Probability, Entropy, and Inference / More About Inference

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Information Theory

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Probability, Entropy, and Inference

Definitions and notation

- An **ensemble** X is a triple $(x, \mathcal{A}_X, \mathcal{P}_X)$, where:
    - x is the **outcome** of a random variable;
    - $\mathcal{A}_X = \{a_1, a_2, \ldots, a_i, \ldots, a_I\}$ is the set of possible values for the random variable, called an **alphabet**;
    - $\mathcal{P}_X = \{p_1, p_2, \ldots, p_I\}$ are the **probability** of each value, with

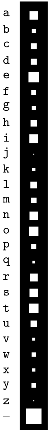
$$\begin{aligned} P(x = a_i) &= p_i, \\ p_i &\geq 0, \qquad \text{and} \\ \sum_{a_i \in \mathcal{A}_X} P(x = a_i) &= 1. \end{aligned}$$

- We may write $P(x = a_i)$ as $P(a_i)$ or $P(x)$ when no confusion is possible.

Definitions and notation

- Example 1 Frequency of letters in *"The Frequently Asked Questions Manual for Linux."*
  The outcome is a letter that is randomly selected from an English document.

i	a_i	p_i
1	a	0.0575
2	b	0.0128
3	c	0.0263
4	d	0.0285
5	e	0.0913
6	f	0.0173
7	g	0.0133
8	h	0.0313
9	i	0.0599
10	j	0.0006
11	k	0.0084
12	l	0.0335
13	m	0.0235
14	n	0.0596
15	o	0.0689
16	p	0.0192
17	q	0.0008
18	r	0.0508
19	s	0.0567
20	t	0.0706
21	u	0.0334
22	v	0.0069
23	w	0.0119
24	x	0.0073
25	y	0.0164
26	z	0.0007
27	–	0.1928



Definitions and notation

- **Probability of a subset.** If T is a subset of $\mathcal{A}_X$ then

$$P(T) = P(x \in T) = \sum_{a_i \in T} P(x = a_i).$$

- $\boxed{\text{Example 1}}$ (Continued)

  If we define V to be vowels, $V = \{\texttt{a}, \texttt{e}, \texttt{i}, \texttt{o}, \texttt{u}\}$, then

$$\begin{aligned} P(V) &= p(\texttt{a}) + p(\texttt{e}) + p(\texttt{i}) + p(\texttt{o}) + p(\texttt{u}) \\ &= 0.0575 + 0.0913 + 0.0599 + 0.0689 + 0.0334 \\ &= 0.3110. \end{aligned}$$

$\triangleleft$

Definitions and notation

- **Joint ensemble.** XY is an ensemble in which each outcome is an ordered pair x, y with $x \in \mathcal{A}_X = \{a_1, \ldots, a_I\}$ and $y \in \mathcal{A}_Y = \{b_1, \ldots, b_J\}$.

  We call $P(x, y)$ the **joint probability** of x and y.

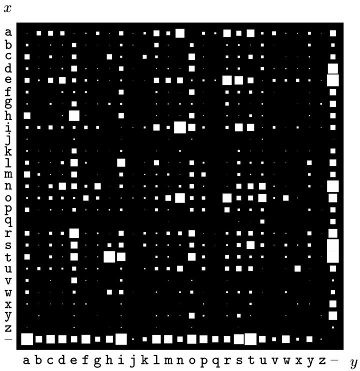
  Commas are optional when writing ordered pairs, so $xy \Leftrightarrow x, y$.

  In a joint ensemble XY the two random variables may or may not be independent.

Definitions and notation

- Example 1 (Continued)

Frequency of bigrams (pair *xy* of letters) in *"The FAQ Manual for Linux."* The outcome is a pair of consecutive letters randomly selected from an English document.



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Definitions and notation

- We can obtain the **marginal probability** $P(x)$ from the joint probability $P(x, y)$ by summation:

  $$P(x = a_i) \equiv \sum_{y \in \mathcal{A}_Y} P(x = a_i, y).$$

  Similarly, using briefer notation, the marginal probability of y is

  $$P(y) \equiv \sum_{x \in \mathcal{A}_X} P(x, y).$$

- **Conditional probability.** If $P(y = b_j) \neq 0$,

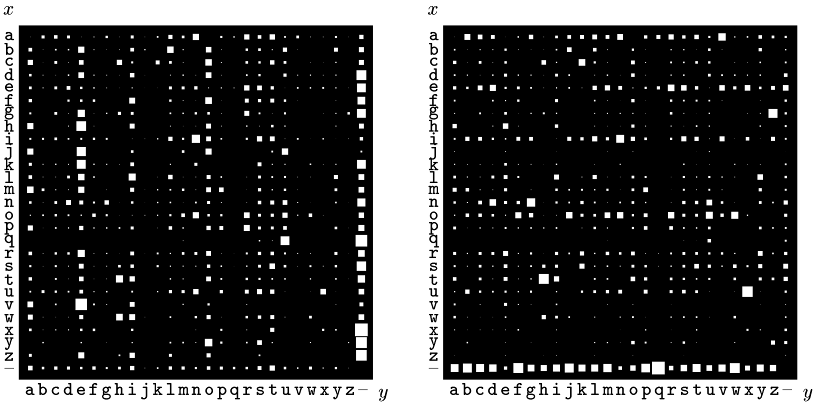
  $$P(x = a_i \mid y = b_j) \equiv \frac{P(x = a_i, y = b_j)}{P(y = b_j)}.$$

  If $P(y = b_j) = 0$, then $P(x = a_i \mid y = b_j)$ is undefined.

Definitions and notation

- Example 1 (Continued)

Conditional probabilities of bigrams in *"The FAQ Manual for Linux."*



(a) $P(y \,|\, x)$

(b) $P(x \,|\, y)$

Definitions and notation

- We will use $\mathcal{H}$ to denote the assumptions on which the probabilities are based.
- **Product rule / Chain rule:**

$$\begin{aligned} P(x, y \mid \mathcal{H}) &= P(x \mid \mathcal{H}) P(y \mid x, \mathcal{H}) \\ &= P(y \mid \mathcal{H}) P(x \mid y, \mathcal{H}). \end{aligned}$$

- **Sum rule:**

$$\begin{aligned} P(x \mid \mathcal{H}) &= \sum_{y} P(x, y \mid \mathcal{H}) \\ &= \sum_{y} P(y \mid \mathcal{H}) P(x \mid y, \mathcal{H}). \end{aligned}$$

- **Independence.** Two random variables X and Y are independent (sometimes written as $X \perp Y$) if, and only if, for all x, y:

$$P(x, y) = P(x) P(y).$$

What *"probability"* means

- Probabilities can be interpreted in two main ways:

  1. The **frequentist interpretation** sees probabilities as frequencies of outcomes in random experiments.

     This is the usual approach we have taken so far.

  2. The **Bayesian interpretation** (also known as **degree-of-belief interpretation**, **knowledge interpretation**, or **subjective interpretation**) sees probabilities as the degree to which someone belief in the truth value of propositions depending on random variables.

     For instance, juries in trials usually try to access *"the probability that Mr. Jones killed his wife, given the evidence"*.

     The degree-of-belief interpretation of probabilities is essential for probabilistic reasoning, in particular, Bayesian reasoning.

Cox axioms for reasoning about belief

- **Notation.** Let
  - "the degree of belief in proposition x" be denoted by $B(x)$,
  - the negation of x be denoted by $\overline{x}$, and
  - the degree of belief in a conditional proposition, "x, assuming proposition y to be true", be denoted by $B(x \mid y)$.

Cox axioms for reasoning about belief

- The **Cox axioms** establish what is essential to reason about belief:

A1 Degrees of belief can be ordered.

Using $\succeq$ to denote "is greater than":

$$\text{if } B(x) \succeq B(y) \text{ and } B(y) \succeq B(z) \text{ then } B(x) \succeq B(z).$$

(As a consequence, beliefs can be mapped onto real numbers.)

A2 The degree of belief in a proposition x and its negation $\overline{x}$ are related.

There is a function f such that

$$B(x) = f(B(\overline{x})).$$

A3 The degree of belief in a conjunction of propositions x, y (i.e., $x \wedge y$) is related to the degree of belief in the conditional proposition $x \mid y$ and the degree of belief in the proposition y.

There is a function g such that

$$B(x, y) = g(B(x \mid y), B(y)).$$

Probabilities as belief

- If a set of beliefs satisfy the Cox axioms, we can map them onto probabilities satisfying

  $$p(\text{FALSE}) = 0, \quad p(\text{TRUE}) = 1, \quad \text{and} \quad 0 \leq p(x) \leq 1,$$

  and also satisfying the rules of probability

  $$p(x) = 1 - p(\overline{x}),$$

  and

  $$p(x, y) = p(x \mid y)p(y).$$

- This is actually what we have been using all along when applying Bayesian reasoning.

- Probabilities as belief or knowledge is a (very beautiful!) generalization of logic.

  I strongly recommend E. T. Jaynes's book *Probability Theory: The Logic of Science* [link].

Forward and inverse probabilities

- Using probabilities as degrees of belief, we can make inference of future outcomes and of past parameters of an experiment.

  We can talk, then, about forward and inverse probabilities.

Forward and inverse probabilities

- **Forward probability** problems involve a generative model that describes a process that is assumed to give rise to some data.

  The task is to compute the probability distribution or expectation of some quantity that depends on the data.

Forward and inverse probabilities

- $\boxed{\text{Example 2}}$ An urn contains K balls, of which B are black and $W = K - B$ are white. Fred draws a ball at random from the urn and replaces it, N times.

  a) What is the probability distribution of the number of times a black ball is drawn, n_B?

  b) What is the expectation of n_B? What is the variance of n_B? What is the standard deviation of n_B? Give numerical answers for the cases $N = 5$ and $N = 400$, when $B = 2$ and $K = 10$.

Solution.

a) Let us define $f_B = {}^{B}/_{K}$ as the fraction of black balls in the urn.

The number of black balls follow has a binomial distribution.

$$p(n_B \mid f_B, N) = \binom{N}{n_B} f_B^{n_B} (1 - f_B)^{N - n_B}.$$

Forward and inverse probabilities

- $\boxed{\text{Example 2}}$ (Continued)

  b) Since we have a binomial distribution, its expecatation is

  $$E(n_B) = Nf_B,$$

  its variance is

  $$Var(n_B) = Nf_B(1 - f_B),$$

  and its standard deviation is

  $$\sigma(n_B) = \sqrt{Nf_B(1 - f_B)}.$$

  Hence, when $B = 2$ and $K = 10$ we have $f_b = {}^1\!/_5$, and

  - for $N = 5$ we have

    $$E(n_B) = 1, \; Var(n_B) = {}^4\!/_5, \text{ and } \sigma(n_B) = 0.89.$$

  - for $N = 400$ we have

    $$E(n_B) = 80, \; Var(n_B) = 64, \text{ and } \sigma(n_B) = 8.$$

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Forward and inverse probabilities

- Like forward probability problems, **inverse probability** problems involve a generative model of a process.

  But instead of computing the probability distribution of some quantity produced by the process, we compute the conditional probability of one or more of the unobserved variables in the process, given the observed variables.

  This invariably requires the use of Bayes' theorem.

Forward and inverse probabilities

- $\boxed{\text{Example 3}}$ Urn A contains three balls: one black, and two white; urn B contains three balls: two black, and one white.

  One of the urns is selected at random an one ball is drawn. The ball is black. What is the probability that the selected urn is A?

  Solution.
  We have two alternative hypotheses: the selected urn is A or it is B.

  We have two possible evidences: the ball drawn is black (b) or it's white (w).

  We can find $p(urn = A \mid ball = black)$ using Bayes's theorem:

$$\begin{aligned}
p(urn = A \mid ball = b) &= \frac{p(urn = A, ball = b)}{p(ball = b)} \\
&= \frac{p(urn = A)p(ball = b \mid urn = A)}{p(urn = A)p(ball = b \mid urn = A) + p(urn = B)p(ball = b \mid urn = B)} \\
&= \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 1/2 \cdot 2/3} \\
&= \frac{1}{3}
\end{aligned}$$

Forward and inverse probabilities

- Like we said, using probabilities as degrees of belief, we can make inference of future outcomes and of past parameters of an experiment.

- **Notation.** Let

  - $\mathcal{H}$ be the overall hypothesis space;

  - θ be the parameters of an experiment; and

  - D be the data obtained as the outcome of the experiment.

- $p(\theta \mid \mathcal{H})$ is the **prior probability** of the parameter θ, that is, the probability of the parameter before the experiment is run.

Forward and inverse probabilities

- The conditional distribution $p(D \mid \theta, \mathcal{H})$ represents how the experiment evolves, relating the parameter θ and the data D given some hypothesis $\mathcal{H}$.

  Depending on what we are reasoning about, this quantity has different names:

  1. If we fix the value of the parameter θ, $p(D \mid \theta, \mathcal{H})$ is the **conditional probability** of observing data D if the parameter assumed value θ under hypothesis $\mathcal{H}$.
  2. If we fix the value of the data D, $p(D \mid \theta, \mathcal{H})$ is the **likelihood** of the parameter θ given we observed data D, under hypothesis $\mathcal{H}$.

Forward and inverse probabilities

- $p(D \mid \mathcal{H})$ is the **probability of evidence**, that is, the probability of producing data D under hypothesis $\mathcal{H}$, $\underline{\text{after the experiment is run}}$.

  Naturally

$$\begin{aligned} p(D \mid \mathcal{H}) &= \sum_{\theta} p(\theta, D \mid \mathcal{H}) \\ &= \sum_{\theta} p(\theta \mid \mathcal{H}) p(D \mid \theta, \mathcal{H}). \end{aligned}$$

- The general equation

$$p(\theta \mid D, \mathcal{H}) = \frac{p(\theta \mid \mathcal{H}) p(D \mid \theta, \mathcal{H})}{p(D \mid \mathcal{H})}$$

  is written as

$$\text{posterior probability} = \frac{\text{prior probability} \times \text{likelihood}}{\text{evidence}}.$$

Probability vs. Likelihood

- We can say:
  - "the probability of the data", meaning the probability of obtaining data D given parameter θ, under hypothesis $\mathcal{H}$.
  - "the likelihood of the parameter", meaning how likely it is that a given parameter θ generated the data D we observed, under assumption $\mathcal{H}$.
- We can not say "the likelihood of the data".

The Likelihood Principle

- Let us consider now a "more complicated" example involving urns and balls.

- $\boxed{\text{Example 4}}$ Urn *A* contains five balls: one black, two white, one green, and one pink; urn *B* contains five hundred balls: two hundred black, one hundred white, 50 yellow, 40 cyan, 30 sienna, 25 silver, 20 gold, and 10 purple. (One fifth of *A*'s balls are black; two-fifths of *B*'s balls are black.)

  One of the urns is selected at random an one ball is drawn. The ball is black. What is the probability that the selected urn is *A*?

  Solution.

  You can do it, guys! I know you can!

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The Likelihood Principle

- In the previous two examples, does each answer depend on the detailed contents of each urn?

  No! All that matters is the probability of the outcome that actually happened (here, that the ball drawn was black) given the different hypotheses.

  We need only to know the likelihood, i.e., how the probability of the data that happened varies with the hypothesis.

  This simple rule about inference is known as the likelihood principle.

- **The likelihood principle:** given a generative model for data D given parameters θ, $p(D \mid \theta)$, and having observed a particular outcome d, all inferences and predictions should depend only on the function $P(d \mid \theta)$.

  In other words, the likelihood principle says that once an outcome d is observed, the probabilities of other outcomes do not have an influence on the inferences we make about the parameter θ.

More About Inference

Bayes Theorem at the heart of inference

- We have seen many examples showing that Bayes' Theorem is very useful when doing inference.

  If we are presented with data D and have a possible explanation E for the data, Bayes' theorem tells us that

  (Recall that $\mathcal{H}$ denotes the assumptions on which the probabilities are based.)

$$p(E \mid D, \mathcal{H}) = \frac{p(E \mid \mathcal{H}) p(D \mid E, \mathcal{H})}{p(D \mid \mathcal{H})}.$$

Bayes Theorem at the heart of inference

- The probability $p(E \mid D, \mathcal{H})$ we will encounter will depend on the assumptions we made, such as:
  - the hypotheses $\mathcal{H}$, and
  - the *a priori* probability $p(E \mid \mathcal{H})$ of the explanation.
- However:

  you can't do inference–or data compression–without making assumptions,

  and Bayesians won't apologize for it.

Why Bayesian inference is good inference

1. Once assumptions are made, the inferences are:
   - objective,
   - unique, and
   - reproducible with complete agreement by anyone who has the same information and makes the same assumptions.

2. Bayesian inference makes assumptions explicit, so you are forced to think clearly about what hypotheses your model is based on.

   Moreover, once assumptions are explicit, they are easier to criticize, and easier to modify.

Why Bayesian inference is good inference

3. When we are not sure which of various alternative assumptions is the most appropriate for a problem, we can treat this question as another inference task.

   Given data D, we can compare alternative assumptions $\mathcal{H}$ using

   $$P(\mathcal{H} \mid D, I) = \frac{P(\mathcal{H} \mid I) P(D \mid \mathcal{H}, I)}{P(D \mid I)},$$

   where I denotes the highest assumptions, which we are not questioning.

4. We can take into account our uncertainty about the assumptions themselves when making subsequent predictions.

   Rather than choosing one particular assumption $\mathcal{H}^*$, and predicting quantity $\mathbf{t}$ using $P(t \mid D, \mathcal{H}^*, I)$, we can do

   $$P(\mathbf{t} \mid D, I) = \sum_{\mathcal{H}} P(\mathcal{H} \mid D, I) P(\mathbf{t} \mid D, \mathcal{H}, I).$$

Model comparison

- Assume we have two competing models for a phenomenon, represented by hypothesis $\mathcal{H}_1$ and hypothesis $\mathcal{H}_2$, and that we run an experiment and observe data D.

  The best model ($\mathcal{H}_1$ or $\mathcal{H}_2$) fitting the data (D) is indicated by the ratio:

$$\begin{aligned}\frac{p(\mathcal{H}_1 \mid D)}{p(\mathcal{H}_2 \mid D)} &= \frac{p(\mathcal{H}_1)p(D \mid \mathcal{H}_1)}{p(D)} \cdot \frac{p(D)}{p(\mathcal{H}_2)p(D \mid \mathcal{H}_2)} \\ &= \frac{p(\mathcal{H}_1)}{p(\mathcal{H}_2)} \cdot \frac{p(D \mid \mathcal{H}_1)}{p(D \mid \mathcal{H}_2)},\end{aligned}$$

  where

  - $\frac{p(\mathcal{H}_1)}{p(\mathcal{H}_2)}$ is the **ratio of the prior probabilities** of the competing models, and
  - the **likelihood ratio** $\frac{p(D \mid \mathcal{H}_1)}{p(D \mid \mathcal{H}_2)}$ is the ratio of the **evidence for each model**.

Inference - An example of legal evidence

- $\boxed{\text{Example 5}}$ Two people have left traces of their own blood at the scene of a crime.

  A suspect, Oliver, is tested and found to have type “O” blood.

  The blood groups of the two traces are found to be:

  - of type “O” (a common type in the local population, having frequency 60%) and,
  - of type “AB” (a rare type, with frequency 1%).

  Do these data (type “O” and “AB” blood were found at the scene) give evidence in favor of the proposition that Oliver was one of the two people present at the crime?

Inference - An example of legal evidence

- $\boxed{\text{Example 5}}$ (Continued)

Solution.

- A careless lawyer might claim that the fact that the suspect's blood type was found at the crime scene is evidence for the theory that he was present.

  This is wrong.

- Let S be the proposition *"the suspect and one unknown person were present"* and the alternative proposition $\overline{S}$ be *"two unknown people from the population were present"*.

- Let D be the data collected (i.e., there is one sample of blood type "O" and one of blood type "AB" in the scene), and let $\mathcal{H}$ be the set of underlying hypotheses.

- We want to evaluate the contribution the data D gives as evidence for each supposition (S or $\overline{S}$), i.e., the ratio

$$\frac{p(D \mid S, \mathcal{H})}{p(D \mid \overline{S}, \mathcal{H})}.$$

Inference - An example of legal evidence

- $\boxed{\text{Example 5}}$ (Continued)

  - Let us call p_{AB} and p_O the probabilities of a person in the general population having blood type "AB" and "O", respectively.
  - We know that

    $$p(D \mid S, \mathcal{H}) = p_{AB},$$

    since given S, we already know that one trace will be of type "O".
  - We also know that

    $$p(D \mid \overline{S}, \mathcal{H}) = 2 p_O p_{AB}.$$

  - Hence, the ratio of the evidences is

    $$\frac{p(D \mid S, \mathcal{H})}{p(D \mid \overline{S}, \mathcal{H})} = \frac{1}{2p_O} = \frac{1}{2 \cdot 0.6} = 0.83,$$

    so the data provides <u>evidence against the supposition</u> that the suspect, Oliver, was present.

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Inference - An example of legal evidence

- $\boxed{\text{Example 6}}$ Assume the same scenario of the example above, but now there is another suspect, Alberto, whose blood type is "AB".

  Does the evidence found (one sample of blood type "O" and one sample of blood type "AB") favor the proposition that Alberto was one of the two people present at the crime?

Inference - An example of legal evidence

- Example 6 (Continued)

Solution.

- Intuitively, it may seem that the data does provide evidence that Alberto was at the crime scene.

  But let's be careful and do the right calculations.

- Let S' be the proposition *"the suspect (Alberto) and one unknown person were present"* and the alternative proposition $\overline{S}$ be *"two unknown people from the population were present"*.

- We know that

$$p(D \mid S', \mathcal{H}) = p_O,$$

  since given S', we already know that one trace will be of type "AB".

- The ratio of the evidences is

$$\frac{p(D \mid S', \mathcal{H})}{p(D \mid \overline{S}, \mathcal{H})} = \frac{p_O}{2 p_O p_{AB}} = \frac{1}{2 p_{AB}} = \frac{1}{2 \cdot 0.01} = 50,$$

  so the data provides evidence for the supposition that Alberto was present.

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Inference - An example of legal evidence

- Example 7 Assume the same scenario of the example above, but now consider that 99% of the population is of blood type "O", and 1% is of blood type "AB" (and that no other blood type exists).

  Intuitively the evidence found (one sample of blood type "O" and one sample of blood type "AB") favor the proposition that Alberto was one of the two people present at the crime.

  On the other hand, is it reasonable to assume that the evidence found favor the proposition that Oliver was one of the two people present at the crime?

Inference - An example of legal evidence

- $\boxed{\text{Example 7}}$ (Continued)

  Solution.

  - Intuitively (since there exist only blood types "AB" and "O") the the evidence cannot increase the probability that both a person of blood type "AB" (such as Alberto) and a person of blood type "O" (such as Oliver) were present.

    That would mean that the evidence increases the probability of all suspects, no matter whom, being at the crime scene, which is nonsense.

  - In other words:

    $\underline{\text{data may be compatible with several theories}}$, but

    $\underline{\text{if some data provides evidence for some theories,}}$
    $\underline{\text{it necessarily provides evidence against some other theories.}}$

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Inference - An example of legal evidence

- $\boxed{\text{Example 8}}$ Assume once again the same scenario of the example above.

  Consider that were found at the crime scene

  - n_O blood stains of individuals of type "O", and
  - n_{AB} blood stains of individuals of type "AB"

  from a total of N blood samples found in total.

  Consider that the frequencies of blood type "O" in the general population is p_O, and the frequency of blood type "AB" in the general population is p_{AB} (there may be other blood types too).

  Find a closed formula for the likelihood ratio of the following two competing hypotheses

  - S: *"the type "O" suspect (Oliver) and $N-1$ unknown others left N stains"*, and
  - $\overline{S}$: *"N unknowns left N stains"*.

Inference - An example of legal evidence

- $\boxed{\text{Example 8}}$ (Continued)

Solution.

- For hypothesis $\overline{S}$ we have

$$P(n_O, n_{AB} \mid \overline{S}) = \frac{N!}{n_O! n_{AB}!} p_O^{n_O} p_{AB}^{n_{AB}}.$$

- For hypothesis S we have

$$P(n_O, n_{AB} \mid S) = \frac{(N-1)!}{(n_O - 1)! n_{AB}!} p_O^{n_O - 1} p_{AB}^{n_{AB}},$$

since because Oliver is fixed, we only need the distribution of the other $N-1$ individuals.

Inference - An example of legal evidence

- $\boxed{\text{Example 8}}$ (Continued)

  - The likelihood ratio is

$$\frac{P(n_O, n_{AB} \mid S)}{P(n_O, n_{AB} \mid \overline{S})} = \frac{(N-1)! p_O^{n_O - 1} p_{AB}^{n_{AB}}}{(n_O - 1)! n_{AB}!} \cdot \frac{n_O! n_{AB}!}{N! p_O^{n_O} p_{AB}^{n_{AB}}}$$
$$= \frac{n_O/N}{p_O}.$$

  - This likelihood ration means that the evidence depends on a comparison on the frequency of Oliver's blood type in the data (n_O/N), and the frequency of Oliver's blood type in the general population (p_O):

    1. If Oliver's blood type is more frequent in the data than in the population, the evidence counts for his presence at the crime scene.
    2. If Oliver's blood type is less frequent in the data than in the population, the evidence counts for his absence from the crime scene.
    3. If Oliver's blood type is as frequent in the data than as it is in the population, the evidence is neutral for his presence in the crime scene.

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Model comparison - Examples

- Example 9 A die is selected at random from two twenty-faced dice on which the symbols 1-10 are written with nonuniform frequency as follows.

Symbol	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Number of faces of die A	6	4	3	2	1	1	1	1	1	0
Number of faces of die B	3	3	2	2	2	2	2	2	1	1

The randomly chosen die is rolled 7 times, with the following outcomes:

$$5, 3, 9, 3, 8, 4, 7.$$

Is this sequence evidence in favor of the die being die A or die B?

Model comparison - Examples

- $\boxed{\text{Example 9}}$ (Continued)

The observed data D is the sequence of outcomes 3, 9, 3, 8, 4, 7.

The concurrent hypotheses are whether the die is A or B.

We can, then, compute the ratio of posterior probabilities:

$$\begin{aligned}\frac{p(A \mid D)}{p(B \mid D} &= \frac{p(A)}{p(B)} \cdot \frac{p(D \mid A)}{p(D \mid B)} \\ &= \frac{1/2}{1/2} \cdot \frac{1/20 \cdot 3/20 \cdot 1/20 \cdot 3/20 \cdot 1/20 \cdot 2/20 \cdot 1/20}{2/20 \cdot 2/20 \cdot 1/20 \cdot 2/20 \cdot 2/20 \cdot 2/20 \cdot 2/20} \\ &= \frac{9}{32}\end{aligned}$$

Since the ratio is smaller than 1, the evidence favors the die being die B.

(Actually, knowing that $p(A \mid D) + p(B \mid D) = 1$, we can compute

$$p(A \mid D) = 9/41, \quad \text{and} \quad p(B \mid D) = 32/41.)$$

Model comparison - Examples

- $\boxed{\text{Example 10}}$ Consider again the same dice A and B from the previous example, but assume now that there is a third twenty-faced die, die C, on which the symbols 1-20 are written once each.

  As above, one of the three dice is selected at random and rolled 7 times, giving the outcomes:

$$3, 5, 4, 8, 3, 9, 7.$$

  What is the probability that the die is

  a) die A?

  b) die B?

  c) die C?

Model comparison - Examples

- $\boxed{\text{Example 10}}$ (Continued)

Solution.

We can compute the quantities directly using the formulas

$$p(A \mid D) = \frac{p(A)p(D \mid A)}{p(D)}, \quad p(B \mid D) = \frac{p(B)p(D \mid B)}{p(D)}, \text{ and } \quad p(C \mid D) = \frac{p(C)p(D \mid C)}{p(D)}.$$

First, let us compute

$$p(D \mid A) = \frac{3}{20} \cdot \frac{1}{20} \cdot \frac{2}{20} \cdot \frac{1}{20} \cdot \frac{3}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} = \frac{18}{20^7}$$

$$p(D \mid B) = \frac{2}{20} \cdot \frac{2}{20} \cdot \frac{2}{20} \cdot \frac{2}{20} \cdot \frac{2}{20} \cdot \frac{1}{20} \cdot \frac{2}{20} = \frac{64}{20^7}$$

$$p(D \mid C) = \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} = \frac{1}{20^7}$$

Model comparison - Examples

- $\boxed{\text{Example 10}}$ (Continued)

  Let us assume that all dice are equally likely, i.e.,

$$p(A) = p(B) = p(C) = {}^{1}\!/\!_{3}.$$

  Then, we can compute

$$\begin{aligned} p(D) &= p(A)p(D \mid A) + p(B)p(D \mid B) + p(C)p(D \mid C) \\ &= \frac{1}{3} \cdot \frac{18}{20^7} + \frac{1}{3} \cdot \frac{64}{20^7} + \frac{1}{3} \cdot \frac{1}{20^7} \\ &= \frac{83}{3 \cdot 20^7} \end{aligned}$$

Model comparison - Examples

- $\boxed{\text{Example 10}}$ (Continued)

  And, finally, we can compute:

$$p(A \mid D) = \frac{p(A) \cdot p(D \mid A)}{p(D)} = \frac{{}^{1}\!/_{3} \cdot {}^{18}\!/_{20^7}}{{}^{83}\!/_{(3 \cdot 20^7)}} = \frac{18}{83}$$

$$p(B \mid D) = \frac{p(B) \cdot p(D \mid B)}{p(D)} = \frac{{}^{1}\!/_{3} \cdot {}^{64}\!/_{20^7}}{{}^{83}\!/_{(3 \cdot 20^7)}} = \frac{64}{83}$$

$$p(C \mid D) = \frac{p(C) \cdot p(D \mid C)}{p(D)} = \frac{{}^{1}\!/_{3} \cdot {}^{1}\!/_{20^7}}{{}^{83}\!/_{(3 \cdot 20^7)}} = \frac{1}{83}$$

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