7.3 Portfolio

March 2023

1 Question 1

1. Write the outline of an existence proof.
2. Write the outline of an existence and uniqueness proof.

2 Question 2

1. What is a frequently met pattern in which an existence statement relates to a conditional statement?
2. Write three examples of famous theorems in mathematics that respect this pattern.

3 Question 3

1. Prove that there exists a real number x such that $x^2 - 6x + 8 = 0$.
2. Prove that there exists uniquely a real number x such that $5x - 15 = 0$.

4 Question 4

1. Prove that there exists a prime number p such that $p + 8$ is also a prime number.
2. Prove that there exists a differentiable function f defined on a real interval I such that $f = f'$ on the interval I.

5 Question 5

1. State the Min-Max Theorem for a continuous function on the compact interval $[a, b]$. Draw a picture that illustrates this property.

2. Give an example of continuous function f defined on a real interval I, that fails to have an absolute max or an absolute min on the interval I, due to the interval I not being compact. Draw a picture to illustrate this situation.

6 Question 6

1. State Rolle's Theorem and draw a picture to illustrate it.

2. Which part of Rolle's Theorem is given through an existence statement (the hypothesis, or the conclusion)?

3. What are the two cases a) and b) considered in the proof of this theorem?

4. What are two fundamental theorems that have been used to prove Rolle's Theorem in case b)?

7 Question 7

1. Prove that the equation $x^3 + 3x + 1 = 0$ has a unique real solution.

8 Question 8

1. Write the definition of a strictly increasing function f on the real interval I. Draw the picture of a strictly increasing function f on an interval I.

2. Write the definition of a one-to-one function f on the real interval I. Draw the picture of a function f defined on the real interval I that is NOT one-to-one.

3. Prove that if a function f is not one-to-one on the interval I, then f is not strictly increasing on the interval I.

9 Question 9

1. Give two different examples of real functions f and g defined on the same real interval I, that are both strictly increasing on I. Draw a picture that illustrates their graphs on the interval I.

2. Give an example of function f defined on a real interval I that is one-to-one on the given interval I, and an example of function g defined on the same real interval I that is NOT one-to-one on the interval I. Draw a picture that contains both the graph of f and the graph of g.

10 Question 10

1. State the Intermediate Value Property of a continuous function f on a given interval $[\,a, b]$ and draw a picture that illustrates it.

11 Question 11

1. Let S be the following set:

   $S = \{a\sqrt{7} + b; a, b \in Z\}$.

   Prove that for every $x \in S$ there exist uniquely two integers a and b such that $x = a\sqrt{7} + b$.