CSE 2500 - Homework 6

Mike Medved

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1. (15 points) Exercise Set 4.6, Question 28
   If true, prove the following statement or find a counterexample if the statement is false, but do not use Theorem 4.6.1 in your proof. For any odd integer n,

   $$\lfloor \frac{n^2}{4} \rfloor = (\frac{n-1}{2})(\frac{n+1}{2})$$

   (a) For an odd integer n, let $n = 2k+1$ for an integer k. Substituting n back into the given expression yields $\lfloor \frac{(2k+1)^2}{4} \rfloor \implies \lfloor \frac{4k^2+4k}{4} + \frac{1}{4} \rfloor \implies \lfloor k^2 + k + \frac{1}{4} \rfloor \implies k(k+1)$.

   Now, moving onto the right side, n can be substituted in to yield $(\frac{(2k+1)-1}{2})(\frac{(2k+1)+1}{2})$ which ultimately simplifies down to $k(k+1)$.

   $k(k+1)$ is a known form for an integer, and since both the left and right hand sides are equal to $k(k+1)$, they must equal each other.

2. (20 points) Exercise Set 4.6, Question 29
   If true, prove the following statement or find a counterexample if the statement is false, but do not use Theorem 4.6.1 in your proof. For any odd integer n,

   $$\lceil \frac{n^2}{4} \rceil = \frac{n^2+3}{4}$$

   (a) For an odd integer n, let $n = 2k+1$ for an integer k. Substituting n back into the given expression yields $\lceil \frac{(2k+1)^2}{4} \rceil \implies \lceil \frac{4k^2+1+4k}{4} \rceil \implies \lceil (k^2+k) + \frac{1}{4} \rceil \implies k^2+k+1$.

   Now, moving onto the right side, n can be substituted in to yield $\frac{(2k+1)^2+3}{4}$ which ultimately simplifies down to k^2+k+1.

   Since both the sides of this expression are equal, the expression must be equal.

3. (20 points) Exercise Set 4.8, Question 23.
   Prove that for any integer $a, 9 \nmid (a^2 - 3)$

   (a) $9|(a^2+3)$ can be expressed as $a^2-3 = 9b$ for an integer b. With this, it can be arranged hat $a^2 = 9b+3$, and further that $a^2 = 3(1+3b)$. This confirms that both a and a^2 are multiples of 3. Therefore $a^2 = 9c^2$ for some integer c.

      This demonstrates that $9|a^2$, and by supposition, that $9|(a^2 - 3)$. However, by the properties of divisibility, it can be expressed as $9|(a^2 - (a^2 - 3))$, which when evaluated, $9|3$, cannot be true.

      This means that the supposition that was just created was false. This discovery contradicts the statement, and as a result proves $9 \nmid (a^2-3)$.

4. (15 points) Exercise Set 5.2, Question 2.
   Use mathematical induction to show that any postage of at least 12¢ can be obtained using 3¢ and 7¢ stamps.

   (a) Let $P(n)$ represent that n¢ can be made up of 3¢ and 7¢ coins; since $P(n)$ should be true for all integers $n \geq 12$, let $a = 12$.

      To demonstrate that $P(12)$ holds true, $12 = 4 * 3$, which clearly demonstrates that 12¢ can be accumulated using four 3¢ coins. Therefore $P(12)$ is true.

      Now with this baseline truth, it must be shown that for all integers $k \geq 12$, $P(k + 1)$ must hold true. To accomplish this, the problem must be visualized; if k¢ can be made up using 7¢ twice or more, then replace two 7¢ stamps by using five 3¢ stamps. In this way, the total amount of cents has increased by one, resulting in $(k + 1)$.

      Using this logic, if k¢ can be made using a single 7¢ stamp, and if less than two 3¢ stamps are used, it result in at the most 10¢. Given at most two 3¢ stamps, you could replace two 3¢ using one 7¢, then the total amount of cents that can be made increased by one, resulting in the aforementioned $(k + 1)$. If k¢ cannot be made using 7¢ stamps, then k¢ can only be made using 3¢ stamps, and at that, at least four 3¢ stamps are required to make the minimum 12¢. Continuing, the four 3¢ stamps can be replaced by a single 7¢ stamp, and two 3¢ stamps. In this way, the total amount of cents also increases by one, yielding the same $(k+1)$ quantity. In this way, $P(k + 1)$ holds true.

      $\therefore$ n¢ can be made of 3¢ and 7¢ stamps for any $n \geq 12$.

5. (15 points) Exercise Set 5.2, Question 14.
   Prove the following statement by mathematical induction.

$\sum_{i=1}^{n+1} i * 2^i = n * 2^{n+2} + 2$, for all integers $n \geq 0$.

(a) Let $P(n)$ represent $\sum_{i=1}^{n+1} i * 2^i = n * 2^{n+2} + 2$; since $P(n)$ should be true for all integers $n \geq 0$, let $a = 0$.

To demonstrate that $P(0)$ holds true, $\sum_{i=1}^{n+1} i*2^i = 0*2^2+2$, wherein the left-hand side equals 0, the right-hand side can be represented as $0 * 2^2 + 2 = 0 + 2$, which also equals 2. This demonstrates that $P(0)$ holds true for this statement.

Now with this baseline truth, it must be shown that for all integers $k \geq 0$, $P(k)$ must hold true. To accomplish this, the discrete sum can be represented as $\sum_{i=1}^{k+1} i * 2^i = k * 2^{k+2} + 2$ by substituting in k for n. Additionally, it can be demonstrated that for all values of $k + 1$ the statement holds true by inserting $k + 1$ into the same discrete sum; $\sum_{i=1}^{(k+1)+1} i * 2^i = (k + 1) * 2^{(k+1)+2} + 2$.

Using the same principle as the baseline truth, the left-hand side of the $k + 1$ substitution can be represented as $\sum_{i=1}^{(k+1)+1} i * 2^i = \sum_{i=1}^{k+2} i * 2^i \implies \sum_{i=1}^{k+1} i * 2^i + (k+2) * 2^{k+2} \implies k * 2^{k+2} + 2 + (k+2) * 2^{k+2} \implies k * 2^{k+2} + (k+2) * 2^{k+2} + 2 \implies 2^{k+2}(2k+2) + 2 \implies (k+1) * 2^{(k+1)+2} + 2$ which equals the right-hand side of the expression. In this way, $P(k + 1)$ is also true.

Since both the baseline and induction truths are valid, the property $P(n)$ must be true for all integers $n \geq 0$.

$\therefore \sum_{i=1}^{n+1} i * 2^i = n * 2^{n+2} + 2$ for all integers $n \geq 0$.

6. (15 points) Exercise Set 5.3, Question 14.
Prove the following statement by mathematical induction.

$n^3 - n$ is divisible by 6, for each integer $n \geq 0$.

(a) Let $P(n)$ represent $n^3 - n|6$, thus $P(n)$ should hold true for all integers $n \geq 0$, let $a = 0$.

To demonstrate that $P(0)$ holds true, $n^3 - n \implies 0^3 - 0 = 0$. In this case, $0|6$, thus $n^3 - n|6$ for $n = 0$, in this way, $P(0)$ holds true.

Now, with this baseline truth, it must be shown that for all integers $k \geq 0$, $P(k)$ must hold true. To accomplish this, the expression can be rewritten as $k^3 - k = 6a$ for some integer a. In order to accomplish this, we must prove that $P(n)$ must be true for any value of $n = k + 1$, so for $k + 1$, $n^3 - n = 6m + 3k(k + 1)$.

With this substitution in mind, and the fact that $k(k+1)$ is the product of two consecutive integers, this implies that either one of k or $k+1$ must be an even integer. Therefore, this product of $k(k+1)$ must be an even integer. In this way, it can be assumed that $k(k+1) = 2b$, for some integer b. By substituting b into the previous expression, we are left with $n^3 - n = 6(m + b)$. By using the rules of integers, it is known that the sum of two integers must also be an integer, therefore by this logic, b must be an integer. Further, this implies that $n^3 - n|6$ for $n = k+1$, and that $P(n)$ must be true for $n = k+1$.

$\therefore$ By mathematical induction, $P(n)$ is true for all integers $n \geq 0$, thus, $n^3 - n|6$ for $n \geq 0$.