CSE 2500 - Homework 5

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1. (10 points) Exercise Set 4.3, Question 10
   Assume that m and n are both integers and that $n \neq 0$. Explain why $\frac{(5m+12n)}{(4n)}$ must be a rational number.

   (a) Since $4 \neq 0$ or $n \neq 0$, this means implies $4n \neq 0$. Additionally, $5m + 12n \neq 0$, therefore $\frac{5m+12n}{4n}$ is the quotient of two non-negative integers with a nonzero denominator. Thus, the quantity $\frac{5m+12n}{4n}$ is a rational number.

2. (10 points) Exercise Set 4.3, Question 28
   Suppose a, b, c and d are integers and $a \neq c$. Suppose also that x is a real number that satisfies the equation

   $$\frac{ax+b}{cx+d} = 1$$

   Must x be rational? If so, express x as a ratio of two integers.

   (a) Firstly, by cross multiplying the quantity $ax + b = cx + d$, we can solve for $x = \frac{d-b}{a-c}$ with $a \neq c$.

   Let $p = d - b$, $q = a - c$, then the quantity can be expressed as $\frac{p}{q}$, with p and q representing integers. Since the difference of any integers is an integer, $q = a - c \implies q \neq 0$.

   $\therefore\ x = \frac{p}{q}$ for some integers p and q, with $q \neq 0$. As a result, x is the ratio of two integers and is considered rational.

3. (10 points) Exercise Set 4.3, Question 11.
   Prove that the negative of any rational number is rational.

   (a) Let x be a rational number, and a, b be integers. If $x = \frac{a}{b}$, then the negative of x must be $\frac{-a}{b}$. With this, p can be represented as $a * -1$, where p is an integer.

   $\therefore\ -x = \frac{p}{b}$ assumes the form of a rational number, and as a result must be rational.

4. (10 points) Exercise Set 4.4, Question 5.
Is $6m(2m + 10)$ divisible by 4? Give reasoning for your answer. Assume m is an integer.

   (a) This expression can be simplified by factoring: $12(m^2 + 5m)$.

   Since it is given that m is an integer, $m^2 + 5m$ must also be an integer. Additionally, since $m^2 + 5m$ is a multiple of twelve, twelve is divided by four.

   Therefore $12(m^2 + 5m)$ is $6m(2m + 10)$ divided by four.

5. (10 points) Exercise Set 4.4, Question 29.
Determine whether the following statement is true or false. If true, prove the statement directly from definitions. If false, give a counterexample.

   For all integers a and b, if $a|b$ then $a^2|b^2$.

   (a) Let a and b are integers. From the definition of divisibility, $b = ak$ for an integer k. Through squaring each side of this statement, $b^2 = a^2k^2$ and letting $c = k^2$, then c is an integer since k is an integer.

   As a result, $b^2 = a^2c$ for an integer c, and by the definition revisiting the definition of divisibility, it can be seen that $a^2|b^2$.

   $\therefore$ For all integers a and b, if $a|b$, then $a^2|b^2$.

6. (10 points) Exercise Set 4.4, Question 45.
Prove that if n is any nonnegative integer whose decimal representation ends in 5, then $5|n$.

   (a) Let n be a nonnegative integer which has a decimal representation ending in 5. By definition, for an integer $k \geq 0$, then n can be represented as $n = d_k * 10^k + d_{k-1} * 10^{k-1} + ... + d_1 * 2 + 1$.

   With this, n can be further simplified by factoring out five, leaving $d_k * 2 * 10^{k-1} + d_{k-1} * 2 * 10^{k-2} + ... + d_1 * 2 + 1$ which is a integer since this is a discrete integer sum.

   $\therefore n|5$

7. (10 points) Exercise Set 4.5, Question 18(a).
Prove that the product of two consecutive integers is even.

   (a) Let a be an integer, by the problem statement, a and $a + 1$ are consecutive integers. With this in mind, if a is even, $a + 1$ must be odd, and the other way around.

First Case
If a is even, then let $a = 2x$ and $b = 2x + 1$ where x is an integer. Knowing this, the form of an even integer is $x(x + 1)$. By rearranging the terms of a and b, it can be seen that $c = 2x^2 + x$, and that $x(x + 1) = 2c$, and since $x(x + 1)$ is the form of even integers, it can be concluded that this product is even.

Second Case
If a is odd, then let $a = 2x+1$ and $b = 2x+2$, where x is an integer. Knowing this, the form of an even integer is $x(x+1)$. By rearranging the terms of a and b, it can be seen that $x(x+1) = (2x+1)(2x+2)$. By fully simplifying this expression, $x(x+1)$ becomes $2x^2+3x+1 = c$, where c is an integer. Since this assumes the form of an even integer, it can be concluded that the product of $x(x + 1)$ must be even.

$\therefore$ Since the product of two consecutive integers is even for values of a that are both even and odd, it can be reasonably concluded that the product of two consecutive integers must be even for all values of a.

8. (10 points) Exercise Set 4.7, Question 15.
Prove the following statement by contradiction: If a, b and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

   (a) Let a and b be odd integers, thus, $a = 2x + 1$ and $b = 2y + 1$ where x and y are integers. With this, $a^2 + b^2 = (2x + 1)^2 + (2y + 1)^2 \implies c^2 = 4x^2 + 4x + 4y^2 + 4y + 2$. By factoring the aforementioned quantity of $a^2 + b^2$, we are left with $2(2(x^2 + x + y^2 + y) + 1)$.

   This quantity can be expressed as d, where d must be an integer due to the closure of addition and multiplication on the set of integers. Now, we can solve for c in terms of d as $c = \sqrt{2(2d + 1)}$. From this, we can conclude that since $\sqrt{2}$ is inherently irrational, $2d + 1$ must be a factor of 2 in order to make this statement rational. However, since $2 \neq 2d + 1$, $\sqrt{2}\sqrt{2d + 1}$ must be irrational.

   $\therefore$ If both a and b are odd, c must be irrational, and as a result, could not be an integer. Therefore, by contradiction, for $a^2 + b^2 = c^2$ where a, b, and c are integers, one of a or b must be even.

9. (10 points) Prove the following by contraposition:
For all integers m and n, if mn is even then m is even or n is even.

   (a) Let $m = 2a + 1$ and $n = 2b + 1$ where a and b are integers, and $mn = (2a + 1)(2b + 1)$ assumes the form of an even integer. By simplifying the above expression we are left with $mn = 2(2ab+a+b)+1$. With this, $c = 2ab + a + b$ where c is an integer, so if either m or n

are odd, $mn = 2c + 1$ which is the form of an odd number.

Due to this being the contrapositive proof to this statement, the original statement is also true due to the nature of a contrapositive.

10. (10 points) Prove the following by contradiction:
For all integers m and n, if mn is even then m is even or n is even.

    (a) Let $m = 2a + 1$ and $n = 2b + 1$ where a and b are integers, and $mn = (2a + 1)(2b + 1)$ assumes the form of an even integer. By simplifying the above expression we are left with $mn = 2(2ab+a+b)+1$. With this, $c = 2ab + a + b$ where c is an integer, so if either m or n are odd, $mn = 2c + 1$ which is the form of an odd number.

    This contradicts the original statement that affirms that mn is an even integer, and as such, if mn is even then either m or n must be even, because both cannot be odd in order for this proof to hold true.