MATH2710 — Proof Presentation

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Introduction

In this report, we will prove that the sum of two independent normal random variables is also a normal random variable, such that:

$$
\begin{gathered}
X \sim \mathcal{N}(\mu_X, \sigma_X^2) \qquad Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \\
X + Y = Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)
\end{gathered}
$$

Preliminaries

The Moment Generating Function

Let X be a random variable with distribution function $F_X(x)$. The moment generating function of X is defined as follows:

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right]$$

When added, the moment generating functions of two independent random variables X and Y are equal to the product of their individual moment generating functions:

$$
\begin{aligned}
\phi_{X+Y}(t) &= \phi_X(t) \cdot \phi_Y(t) \\
&= \mathbb{E}\left[e^{itX}\right] \cdot \mathbb{E}\left[e^{itY}\right] \\
&= \mathbb{E}\left(e^{it(X+Y)}\right)
\end{aligned}
$$

Proof

Theorem. *Let X and Y be independent normal random variables with mean μ and variance σ^2. Prove that the random variable $X+Y=Z$ results in another normal random variable $Z \sim \mathcal{N}(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$.*

Proof. Let X, Y be two independent normal random variables such that:

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \quad Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$$

We aim to prove that the random variable $Z = X+Y$ is also a normal random variable with mean $\mu_X+\mu_Y$ and variance $\sigma_X^2+\sigma_Y^2$.

In this case, we know that X, Y are both independent of one another, and are normal random variables. Therefore, we can use the sum of their moment generating functions to prove that Z is also a normal random variable.

Let $t \in \mathbb{R}$, then the moment generating function of $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ is the following:

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right] = \mathbb{E}\left[e^{it(\mu_X+\sigma_X)}\right] = \exp\left(it\mu_X - \frac{\sigma_X^2 t^2}{2}\right)$$

Therefore, the moment generating function of $X+Y$ must be:

$$\begin{aligned}
\phi_{X+Y}(t) &= \phi_X(t) \cdot \phi_Y(t) \\
&= \exp\left(it\mu_X - \frac{\sigma_X^2 t^2}{2}\right) \cdot \exp\left(it\mu_Y - \frac{\sigma_Y^2 t^2}{2}\right) \\
&= \exp\left(it\left(\mu_X+\mu_Y\right) - \frac{\left(\sigma_X^2+\sigma_Y^2\right)\cdot t^2}{2}\right)
\end{aligned}$$

Hence, the moment generating function of Z has mean $\mu_X+\mu_Y$ and variance $\sigma_X^2+\sigma_Y^2$, and as no two distinct distributions can have the same moment generating function, the distribution of $X+Y$ must be a normal distribution. $\square$