequency**.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \tag{6}$$

The **inverse transform** is then obtained via

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \tag{7}$$

In the case of spatial coordinates, x denotes length and ξ denotes inverse wavelength: $\xi = \dfrac{1}{\lambda}$. In the time domain, x denotes time and ξ denotes frequency. In the case that $x = t$ is in seconds, but ξ is **angular** frequency ω then a factor of 2π appears to get the normalization correct.

Fourier Transforms (II)

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$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{8}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \tag{9}$$

Since $\omega = 2\pi\xi = \dfrac{2\pi}{\lambda}$.

The $\dfrac{1}{\sqrt{2\pi}}$ factor in both these integrals is a common normalization in quantum mechanics but maybe not in engineering where only a single $\dfrac{1}{2\pi}$ factor is often used.

Discrete Fourier Transforms (I)

If $\hat{f}(\omega)$ or $f(t)$ are known analytically or numerically, the Fourier transform integrals can be evaluated using the integration techniques studied earlier.

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In this way the **DFT can be thought of as a technique for interpolating, compressing, and extrapolating data**.

Discussion

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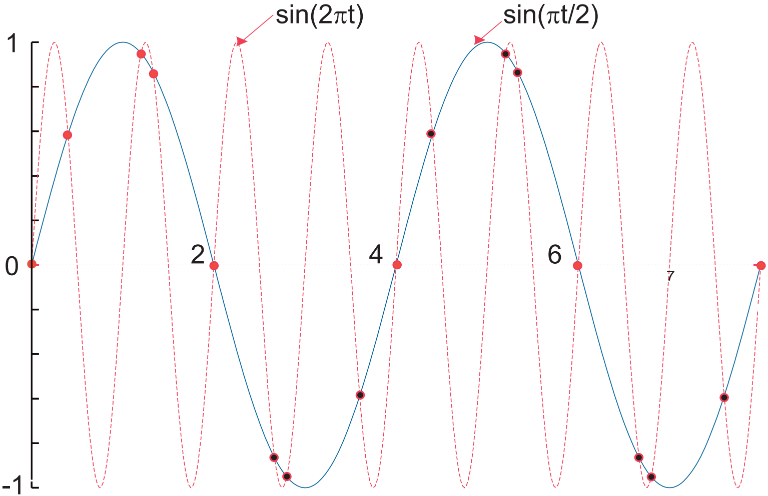

Discrete Fourier Transforms (II)

The DFT algorithm results from evaluating the integral not from 1 to $+1$ but rather from time 0 to time T over which the signal is measured, and from approximating the integration of the integral by computing a discrete sum:

$$\hat{f}(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega_n t} dt \tag{10}$$

$$\simeq \frac{1}{\sqrt{2\pi}} \int_{0}^{T} f(t) e^{-i\omega_n t} dt \tag{11}$$

$$\simeq \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} h f(t_k) e^{-i\omega_n t_k} \qquad (h \equiv \text{stepsize}) \tag{12}$$

$$\simeq \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n/N} \tag{13}$$

$$\hat{f}_n \equiv \frac{\hat{f}(\omega_n)}{h} = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n/N} \tag{14}$$

Discrete Fourier Transforms (III)

We then need the inverse as well, which we can obtain with $d\omega \rightarrow 2\pi/Nh$ we invert the $\hat{f}_n$

$$f_k \quad = \quad \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \frac{2\pi}{Nh} \hat{f}_n e^{i\omega_n t} \tag{15}$$

Once we know the N values of the transform $\hat{f}_n$, we can use this expression to evaluate $f(t)$ for any time t. The frequencies ωn are determined by the number of samples taken and by the total sampling time $T = Nh$ as

$$\omega_n = n\frac{2\pi}{Nh} \tag{16}$$

Clearly, the larger we make the time $T = Nh$ over which we sample the function, the smaller will be the frequency steps or resolution. Accordingly,if you want a smooth frequency spectrum, you need to have a small frequency step $2\pi/T$.

Discrete Fourier Transforms (IV)

Lastly, we can simplify this expression to yield a clear computational approach:

$$f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^{N} Z^{-nk} \hat{f}_n \qquad (Z = e^{-2\pi i/N}) \tag{17}$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} Z^{nk} f_k \qquad (n = 0, 1, \cdots, N) \tag{18}$$

With this formulation, the computer needs to compute only powers of Z.