MATH3160 — Portfolio 6.3

Mike Medved

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1 Deliverables

1.1 Standard Normal Random Variable

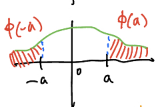
- Density: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}, \; x \in \mathbb{R}$

Peak at $x=0$

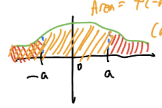
- Area under $f = 1$
- Graph of f is symmetric
- Peaks at $x=0$
- $\lim_{x \to \pm\infty} f = 0$

- $\phi(-a) = \phi(a) \; \forall a \in \mathbb{R}$ (due to symmetry)

Geometric proof:



$\phi(-a) = \phi(a)$ due to graph symmetry

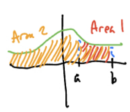


Area $= \phi(-a) + \phi(a) = 1$ (area under f)

So, $\phi(-a) = 1 - \phi(a)$

- $P(a < Z < b) = \phi(b) - \phi(a)$

Geometric Proof:



$P(a < Z < b) = \text{Area 1}$

$P(Z \in (-\infty, b)) = \text{Area 2}$

$P(Z \in (-\infty, a)) = \text{Area 3}$

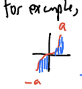
$P(a < Z < b) = A_1 = A_2 - A_3$

$\Rightarrow A_1 = \phi(b) - \phi(a)$

- $E[Z] \overset{\text{def}}{=} \int_{\mathbb{R}} x \cdot f(x)\, dx \Rightarrow \int_{\mathbb{R}} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2} dx \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x \cdot e^{-x^2})\, dx$

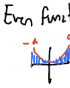
Odd function: $f(x) = -f(x) \; \forall x \in \mathbb{R}$

for example, $f(x) = x^3$



$\int_{-a}^{a} f(x)\, dx = 0$

Even function: $f(x) = f(-x)$



$\int_{-a}^{a} f(x)\, dx = 2 \cdot \int_{-a}^{a} f(x)\, dx$

$f(-x) = x \cdot e^{-x^2} = -f(x)$

(by definition of an odd function)

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-x^2} = 0 \quad \therefore E[Z] = 0$

- $\mathrm{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$

$$\mathbb{E}[Z^2] \xrightarrow[\text{formula}]{\text{transport}} \int_{\mathbb{R}} x^2 \cdot f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2}\,dx$$

Even function

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2}\,dx \Rightarrow \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\left(-x \cdot e^{-x^2/2}\right)dx$$

$$\overset{\text{Chain}}{\underset{\text{rule}}{\Rightarrow}} \left(e^{-x^2/2}\right)' = -x \cdot e^{-x^2/2}$$

faster

0 when $x \to \infty$

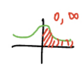
$$\overset{\text{int by}}{\underset{\text{parts}}{\Rightarrow}} -\frac{2}{\sqrt{2\pi}} \cdot \left[\left(x \cdot e^{-x^2/2}\right)\right]_0^{\infty} - \int_0^{\infty} 1 - e^{-x^2/2}\,dx$$

∞ when $x \to \infty$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} e^{-x^2/2}\,dx \Rightarrow 2 \cdot \int_0^{\infty} \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right) e^{-x^2/2}}_{f(x)}\,dx$$

but for half of whole area

$$\Rightarrow \cancel{2} \cdot \left(\frac{1}{\cancel{2}} \text{ area under } f\right) = 1$$



$$\therefore \mathbb{E}[Z^2] = 1, \quad \underset{1}{\mathbb{E}[Z^2]} - (\underset{0}{\mathbb{E}[Z]}) = 1$$

1.2 General Normal → Standard Normal

Let $X \sim \mathcal{W}(\mu, \sigma^2)$, then $\frac{x-\mu}{\sigma} \sim \mathcal{N}(0,1)$. This is called the reduction to a standard normal random variable, and can be accomplished by transforming the random variable X into a standard normal random variable Z by the above reduction.

The opposite holds, given $Z \sim \mathcal{N}(0,1)$, we can transform Z into a random variable X with a general normal distribution by the following transformation to revert back to a general normal random variable:

$$X = \mu + \sigma Z$$

1.3 General Normal Random Variable

- $\operatorname{Im} X = \mathbb{R}$ (As $f \neq 0$ $\forall x \in \mathbb{R}$)
- Density:

① Get F_X ⟶ $F_X(x) \overset{\text{def}}{=} P(X \le x) = P(c \cdot Z + \mu \le x)$

$$\Rightarrow \underbrace{P\left(Z \le \frac{x-\mu}{c}\right)}_{\Phi\left(\frac{x-\mu}{c}\right)} = \Phi\left(\frac{x-\mu}{c}\right)$$

② Get $f_X = F'_X \rightarrow f_X = \left(\Phi\left(\frac{x-\mu}{c}\right)\right)'$

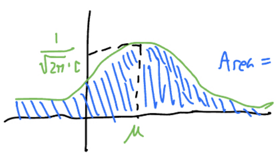
$$\overset{\text{chain}}{\underset{\text{rule}}{\Longrightarrow}} \Phi'\left(\frac{x-\mu}{c}\right) \cdot \left(\frac{x-\mu}{c}\right)' \qquad \boxed{\frac{x}{c} - \frac{\mu}{c}}$$

$$\Rightarrow f_Z\left(\frac{x-\mu}{c}\right) \cdot \frac{1}{c}$$

$$\boxed{f_Z(x) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi} \cdot c} e^{-\frac{(x-\mu)^2}{2c^2}}, \quad x \in \mathbb{R}$$

plot:



- $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \mathbb{E}[\underset{\text{std normal r.v.}}{c Z} + \mu] \overset{\text{linearity}}{=\!=} c \cdot \underset{0}{\mathbb{E}[Z]} + \mu$$

$$= c \cdot 0 + \mu = \mu$$

- $Var(X)$:

$$Var(X) = Var(\underset{\text{scaling}}{c} \cdot Z + \overset{\text{shifting}}{\mu})$$

$$\boxed{Var(aX+b) = a^2 \cdot Var(X)}$$

$$\Rightarrow c^2 \cdot \underset{1}{Var(Z)} = c^2$$

(so, if $X \sim N(\mu, c^2)$, then $\mu = \mathbb{E}[X]$ and $c^2 = Var(X)$)