CSE 2500 - Introduction to Discrete Systems Fall 2021

Midterm Exam 2

For full credit, please adhere to the following:

- There are 3 problems with total 100 points.
- Unsupported answers receive no credit.
- All answers can be typed or handwritten, and should be readable.
- Submit the assignment in one file (.pdf, .doc, etc.) via HuskyCT.
- You have 15 minutes to upload your solutions after the exam.

1. (a) (15 points) Assume that m and n are particular integers.
   - Is $(6m + 8n)$ even? Why?
   - Is $10mn + 7$ odd? Why?
   - If $m > n > 0$, is $m^2 - n^2$ composite? Why?

   Solution:

   - Yes. [3 points] $(6m+8n) = 2(3m+4n)$ and $3m+4n$ is an integer [2 point]
   - Yes. [3 points] $10mn + 7 = 2(5mn + 3) + 1$ and $5mn + 3$ is an interger [2 point]
   - Not necessarily. [3 points] $m^2-n^2 = (m+n)(m-n)$. [1 point] If $m-n = 1$, then $m^2 - n^2$ is a prime number. [1 point] (or If $m - n > 1$, then $m^2 - n^2$ is a composite number.)

   (b) (5 points) Is 605.83 a rational number? Justify your answer

   Solution: Yes. [3 points] Since $605.83 = \frac{60583}{100}$ [2 points].

   (c) (10 points) Prove the following statement directly from the definitions of the terms. Do not use any other facts previously proved in class or in the text or in the exercises.

   For all integers a, b, and c, if $a|b$ and $a|c$, then $a|(5b + 3c)$.

   Solution:
   Proof: Suppose a, b, and c are any integers such that $a|b$ and $a|c$. [We must show that $a|(5b + 3c)$].

By definition of divisibility, b = ar and c = as for some integers r and s.

$$\begin{aligned} 5b + 3c &= 5(ar) + 3(as) \text{ by substitution} \\ &= a(5r + 3s) \text{ by the commutative and associative laws of algebra.} \end{aligned} \tag{1}$$

Let $t = 5r + 3s$. Then t is an integer because products and sums of integers are integers, and $5b + 3c = at$. Thus, by definition of divisibility, $a|(5b + 3c)$.

2. (a) (20 points) Prove directly from the definitions that for every integer n, $n^2 - n + 3$ is odd. Use division into two cases: n is even and n is odd.

Solution:
Proof: Suppose n is a integer. n can be either even $(2k)$ or odd$(2k + 1)$ for some integer k.
Case 1 $(n = 2k)$: we must show that $n^2 - n + 3$ is odd.

$$\begin{aligned} (2k)^2 - 2k + 3 &= 4k^2 - 2k + 3 \\ &= 2(2k^2 - k + 1) + 1 \end{aligned} \tag{2}$$

$t = 2k^2 - k + 1$ is a integer since the multiplication and addition of integer are closed on the set of integer. Therefore, $n^2 - n + 3$ can be written as $2t + 1$ and is an odd number.
Case 2 $(n = 2k + 1)$: we must show that $n^2 - n + 3$ is odd.

$$\begin{aligned} (2k + 1)^2 - (2k + 1) + 3 &= 4k^2 + 4k + 1 - (2k + 1) + 3 \\ &= 2(2k^2 + k + 1) + 1 \end{aligned} \tag{3}$$

$t = 2k^2 + k + 1$ is a integer since the multiplication and addition of integer are closed on the set of integer. Therefore, $n^2 - n + 3$ can be written as $2t + 1$ and is an odd number.
Hence for both two cases, we showed that $n^2 - n + 3$ is odd.

(b) (10 points) Consider the following statement: For all integers n, if n^3 is even then n is even. Prove the statement by **contradiction**.

Solution:
Proof: (by contradiction):Suppose not. That is, suppose there is an integer n such that n^3 is even and n is odd. [We must show that this supposition leads logically to a **contradiction**.]
By definition of odd, $n = 2a + 1$ for some integer a. Thus, by substitution and algebra, $n^3 = (2a + 1)^3 = (2a + 1)^2(2a + 1) = (4a^2 + 4a + 1)(2a + 1) = 8a^3 + 12a^2 + 6a + 1 = 2(4a^3 + 6a^2 + 3a) + 1$.

Let $t = 4a^3 + 6a^2 + 3a$. Then, $n^3 = 2t + 1$ and t is an integer because it is a sum of products of integers. It follows that n^3 is odd by definition of odd, which contradicts the supposition that n^3 is even. [Hence the supposition is false and the given statement is true.]

(c) (10 points) Consider the following statement: For all integers n, if n^3 is even then n is even. Prove the statement by **contraposition**.

Solution:
Proof (by contraposition): Suppose n is any integer such that n is odd. [We must show that n^3 is odd.]
By definition of odd, $n = 2a + 1$ for some integer a. Thus $n^3 = (2a + 1)^3 = (2a+1)^2(2a+1) = (4a^2+4a+1)(2a+1) = 8a^3+12a^2+6a+1 = 2(4a^3+6a^2+3a)+1$. Let $t = 4a^3 + 6a^2 + 3a$. Then, $n^3 = 2t + 1$ and t is an integer because it is a sum of products of integers. It follows that n^3 is odd by definition of odd [as was to be shown]

3. (30 points) (a) 10 points; (b) 10 points; (c) 10 points

For each integer $n \geq 3$, let $P(n)$ be the equation

$$\boxed{3 + 4 + 5 + \cdots + n = \frac{(n-2)(n+3)}{2}.} \leftarrow P(n)$$

(Recall that by definition $3 + 4 + 5 + \cdots + n = \sum_{i=3}^{n} i$.)

(a) Is $P(3)$ true? Justify your answer.

(b) In the inductive step of a proof that $P(n)$ is true for all integers $n \geq 3$, we suppose $P(k)$ is true (this is the inductive hypothesis), and then we show that $P(k+1)$ is true. Fill in the blanks below to write what we suppose and what we must show for this particular equation.

Proof that for all integers $k \geq 3$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer that is greater than or equal to 3, and suppose that ______.
We must show that ______.

(c) Finish the proof started in (b) above.

Solution:

a. $P(3) : \sum_{i=3}^{3} i = \frac{(3-2)(3+3)}{2}$

$P(3)$ is true because the left-hand side equals $\sum_{i=3}^{3} i = 3$ and the right-hand side equals $\frac{(3-2)(3+3)}{2} = \frac{1 \cdot 6}{2} = 3$ also.

b. (i) $P(k)$: $3+4+5+\cdots+k = \frac{(k-2)(k+3)}{2}$;

(ii) $P(k+1)$: $3+4+5+\cdots+(k+1) = \frac{((k+1)-2)((k+1)+3)}{2}$

Or, equivalently, $P(k+1)$ is $3+4+5+\cdots+(k+1) = \frac{(k-1)(k+4)}{2}$

c. *Proof that for all integers* $k \geq 3$, *if* $P(k)$ *is true then* $P(k+1)$ *is true*:

Let k be any integer that is greater than or equal to 3, and suppose that

$$3+4+5+\cdots+k = \frac{(k-2)(k+3)}{2} \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$3+4+5+\cdots+(k+1) = \frac{((k+1)-2)((k+1)+3)}{2},$$

or, equivalently,

$$3+4+5+\cdots+(k+1) = \frac{(k-1)(k+4)}{2}. \qquad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is

$$\begin{aligned}
3+4+5+\cdots+(k+1) &= 3+4+5+\cdots+k+(k+1) \\
&\qquad \text{by making the next-to-last term explicit} \\
&= \frac{(k-2)(k+3)}{2} + (k+1) \\
&\qquad \text{by inductive hypothesis} \\
&= \frac{(k-2)(k+3)}{2} + \frac{2(k+1)}{2} \\
&\qquad \text{by creating a common denominator} \\
&= \frac{k^2+k-6}{2} + \frac{2k+2}{2} \\
&\qquad \text{by multiplying out} \\
&= \frac{k^2+k-6+2k+2}{2} \\
&\qquad \text{by adding the fractions} \\
&= \frac{k^2+3k-4}{2} \\
&\qquad \text{by combining like terms.}
\end{aligned}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k-1)(k+4)}{2} = \frac{k^2+4k-k-4)}{2} = \frac{k^2+3k-4)}{2}$$

Thus the left-hand and right-hand sides of $P(k+1)$ are equal [as was to be shown].