MATH3160 — Portfolio 4.3

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1 Deliverables

1.1 Probability Density Function (pdf)

The density of a continuous random variable X is by definition the probability that X takes a value in an open or closed interval $[a, b]$. The probability density function (pdf) itself is on the domain of $f : \mathbb{R} \to [0, \infty]$, and can be simply defined as the following integral:

$$P(X \in I) = \int_I f(x)\,dx$$

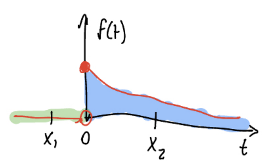
An example of using it to find if X takes values in a given interval is shown below.

This solution refers to a random variable whose density is given as the following:

$$\begin{cases} \frac{1}{10}e^{-\frac{1}{10}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

b) Compute F (pdf)

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$



Cases x_1 and x_2

I $x \in (-\infty, 0)$

II $x \in [0, \infty]$

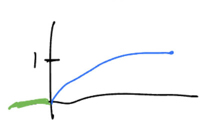
I $F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{0}\,dt = \int_{-\infty}^{x} 0\,dt$

II $F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{0}\,dt + \int_{0}^{x} f(t)\,dt$

$$\Rightarrow \int_0^x f(t)\,dt \Rightarrow \frac{1}{10}\int_0^x e^{-\frac{1}{10}t}\,dt \Rightarrow \frac{1}{\cancel{10}}(\cancel{-10}e^{-\frac{1}{10}t})$$

$$\Rightarrow -\left[e^{-1/10\,x} - \underbrace{e^{-1/10\,0}}_{1}\right] = \boxed{1 - e^{-1/10\,x}}$$

$$F(x) = \begin{cases} 1 - e^{-1/10\,x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Approaching ∞:

$1 - e^{-\frac{\infty}{10}} = 1 - 0 = 1$

Approaching 0:

$1 - e^{\frac{0}{10}} = 1 - 1 = 0$

1.2 Cumulative Distribution Function (cdf)

The distribution of a continuous random variable X is by definition the probability that X takes a value less than or equal to a given cutoff x. The cumulative distribution function (cdf) itself is on the domain of $F\colon \mathbb{R} \to [0,1]$, and can be simply defined as the following integral:

$$F(X) = P(X \leq x) = P(X \in (-\infty, x]) = \int_{-\infty}^{x} f(t)\,dt$$

The following are the four properties of the continuous distribution function:

1. $\lim\limits_{x \to -\infty} f(x) = 0$
2. $\lim\limits_{x \to +\infty} f(x) = 1$
3. F is increasing
4. F is continuous

1.3 Example of finding distribution using density

Using the example from Section 1.1, we can see that the density function is given as:

$$\begin{cases} \frac{1}{10}e^{-\frac{1}{10}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

From this, we computed the the pdf to be $1 - e^{\frac{1}{10}x}$ for an arbitrary input x. Given this, we can differentiate the pdf to find the cdf:

$$\begin{aligned} F(X) &= 1 - e^{\frac{1}{10}x} \\ &= -(-\frac{1}{10})e^{-\frac{1}{10}} \\ &= \frac{1}{10}e^{-\frac{1}{10}x} \end{aligned}$$

1.4 Continuous formulas for E[X], Var[X], and mgf(X)

The formula for the expectation, $E[X]$, of a continuous random variable X is given as:

$$E[X] = \int_{\mathbb{R}} x \cdot f(x)\,dx$$

The formula for the variance of a continuous random variable X is unchanged from the discrete case:

$$\mathit{Var(X)} = E[(X - E[X])^2]$$

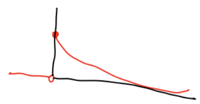
The formula for the moment generating function of a continuous random variable X is given on the domain of $M_X\colon \mathbb{R} \to [-\infty, \infty]$, and has the following formula:

$$M_X(t) = E\left[e^{tX}\right]$$

1.5 Example of using density to find E[X], Var(X), and mgf(X)

Below is a problem that solves for the expectation, variance, and moment generating function of a continuous random variable X.

- $f(x) = \begin{cases} 2e^{-2x}, & t \geq 0 \\ 0, & x < 0 \end{cases}$



a) $M_X(t)$ b) $E[X]$, c) $Var(X)$

a) $M_X(t) = \int_{\mathbb{R}} e^{tx} \cdot f(x)\, dx \Rightarrow \int_0^\infty e^{tx} \cdot 2e^{-2x}\, dx$

$\Rightarrow 2\int_0^\infty e^{tx} \cdot e^{-2x}\, dx \Rightarrow 2\int_0^\infty e^{(t-2)x}\, dx$

$\boxed{e^x \cdot e^y = e^{x+y}}$

$\Rightarrow 2\left[\frac{1}{t-2} \cdot e^{(t-2)x}\right]_0^\infty \Rightarrow 2 \cdot (t-2)\left[e^{(t-2)x}\right]_0^\infty$

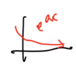
for $t-2 \neq 0$

$\Rightarrow 2(t-2)\left[e^{(t-2)\infty} - e^{(t-2)\cdot 0}\right]$

for $t-2<0$ to give a finite number

$\underset{0}{e^{(t-2)\infty}}$, $\underset{1}{e^{(t-2)\cdot 0}}$

For $a>0$, For $a<0$



Thus, for $a=\infty$

$\Rightarrow \frac{2}{t-2} \cdot (0-1) = \frac{2}{2-t}$

for $t-2<0$ $(t<2)$

Domain $M_X \rightarrow (-\infty, 2)$

b) $E[X] = M_X'(0)$

$M_X'(t) = \left(\frac{2}{2-t}\right)'$

$\frac{d}{dx}\left[\frac{1}{t}\right] = (t^{-1})' = (-1)\cdot t^{-2}$

$\Rightarrow -\frac{1}{t^2}$

$\left(\frac{2}{2-t}\right)' \Rightarrow 2\cdot\left(\frac{1}{2-t}\right)' \Rightarrow 2\cdot\left(-\frac{1}{(2-t)^2}\right)\cdot \underset{-1}{(2-t)'}$

chain rule

$\Rightarrow \frac{2}{(2-t)^2}$ for $t=0$, $\frac{2}{(2-0)^2} \Rightarrow \frac{1}{2}$

$E[X] = \frac{1}{2}$

c) $Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = M_X''(0) = (M_X'(0))' = \left(\frac{2}{(2-t)^2}\right)' \Rightarrow 2\cdot\left(\frac{1}{(2-t)^2}\right)'$

$\Rightarrow 2\cdot\left(-\frac{1}{(2-t)^{4\,3}}\right)\cdot\left((2-t)^2\right)'$

chain rule

$2\cancel{(2-t)} \cdot \underset{-1}{(2-t)'}$

$\Rightarrow \frac{4}{(2-t)^3}$, for $t=0$, $\frac{4}{(2-0)^3} \Rightarrow \frac{4}{8} = \frac{1}{2}$

$\therefore E[X^2] = \frac{1}{2}$, $E[X] = \frac{1}{2}$

$\Rightarrow E[X^2] - (E[X])^2 \Rightarrow \frac{1}{2} - \left(\frac{1}{2}\right)^2 \Rightarrow \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$Var(X) = \frac{1}{4}$