MATH3160 — Portfolio 4.4, 5.2-5.4

Mike Medved

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1 Deliverables

1.1 Continuous vs Discrete Tools

	Discrete	**Continuous**
Image	Finite / Countable	Interval (Infinite + Uncountable)
Probabilities of Interest	$P(X = a)$, $a \in \text{Im X}$	$P(X \in I)$, where I is an interval on $\mathbb{R}$
Density	$\sum_{x \in \text{Im X}} f(x) = 1$	$\int_{\mathbb{R}} f(x)dx = 1$
Distribution	$F(X) = \sum_{t \leq x} f(t)$, $t \in \text{Im X}$	$F(X) = \int_{-\infty}^{-x} f(t)dt$
Expectation	$\sum_{x \in \text{Im X}} x * f(x)$	$\int_{\mathbb{R}} x * f(x)dx$
Moment Generating Function	$\sum_{x \in \text{Im X}} g(x) * f(x)$	$\int_{\mathbb{R}} g(x) * f(x)dx$
Variance	$E\left[X^2\right] - (E\left[X\right])^2$	$E\left[X^2\right] - (E\left[X\right])^2$

1.2 Uniform Random Variable Example

X = Toss a fair die

- $\text{Im } X = \{1,2,3,4,5,6\}$
- pmf $X = f(k) = \frac{1}{6} \; \forall k \in \text{Im} X$

$x \in \text{Im} X$	1	2	...	6
$f(x)$	$1/6$	$1/6$	...	$1/6$

- $E[X] = \sum_{x \in \text{Im} X} x \cdot f(x) \Rightarrow 1 \cdot 1/6 + \dots + 6 \cdot 1/6$

$\therefore E[X] = \frac{7}{2}$

- $Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = 1^2 \cdot 1/6 + \dots + 6^2 \cdot 1/6$

$\therefore E[X^2] = \frac{91}{6}$

$\overset{\text{def}}{\Longrightarrow} \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{3}$

- $M_X(t) = e^{t \cdot 1} \cdot 1/6 + \dots + e^{t \cdot 6} \cdot 1/6$

$\Rightarrow \frac{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}$

1.3 Bernoulli Random Variable Example

$X =$ Toss a fair die,

Success = Roll a 5

Failure = Roll anything else

Thus, $X \sim \text{Bern}\left(\frac{1}{5}\right)$

- $\text{Im}\, X = \{0, 1\}$

- pmf X

$x \in \text{Im}\, X$	0	1
$f(x)$	$5/6$	$1/6$

- $E[X] = \cancel{0 \cdot \frac{5}{6}} + 1 \cdot \frac{1}{6} = \frac{1}{6}$

- $\text{Var}(X) = E[X^2] - (E[X])^2$

$E[X^2] = \cancel{0^2 \cdot 5/6} + 1^2 \cdot \frac{1}{6} = \frac{1}{6}$

$\Rightarrow \frac{1}{6} - \left(\frac{1}{6}\right)^2 = 0$

$\therefore \text{Var}(X) = 0$

- $M_X(t) = e^{t \cdot 0} \cdot 5/6 + e^{t \cdot 1} \cdot 1/6$

$= \frac{5}{6} + e^t \cdot \frac{1}{6}$

$= \frac{5}{6} + \frac{e^t}{6}$

1.4 Binomial Random Variables

A binomial random variable of parameters n, p is the number of successes in n independent *Bern(p)* trials. This is denoted as $X \sim \text{Bin}(n, p)$, and is actually one of the representations of the Bernoulli random variable, as such: $\text{Bern}(p) \iff \text{Bin}(1, p)$.

Further, as the Bernoulli random variable is a special case of the binomial random variable, we can use the same formula for the expectation and variance of the binomial random variable as we did for the Bernoulli random variable.

In this way, we can express binomial random variables as their Bernoulli components:

$$X = \sum_{i=1}^{n} X_i$$

where $X_i \sim \text{Bern}(p)$. In this way, we can use the same formula for the expectation and variance of the binomial random variable as we did for the Bernoulli random variable.

1.4.1 Example of computation

X = # of costumes sold within first 10 clients

(Each client: $P(\text{Buys witch}) = \frac{1}{3}$)

$X \sim \text{Bin}(10, 1/3)$

a) $P(X=3)$

b) $P(X \geq 5)$

c) $P(X<6)$

d) M_X

e) $E[X^2]$

f) $\sigma(X)$

a) $f(3) = \binom{n}{3} p^3 \cdot q^{n-3} \Rightarrow \binom{10}{3}(1/3)^3 \cdot (1-1/3)^{10-3}$

$\Rightarrow \boxed{\binom{10}{3} \cdot \frac{1}{3^3} \cdot \left(\frac{2}{3}\right)^7}$

b) $P(X \geq 5) = P(X=5) + P(X=6) + \ldots + P(X=10)$

$\Rightarrow \sum_{k=5}^{10} P(X=k) \Rightarrow \sum_{k=5}^{10} \binom{n}{k} p^k \cdot q^{n-k}$

$\Rightarrow \boxed{\sum_{k=5}^{10} \binom{10}{k} \cdot \left(\frac{1}{3}\right)^k \cdot \left(\frac{2}{3}\right)^{10-k}}$

c) $P(X<6) = 1 - P(X \geq 6) = 1 - [P(X=6) + \ldots + P(X=10)]$

$\Rightarrow \boxed{1 - \sum_{k=6}^{10} \binom{10}{k} \cdot \left(\frac{1}{3}\right)^k \cdot \left(\frac{2}{3}\right)^{10-k}}$

d) $M_X(t) = (pe^t + q)^n = \boxed{\left(\frac{1}{3}e^t + \frac{2}{3}\right)^{10}}$ Domain $M_X = \mathbb{R}$

e) $E[X^2]$

Choice 1

$E[X^2] \xrightleftharpoons[\text{formula}]{\text{transport}} \sum_{x \in \text{Im}X} x^2 \cdot f(x)$

Choice 2

$E[X^2] = M_X''(0)$

Choice 3

$Var(X) = E[X^2] - (E[X])^2$

$\xRightarrow{\text{Choice 3}} E[X^2] = \underbrace{Var(X)}_{npq} + (\underbrace{E[X]}_{np})^2$

$\Rightarrow npq + np^2$

$\Rightarrow (np)(q+np)$

$\Rightarrow (10 \cdot \frac{1}{3})(\underbrace{\tfrac{2}{3} + 10 \cdot \tfrac{1}{3}}_{12/3 = 4})$

$\Rightarrow 10 \cdot \frac{1}{3} \cdot 4 = \frac{40}{3}$