MATH2710 — Portfolio 4.1 - 7.1

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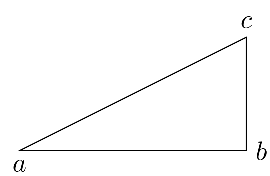
Mathematical Components

Lemma

Definition: A true and simple mathematical statement whose main purpose is to help a theorem.
Example: *If x is a real number, then x^2 is a real number.*

Theorem

Definition: A true mathematical statement of significant importance that has been proved to be true.
Example: The Pythagorean Theorem is famous example of a theorem, it is shown below.

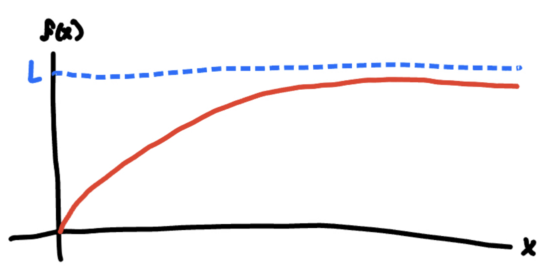


$$a^2 + b^2 = c^2$$

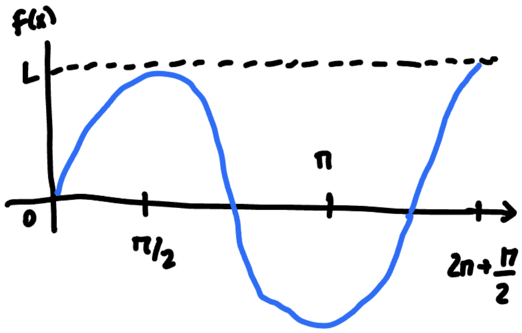
Corollary

Definition: A true mathematical statement that is an immediate consequence of a theorem or proposition.
Example: Take the below theorem as an example. Below, you will see a corollary that is an immediate consequence of the theorem.

Theorem. *If $\lim_{x\to\infty} f(x) = \ell$, then $\forall (a_n) m \geq 1, a_n \xrightarrow{n\to\infty} \infty, f(a_n) \xrightarrow{n\to\infty} \ell$.*



Corollary. $\lim_{x\to\infty} sin(x)$ DNE.



Proof. We aim to prove the above corollary.

For $a_n = 2n\pi + \frac{\pi}{2} \to \infty$, we have $f(a_n) = 1 \xrightarrow{n\to\infty} 1$
For $b_n = 2n\pi - \frac{\pi}{2} \to \infty$, we have $f(a_n) = -1 \xrightarrow{n\to\infty} -1$

$\xRightarrow{Theorem\ 1}$ $sin(x)$ does not have a limit as $x \to \infty$. □

Direct Proof

Outline

A direct proof for a statement $P \Rightarrow Q$ takes the following form:

1. Assume P is true, to prove: Q.
2. To prove: Q_1.
3. ...
4. To prove Q_n.

In this way, Q will be transformed from $Q_1 \to Q_n$ through a series of logical implications.

Examples

Divisibility of Three Integers

Let $(a, b, c) \in \mathbb{Z}$, prove that if $a|b$ and $b|c$, then $a|c$.

Reminder: $a|b$ refers to the fact that $\exists n \in \mathbb{Z}, a \cdot n = b$.

Proof. We aim to prove that a divides c, thus: $\exists n \in \mathbb{Z}, a \cdot n = c$.

1. As $a|b$, we have $\exists p \in \mathbb{Z}, a \cdot p = b$
2. As $b|c$, we have $\exists m \in \mathbb{Z}, b \cdot m = c$

From (1) and (2) we get the following: $\exists p, m \in \mathbb{Z}, (a \cdot p) \cdot m = c$. As we already know that $a \cdot p = b$, we may rewrite the above with $a \cdot p = n \in \mathbb{Z}$.

Thus, we have found that $\exists n \in \mathbb{Z}, a \cdot n = c$. □

Union of Two Bounded Sets

Let $A, B \subseteq \mathbb{R}$ be bounded sets, prove that $A \cup B$ is bounded.

Proof. As A, B are bounded, let's assume that $\exists m, \forall a \in A, m > a$. Similarly, let's assume that $\exists n, \forall b \in B, n > b$. From this, we can say that $k = \max(m, n)$, which means $\forall(a, b), k > a, k > b$. This means k upper-bounds both A and B.

As k upper-bounds both A and B, we can say that $A \cup B$ is bounded. □

Proof by Contrapositive

Outline

A proof by contrapositive of a statement $P \Rightarrow Q$ is the direct proof of $\neg Q \Rightarrow \neg P$. Thus, the proof by contrapositive takes the following form:

1. Assume $\neg Q$ is true, to prove: $\neg P$.
2. Translate $(\neg P)_1 \to (\neg P)_n$.
3. Assume $\neg Q$, using logical implications show $(\neg P)_n$ is true.

Examples

Perfect Squares

Let $n \in \mathbb{N}$, prove that if n is $M_4 + 2$ or n is $M_4 + 3$, then n is not a perfect square.

Definition 1. A perfect square k is an integer k such that $\exists n \in \mathbb{N}, n^2 = k$.
Definition 2. $a|b$ is equivalent to $\exists q \in \mathbb{Z}, b = aq$

Proof. Assume n is a perfect square, to prove: n is not $M_4 + 2$ and n is not $M_4 + 3$.

1. **Case.** n is a $M_4 + 2$

   Then, $n = 4k + 2$ for some integer k. We can rewrite n as $n = 2(2k + 1)$.

   Notice that $2k + 1$ is an odd integer. We know that the square of an odd integer is always odd, so let $2k + 1 = 2m + 1$ for some integer m. Then, $n = 2(2m + 1)$.

   We can see that n has a factor of 2 raised to the power of 1, but no other factors of 2 in its prime factorization. Therefore, n cannot be a perfect square.

2. **Case.** n is a $M_4 + 3$

   Then, $n = 4k + 3$ for some integer k. We can rewrite n as $n = 1 + 4k + 2$.

   Using the same logic as in Case 1, we can see that n has a factor of 2 raised to the power of 1, but no other factors of 2 in its prime factorization. Therefore, n cannot be a perfect square.

   Therefore, if n is a $M_4 + 2$ or $M_4 + 3$, then n is not a perfect square.

□

Divisibility of Two Integers

Let $x, y \in \mathbb{Z}$, prove that if $\neg(xy|11)$, then $\neg(x|11)$ and $\neg(y|11)$.

Proof. Assume $xy|11$, to prove: $x|11$ or $y|11$.

1. **Case.** If $x = 11c, c \in \mathbb{Z}$, then $xy = 11cy$, thus $xy|11$.
2. **Case.** If $y = 11d, d \in \mathbb{Z}$, then $xy = 11xd$, thus $xy|11$.

□

Clarity

Proof by Contradiction

Contradiction of P

A proof by contradiction on a statement of type P is the direct proof of $\neg P \Rightarrow c$ for some initially unknown contradiction c. Thus, the proof by contradiction on P takes the following form:

1. To prove: P.
2. To prove: $\neg P \Rightarrow c$.
3. Assume $\neg P$, translate $(\neg P)_1 \rightarrow (\neg P)_n$ until you arrive at c.

Examples

Irrationality of $\sqrt{5}$

Proof. Assume absurdly that $\sqrt{5}$ is rational. $q \in \mathbb{Q}$ take the form $\frac{a}{b}, (a, b) \in \mathbb{Z}, b \neq 0$. Thus, $\sqrt{5} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$. We can then square both sides, giving us: $\frac{a^2}{b^2} = 5$. Further, we are able to multiply both sides by y^2 to isolate x^2, $5y^2 = x^2$.

Since $5y^2 = x^2$, they must have the same number of prime factors. This shows that both x^2, y^2 have an even number of prime factors, and $5y^2$ has an odd number of prime factors. This is a contradiction, as $5y^2 = x^2$, yet they have a different amount of prime factors. Thus, our assumption was invalidated, and $\sqrt{5}$ is irrational. □

Finding $x, y \in \mathbb{Z}$ for $x^2 + 3y^2 = n$

Let n be an even integer that is not M_4, prove by contradiction that we cannot find $x, y \in \mathbb{Z}$ such that $x^2 + 3y^2 = n$.

Proof. Assume absurdly that for some $n \in \mathbb{Z}, n \in M_4$, $\exists (x, y) \in \mathbb{Z}, x^2 + 3y^2 = n$. Since n is M_4, we can write $n = 4k$ for some $k \in \mathbb{Z}$. Thus, $x^2 + 3y^2 = 4k$, and $x^2 + 3y^2 = 4k + 1$, and $x^2 + 3y^2 = 4k + 2$, and $x^2 + 3y^2 = 4k + 3$. Thus, $x^2 + 3y^2$ is congruent to $0, 1, 2, 3$ modulo 4. This is a contradiction, as $x^2 + 3y^2$ is congruent to $0, 1, 2, 3$ modulo 4, yet n is M_4. Thus, our assumption was invalidated, and we cannot find $x, y \in \mathbb{Z}$ such that $x^2 + 3y^2 = n$. □

Contradiction of $P \Rightarrow Q$

A proof by contradiction on a statement of type $P \Rightarrow Q$ is the direct proof of $\neg(P \Rightarrow Q) \Rightarrow c$ for some initially unknown contradiction c. Thus, the proof by contradiction on $P \Rightarrow Q$ takes the following form:

1. To prove: $P \Rightarrow Q$.
2. To prove: $\neg(P \Rightarrow Q) \Rightarrow c$.
3. To prove: $(P \wedge (\neg Q)) \Rightarrow c$.
4. Assume $P \wedge (\neg Q)$, translate through logical implications until c is discovered.

Similarities with Proof by Contrapositive

One similarity between the Proof by Contradiction of $P \Rightarrow Q$ and that of the Proof by Contrapositive is that we assume $\neg Q$ in both proofs.

Differences with Proof by Contrapositive

One difference between the Proof by Contradiction of $P \Rightarrow Q$ and that of the Proof by Contrapositive is that prove $\neg P$ in the contrapositive proof, whereas in the contradiction proof we prove a contradiction c.

Biconditionality

Ways to Read

1. $P \Leftrightarrow Q$ can be read as "P if and only if Q".
2. $P \Leftrightarrow Q$ can be read as "P is a necessary and sufficient condition for Q".
3. $P \Leftrightarrow Q$ can be read as "P is equivalent to Q".

Outline

Use any means necessary to prove the below statements.

1. To prove: $P \Rightarrow Q$.
2. To prove: $Q \Rightarrow P$.

Example 1

Let $x, y \in \mathbb{Z}$, prove that $4|x^2 - y^2$ iff x, y have the same parity.

Proof. First, we must prove $P \Rightarrow Q$. That is, that assuming $4|x^2 - y^2$, we can conclude that x, y have the same parity.

Assume $4|x^2 - y^2$, then $x^2 - y^2 = 4k$ for some $k \in \mathbb{Z}$. Thus, $x^2 = 4k + y^2$ or $y^2 = 4k + x^2$. Since x^2, y^2 are both even, they must both be congruent to 0 modulo 4. Thus, $x^2 = 4k + y^2$ or $y^2 = 4k + x^2$ implies that x^2, y^2 are both congruent to 0 modulo 4. Thus, x, y have the same parity.

Now, we must prove the converse, that $Q \Rightarrow P$. That is, that assuming x, y have the same parity, we can conclude that $4|x^2 - y^2$.

Assume x, y have the same parity. Since x, y have the same parity, they must both be even or both be odd. Thus, x^2, y^2 are both even or both odd. Since x^2, y^2 are both even or both odd, they must both be congruent to 0 modulo 4. Thus, $x^2 = 4k + y^2$ or $y^2 = 4k + x^2$ implies that x^2, y^2 are both congruent to 0 modulo 4. Thus, $x^2 - y^2 = 4k$ for some $k \in \mathbb{Z}$, and $4|x^2 - y^2$. $\square$

Example 2

Let $x, y \in \mathbb{Z}$, prove that $x^2 = y^2$ iff $x = y$ or $x = -y$.

Proof. In the case of this proof, we are able to immediately show that the inequality holds for both $x = y$ and $x = -y$ since $|x| = |y| \Leftrightarrow x = \pm y$. Thus, we do not need to evaluate both $P \Rightarrow Q$ and $Q \Rightarrow P$ explicitly. $\square$