MATH2710 — Portfolio 2.4 - 2.9

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1 Biconditional Statements

1.1 Conjunction of Two Conditional Statements

A biconditional statement involving P and Q takes the form $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

1.2 Iff Form

In it's *iff* form, a biconditional statement takes the form $P \iff Q$.

1.3 Necessary and Sufficient Form

A statement is said to be both necessary and sufficient if it is $P \iff Q$. Furthermore, P being a sufficient condition for Q implies $P \Rightarrow Q$, whereas P being a necessary condition for Q implies $Q \Rightarrow P$. Thus, being necessary and sufficient is equivalent to saying that $P \iff Q$.

1.4 Equivalent Form

A statement is said to be equivalent if it is logically equivalent to another statement, that is, all truth values for the value of P and Q are the same. For example, $P \iff Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

1.5 Truth Table

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
F	F	T	T	T
T	F	F	T	F
F	T	T	F	F

1.6 Proof

In order to prove a biconditional statement, we must prove both directions of the implication. Thus, we must prove both $P \Rightarrow Q$ and $Q \Rightarrow P$ in order to prove $P \iff Q$.

1.7 Examples

1.7.1 Iff Form

1. $P = |x| = 3$, $Q = x \in \{-3, 3\}$
2. $P = n$ is M_6, $Q = (n \text{ is } M_3) \wedge (n \text{ is } M_2)$

Thus, for both of the above examples, $P \iff Q$.

1.7.2 Necessary and Sufficient Form

1. For the matrix A to be invertible, it is necessary and sufficient that $det(A) \neq 0$.
2. A necessary and sufficient condition for a triangle T to be right is that the square of one side equals the sum of the squares of the other two sides.

1.7.3 Equivalent Form

1. The matrix A being invertible is equivalent to it's determinant being non-zero.
2. The function f having a constant derivative is equivalent to the function being linear.

2 Logical Equivalence

Definition: Two statements P and Q are said to be logically equivalent if they have the same truth value for all possible values of P and Q.

2.1 Compound Statements

The compound statement $(P \vee Q) \wedge (\neg(P \wedge Q))$ is equivalent to $P \oplus Q$, where $\oplus$ is the exclusive-or (XOR) operator. Thus, the statement equates to "either P or Q, but not both."

Similarly, the compound statement $((\neg P) \vee Q)$ is equivalent to $\neg(P \wedge (\neg Q))$, which is equivalent to $\neg(P \wedge Q)$.

2.2 De Morgan's Laws for Logic

In logic, De Morgan's laws are the following two statements:

1. $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
2. $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

2.2.1 Negation of $P \Rightarrow Q$

The negation of $P \Rightarrow Q$ is $P \wedge (\neg Q)$, this is because applying De Morgan's laws to $P \Rightarrow Q$ yields:

$$\begin{aligned}\neg(P \Rightarrow Q) &\Rightarrow \neg((\neg P) \vee Q) \\ &\Rightarrow (\neg(\neg P)) \vee (\neg Q) \\ &= P \wedge (\neg Q)\end{aligned}$$

2.2.2 Negation of $\forall x \in X : P(x)$

The negation of $\forall x \in X : P(x)$ is $\exists x \in X, \neg P(x)$.

Examples:

1. P: "$\forall x \geq 0, \exists y \in \mathbb{R}, x = y^2$"
   $\neg P$: "$\exists x \geq 0, \forall y \in \mathbb{R}, x \neq y^2$"
2. Q: "$\forall$ triangles, $\exists$ an angle which is acute"
   $\neg Q$: "$\exists$triangle, $\forall$ angles in that triangle are not acute"

2.2.3 Negation of $\exists x \in X \colon P(x)$

Conversely, the negation of $\exists x \in X \colon P(x)$ is $\forall x \in X, \neg P(x)$.

Examples:

1. P: "$\exists$ function $f, \forall$ function $g, f + g = g$."
   $\neg P$: "$\forall$ function $f, \exists$ function $g, f + g \neq g$."

2. Q: "$\exists (x, y) \in \mathbb{R}^2, x \cdot y > 0$"
   $\neg Q$: "$\forall (x, y) \in \mathbb{R}^2, x \cdot y \neq 0$"

2.2.4 Negation of $\forall x \in X, \exists y \in Y \colon P(x)$

The negation of $\forall x \in X, \exists y \in Y \colon P(x)$ is $\exists x \in X, \forall y \in Y \colon \neg P(x)$.

Examples:

1. P: "$\forall \epsilon > 0, \exists \delta > 0, \forall x$ with $|x| < \delta \colon f(x) > \epsilon$"
   $\neg P$: "$\exists \epsilon > 0, \forall \delta > 0, \exists x$ with $|x| < \delta \colon f(x) \leq \epsilon$"

2. Q: "$\forall x \in X, \exists y \in Y, \forall z \in Z \colon f(x, y, z) > \delta$"
   $\neg Q$: "$\exists x \in X, \forall y \in Y, \exists z \in Z \colon f(x, y, z) \leq \delta$"

2.2.5 Negation of $\exists x \in X, \forall y \in Y \colon P(x)$

The negation of $\exists x \in X, \forall y \in Y \colon P(x)$ is $\forall x \in X, \exists y \in Y \colon P(x)$.

Examples:

1. P: "$\exists x \in X, \forall y \in Y, x + y > 0$"
   $\neg P$: "$\forall x \in X, \exists y \in Y, x + y \leq 0$"

2. Q: "$\exists x \in X, \forall y \in Y, x \cdot y = 0$"
   $\neg Q$: "$\forall x \in X, \exists y \in Y, x \cdot y \neq 0$"

2.3 Mechanics of $P(x) \Rightarrow Q(x)$

There is a hidden "for all x" before the statement $P(x) \Rightarrow Q(x)$.

2.3.1 Negation

The negation of $P(x) \Rightarrow Q(x)$ is $P(x) \wedge (\neg Q(x))$.

2.3.2 Truth Table

P	Q	$\neg Q$	$P \wedge (\neg Q))$
T	T	F	F
F	F	T	F
T	F	T	T
F	T	F	F

2.3.3 Every Form of $(x \in X) \Rightarrow Q(x)$

The "every form" of $(x \in X) \Rightarrow Q(x)$ is $\forall x \in X \colon Q(x)$.

2.3.4 Examples of "If" form → "Every" form

1. P_{if}: "If f is a polynomial of degree ≥ 2, then f' is not constant."
   P_{every}: "(function f such that $deg(f) \geq 2$) $\Rightarrow$ (f' is not constant)"

2. P_{if}: "If P is $M_4 + 1$, then P is odd."
   P_{every}: "(P is $M_4 + 1$) $\Rightarrow$ (P is odd)"

2.3.5 Examples of "Every" form → "If" form

1. P_{every}: "(f is continuous on $[a, b]$) $\Rightarrow$ (f is Riemann Integrable on $[a, b]$)"
   P_{if}: "If f is continuous on $[a, b]$, then f is Riemann Integrable on $[a, b]$."

2. Q_{every}: "(n is M_4) $\Rightarrow$ (n is even)"
   Q_{if}: "If n is M_4, then n is even."