MATH3160 — Portfolio 4.2

Mike Medved

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1 Deliverables

1.1 Tools for Discrete Random Variables

The following are a few of the available tools to work with discrete random variables.

1.1.1 Probability Mass Function (pmf)

The probability mass function, or *pmf* for short is a function that maps a discrete random variable to its probability. The pmf is in the domain of the image of the random variable X and the range of the probability.

$$f(x) = P(X = x)$$

Further, the sum of all pmf of X must be equal to one in order to satisfy the law of total probability:

$$\sum_{x \in Im(X)} f(x) = 1$$

1.1.2 Cumulative Distribution Function (cdf)

The cumulative distribution function, or *cdf* for short is a function that maps a discrete random variable to its cumulative probability. The cdf is in the domain of the image of the random variable X and the range of the cumulative probability.

$$F(x) = P(X \leq x)$$

1.1.3 Example of 1.1.1 and 1.1.2

X = # of days/week that student comes to UConn

Tues (with 0.8 probability)

Thurs (with 0.8 probability)

(independent)

$\operatorname{Im} X = \{0, \overset{\text{Tu or Th}}{1}, 2\}$

$\underset{\text{no days}}{0}$, $\underset{\text{Both}}{2}$

$x \in \operatorname{Im} X$	0	1	2
$f(x) = P(X=x)$	0.04	0.32	0.64

Notice they add up to 1.0.

$P(X=0) \to P(\text{Tues}^c \cap \text{Thurs}^c) = P(\text{Tues}^c) \cdot P(\text{Thurs}^c)$

$0.2 \cdot 0.2 = 0.04$

$P(X=1) \to P((\text{Tues} \cap \text{Thurs}^c) \cup (\text{Tues}^c \cap \text{Thurs})) =$

disjoint

$\Rightarrow P(\text{Tues} \cap \text{Thurs}^c) + P(\text{Tues}^c \cap \text{Thurs})$

$\Rightarrow P(\text{Tues}) \cdot P(\text{Thurs}^c) + P(\text{Tues}^c) \cdot P(\text{Thurs})$

$0.8 \cdot 0.2 + 0.2 \cdot 0.8$

$0.16 + 0.16 = 0.32$

$P(X=2) \to P(\text{Tues} \cap \text{Thurs}) = P(\text{Tues}) \cdot P(\text{Thurs})$

$0.8 \cdot 0.8 = 0.64$

Now, find F.

Case 1

$F(x) = P(X \le x)$

$= 0$

No values < 0 in $\operatorname{Im} X$

Case 2

$F(x) = P(X \le x)$

$= P(0)$

$= 0.04$

Only value < 1 in $\operatorname{Im} X = P(0)$

Case 3

$F(x) = P(X \le x)$

$= P(0) + P(1)$

$= 0.36$

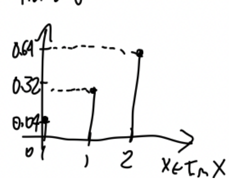
Values 0, 1 are only values in $\operatorname{Im} X$ less than 2.

Case 4 contains all values

$P(0) + P(1) + P(2) = 1$

$$F(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 0.04, & x \in [0, 1) \\ 0.36, & x \in [1, 2) \\ 1, & x \in [2, \infty) \end{cases}$$

Plot of f



1.1.4 Expectation

The expectation of a discrete random variable X is the weighted average of the random variable.

$$E[X] = \sum_{x \in Im(X)} x \cdot P(X = x)$$

The intuition behind the expectation of X, is that it is the value around which we *expect* the random variable X to be situated.

1.1.5 Variance

The variance of a discrete random variable X is the weighted average of the squared deviation of the random variable from its expectation.

$$Var(X) = E\left[\left(X - E[X]^2\right)\right]$$

The intuition behind the variance of X, is that it is the average spread/deviation of the possible values of X around the expectation $E[X]$.

1.1.6 Example of 1.1.4 and 1.1.5

Example:

X = Number of days/week that student comes to UConn

$$E[X] = 0 \cdot 0.04 + 1 \cdot 0.32 + 2 \cdot 0.64 \quad \text{(multiply rows)}$$
$$= 1.6$$

pmf X:

$x \in Im X$	0	1	2
pmf	0.04	0.32	0.64

pmf X^2:

$x \in Im X^2$	0^2	1^2	2^2
pmf	0.04	0.32	0.64

$$Var(X) = E[X^2] - (E[X])^2$$
$$= 2.88 - 1.6^2$$
$$= 0.32$$

$$E[X^2] = 0 \cdot 0.04 + 1^2 \cdot 0.32 + 2^2 \cdot 0.64$$
$$= 2.88$$

$$\sigma(X) = \sqrt{32} = 0.57$$

1.1.7 Properties of Expectation and Variance

There are several important properties that the expectation and variance functions hold:

1. The expectation of a constant is that constant, $E[C] = C$.
2. The expectation of a random variable X must be positive, $E[X] \geq 0$.
3. The sum of two random variables is the sum of their expectations, $E[aX + bY] = a \cdot E[X] + b \cdot E[Y]$.
   - If $a, b = 1$: $E[X + Y] = E[X] + E[Y]$.
4. If $X \leq Y$, then the expectations follow this property, $E[X] \leq E[Y]$.
5. The product of two expectations does not equal the individual products, $E[X \cdot Y] \neq E[X] \cdot E[Y]$.
   - The only exception to this property is if X and Y are independent, then the product of the expectations *is* equal to the individual products, $E[X \cdot Y] = E[X] \cdot E[Y]$.
6. The transport formula reveals the process of transferring a random variable into a normal variable:

$$E[g(X)] = \sum_{x \in Im(X)} g(x) \cdot P(X = x)$$

   The composition between a random variable X and a real function g is a random variable $g(X)$.

7. Generally, the variance of the sum of two random variables, does not equal the sum of it's individual components, $\mathit{Var(X+Y)} \neq \mathit{Var(X)} + \mathit{Var(Y)}$.
   - Similarly to the product of two expectations, the only exception to this property is if X and Y are independent, then the sum of the variances *is* equal to the individual sums such that, $\mathit{Var(X+Y)} = \mathit{Var(X)} + \mathit{Var(Y)}$.