CSE 2500 - Homework 8

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1. (10 points) Exercise Set 5.2, Question 34
   Find a formula a, r, m and n for the sum
   $ar^m + ar^{m+2} + ... + ar^{m+n}$
   where m and n are integers, n ≥ 0, and a and r are real numbers. Justify your answer.

   (a) Given the sequence $ar^m + ar^{m+2} + ... + ar^{m+n}$, a can be factored out, producing $a(r^m + r^{m+1} + r^{m+2} + ... + r^{m+n})$. With this, we can use the properties of exponents, specifically that $x^{a+b} = x^a x^b$, in order to rewrite the formula as $ar^m(1 + r + r^2 + ... + r^n)$. This formula can be rewritten as $ar^m(\frac{r^{n+1}-1}{r-1})$.

   $\therefore$ The formula for this sum is $ar^m(\frac{r^{n+1}-1}{r-1})$

2. (10 points) Exercise Set 5.4, Question 5
   Suppose that $e_0, e_1, e_2, ...$ is a sequence defined as follows:
   $e_0 = 12, e_1 = 29$
   $e_k = 5e_{k-1} - 6e_{k-2}$ for each integer $k \geq 2$.
   Prove that $e_n = 5 * 3^n + 7 * 2^n$ for every integer $n \geq 0$.

   (a) By utilizing mathematical induction, we can see that $P(n) = e_n = 5 * 3^n + 7 * 2^n$, we must show that $P(0)$ and $P(1)$ are both true. $e_0 = 12$, and by plugging in 0 into the initial e_n equation, we can see that $5 * 3^0 + 7 * 2^0 = 12$. Thus, both the left and right sides of the equation are true for $P(0)$.

   Moving on, we must show that $P(1)$ is also true, e_1 is given to be 29, and by plugging in 1, we can see that $5 * 3^1 + 7 * 2^1 = 29$. Thus, the initial step of the mathematical induction proof is true, since $P(0)$ and $P(1)$ are both true.

   With this in mind, we must show that $P(i)$ must be true for all integers $k \geq 1$ for $0 \leq i \leq k$, and with this, to additionally show that $P(k + 1)$ is true. To accomplish this, we must suppose that $P(i)$ is true for all integers in the range of i, and thus, to show that $e_{k+1} = 5 * 3^{k+1} + 7 * 2^{k+1}$.

Considering $e_{k+1} = 5e_{k+1-1} - 6e_{k+1-2}$ by the definition of e_n, we can substitute in $5(5*3^k + 7*2^k) - 6(5*3^{k-1} + 7*2^{k-1})$ since it is known that both $P(k-1)$ and $P(k)$ are true. Now, it can be algebraically evaluated that $5(5*3^k+7*2^k)-6(5*3^{k-1}+7*2^{k-1})$ is fundamentally equivalent to $5*3^{k+1}+7*2^{k+1}$, and thus, that $P(k+1)$ is true, since $e_{k+1} = 5*3^{k+1} + 7*2^{k+1}$.

$\therefore$ Since both the initial and induction steps were proven to be true, $P(n)$ has been shown to hold true for all integers $n \geq 0$. This demonstrates that for $e_n = 5*3^n + 7*2^n$ for all integers $n \geq 0$, where $e_0 = 12, e_1 = 29$, and $e_k = 5e_{k-1} - 6e_{k-2}$ for all integers $k \geq 2$.

3. (10 points) Exercise Set 5.7, Question 9.
   Use iteration to guess an explicit formula for the sequence.
   $g_k = \frac{g_{k-1}}{g_{k-1}+2}$, for each integer $k \geq 2$
   $g_1 = 1$

   (a) The recurrence relation of a sequence is by definition $g_k = \frac{g_{k-1}}{g_{k-1}+2}$ for all integers $k \geq 2, g_1 = 1$. With this, we will substitute values $k \geq 2$ into the equation and get

   $g_2 = \frac{g_1}{g_1+2} \implies \frac{1}{1+2} \implies \frac{1}{3}$
   $g_3 = \frac{g_2}{g_2+2} \implies \frac{\frac{1}{3}}{\frac{1}{3}+2} \implies \frac{1}{7}$
   $g_4 = \frac{g_3}{g_3+2} \implies \frac{\frac{1}{7}}{\frac{1}{7}+2} \implies \frac{1}{15}$

   Through observing the pattern, we can see that g_n will equal $\frac{1}{1+2+2^2+2^3+...+2^{n-1}}$ for all integers $n \geq 1$. A geometric sequence that follows $1+r+r^2+r^3+...+r^n$ can be expressed as $\frac{r^{n+1}-1}{r-1}$, and by substituting $r = 2, n = n-1$, we will get $1+2+2^2+2^3+...+2^{n-1} = \frac{2^{n-1+1}-1}{2-1} \implies 2^n - 1$.

   $\therefore$ By substituting this found value into g_n, we will find that the explicit formula for this series is $g_n = \frac{1}{2^n-1}$ for all integers $n \geq 1$.

4. (10 points) Exercise Set 5.7, Question 11.
   Use iteration to guess an explicit formula for the sequence.
   $p_k = p_{k-1} + 2*3^k$, for each integer $k \geq 2$, and $p_1 = 2$

   (a) Given that $p_k = p_{k-1} + 2*3^k$, we can evaluate the first few items in the sequence, wherein

   $p_1 = 2$
   $p_2 = p_1 + 2*3^2 = 2 + 2*3^20$
   $p_3 = p_2 + 2*3^3 = 2 + 2*3^2 + 2*3^3$
   $p_4 = p_3 + 2*3^4 = 2 + 2*3^2 + 2*3^3 + 2*3^4$
   $p_n = p_{n-1} + 2*3^n \implies p_n = 2 + 2*3^n + 2*3^3 + ... + 2*3^{n-1} + 2*3^n$

With p_n now known, we can simplify the expression by factoring out the common term $2 + 2 * 3^2$, leaving us with $2 + 2 * 3^2(1 + 3 + 3^2 + ... + 3^{n-1})$.

The sum of a geometric series can be expressed as $S = \frac{a(r^{n-1}-1)}{r-1}$. This can be applied to our p_n term, and as such we will get the following expression $2 + 2 * 3^2(\frac{3^{n-1}-1}{3-1})$, which can be simplified down to $\frac{2+2*3^2}{2} \implies 2 + 3^2(3^{n-1} - 1) \implies 2 + 3^{n+1} - 9 \implies 3^{n+1} - 7$.

$\therefore$ The explicit formula for this sequence is $p_n = 3^{n+1} - 7$.

5. (12 points) Exercise Set 5.8, Question 2.
   Which of the following are second-order linear homogeneous recurrence relations with constant coefficients.

   - $a_k = (k-1)a_{k-1} + 2ka_{k-2}$
   - $b_k = -b_{k-1} + 7b_{k-2}$
   - $c_k = 3c_{k-1} + 1$
   - $d_k = 3d_{k-1}^2 + d_{k-2}$
   - $r_k = r_{k-1} + 6r_{k-3}$
   - $s_k = s_{k-1} + 10s_{k-2}$

   (a) The second and last equations express second-order linear homogeneous recurrence relations with constant coefficients.

6. (18 points) Exercise Set 6.1, Question 11.
   Let the universal set be $\mathbb{R}$, the set of all real numbers, and let $A = \{x \in \mathbb{R} \mid 0 < x \leq 2\}$, $B = \{x \in \mathbb{R} \mid 1 \leq x < 4\}$, $C = \{x \in \mathbb{R} \mid 3 < x \leq 9\}$. Find each of the following:

   - $A \cup B$
   - $A \cap B$
   - A^c
   - $A \cup C$
   - $A \cap C$
   - B^c
   - $A^c \cap B^c$
   - $A^c \cup B^c$
   - $(A \cap B)^c$

(a) From the definition of set union, $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$. Knowing this, we can rewrite the original statement $A \cup B = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ and } 1 \leq x < 4\}$ to follow the form of set union: $A \cup B = \{x \in \mathbb{R} \mid x \in (0, 2] \text{ or } x \in [1, 4)\} \implies \{x \in \mathbb{R} \mid x \in (0, 4)\}$.

$\therefore A \cup B = \{x \in \mathbb{R} \mid 0 < x < 4\}$

(b) From the definition of set intersection, $A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$. Knowing this, we can rewrite the original statement $A \cap B = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ and } 1 \leq x < 4\}$ to follow the form of set intersection: $A \cap B = \{x \in \mathbb{R} \mid x \in (0, 2] \text{ and } x \in [1, 4)\} \implies \{x \in \mathbb{R} \mid x \in [1, 2]\}$

$\therefore A \cap B = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$

(c) From the definition of set compliments, $A^c = \{x \in U \mid x \notin A\}$. Knowing this, we can rewrite the original statement as $A^c = \{x \in \mathbb{R} \mid x \notin (0, 2]\} \implies \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x > 2\} \implies (-\infty, 0] \cup (2, \infty)$.

$\therefore A^c = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x > 2\}$

(d) From the definition of set union, $A \cup C = \{x \in U \mid x \in A \text{ or } x \in C\}$. Knowing this, we can rewrite the original statement $A \cup C = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$ to follow the form of set union: $A \cup C = \{x \in \mathbb{R} | x \in (0, 2] \text{ or } x \in [3, 9)\} \implies \{x \in \mathbb{R} | 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$ because there is nothing shared by the two sets.

$\therefore A \cup C = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$

(e) From the definition of set intersection, $A \cap C = \{x \in U | x \in A \text{ and } x \in C\}$. Knowing this, we can rewrite the original statement $A \cap C = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$ to follow the form of set intersection: $A \cap C = \{x \in \mathbb{R} \mid x \in (0, 2] \text{ and } x \in [3, 9)\} \implies \emptyset$ because there is nothing in common between the two sets.

$\therefore A \cap C = \emptyset$

(f) From the definition of set compliments, $B^c = \{x \in U \mid x \notin B\}$. Knowing this, we can rewrite the original statement as $B^c = \{x \in \mathbb{R} \mid x \notin [1, 4)\} \implies \{x \in \mathbb{R} \mid x \in (-\infty, 1) \text{ or } x \in [4, \infty)\} \implies \{x \in \mathbb{R} \mid x < 1 \text{ or } x \geq 4\}$

$\therefore B^c = \{x \in \mathbb{R} \mid x \notin (0, 2]\}$

(g) From the definition of set intersection, $A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$. Knowing this, we can rewrite the original statement $A^c \cap B^c = \{x \in \mathbb{R} \mid x \in (-\infty, 0] \text{ or } x \in (2, \infty) \text{ or } x \in (-\infty, 1) \text{ or } x \in [4, \infty)\}$ to follow the form of set union: $\{x \in \mathbb{R} \mid x \in (-\infty, 0] \cup x \in [4, \infty]\} \implies \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$

$\therefore A^c \cap B^c = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$

(h) From the definition of set union, $A \cup B = \{x \in U \mid x \in A$ or $x \in B\}$. Knowing this, we can rewrite the original statement $\{x \in \mathbb{R} \mid x \in (-\infty, 0]$ or $x \in (2, \infty)$ or $x \in (-\infty, 1)$ or $x \in [4, \infty)\}$ to follow the form of set union: $A^c \cup B^c = \{x \in \mathbb{R} \,|\, x \in (-\infty, 1)$ or $x \in (2, \infty)\}$ $\implies \{x \in \mathbb{R} \mid x < 1$ or $x > 2\}$.

$\therefore A^c \cup B^c = \{x \in \mathbb{R} \mid x < 1$ or $x > 2\}$

(i) From the definition of set compliments, $A^c = \{x \in U \,|\, x \notin A\}$. Knowing this, we can rewrite the original statement as $(A \cap B)^c = \{x \in \mathbb{R} \mid x \notin [1, 2]\} \implies \{x \in \mathbb{R} \mid x \in (-\infty, 1)$ or $x \in (2, \infty)\}$

$\therefore (A \cap B)^c = \{x \in \mathbb{R} \mid x < 1$ or $x > 2\}$

7. (10 points) Exercise Set 6.2, Question 9.
Use an element argument to prove the following statement. Assume that all sets are subsets of a universal set U.
For all set A, B, and C,
$(A - B) \cup (C - B) = (A \cup C) - B$

(a) First we must prove that $(A - B) \cup (C - B) \subseteq (A \cup C) - B$. In order to accomplish this, we must suppose that there exists $x \in (A - B)$ or $x \in (C - B)$.

Case 1
If $x \in (A - B)$, then by the definition of set difference, $x \in A$ and $x \notin B$. By this logic, since $x \in A$, by set union, $x \in A \cup C$. In this way, $x \in A \cup C$ and $x \notin B$, and by set difference, this implies that $x \in (A \cup C) - B$, meaning $x \in (A - B) \implies x \in (A \cup C) - B$.

Case 2
If $x \in (C - B)$, then by the definition of set difference, $x \in C$ and $x \notin B$. By this logic, since $x \in C$, by set union, $x \in A \cup C$. In this way, $x \in A \cup C$ and $x \notin B$, and by set difference, this implies that $x \in (A \cup C) - B$, meaning $x \in (C - B) \implies x \in (A \cup C) - B$.

(b) Since both cases produce the same result, $(A - B) \cup (C - B)$ must be a proper subset of $(A \cup C) - B$. With this known, we must now prove that $(A \cup C) - B \subseteq (A - B) \cup (C - B)$. In order to prove this statement, we can utilize the definition of set difference and suppose that there exists $x \in (A \cup C) - B$, and that $x \in A \cup C$ but $x \notin B$. From the definition of set union for $x \in A \cup C$, $x \in A$ or $x \in C$.

Case 1
Set difference implies $x \in (A - B)$ if $x \in A$, $x \notin B$.

Case 2
Set difference implies $x \in (C - B)$ if $x \in C$, $x \notin B$.

By what we can tell from the two cases, and the definition of set union, we can see that the set follows $x \in (A - B) \cup (C - B)$, and further, that $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

$\therefore$ By the definition of equality, we can conclude the fact that $(A - B) \cup (C - B) = (A \cup C) - B$.

8. (10 points) Exercise Set 6.3, Question 32.
Construct an algebraic proof for the given statement. Cite a property from Theorem 6.2.2 for each step.
$(A - B) \cup (A \cap B) = A$

   (a) $(A - B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B)$ [Set Difference Law]
   $A \cap (B^c \cup B)$ [Distributive Law]
   $A \cap (B \cup B^c)$ [Commutative Law for Set Union]
   $A \cap (U)$ [Compliment Law for Set Union]
   A [Identity Law for Set Intersection]

   $\therefore (A - B) \cup (A \cap B) = A$

9. (10 points) Suppose we are given the letters to the word $ALGORITHM$ for which we'd like to study the number of permutations and combinations of various subsets of this set of letters.

   (a) (3 points) How many ways can three of the letters of the word $ALGORITHM$ be selected and written in a row?

      i. Three of the letters of the word $ALGORITHM$ can be selected in $\frac{9!}{(9-3)!} = 504$ different ways.

   (b) (3 points) How many ways can three of the letters of the word $ALGORITHM$ be selected and written in a row but where we do wish to account for the order of the 3 letters? That is, the selection ALG is considered the same as GLA and the also the same as LGA, and thus, these 3 orderings should only be counted once.

      i. This can be accomplished in $\frac{9!}{(9-6)!} = 60480$ different ways.

   (c) (2 points) How many ways can six letters of the word $ALGORITHM$ be selected and written in a row if the first letter must be an A?

      i. Six letters of the word $ALGORITHM$ can be selected with A being the first letter in $\frac{8!}{(8-5)!} = 6720$ different ways.

   (d) (2 points) How many ways can six letters of the word $ALGORITHM$ be selected and written in a row if the first letter must be an A?, but where order is not important.

i. This can be accomplished in $\frac{7!}{(7-4)!} = 840$ different ways.