MATH2710 — Portfolio 1.1 - 1.4

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1 Deliverables

1.1 Sets

1.1.1 Finite Sets

A finite set is a set that contains a finite amount of elements. For example, the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ are both finite sets.

1.1.2 Countably Infinite Sets

A countably infinite set is a set that contains an infinite amount of elements, but can be put into a one-to-one correspondence with the natural numbers. For example, the set $C = \{1, 2, 3, 4, 5, \ldots\}$ is a countably infinite set. Similarly, the natural numbers $\mathbb{N}$ themselves are a countably infinite set.

1.1.3 Uncountably Infinite Sets

A uncountably infinite set is a set that contains an infinite amount of elements, but cannot be put into a one-to-one correspondence with the natural numbers. For example, the set $D = \{1, 2, 3, 4, 5, \ldots, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is a uncountably infinite set. Similarly, the real numbers $\mathbb{R}$ themselves are a uncountably infinite set.

1.2 Sets Inclusivity with respect to the Reals

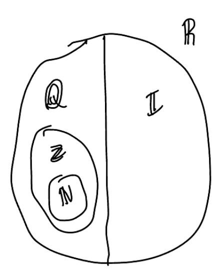

Figure 1: Inclusivity of $\mathbb{N}, \mathbb{Z},$ and $\mathbb{Q}$, with respect to $\mathbb{R}$

1.3 Intervals

1.3.1 (a, b)

An interval (a, b) is a set of all real numbers x such that $a < x < b$. Some examples of intervals on (a, b) are $(0, 1)$ and $(10, 100)$.

1.3.2 (a, ∞)

An interval (a, ∞) is a set of all real numbers x such that $a < x$. Some examples of intervals on (a, ∞) are $(0, \infty)$ and $(10, \infty)$.

1.4 Cartesian Products

1.4.1 Two Sets

The Cartesian Product of two sets, A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Specifically, $A \times B = \{(x, y); x \in A, y \in B\}$

For example, the Cartesian Product of the sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ is the set $A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$. The cardinality of $A \times B$ is $|A| \times |B|$, so in this example, $|A \times B| = 9$ as seen above.

1.4.2 Three Sets

The Cartesian Product of three sets, A, B, and C is the set of all ordered triples (a, b, c) such that $a \in A$, $b \in B$, and $c \in C$. Specifically, $A \times B \times C = \{(x, y, z); x \in A, y \in B, z \in C\}$

For example, the Cartesian Product of the sets $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{7, 8, 9\}$ is the set:

$$A \times B \times C = \{(1, 4, 7), (1, 4, 8), (1, 4, 9), (1, 5, 7), \dots\}$$

The cardinality of $A \times B$ is $|A| \times |B|$, and the cardinality of $A \times B \times C$ is $|A| \times |B| \times |C|$, so in this example, $|A \times B \times C| = 27$ as seen above.

1.4.3 Generalized Cartesian Product

The Generalized Cartesian Product for n sets is defined as such:

$$A_1 \times A_2 \times \dots A_n = \{(x_1, x_2, \dots, x_n) | x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n\}.$$

For example, the Cartesian Product on $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R} = \{(x_1, \dots, x_n) | x_i \in \mathbb{R}, \forall i\}$. The cardinality of the resultant set is $|\mathbb{R}|^n$.

1.5 Equality of Sets

1.5.1 Definition of $A = B$

Two sets A and B are equal if and only if $A = B \iff A \subseteq B$ and $B \subseteq A$. In other words, $A = B$ if and only if A is a subset of B and B is a subset of A.

1.5.2 Difference between $A \subseteq B$ and $A \subset B$

The difference between $A \subseteq B$ and $A \subset B$ is that $A \subseteq B$ is true if and only if $A = B$ or $A \subset B$. In other words, $A \subseteq B$ is true if and only if A is a subset of B or $A = B$. On the other hand, $A \subset B$ is true if and only if A is a subset of B and $A \neq B$. In other words, $A \subset B$ is true if and only if A is a subset of B and A is not equal to B.

1.6 Power Sets

The Power Set of a set A, $P(A)$, is the set of all subsets of A.

Theorem. *Let A be a set, then $|A| = n$, and $|P(A)| = 2^n$*

Examples:

1. Given $A = \{1, 2\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Thus, $|P(A)| = 2^2 = 4$.
2. Given $A = \{1, 2, 3\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Thus, $|P(A)| = 2^3 = 8$.

1.7 n-choose-k

n-choose-k is a formula used to compute the number of ways to choose k elements from an n-element set. It is denoted $\binom{n}{k}$, and is computed as such:

$$\frac{n!}{k!(n-k)!}$$

Examples:

1. Given $n = 5$ and $k = 3$, $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = \frac{5!}{3!2!} = \frac{60}{6} = 10$.
2. Given $n = 10$ and $k = 5$, $\binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10!}{5!(5)!} = \frac{10!}{5!5!} = \frac{3628800}{14400} = 252$

1.8 Binomial Theorem

The Binomial Theorem is a formula used to expand $(a + b)^n$ into a sum of terms.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad \forall a, b \in \mathbb{R} \text{ and } \forall n \in \mathbb{N}$$

Example:
Write the expansion of $(2 + 3)^4$ using the Binomial Theorem:

$$\begin{aligned}
(2+3)^4 &= \sum_{k=0}^{4} \binom{4}{k} 2^{k-1} 3^k \\
&= \binom{4}{0} 2^4 \cdot 3^0 + \binom{4}{1} 2^3 \cdot 3^1 + \binom{4}{2} 2^2 \cdot 3^2 + \binom{4}{3} 2^1 \cdot 3^3 + \binom{4}{4} 2^0 \cdot 3^4 \\
&= 16 + 96 + 216 + 216 + 81 \\
&= 625
\end{aligned}$$