MATH2710 — Portfolio 7.3

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Existence/Uniqueness

Outline of Existence Proof

An existence proof uses any method to show that a solution exists such that $\exists x \in X, P(x)$.

Outline of Existence + Uniqueness Proof

In the case of an existence + uniqueness proof, you must first prove that $\exists x \in X, P(x)$ using any method, and then prove that x is a unique solution for the given statement.

Proof Examples

1. Prove that $\exists x \in \mathbb{R}, x^2 - 6x + 8 = 0$.
2. Prove that $\exists! x \in \mathbb{R}, 5x - 15 = 0$.
3. Prove that $\exists p \in$ Primes, such that $p + 8$ is also a prime number.
4. Prove that $\exists f$ differentiable function on real interval I, such that $f = f'$ on I.
5. Prove that if a function f is not one-to-one on the real internal I, then f is not strictly increasing.
6. Let $S = \left\{a\sqrt{7} + b; (a, b) \in \mathbb{Z}\right\}$, prove that $\forall x \in S$, there exists uniquely $(a, b) \in \mathbb{Z}$ such that $x = a\sqrt{7}+b$.

Example 1.

We can use the quadratic formula to solve for x in this case. The quadratic formula is $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a = 1$, $b = -6$, and $c = 8$. This gives us $x = \frac{6\pm\sqrt{6^2-4(1)(8)}}{2(1)} = \frac{6\pm\sqrt{36-32}}{2} = \frac{6\pm\sqrt{4}}{2} = \frac{6\pm2}{2} = 4 \pm 1$. Since x is a real number, we can conclude that $\exists x \in \mathbb{R}, x^2 - 6x + 8 = 0$.

Example 2.

Firstly, we must show that there exists an x such that $5x - 15 = 0$. In this case, we can algebraically solve the equation to find $x = 3$. Now, we must show that x is a unique solution. In order to do this, we must show that $5x - 15 = 0$ is not true for any other x. In this case, we can show that $5x - 15 = 0$ is not true for any other x by showing supposing there are x_1, x_2 such that $5x_1 - 15 = 0$ and $5x_2 - 15 = 0$. Then, we can show that $x_1 = x_2$ by dividing by 5 for each equation, and then isolating x_1 and x_2 on each side of the equation. This gives us $x_1 = \frac{15}{5} = 3$ and $x_2 = \frac{15}{5} = 3$, so $x_1 = x_2$. Therefore, we can conclude that $\exists! x \in \mathbb{R}, 5x - 15 = 0$.

Example 3.

Let's use $p = 3$, thus $p + 8 = 11$. In this case, both $p, p + 8$ are primes. Thus, $\exists p \in$ Primes , such that $p + 8$ is also a prime number.

Example 4.

Consider $f(x) = ce^x$ for some $c \in \mathbb{R}$. We can differentiate $f(x)$ to obtain $f'(x) = ce^x$. In this case $f(x) = f'(x)$ when iff $c = 1$, thus we can conclude that $\exists f$ differentiable function on real interval I, such that $f = f'$ on I.

Example 5.

Assume that f is one-to-one on the real interval I, we aim to prove that f is strictly increasing. That is, the following: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Let x_1, x_2 be $x_1 \neq x_2$, to prove: $f(x_1) \neq f(x_2)$. Without loss of generality, let's assume $x_1 < x_2$. As f is strictly increasing, we have $f(x_1) < f(x_2)$. Hence $f(x_1) \neq f(x_2)$.

Example 6.

Existence Proof:

Proof. As $x \in S$, by definition $\exists(a, b) \in \mathbb{Q}, x = a \cdot \sqrt{7} + b$. □

Uniqueness Proof:

Proof. Assume absurdly that $\exists((a_1, b_1), (a_2, b_2))$ couples of rational numbers, such that:

$$\begin{cases} x = a_1 \cdot \sqrt{7} + b_1 & (1) \\ x = a_2 \cdot \sqrt{7} + b_2 & (2) \\ (a_1, b_1) \neq (a_2, b_2) & (3) \end{cases}$$

By (1) and (2), we can see that:

$$\begin{aligned} a_1 \cdot \sqrt{7} + b_1 &= a_2 \cdot \sqrt{7} + b_2 \\ a_1 \cdot \sqrt{7} - a_2 \cdot \sqrt{7} &= b_2 - b_1 \\ (a_1 - a_2) \cdot \sqrt{7} &= b_2 - b_1 \end{aligned}$$

Thus, we can now check for the two cases wherein $a_1 - a_2$ is equal to zero or not.

Case 1. $a_1 - a_2 \neq 0$
$\sqrt{2} = \frac{b_2 - b_1}{a_1 - a_2} \Rightarrow \sqrt{2} \in \mathbb{Q}$. The fact that $\sqrt{2} \in \mathbb{Q}$ is a contradiction.

Case 2. $a_1 - a_2 = 0$
As $a_1 - a_2 = 0$, we have that $b_2 - b_1 = 0$, so $b_1 = b_2$ and since $a_1 - a_2 = 0$ the same logic applies. This is a contradiction with respect to statement (3). □

Min-Max Theorem

If f is continuous on $[a, b]$, then $\exists M, m \in [a, b]$ such that $f(M) \geq f(m) \forall x \in [a, b]$ and $f(m) \leq f(M) \forall x \in [a, b]$.

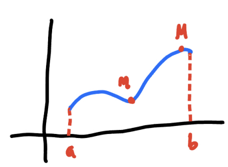

Figure 1: Visualization of The Min-Max Theorem

Example

An example of when the Min-Max Theorem fails to hold is when $f(x) = x^2$ on $[-1, 1]$. In this case, $f(-1) = 1$ and $f(1) = 1$, so $f(-1) \leq f(x) \leq f(1)$ for all $x \in [-1, 1]$, but $f(x) = 0$ has no solution in $[-1, 1]$.

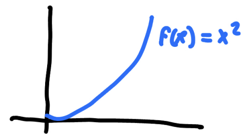

Figure 2: Visualization of The Min-Max Theorem Failing for $f(x) = x^2$

Rolle's Theorem

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

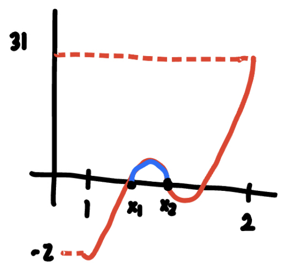

Figure 3: Visualization of Rolle's Theorem

In order to accomplish this proof, there are several major theorems utilized. These are the Min-Max Theorem, Fermat's Theorem, One-To-One Property, and the Intermediate Value Theorem.

Components of Rolle's Theorem

The hypothesis of Rolle's Theorem is proved using an existence proof, and the conclusion is proved using a uniqueness proof. This is because the hypothesis is that $\exists c \in (a, b)$ such that $f'(c) = 0$, and the conclusion is that c is unique.

Cases

There are two cases that must be proved in order to prove Rolle's Theorem. The first case is when f is constant on $[a, b]$, and the second case is when f is not constant on $[a, b]$.

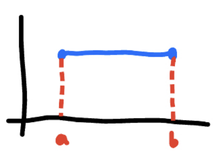

Figure 4: Constant Case

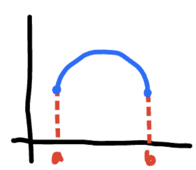

Figure 5: Non-Constant Case

Strictly Increasing Functions

If f is strictly increasing on the real interval I, then f must be one-to-one on I. This means that both:

1. **Strictly Increasing:** Let $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \forall (x_i, x_j) \in I$.
2. **One-to-One:** Let $x_1, x_2 \in I, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2), \forall (x_i, x_j) \in I$.

A visual example of a strictly increasing function is $f(x) = x^2$, shown below:

Figure 6: Example of a Strictly Increasing Function

One-To-One Property

A function being one-to-mean means that no two x-values can have the same y-value. Let $x_1, x_2 \in I, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2), \forall (x_i, x_j) \in I$.

An example of a function that is *not* one-to-one, is $f(x) = (x - 2)^2$. This is because $\forall x \in I, f(x)$ is not unique.

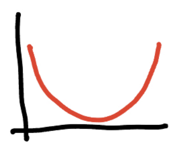

Figure 7: Example of a Function that is Not One-To-One

Proof of One-to-One Property

Theorem. *We aim to show that f is one-to-one on the real interval I.*

Proof. To be one-to-one means that $\forall (x_1, x_2), x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Let x_1, x_2 be $x_1 \neq x_2$, we aim to prove that $f(x_1) \neq f(x_2)$. Thus, without loss of generality, let $x_1 < x_2$. As f is strictly increasing in this case, we have that $f(x_1) < f(x_2)$, and thus $f(x_1) \neq f(x_2) \forall (x_1, x_2) \in I$. $\square$

Intermediate Value Property

Let f be a continuous function on $[a, b]$, then $\forall \lambda \in (f(a), f(b)), \exists c \in (a, b)$ such that $f'(c) = \lambda$. This is shown below in the following figure.

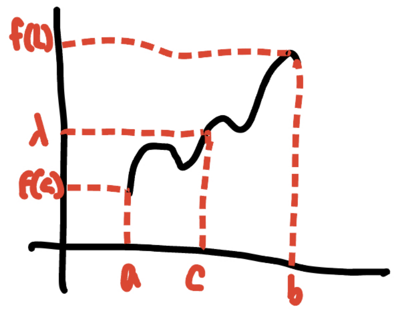

Figure 8: Visualization of the Intermediate Value Theorem