CSE 2500 - Fall 2021
Final Exam

1. Logic and Forms of Argument

   (a) [10 points] Write the (1) converse, (2) inverse, (3) contrapositive and (4) negation of the following universal conditional statement:

   $$\forall x \in \mathbb{R}, \text{ if } 2 < x < 5, \text{ then } x^3 < 125$$

   Solution:

   1. **converse** - (2 points) $\forall x \in \mathbb{R}$, if $x^3 < 125$ then $2 < x < 5$
   2. **inverse** - (2 points) $\forall x \in \mathbb{R}$, if $x \leq 2$ or $x \geq 5$, then $x^3 \geq 125$
   3. **contrapositive** - (3 points) $\forall x \in \mathbb{R}$, if $x^3 \geq 125$, then $x \leq 2$ or $x \geq 5$
   4. **negation** - (3 points) $\exists x \in \mathbb{R}$, $2 < x < 5$ and $x^3 \geq 125$

   (b) Use Universal Modus Ponens and Universal Modus Tollens to fill in valid conclusions for the arguments in i and ii. State which rule of Universal Modus Ponens or Unviersal Modus Tollens you use to reach the conclusion.

   i. [5 points] If the product of two real numbers, a and b, is nonzero, then a and b are both nonzero.
   At least one of the real numbers, a or b equals 0.

   Solution: If the product of two real numbers, a and b, is nonzero, then a and b are both nonzero.
   At least one of the real numbers, a or b equals 0.
   (2 points) $\therefore$ The product of a and b equals 0, (3 points) by Universal Modus Tollens.

   ii. [5 points] If the sum of two integers, a and b, is even, then either both a and b are even or both a and b are odd.
   For integers, a and b, $a + b$ is even.

   Solution: If the sum of two integers, a and b, is even, then either both a and b are even or both a and b are odd.
   For integers, a and b, $a + b$ is even.
   (2 points) $\therefore$ Either a and b are both even or both odd, (3 points) by Universal Modus Ponens.

2. Methods of Mathematical Proof

   (a) Consider the following statement,

      Statement A: $\forall$ integers m and n, if $2m+n$ is odd then m and n are both odd.

      i. [5 points] Write a negation for Statement A.

Solution: $\exists$ integers m and n such that $2m+n$ is odd and m and n are not both odd. In other words, $\exists$ integers m and n such that $2m+n$ is odd and at least one of m and n is even.

      ii. [10 points] Disprove Statement A. That is, show that Statement A is false.

Solution: To disprove a statement of the form "$\forall x \in D$, if $P(x)$ then $Q(x)$", we need find a value of x in D for which the hypothesis $P(x)$ is true and the conclusion $Q(x)$ is false Statement A can be shown to be false by giving a counterexample *Counterexample*: Let $m=2$ and $n=1$. Then, $2m+n=2\cdot 2+1=5$, which is odd. However, it is not the case that both m and n are odd because m is even. (This is one counterexample among many.)

   (b) [15 points] Prove the following statement by contradiction: For any integer n, if n^2 is even, then n is even.

Solution:

- We need to show the following: $\forall n \in \mathbb{Z}\, \text{even}(n^2) \implies \text{even}(n)$.
- (2 points) Let's assume $\exists n \in \mathbb{Z}\, \text{even}(n^2) \wedge \text{odd}(n)$.
- (4 points) We need to show that this statement leads to a contradiction. Since n is odd, there exists an integer k such that $n=2k+1$. So, we have the following:

$$
\begin{aligned}
n^2 &= (2k+1)^2 \\
&= 4k^2+4k+1 \\
&= 2(2k^2+2k)+1 \\
&= 2u+1 \text{ where } u = 2k^2+2k
\end{aligned} \tag{1}
$$

- (4 points) By closure of multiplication and addition on the set of integers, we know that u is an integer. Therefore, since $n^2 = 2u+1$ where u is an integer, we know that n^2 is odd. But this contradicts our original assumption that n^2 is even. Thus, we conclude that our supposition is false and that the original claim is true, by contradiction.

3. [20 points] Sequences, Mathematical Induction, and Recursion: Use mathematical induction to prove that for all integers $n > 1$

$$4 + 8 + 12 + \cdots + 4n = 2n^2 + 2n$$

Solution:

Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$P(n) : 4 + 8 + 12 + \cdots + 4n = 2n^2 + 2n$$

Note the following:

$$4 + 8 + 12 + \cdots + 4n = \sum_{i=1}^{n} 4i$$

Base case: Show that $P(2)$ is true. Note that we start with $P(2)$ since the proof requries we show $P(n)$ for all values of n *greater than* 1, (not *greater than or equal to* 1).

$P(2)$ is true because we have the following:

$$\sum_{i=1}^{2} 4i = 4 \cdot 1 + 4 \cdot 2 = 4 + 8 = 12 = 2(2^2) + 2(2) = 2n^2 + 2n \text{ where } n = 2.$$

Inductive step: Show that for all integers $k > 1$, if $P(k)$ is true then $P(k+1)$ is true.

Assume the inductive hypothesis, i.e., assume the following is true for some $k > 1$:

$$4 + 8 + 12 + \cdots + 4k = 2k^2 + 2k$$

We must show that

$$4 + 8 + 12 + \cdots + 4k + 4(k+1) = 2(k+1)^2 + 2(k+1)$$

Consider the following:

$$\begin{aligned}
4 + 8 + 12 + \cdots + 4k + 4(k+1) &= [4 + 8 + 12 + \cdots + 4k] + 4(k+1) \\
&= (2k^2 + 2k) + 4(k+1) && \text{(by the inductive hypothesis)} \\
&= 2k^2 + 2k + 4k + 4 && \text{(by applying the distributive property)} \\
&= 2k^2 + 4k + 2 + 2k + 2 && \text{(by rearranging terms)} \\
&= 2(k^2 + 2k + 1) + 2k + 2 && \text{(by factoring out 2 from the first 3 terms)} \\
&= 2(k^2 + 2k + 1) + 2(k+1) && \text{(by factoring out 2 from the last 2 terms)} \\
&= 2(k+1)^2 + 2(k+1) && \text{(by factoring the square)}
\end{aligned}$$

Thus, we have shown that if $P(k)$ is true for some $k > 1$, then $P(k+1)$ must also be true. Thus, we have shown $P(n)$ is true for all $n > 1$ by mathematical induction.

4. Counting
   (a) Suppose there are three routes from North Point to Boulder Creek, two routes from Boulder Creek to Beaver Dam, two routes from Beaver Dam to Star Lake, and four routes directly from Boulder Creek to Star Lake. (For the following questions, you may find it helpful to draw a sketch.)
      i. [5 points] How many routes from North Point to Star Lake pass through Beaver Dam?

      Solution: $3 \times 2 \times 2 = 12$ routes from North Point to Star Lake through Beaver Dam

      ii. [5 points] How many routes from North Point to Star Lake bypass Beaver Dam?

      Solution: $3 \times 4 = 12$ routes from North Point to Star Lake bypass Beaver Dam

   (b) A coin is tossed four times. Each time the result H for heads or T for tails is recorded. An outcome of $HHTT$ means that heads were obtained on the first two tosses and tails on the second two. Assume that heads and tails are equally likely on each toss.

i. [10 points] How many distinct outcomes are possible?

Solution: # possible outcomes $= 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$ possible outcomes

ii. [10 points] What is the probability that exactly two heads occur?

Solution: Let $E \equiv$ the event in which exactly two heads occur.
$E = \{\ HHTT, HTHT, HTTH, THHT, THTH, TTHH\ \}$

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{16} = \frac{3}{8} \tag{2}$$

End of exam