Vision Research 51 (2011) 397–399

Contents lists available at ScienceDirect

Vision Research

journal homepage: www.elsevier.com/locate/visres

The geometric horopter

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ARTICLE INFO

Article history:
Received 1 December 2010
Available online 21 January 2011

Keywords:
Horopter
Binocular vision
Single vision

ABSTRACT

Two aspects of the geometric horopter, which here is based on the criterion of equality of angle, are clarified. The first is that in the fixation plane (containing the nodal points and the fixation point) the locus of points lying on the horopter is the larger arc of a circle, and not a full circle as has been previously accepted. The second is that elsewhere, the locus of these points is a straight line perpendicular to this plane and midway between the eyes. These rules hold for both symmetric and asymmetric convergence, and for fixation elevated or depressed from the horizontal.

1. Introduction

Although we have two eyes, we are not normally aware of seeing the world double. This is primarily because the two foveas have associated with them the same perceived visual direction, as Hering demonstrated (as had Ptolemy in the second century and Alhazen in the eleventh, according to Howard (1999)), and so fixated objects appear single. In general, non-fixated objects will subtend different angles at the two eyes with the respective fixation lines, and would be expected always to be perceived as double were it not for sensory fusion. For purely geometric reasons, however, some points in space will subtend the same angles at the two eyes with the fixation lines, and might be expected to be seen singly as a consequence. These points are said to lie on, and define, the horopter (Aguilonius, 1613).

When considering the geometrical aspects of the horopter it is usual to make the assumption that corresponding points on the two retinas are isomorphic, i.e. they have the same x, y retinal co-ordinates, as did Helmholtz and others before him. This assumption follows from the defining of the horopter as the locus of points in space which are seen singly, rather than defining it as the locus of points which make the same angle at the two eyes. However, empirical measurement of the positions in real space of points which are seen singly (without fusion) invariably results in a figure which deviates from the expected form (e.g. Shipley and Rawlings, 1970). This deviation has been said to indicate a failure of the isomorphism of the retinas.

In considering the theoretical form of the horopter it is usual to take single vision as the starting point, and to assume isomorphism, but when examining the issue of isomorphism one needs to adopt a different criterion to avoid any analysis being circular. In order to compare empirical results with theoretical expectations the criterion of equal angle fits the bill, and this criterion determines what shall be called here the "geometric horopter". A failure of isomorphism is but one of a number of explanations which have been put forward to explain the difference between the predictions of the geometric horopter and the empirical findings.

Recently, Schreiber, Tweed, and Schor (2006) have extended the original work of Helmholtz (1910), and of Solomons (1975a, 1975b), in this area by considering ocular kinematics and vertical sensory fusion. They report that, when isomorphism is assumed, the theoretical horopter consists of a circle and also a vertical line through the fixation point, as described earlier by Prévost (1843) and Burckhardt (Helmholtz, 1910). In considering the geometry of the points in space expected to be seen singly on the basis of their location alone, two aspects of this model are clarified here, both of which are independent of the location of the fixation point relative to the eyes. The first is that in the fixation plane (defined by the two nodal points and the fixation point) the locus of the points which lie on the horopter is the larger arc of a circle, and not a full circle as has been universally stated since it was first described this way by Vieth and Müller (Helmholtz, 1910). The second is that elsewhere, the locus of these points is, in general, a straight line perpendicular to the fixation plane, midway between the eyes, and that this line is only vertical in the special case of when the fixation plane is horizontal.

2. Visual direction

For simplicity of explanation, let us assume that the optics of the eye provide us with a single nodal point. Although it is not a requirement, let us also start with the assumption for our current purposes that the plane containing the fixation point, the nodal point and the fovea is horizontal. We can then define the fixation line as a horizontal line from the fovea to the fixation point through the eye's nodal point.

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doi:10.1016/j.visres.2010.12.018

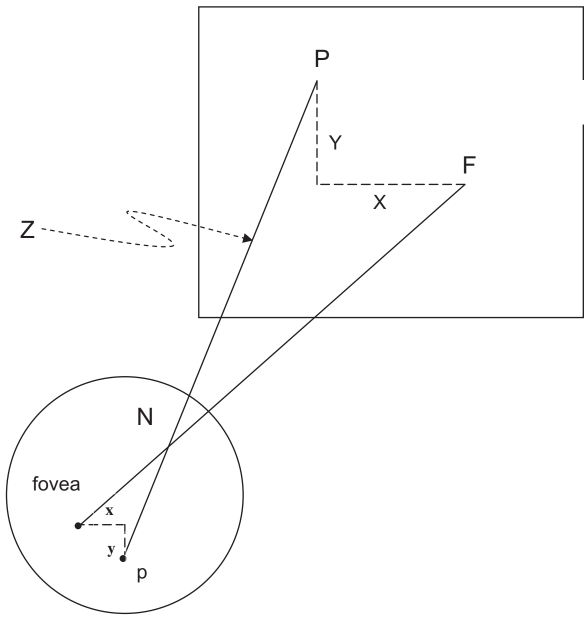

Fig. 1. The fovea, a retinal point *p*, the fixation point *F*, the nodal point *N* and the point in space *P* which is imaged on *p*. The distances *X* and *Y* are shown on the plane perpendicular to the fixation line, and the distances *x* and *y* define the position of the retinal point *p* relative to the fovea. The position *p* is uniquely defined by *x* and *y*, but to uniquely define the point *P*, in terms of *X* and *Y*, the viewing distance *Z* needs to be included.

The angle made with the fixation line by a line going from any, arbitrary, point "*p*" on the retina through the nodal point and out into space can be specified in terms of *X* and *Y*, where *X* is the horizontal distance and *Y* the vertical distance in a plane perpendicular to this line *pP*. If we now consider the viewing distance *Z* along this line, then the values of Y/Z and X/Z provide the tangents of the vertical and horizontal components of the angle respectively (see Fig. 1).

3. The location in space of points on the horopter

For a single point in space to be on the horopter, the only requirements are that the value Y/Z is the same for the two eyes, and that the value of X/Z is the same for the two eyes, irrespective of whether or not the values of *X*, of *Y* or of *Z* are themselves identical in the two eyes.

When *F* and *P* are both in the horizontal plane, the value of *Y* for each eye is zero. It has long been argued that in these circumstances the locus of points in space that fulfil the requirement that the values of value of X/Z for the two eyes should match is a circle, the Vieth–Müller circle. This is incorrect, as the requirement is only met by the arc of the circle which stretches from one nodal point to the fixation point to the other nodal point. In general, points on this arc are at different distances from the eyes, but the value of X/Z is the same for the two eyes. Points on the other arc of the circle are *not* on the horopter because, although numerically their angles are equal, they lie in *opposite* directions relative to the fixation point; the value of X/Z for one eye being the negative of that for the other eye.

Taking away the restriction of considering the horizontal plane alone, extending this arc into three dimensions, gives us a portion of a cylinder. Any point on it will have the same value of X/Z at each eye and so potentially lies on the horopter. However, for a point to lie on the horopter the value of Y/Z for the two eyes must also match. The only points on this surface that fulfil this requirement are, as we have seen, those in the horizontal plane, and in addition

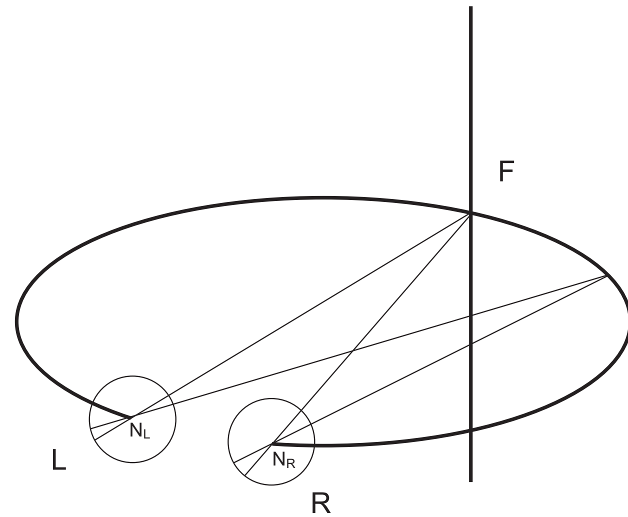

Fig. 2. In the general case, the geometric horopter (bold line) takes the form of the larger arc of a circle incorporating the two nodal points and the fixation point, and a perpendicular line passing through the fixation point. A change of fixation from the point *F* to the point *P* leaves the geometric horopter, including the perpendicular line, unchanged.

those in the vertical line which passes through the arc equidistant from the two eyes. Any other point on this surface will be closer to one eye than the other, and thus cannot have the same value of Y/Z. This line has previously been called the Prévost–Burckhardt line (Helmholtz, 1910; von Tschermak-Seysenegg, 1952).

This analysis holds whether fixation is symmetric or asymmetric – in both cases the horopter consists of the larger arc of a circle, and a perpendicular line midway between the two eyes. However, if fixation is asymmetric the Prévost–Burckhardt line no longer goes through the fixation point. As a consequence, the set of retinal points which are stimulated by points on the Prévost–Burckhardt line in asymmetric vergence are not those stimulated during symmetric vergence.

The above argument has been made with the assumption that the fixation point is in the horizontal plane, but the same geometry applies when it is not. If the fixation point is above or below the horizontal plane containing the nodal points, and the eyes are elevated or depressed, the horopter will still consist of the larger arc of a circle running from one nodal point to the fixation point to the other nodal point, and a single line perpendicular to this plane and midway between the eyes. However, this line will no longer be in the vertical meridian, but will be tilted. In effect, the whole horopter as shown in Fig. 2 will have been rotated around the line joining the two nodal points. Similarly, if the head is tilted to one side, the whole of the geometric horopter will be tilted but with its form unchanged.

4. Discussion

It has long been known that the measured horopter does not precisely match the prediction made here. Empirical measurements of the horopter in the fixation plane (e.g. Shipley & Rawlings, 1970, Hillis & Banks, 2001) generally report a deviation from the expected arc. However, these experiments, by the very nature of the response required, inherently involve the use of a different criterion (equality of perceived visual direction, or single vision) from that used here (equality of angle). One would expect the measured and equal-angle horopters to coincide only if corresponding points (i.e. points giving rise to the same perceived visual direction) in the two eyes were isomorphic and the eyes had not undergone any torsion. The failure of the empirical results to match the theoretical prediction indicates that one or both of these assumptions are not valid. The Hering–Hillebrand deviation has usually been taken as evidence of an asymmetry between the locations of corresponding points on the nasal and temporal retinas, with corresponding points not being truly equidistant from the foveas (although Hallden (1956) provided an optical explanation). In the same way, the backward tilt of the measured "vertical" horopter has been said to be a consequence of a shear in retinal correspondence in the vertical meridian (Helmholtz, 1910; Siderov, Harwerth, & Bedell 1999).

Ocular torsion only becomes an issue when the location of corresponding points on the retina is considered. The location of the points in space which make equal angles at the two eyes is unchanged if the eyes have undergone torsion, as long as this is around the fixation line and the position of the nodal point remains unaltered. However, as Schreiber et al. (2006) have shown, torsion is a crucial factor once retinal correspondence is brought into play. For example, in a situation where the retinas are isomorphic torsion will move the retinal corresponding points, and paradoxically the points in space which lie on the geometric horopter, and make equal angles at the two eyes, could even be seen double.

This cannot be the whole story, however, because comparison between theory and practice in relation to the spectral content of the stimulus reveals a further assumption that is not valid. In the analysis presented, a fixed nodal point is postulated. However, the refractive indices of the eye's media are known to be wavelength-dependent, and as a consequence one would expect the position of nodal points of the eyes to similarly vary with wavelength. Two adjacent lamps, emitting light from different ends of the spectrum, will appear together when foveally-viewed, but will be perceived laterally separated, in different positions in space, when viewed eccentrically (Howarth, 1984) because of ocular lateral chromatic aberration. One would thus expect that the measured horopter would differ for these different wavelengths, which is precisely what is seen (von Tschermak-Seysenegg, 1952). Ocular torsion, and the location of corresponding points, cannot therefore by themselves provide a complete explanation for the deviation of the measured horopter from the prediction made here on the grounds of geometry.

Acknowledgment

The ideas presented here were formed during discussions with Janet Wolf about the perception of visual direction. Her help is gratefully acknowledged, as is that of a knowledgeable anonymous reviewer.

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