University of Engineering & Management (UEM), Jaipur | Kolkata

Institute of Engineering & Management (IEM), Jaipur | Kolkata

Topic: Numerical Methods

Instructor: Doyel Sarkar

Email: doyel.sarkar@uem.edu.in

Affiliation: Department of Basic Science and Humanities, University of Engineering and Management - [UEM], Kolkata

❑Topics of Discussion:

- ✓Difference Operators
- ✓Shift Operator
- ✓Missing Term(s)
- ✓Interpolation:
    - ❑ Newton’s Forward Interpolation
    - ❑ Newton’s Backward Interpolation
    - ❑ Lagrange’s Interpolation

Finite differences.

Let $y = f(x)$ be a real-valued function of x defined in an interval $[a,b]$ and its values are known for $(n+1)$ equally spacing points x_i $(i = 0,1,2,\cdots,n)$ such that $x_i = x_\circ + ih$ $(i = 0,1,2,\cdots,n)$ where $x_\circ = a, x_n = b$ and h is the *space length*. Then x_i $(i = 0,1,2,\cdots,n)$ are called *nodes* and the corresponding values y_i are termed as *entries*.

We now introduce the concept of various type differences in order to find the values of $f(x)$ or its derivative for some intermediate values of x in $[a,b]$.

Forward Differences.

$$\Delta f(x) = f(x+h) - f(x).$$

$$\Delta y_i = y_{i+1} - y_i \quad (i = 0, 1, 2, \cdots, n-1)$$

Similarly, the higher order forward differences are define as

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i \qquad \cdots \qquad (3)$$

$$\cdots \qquad \cdots$$

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

where $i = 0, 1, 2, \cdots, n-1$ and r $(1 \le r \le n)$ is a positive integer.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\begin{aligned}\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 &= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Difference Table:

Table : Forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	0			
		1		
3	1		-4	
		-3		22
5	-2		18	
		15		-35
7	13		-17	
		-2		
9	11			

2.4. Some properties of Δ.

If a and b be any two constants, then

(i) $\Delta a = 0$

(ii) $\Delta\{af(x)\} = a\Delta f(x)$

(iii) $\Delta\{af(x) \pm bg(x)\} = a\Delta f(x) \pm b\Delta g(x)$

(iv) $\Delta[f(x)g(x)] = f(x)\Delta g(x) + \Delta f(x).g(x+h)$

$$= f(x+h)\Delta g(x) + \Delta f(x)g(x)$$

(v) $\Delta\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Backward differences :

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ are called the *first backward differences* and denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively.

Thus we have

$$\nabla y_i = y_i - y_{i-1}, \quad i = 1, 2, \ldots, n \qquad \ldots \qquad (5)$$

where ∇ is called the *backward difference operator.* In general backward differece operator is defined as

$$\nabla f(x) = f(x) - f(x - h)$$

Similarly, the higher order backward differences are defined as

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$$

$$\nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1}$$

and so on.

Example 3. Construct the backward difference table of $y = x^2 + 4$ for $x = 1, 3, 5, 7, 9$ and find the values of $\nabla^2 f(5), \nabla^2 f(7)$ and $\nabla^3 f(9)$.

Solution. The backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	5			
3	13	8		
5	29	16	8	
7	53	24	8	0
9	85	32	8	0

From the table, we have

$$\nabla^2 f(5) = 8,\ \nabla^2 f(7) = 8, \nabla^3 f(9) = 0$$

Show that $\Delta.\nabla \equiv \Delta - \nabla$

Shift Operator

$Ef(x) = f(x+h)$, h being the spacing

$$\therefore E^2 f(x) = Ef(x+h) = f(x+2h)$$

$$E^3 f(x) = Ef(x+2h) = f(x+3h)$$

In this way, in general, we have

$$E^n f(x) = f(x+nh)$$

The *inverse shift operator* E^{-1} is defined by

$$E^{-1} f(x) = f(x-h)$$

and in general, we have

$$E^{-n} f(x) = f(x-nh)$$

Since $\Delta f(x) = f(x+h) - f(x)$, it follows that

$$Ef(x) = f(x+h) = \Delta f(x) + f(x) = (\Delta + 1) f(x)$$

so that

$$E \equiv \Delta + 1$$

i.e. $\Delta \equiv E - 1$

Prob :- Find the missing terms in the following table :

x	0	5	10	15	20	25
y	6	10	?	17	?	31

In the given table, only 4 values of y are given So by using these 4 point we can construct a poly. of deg. 3 where 4th order difference will be '0'.

Difference table :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6				
		4			
5	10		$a-14$		
		$a-10$		$41-3a$	
10	a		$27-2a$		$6a+b-102$
		$17-a$		$3a+b-61$	
15	17		$a+b-34$		$4a+4b-143$
		$b-17$		$82-3b-a$	
20	b		$48-2b$		
		$31-b$			
25	31				

$\therefore \quad 6a + b - 102 = 0 \;\&\; 4a + 4b - 143 = 0$

i.e, $\quad 6a + b = 102 \text{ --- (i)} \;\&\; 4a + 4b = 143 \text{ --- (ii)}$

<u>Solving (i) & (ii)</u>: $\quad a = 53/4 \;\&\; b = 45/2$

$\therefore \quad y(10) = 53/4 = 13.25 \;\&\; y(20) = 22.5$

Example 9. Estimate the missing term in the following tables:

x :	0	1	2	3	4
$f(x)$:	1	3	9	–	81

Solution. Since we are given four values of y, so we take $y=f(x)$ to be a polynomial of degree 3 in x so that

$$\Delta^4 f(x)=0$$

i.e., $(E-1)^4 f(x)=0$

i.e., $E^4 f(x)-4E^3 f(x)+6E^2 f(x)-4Ef(x)+f(x)=0$

$$f(x+4)-4f(x+3)+6f(x+2)-4f(x+1)+f(x)=0$$

$$\left[\because E^n f(x)=f(x+nh), h=1\right]$$

Putting $x=0$, we get

$$f(4)-4f(3)+6f(2)-4f(1)+f(0)=0$$

or, $81-4f(3)+6\times 9-4\times 3+1=0$

$\therefore\ f(3)=31$

Ex.24 Find the missing value in the following table :

x	:	2	4	6	8	10
y	:	5.6	8.6	13.9	–	35.6

Ex.25. Find the missing term in the following table:

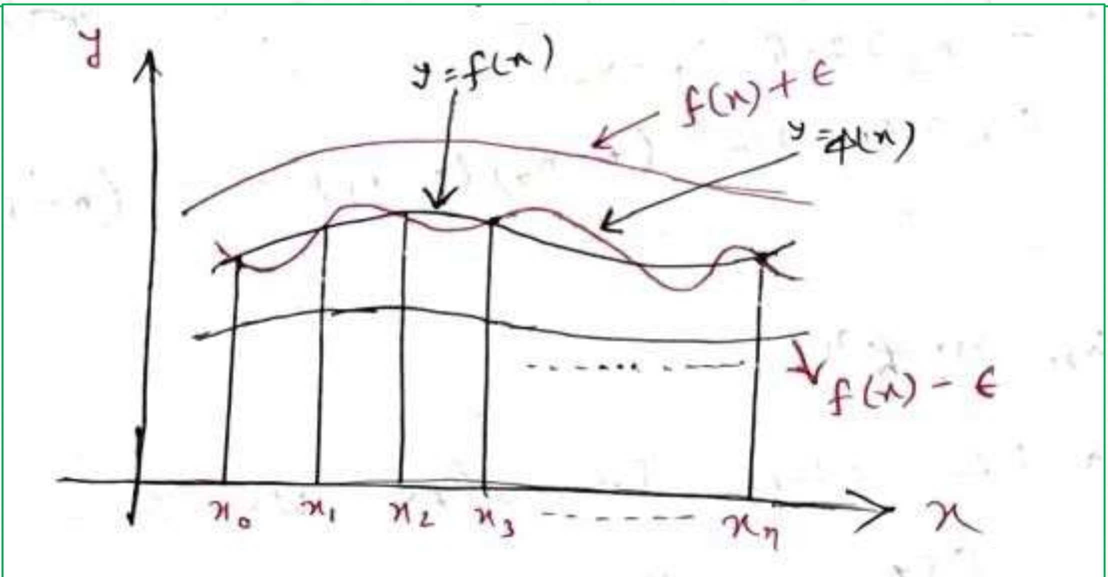
x	:0	1	2	3	4	5
y	:0	–	8	15	–	35

[W.B.U.T., C.S-312, 2007, M(CS)-301,2015]

Interpolation:

What is interpolation? What is extrapolation?

⇒ Interpolation is the process (or method) by which we can approximate a function $y = f(x)$ by another function of relatively simple known form.



Let $y = f(x)$ be a real valued function defined in $x_0 \le x \le x_n$ whose analytical form is not known. Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x :	x_0	x_1	x_2	------	x_n
$y = f(x)$:	y_0	y_1	y_2	------	y_n.

* The process of computing the value of y corresponding to any value x $(\neq x_i \,;\ i = 0(1)n)$ in $[x_0, x_n]$ is called interpolation.

* The process of computing the value of y corresponding to any value of x outside the interval $[x_0, x_n]$ but in the vicinity of $[x_0, x_n]$ is called ~~in~~ extrapolation.

* The Newton Forward Interpolation formula is

$$\phi(x) = f(x) \simeq y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

When $u = \dfrac{x - x_0}{h}$.

Remark(1) :- The N.F.I.F is used when the interpolating point x is near to the begining of the table.

R(2) :- To get better approximate result, the starting argument x_0 should be so choosen such that $0 < u < 1$. More smaller value of u gives us more accurate result.

* The Newton Backward Interpolation formula with x_n & y_n respectively as starting node & entry is —

$$f(x) \simeq y_n + \frac{u}{1!}\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \cdots$$

When $u = \frac{x - x_n}{h}$.

Remark(3) :- It is used for an interpolating point x is near to the end of the table.

R(4) :- To get better approximate result, the starting point x_n should be so choosen that $-1 < u < 0$.

More smaller value of u, gives us more accurate result.

Ex. If $y(10)=35.3$, $y(15)=32.4$, $y(20)=29.2$, $y(25)=26.1$, $y(30)=23.2$, and $y(35)=20.5$, find $y(12)$ using Newton's forward interpolation formula. [*W.B.U.T., M(CS)-301, 2009*]

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	35.3					
		−2.9				
15	32.4		−0.3			
		−3.2		0.4		
20	29.2		0.1		−0.3	
		−3.1		0.1		0.2
25	26.1		0.2		−0.1	
		−2.9		0.0		
30	23.2		0.2			
		−2.7				
35	20.5					

To find $y(12)$, we choose $x_0=10$

Here $n=12$, $h=5$

$$\therefore\ s=\frac{x-x_0}{h}=\frac{12-10}{5}=0.4$$

$\therefore$ From Newton's forward difference interpolation formula,

$$y(x) \simeq y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots,$$

we have

$$y(12) = 35.3 + 0.4(-29) + \frac{0.4(0.4-1)}{2!} \times (-0.3)$$

$$+ \frac{0.4(0.4-1)(0.4-2)}{3!} \times (0.4)$$

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-0.3) = 34.21408$$

Prob^m :- Compute $f(0.23)$ & $f(0.29)$, using suitable formula, from the table given below:

$x \rightarrow$	0.20	0.22	0.24	0.26	0.28	0.30
$y = f(x) \rightarrow$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

The difference table is :-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.20	1.6596					
		0.0102				
0.22	1.6698		0.0004			
		0.0106		−0.0002		
0.24	1.6804		0.0002		0.0004	
		0.0108		+0.0002		−0.0007
0.26	1.6912		0.0004		0.0003	
		0.0112		−0.0001		
0.28	1.7024		0.0003			
		0.0115				
0.30	1.7139					

⇒ For $f(0.23)$:- since all the values of x are equidistant & interpolating point $x = 0.23$ lies in the begining in the table, we use (apply) Newton Forward interpolation formula :-

$$f(x) \simeq y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{x - x_0}{h}$.

Here : $x = 0.23$, $h = 0.02$. Let $x_0 = 0.22$.

$\therefore u = \frac{x - x_0}{h} = \frac{0.23 - 0.22}{0.02} = 0.5 \; [0 < u < 1]$

$$\therefore f(0.23) = 1.6698 + \frac{0.5}{1!}(0.0106) + \frac{0.5(0.5-1)}{2!}(0.0002)$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{3!}(0.0002) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(-0.0003)$$

$$= 1.6698 + 0.0053 - 0.000025 + 0.0000125 - 0.00001172$$

$$\downarrow$$

$$1171875$$

$$= 1.67509922$$

$$\simeq 1.6751$$

Hence $f(0.23) = 1.6751$ correct upto 4D.

For $f(0.29)$:-

Since all the values of x are equidistant & interpolating point $x = 0.29$ lies at the end in the table, we use N.B.I.F :-

let $x_n = 0.30$, $x = 0.29$, $h = 0.02$.

$\therefore u = \frac{x - x_n}{h} = \frac{0.29 - 0.30}{0.02} = -0.5$

According to N.B.I.F formula :-

$$f(0.29) \simeq f(\overset{=y_n}{x_n}) + \frac{u}{1!}\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 y_{n-4} + \text{———}$$

$$= 1.7139 + (-0.5)(0.0115) + \frac{(-0.5)(-0.5+1)}{2} \times (0.0003)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(-0.0001) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(-0.0003)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{5!}(-0.0007)$$

$$= 1.7139 - 0.00575 - 0.0000375 + 0.00000625 + 0.000011718 + 0.00001914$$

$$= 1.708149608 \simeq 1.7081 \text{ (Correct upto 4D)}$$

Task:

Ex. Compute the valus of $f(3.5)$ and $f(7.5)$ using Newtan's interpolation from the following table:

x	:	3	4	5	6	7	8
$f(x)$	:	27	64	125	216	343	512

[*W.B.U.T.*, *CS*-312, 2008]

Lagrange's interpolation formula.

Let $y = f(x)$ be a function of x, continuous and $(n+1)$ times continuously differentiable in $[a, b]$. Let us divide the interval $[a, b]$ by $(n+1)$ points $a = x_0, x_1, \ldots, x_n = b$ which are not necessarily equispaced and the corresponding entries are $y_i = f(x_i)$ $(i = 0, 1, 2, \ldots, n)$. We now wish to find a polynomial $L_n(x)$ in x of degree n such that

$$L_n(x_i) = f(x_i) = y_i (i = 0, 1, 2, \ldots, n) \quad \ldots \quad (28)$$

$$L_n(x) = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ...$$

$$+ \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n$$

Example. Find the polynomial of degree ≤ 3 passing through the points $(-1, 1)$, $(0, 1)$, $(1, 1)$ and $(2, -3)$.

$$L_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_1-x_2)(x_0-x_3)}.y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}.y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}.y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.y_3$$

$$= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}.1 + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}.1$$

$$+\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}.1 + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}(-3)$$

$$= \tfrac{1}{3}\left(-2x^3 + 2x + 3\right).$$

Hence the required polynomial is

$$\tfrac{1}{3}\left(-2x^3 + 2x + 3\right)$$

x	0	2	3	4	7	8
$f(x)$	4	26	58	112	466	668

Find f(5)

Ans : 194 (near about)