Class #7 – part 2

Introduction no Noncooperative Game Theory: Other solution concepts

Other solution concepts

- We reason about multiplayer games using solution concepts
  - interesting subsets of the outcomes of a game
- While the most important solution concept is the Nash equilibrium, there are also a large number of others

Other solution concepts

- Maxmin and minmax strategies
- Minimax regret
- Removal of dominated strategies
- Correlated Equilibrium
- Trembling-hand perfect equilibrium
- ξ-Nash equilibrium

Maxmin and minmax strategies

- Security level

  The maxmin strategy of player ***i***

  in an n-player, general-sum game

  is a strategy that maximizes ***i***'s worst-case payoff,

  in the situation where all the other players happen to play the strategies which cause the greatest harm to ***i***

  - Minimum amount of payoff guaranteed by a maxmin strategy

Maxmin and minmax strategies

- The maxmin strategy for player ***i*** is

$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- and the maxmin value for player ***i*** is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

Maxmin and minmax strategies

- The maxmin strategy is ***i***'s best choice when first ***i*** must commit to a strategy,
- and then the remaining agents ***−i*** observe this strategy - but not ***i***'s action choice -
- and choose their own strategies to minimize ***i***'s expected payoff

Maxmin and minmax strategies

- What is the maxmin strategy?

Husband / Wife	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

Maxmin and minmax strategies

- What is the maxmin strategy?

  Wife's strategy $s_w = (\boldsymbol{p}, 1 - \boldsymbol{p})$

  Husband's strategy $s_h = (\boldsymbol{q}, 1 - \boldsymbol{q})$

  $\boldsymbol{u_w(p,q) = 2pq + (1 - p)(1 - q) = 3pq - p - q + 1}$

  For any fixed $\boldsymbol{p}$, $u_w(\boldsymbol{p},\boldsymbol{q})$ is linear in $\boldsymbol{q}$

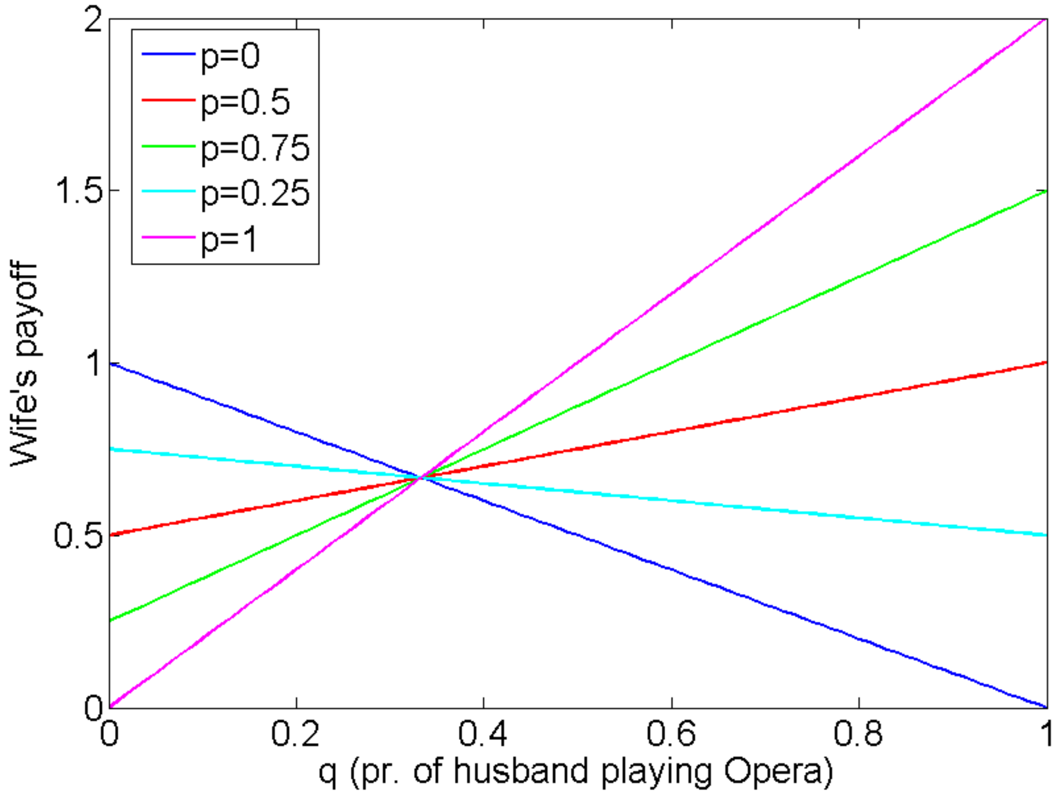
		q	$1-q$
	Husband / Wife	Opera	Football
p	Opera	2, 1	0, 0
$1-p$	Football	0, 0	1, 2

Maxmin and minmax strategies

- What is the maxmin strategy?

$$u_w(p,q) = 3pq - p - q + 1$$

For any fixed p, $u_w(p,q)$ is linear in q

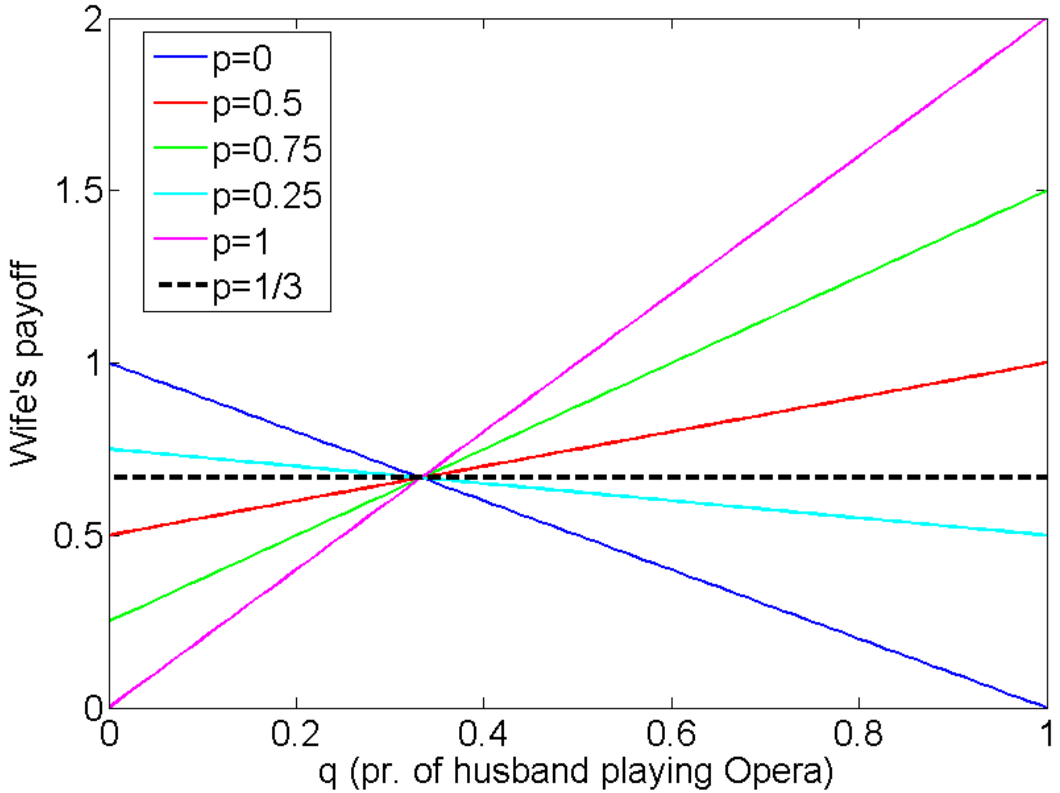


Maxmin and minmax strategies

- What is the maxmin strategy?

$u_w(p,q) = 3pq - p - q + 1$

For any fixed p, $u_w(p,q)$ is linear in q



Maxmin and minmax strategies

Wife \ Husband	Opera (q)	Football ($1-q$)
Opera (p)	2, 1	0, 0
Football ($1-p$)	0, 0	1, 2

- What is the maxmin strategy?

  Wife's strategy $s_w = (p, 1 - p)$

  Husband's strategy $s_h = (q, 1 - q)$

  $u_w(p,q) = 2pq + (1 - p)(1 - q) = 3pq - p - q + 1$

  For any fixed p, $u_w(p,q)$ is linear in q and

  since $0 \le q \le 1$, the *min* must be at $q = 0$ or $q = 1$

  $min_q u_w(p,q) = min\,(u_w(p,0), u_w(p,1))$

  $= min(1 - p, 2p)$

Maxmin and minmax strategies

- What is the maxmin strategy?

  Since

  $\min_q u_w(p,q) = \min(1 - p, 2p)$

  Now we have to find

  $\arg\max_p(\min(1 - p, 2p))$

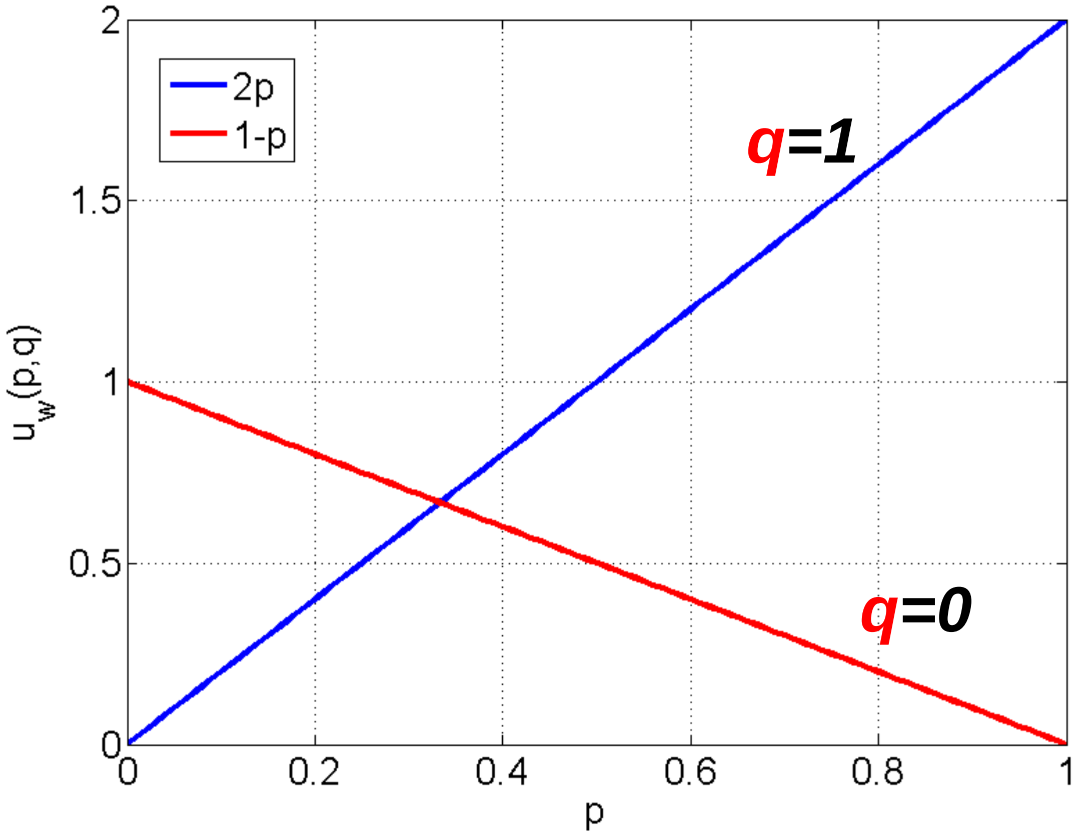
Wife \ Husband		Opera (q)	Football ($1-q$)
p	Opera	2, 1	0, 0
$1-p$	Football	0, 0	1, 2

Maxmin and minmax strategies

- What is the maxmin strategy?

$$\arg\max_p(\min(1 - p, 2p))$$

		q	$1-q$
	Husband / Wife	Opera	Football
p	Opera	2, 1	0, 0
$1-p$	Football	0, 0	1, 2

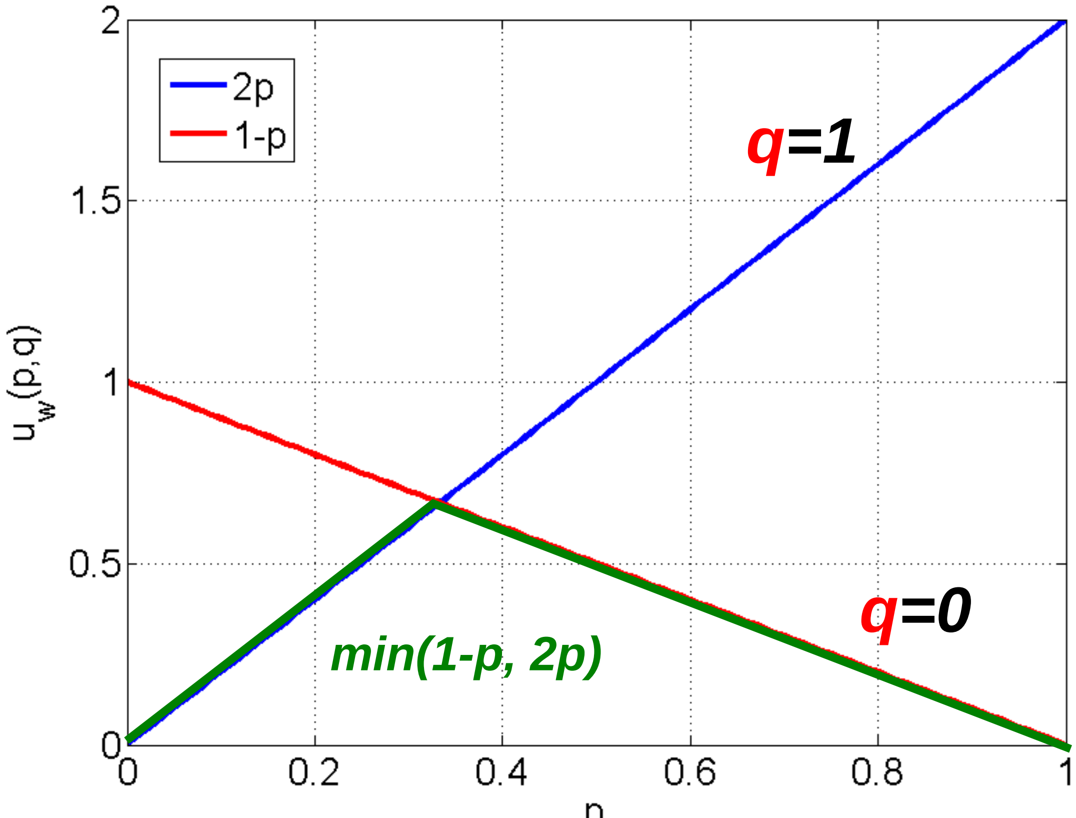


Maxmin and minmax strategies

- What is the maxmin strategy?

$$\arg\max_p(\min(1-p, 2p))$$

		q	$1-q$
	Husband / Wife	Opera	Football
p	Opera	2, 1	0, 0
$1-p$	Football	0, 0	1, 2



Maxmin and minmax strategies

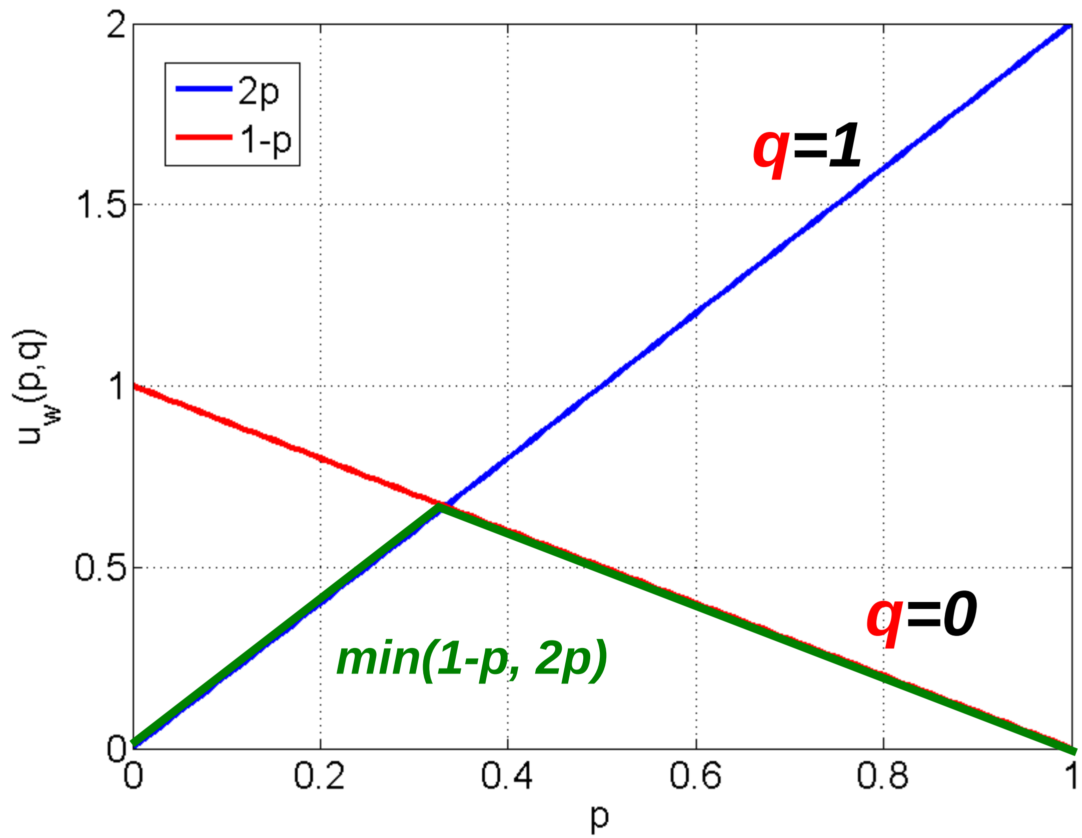
- What is the maxmin strategy?

  arg max$_p$(min(1 – p, 2p))

  2p = 1-p

  p = 1/3

		q	*1-q*
	Wife \ Husband	Opera	Football
p	Opera	2, 1	0, 0
1-p	Football	0, 0	1, 2



Maxmin and minmax strategies

- Why might an agent ***i*** want to use a **maxmin** strategy?

		Hunter 2	
		Hunt Stag	*Hunt Hare*
Hunter 1	*Hunt Stag*	4, 4	0, 3
	Hunt Hare	3, 0	3, 3

Figure 6.10: Stag Hunt

Maxmin and minmax strategies

- Why might an agent ***i*** want to use a **maxmin** strategy?
  - Useful if ***i*** is cautious (wants to maximize his/her worst-case utility) and doesn’t have any information about the other agents
    - whether they are rational
    - what their payoffs are
    - whether they draw their action choices from known distributions
  - Useful if ***i*** has reason to believe that the other agents’ objective is to minimize ***i***’s expected utility

Maxmin and minmax strategies

- The minmax strategy and minmax value play a dual role to their maxmin counterparts
- In two-player games the minmax strategy for player ***i*** against player ***−i*** is a strategy that keeps the maximum payoff of ***−i*** at a minimum
  - The minmax value of player ***−i*** is that minimum
- The amount that one player can punish another without regard for his own payoff

Maxmin and minmax strategies

- In a two-player game, the **minmax strategy** for player ***i*** against player ***−i*** is

$$\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

- Player ***−i***'s **minmax value** is

$$\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

Maxmin and minmax strategies

- In n-player games with ***n > 2***, defining player ***i***'s minmax strategy against player ***j*** is a bit more complicated
  - Why?
  - ***i*** will not usually be able to guarantee that ***j*** achieves minimal payoff by acting unilaterally
- However, if we assume that all ***-i*** players <u>choose</u> to "gang up" on ***j***, then we can define minmax strategies for the n-player case

Maxmin and minmax strategies

- In an n-player game, the **minmax strategy** for player ***i*** against player ***j*** **≠** ***i*** is ***i***'s component of the mixed-strategy profile s_{-j} in the expression

$$\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$$

- where ***−j*** denotes the set of players other than ***j***
- The **minmax value** for player ***j*** is

$$\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$$

Maxmin and minmax strategies

- **Minmax Theorem** (von Neumann, 1928)
  - Let ***G*** be any finite two-player zero-sum game
  - For each player ***i***, ***i***'s expected utility in any Nash equilibrium

    = ***i***'s maxmin value

    = ***i***'s minmax value

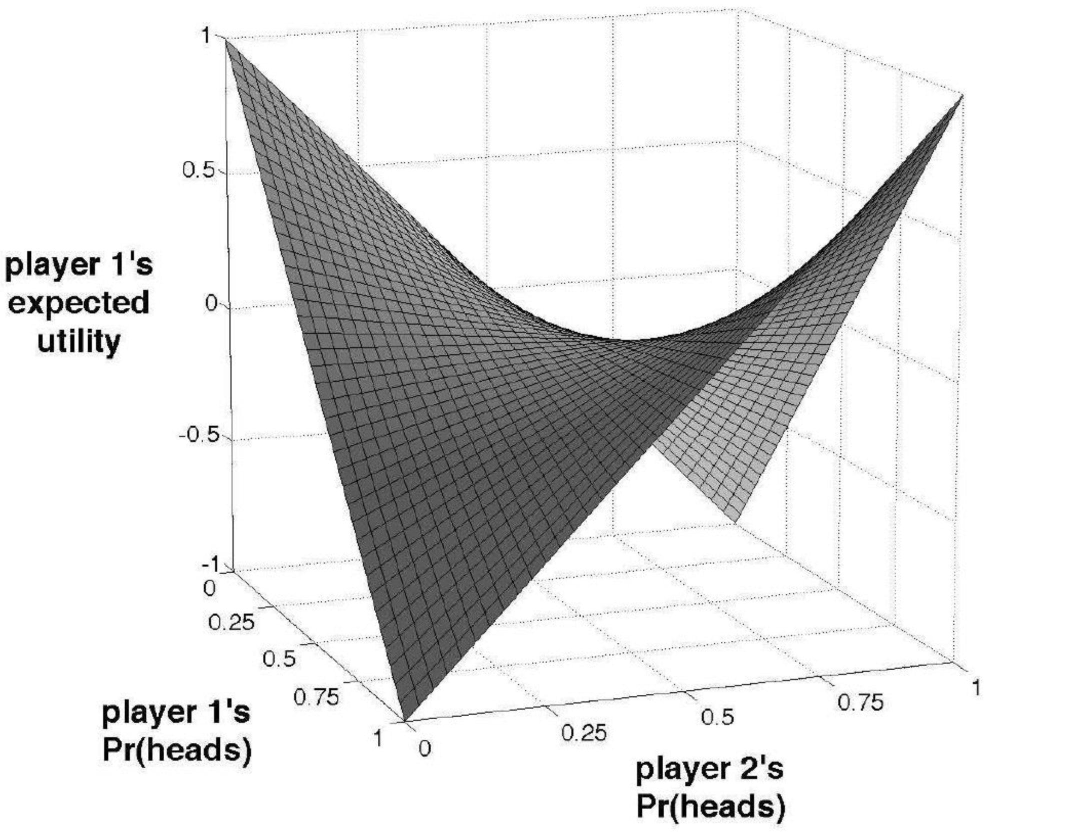
Maxmin and minmax strategies

- In two players zero sum game
  - The maxmin value for player ***1*** is called the value of the game
  - For both players, the set of maxmin strategies coincides with the set of minmax strategies
  - Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium

Maxmin and minmax strategies

- Are there any other Nash Equilibria besides (1/2, 1/2)?

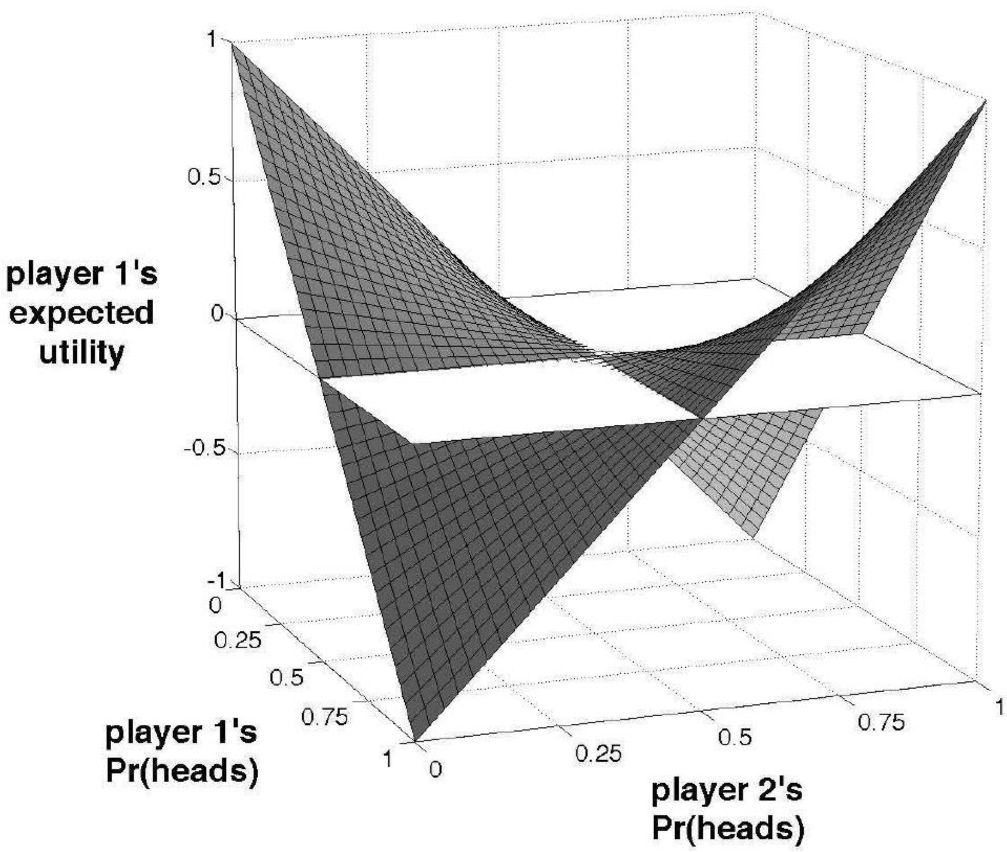
	Heads	Tails
Heads	1	−1
Tails	−1	1



Maxmin and minmax strategies

- Saddle point

	Heads	Tails
Heads	1	−1
Tails	−1	1



Maxmin and minmax strategies

- **Theorem (Julia Robinson) [Robinson 1951]**: In two players zero-sum repeated games in normal form
- if in each round each player chooses the best response pure strategy against the observed mixed strategy of the total history of the other player
- then the mixed strategies of the whole history converge to a pair of mixed strategies forming a Nash equilibrium

Maxmin and minmax strategies

- How does it work in non-zero sum games?

	a	b	c
a	1, 1	3, 3	0, 1
b	3, 1	3, 3	3, 4
c	3, 3	0, 3	4, 0

Is it a Nash equilibrium?

Maxmin and minmax strategies

- How does it work in non-zero sum games?

	a	b	c	$trick$
a	$1, 1$	$3, 3$	$0, 1$	$-\epsilon, 0$
b	$3, 1$	$3, 3$	$3, 4$	$-\epsilon, 0$
c	$3, 3$	$0, 3$	$4, 0$	$-\epsilon, 0$
$trick$	$0, -\epsilon$	$0, -\epsilon$	$0, -\epsilon$	$\boxed{0, 0}$

Maxmin and minmax strategies

- Let's play another game...

	L	R
T	***100, 2***	***0.9, 5***
B	***2, 1***	***1, 0***

Maxmin and minmax strategies

- Let's play another game...
  - What are the maxmin and minmax strategies?

	L	R
T	***100, 2***	***0.9, 5***
B	***2, 1***	***1, 0***

Maxmin and minmax strategies

- Let's play another game...
  - What are the maxmin and minmax strategies?

	L	R
T	$100, a$	$1-\epsilon, b$
B	$2, c$	$1, d$

Regret

- An agent ***i***'s **regret** for playing an action $\boldsymbol{a_i}$ if the other agents adopt action profile $\boldsymbol{a_{-i}}$ is defined as

$$\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i})\right] - u_i(a_i, a_{-i})$$

- The amount that ***i*** loses by playing $\boldsymbol{a_i}$, rather than playing his best response to $\boldsymbol{a_{-i}}$
- Considers those actions that would give him the highest regret for playing $\boldsymbol{a_i}$

Max regret

- An agent ***i***'s **maximum regret** for playing an action $\boldsymbol{a_i}$ is defined as

$$\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

- This is the amount that ***i*** loses by playing $\boldsymbol{a_i}$ rather than playing his best response to $\boldsymbol{a_{-i}}$, if the other agents chose the $\boldsymbol{a_{-i}}$ that makes this loss as large as possible

Minimax regret

- **Minimax regret** actions for agent ***i*** are defined as

$$\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right]$$

- An action that yields the smallest maximum regret

Minimax regret

$$\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right]$$

	L	R
T	***100, a***	***0.5, b***
B	***2, c***	***1, d***

Minimax regret

$$\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \boxed{\left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)} \right]$$

	L	R
T	**0, a**	**0.5, b**
B	**98, c**	**0, d**

Minimax regret

$$\arg\min_{a_i \in A_i} \boxed{\left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i})\right] - u_i(a_i, a_{-i}) \right)\right]}$$

	L	R
T	**0, a**	**0.5, b**
B	**98, c**	**0, d**

Minimax regret

$$\boxed{\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right]}$$

	L	R
T	***0*, a**	***<u>0.5</u>*, b**
B	***<u>98</u>*, c**	***0*, d**

Minimax regret

- **Minimax regret** can be extended to a solution concept in the natural way
  - Identification of action profiles that consist of minimax regret actions for each player
  - Note that we can safely restrict ourselves to actions rather than mixed strategies because of the linearity of expectation

Minimax regret

- Who in the world plays minimax regret?
  - I don't care about receiving my worst-case payoff, I care about not receiving my best-case payoff

Minimax regret

- What is the strategical flaw in playing minimax regret?
  - Players don't care about other players' payoffs

Another game to be played

- Two firms that are each planning to produce and market a new product
- Two market segments
  - people who would only buy a low-priced version of the product
  - people who would only buy an upscale version
- the profit any firm makes on a sale of either a low price or an upscale product is the same
- Each firm wants to maximize its profit

Another game to be played

- People who would prefer a low-priced version account for 60% of the population
- People who would prefer an upscale version account for 40% of the population
- Firm 1 is the much more popular brand
  - Firm 1 gets 80% of the sales and Firm 2 gets 20% of the sales of the same product
- If a firm is the only one to produce a product for a given market segment, it gets all the sales

Another game to be played

- If the two firms market to different market segments, they each get all the sales in that segment
  - So the one that targets the low-priced segment gets a payoff of .6 and the one that targets the upscale segment gets .4.
- If both firms target both segments
  - For the low-priced segment, Firm 1 gets 80% of it, for a payoff of (.8)(.6)=.48, and Firm 2 gets 20% of it, for a payoff of .(.2)(.6)=.12
  - For the upscale segment, Firm 1 gets a payoff of (.8)(.4) = .32 and Firm 2 gets (.2)(.4) = .08

Another game to be played

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Figure 6.5: Marketing Strategy

A Game in Which Only One Player Has a Strictly Dominant Strategy

- The players have common knowledge of the game
  - they know the structure of the game, they know that each of them know the structure of the game, they know that each of them know that each of them know, and so on

Removal of dominated strategies

- Let s_i and s_i' be two strategies of player i, and S_{-i} the set of all strategy profiles of the remaining players. Then
- s_i **strictly dominates** s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- s_i **weakly dominates** s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, and for at least one $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

Removal of dominated strategies

- A strategy is **strictly** (resp., **weakly**) **dominant** for an agent if it **strictly** (**weakly**) dominates **any other** strategy for that agent
  - Is a strategy profile $(s_1, . . . , s_n)$ in which every s_i is dominant for player i (whether strictly, weakly, or very weakly) a Nash equilibrium?
    - ***equilibrium in (strictly, weakly) dominant strategies***
  - Is an equilibrium in strictly dominant strategies necessarily the unique Nash equilibrium?
  - Is it Pareto optimal?

Removal of dominated strategies

- Dominant strategies are rare, but dominated strategies are not
- A strategy s_i is **strictly** (**weakly**) **dominated** for an agent i if some other strategy s_i' **strictly** (**weakly**) dominates s_i

A Game in Which Only One Player Has a Strictly Dominant Strategy

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Figure 6.5: Marketing Strategy

Strict dominant strategy for Firm 1
(all first row payoffs are > second row payoffs)

A Game in Which Only One Player Has a Strictly Dominant Strategy

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Figure 6.5: Marketing Strategy

Firm 2 knows Firm 1 is going to pick Low-Priced.
What should it do, then?

A Game in Which Only One Player Has a Strictly Dominant Strategy

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	$\boxed{.60, .40}$
	Upscale	.40, .60	.32, .08

Figure 6.5: Marketing Strategy

Firm 2 knows Firm 1 is going to pick Low-Priced.
What should it do, then?
Upscale is its **best response** when Firm 1 plays Low Priced

A Game in Which Only One Player Has a Strictly Dominant Strategy

- Both firms are developing their marketing strategies concurrently and in secret
- The intuitive message of this prediction
  - Firm 1 is so strong that it can proceed without regard to Firm 2's decision
  - Firm 2's best strategy is to stay safely out of the way of Firm 1

Another solution concept

- Let's play another game...

	L	*C*	*R*
U	3,1	0,3	0,0
M	1,5	1,1	10,0
B	0,½	4,2	5,0

Another solution concept

- Let's play a game...

	L	***C***	***R***
U	3,1	0,3	0,0
M	1,5	1,1	10,0
B	0,½	4,2	5,0

Another solution concept

- Let's play a game...

	L	*C*
U	3,1	0,3
M	1,5	1,1
B	0,½	4,2

Another solution concept

- Let's play a game...

	L	*C*
U	3,1	0,3
M	1,5	1,1
B	0,½	4,2

Why is the action dominated?

Another solution concept

- Let's play a game...

	L	***C***
U ***(p)***	3,1	0,3
M	1,5	1,1
B ***(1-p)***	0,½	4,2

Why is the action dominated?

When P2 plays **L**:
3p+0(1-p) > 1, p>1/3

When P2 plays **C**:
0p+4(1-p) > 1, p<3/4

So, for all ***1/3<p<3/4***, **M** should never be played, regardless of what P2 does

Another solution concept

- Let's play a game...

	L	***C***
U	3,1	0,3
B	0,½	4,2

Another solution concept

- Let's play a game...

	L	*C*
U	3,1	0,3
B	0,½	4,2

Another solution concept

- Let's play a game...

	C
U	0,3
B	4,2

Another solution concept

- Let's play a game...

	C
U	0,3
B	4,2

Another solution concept

- Let's play a game...

	C
B	4,2

Removal of dominated strategies

- Games solvable by iterated elimination of dominated strategies
  - Might the order of elimination affect the final outcome?
    - for strictly dominated strategies, no!
      - **Church–Rosser property**
    - for weakly dominated strategies, yes

Removal of dominated strategies

- Experiment by B.A. Baldwin and G.B. Meese (1979) “Social Behavior in Pigs Studied by Means of Operant Conditioning,” Animal Behavior, Vol 27, pp 947-957
- Two pigs in a cage, one is larger: “dominant’’ (sorry for the terminology...)
- need to press a lever to get food to arrive
- food and lever are at opposite sides of cage
- run to press and the other pig gets the food

Removal of dominated strategies

- 10 units of food - the typical split:
  - if large gets to food first then ***(1,9)*** split
    - 1 for small, 9 for large
  - if small gets to food first then ***(4,6)*** split
  - if get to food at the same time then ***(3,7)*** split
- Pressing the lever costs ***2*** units of food in energy

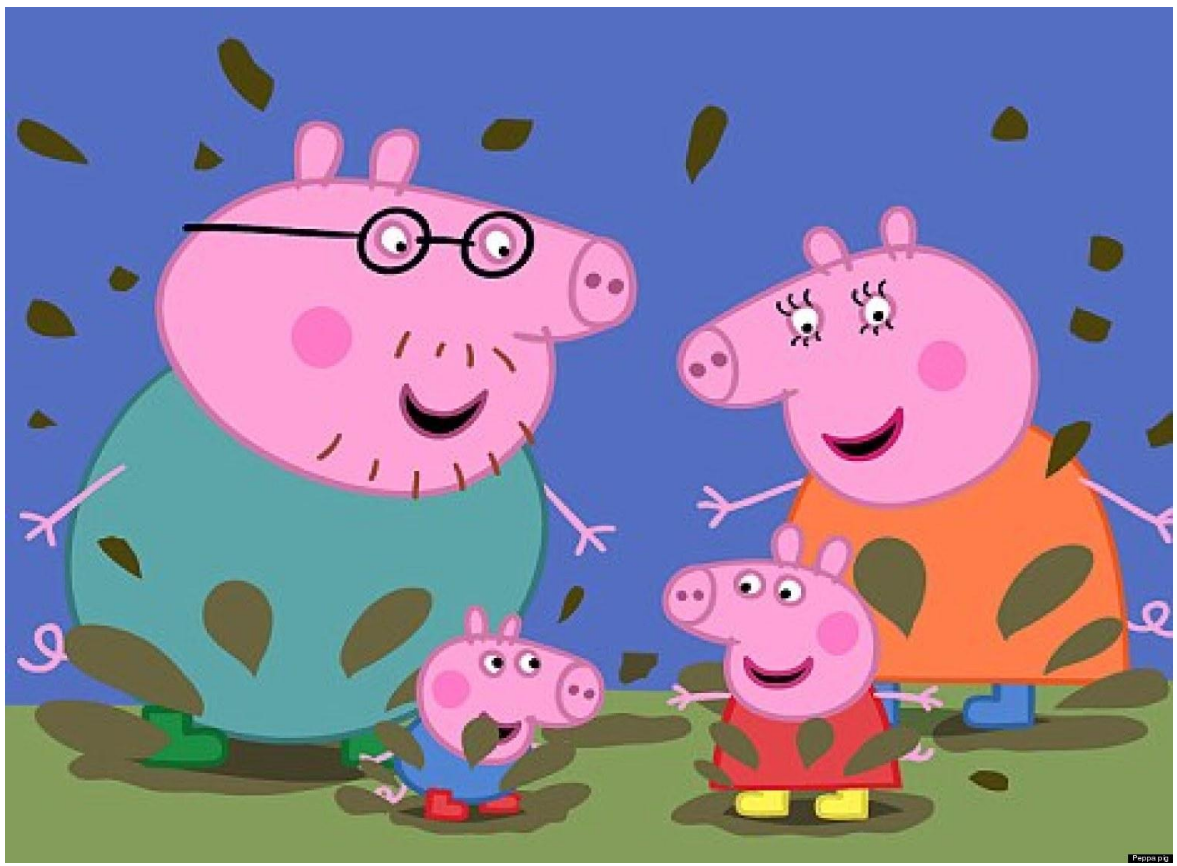
Removal of dominated strategies

- Payoff matrix

$Small/Large$	$Press$	
$Wait$	$4, 4$	

Removal of dominated strategies

- Do pigs eliminate their dominated strategies?



Removal of dominated strategies

- Pigs Behavior: Frequency of pushing the lever per 15 minutes, after ten tests (learning...)

	$Alone$	
$LargePigs$	75	
$SmallPigs$	70	

Battle of Sexes: replay

- What is the Nash Equilibrium for this game?
  - s = (2/3, 1/3), u(s) = (2/3, 2/3)

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

Battle of Sexes: replay

- Alternative setting
  - I will flip a coin and let you know the result
  - You can communicate before the coin toss

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

Correlated Equilibrium

- If ***heads***, then both play ***Opera***
- If ***tails***, then both play ***Football***
- What is the expected payoff for each player?
  - (3/2, 3/2) > (2/3, 2/3)

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

Correlated Equilibrium

- "If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium"
- Roger Myerson (Nobel prize winner)

Correlated Equilibrium

- Two animals are engaged in a contest to decide how a piece of food will be divided
- Each animal can choose to behave aggressively (the Hawk strategy) or passively (the Dove strategy)
- If the two animals both behave passively, they divide the food evenly
- If one behaves aggressively while the other behaves passively, then the aggressor gets most of the food, while the passive one gets a little
- If both animals behave aggressively, then they destroy the food (and possibly injure each other)

Correlated Equilibrium

- How the payoff matrix would look like for this game?

Correlated Equilibrium

- How the payoff matrix would look like for this game?

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

Figure 6.12: Hawk-Dove Game

Correlated Equilibrium

- Let's play it!

		Animal 2: D	Animal 2: H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

Figure 6.12: Hawk-Dove Game

Correlated Equilibrium

- Is there any pure strategy Nash equilibria?
  - YES

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$\boxed{1, 5}$
	H	$\boxed{5, 1}$	$0, 0$

Figure 6.12: Hawk-Dove Game

Correlated Equilibrium

- Is there any mixed strategy Nash equilibria?
  - Of course!
  - $3q + 1 - q = 5q$, $q = 1/3$
  - $s = (1/3, 1/3)$, $u(s) = (5/3, 5/3)$

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

Figure 6.12: Hawk-Dove Game

Correlated Equilibrium

- Another example: the game of chicken

Correlated Equilibrium

- Hawk or Dove (game of chicken)
  - Very hard to predict!
    - Ultimate example: countries at war
    - If a country is being aggressive, the best response of the other is to be passive
    - Where else?
  - Another name: Snowdrift Dilemma

Correlated Equilibrium

- Let's play an alternative setting...
  - I have three cards, of which I will randomly pick one
  - Then, I will secretly tell you what you should do

d, d	h, d	d, h

	D	H
D	3,3	1,5
H	5,1	-2,-2

Correlated Equilibrium

- Let's play an alternative setting...
  - I have three cards, of which I will randomly pick one
  - Then, I will secretly tell you what you should do
  - The expected payoff is:
    - $u(s) = 1/3 * 3 + 1/3 * 5 + 1/3 * 1 = 3 > 5/3$

d, d	h, d	d, h

	D	*H*
D	3,3	1,5
H	5,1	-2,-2

Correlated Equilibrium

- What would have happened if I had a single card?

d, d

	D	H
D	3,3	1,5
H	5,1	-2,-2

Correlated Equilibrium

- Given an n-agent game $\boldsymbol{G = (N,A,u)}$, a **correlated equilibrium** is a tuple $\boldsymbol{(v, \pi, \sigma)}$,
- where $\boldsymbol{v}$ is a tuple of random variables $\boldsymbol{v = (v_1, \ldots, v_n)}$ with respective domains $\boldsymbol{D = (D_1, \ldots, D_n)}$,
- $\boldsymbol{\pi}$ is a joint distribution over $\boldsymbol{v}$, $\boldsymbol{\sigma = (\sigma_1, \ldots, \sigma_n)}$ is a vector of mappings $\boldsymbol{\sigma_i : D_i \rightarrow A_i}$,
- and for each agent $\boldsymbol{i}$ and every mapping $\boldsymbol{\sigma_i' : D_i \rightarrow A_i}$ it is the case that

$$\sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right)$$

$$\geq \sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i'(d_i), \ldots, \sigma_n(d_n)\right)$$

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}, D_2 = \{d, h\}$

$v_1 = [⅔, ⅓], v_2 = [⅔, ⅓]$

$\pi =$

	d	*h*
d	1/3	1/3
h	1/3	0

$\sigma_1 =$

D_1	A_1
d	***D***
h	***H***

	D	*H*
D	3,3	1,5
H	5,1	-2,-2

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}, D_2 = \{d, h\}$

$v_1 = [⅔, ⅓], v_2 = [⅔, ⅓]$

$\pi =$

	d	h
d	1/3	1/3
h	1/3	0

$\sigma_1 =$

D_1	A_1
d	D
h	H

$\sigma'_1 =$

D_1	A_1
d	H
h	H

$u(\sigma_1) = 1/3 * 3 + 1/3 * 1 + 1/3 * 5 = 3$

$u(\sigma'_1) = 1/3 * 5 + 1/3 * -2 + 1/3 * 5 = 8/3$

1 should **not** deviate to ***H***!

	D	H
D	3,3	1,5
H	5,1	-2,-2

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}, D_2 = \{d, h\}$

$v_1 = [⅔, ⅓], v_2 = [⅔, ⅓]$

$\pi =$

	d	*h*
d	1/3	1/3
h	1/3	0

$\sigma_1 =$

D_1	A_1
d	***D***
h	***H***

	D	*H*
D	3,3	1,5
H	5,1	**-2,-2**

$u_1(D|d) = (3 + 1) / 2 = 2$
$u_1(H|d) = (5 - 2) / 2 = 1.5$

1 should **not** deviate to ***H***!

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}, D_2 = \{d, h\}$

$v_1 = [⅔, ⅓], v_2 = [⅔, ⅓]$

$\pi =$

	d	*h*
d	1/3	1/3
h	1/3	0

$\sigma_1 =$

D_1	A_1
d	***D***
h	***H***

?

	D	*H*
D	3,3	1,5
H	5,1	**0,0**

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}$, $D_2 = \{d, h\}$

$v_1 = [⅔, ⅓]$, $v_2 = [⅔, ⅓]$

$\pi =$

	d	*h*
d	1/3	1/3
h	1/3	0

$\sigma_1 =$

D_1	A_1
d	***D***
h	***H***

?

	D	*H*
D	3,3	1,5
H	5,1	**0,0**

$u_1(D|d) = (3 + 1) / 2 = 2$
$u_1(H|d) = (5 + 0) / 2 = 2.5$

1 should deviate to ***H***!

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}$, $D_2 = \{d, h\}$

$v_1 = [?, ?]$, $v_2 = [?, ?]$

$\pi =$

	d	h
d	?	?
h	?	?

$\sigma_1 =$

D_1	A_1
d	D
h	H

	D	H
D	3,3	1,5
H	5,1	**0,0**

Correlated Equilibrium

d, d	h, d	d, h

$D_1 = \{d, h\}, D_2 = \{d, h\}$

$v_1 = v_2 = [4/7, 3/7]$

$\pi =$

	d	h
d	1/7	3/7
h	3/7	0

$\sigma_1 =$

D_1	A_1
d	D
h	H

	D	H
D	3,3	1,5
H	5,1	0,0

$u_1(D|d) = (3 + 3*1) / 4 = 6/4$
$u_1(H|d) = (5 + 3*0) / 4 = 5/4$

1 should not deviate to ***H***!

Correlated Equilibrium

- Standard games can be viewed as the degenerate case in which the signals of the different agents are probabilistically independent

- Do we play games in which we achieve a Correlated Equilibrium?

Correlated Equilibrium

- **Theorem**
  - For every Nash equilibrium $\boldsymbol{s*}$ there exists a corresponding correlated equilibrium $\boldsymbol{\sigma}$
    - let $\boldsymbol{D_i = A_i}$
    - let $\boldsymbol{\pi(d) = \prod_{i \in N} s_i^*(d_i)}$
    - $\boldsymbol{\sigma_i}$ maps each $\boldsymbol{d_i}$ to the corresponding $\boldsymbol{a_i}$
    - Thus, correlated equilibria always exist

Correlated Equilibrium

o, o	o, f	f, o	f, f

- $D_1 = D_2 = \{o, f\}$ (could be other signals)
- $v_1 = v_2 =$ [⅔, ⅓]

$\pi =$

	o	f
o	2/9	4/9
f	1/9	2/9

$\sigma_1 = \sigma_2 =$

D_1	A_1
o	O
f	F

	O	F
O	2,1	0,0
F	0,0	1,2

$u_1(O|o) = (1*2 + 2*0) / 3 = 2/3$
$u_1(F|o) = (1*0 + 2*1) / 3 = 2/3$

1 does not gain by deviating to ***F***!

Correlated Equilibrium

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash

Correlated Equilibrium

- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined

Traveler's Dilemma

Traveler's Dilemma

- Two travelers purchase identical African masks while on a tropical vacation. Their luggage is lost on the return trip, and the airline asks them to make independent claims for compensation. In anticipation of excessive claims, the airline representative announces: “We know that the bags have identical contents, and we will entertain any claim between $180 and $300, but you will each be reimbursed at an amount that equals the minimum of the two claims submitted. If the two claims differ, we will also pay a reward ***R>1*** to the person making the smaller claim and we will deduct a penalty ***R>1*** from the reimbursement to the person making the larger claim

Traveler's Dilemma

- **Action**: choose an integer between **180** and **300**
  - If both players pick the same number, they both get that amount as payoff
  - If players pick a different number:
    - the low player gets his number (**L**) plus some constant **R**
    - the high player gets **L – R**
    - **R = 5**
    - **R = 180**

Traveler's Dilemma

- What is the equilibrium?
  - ***(180; 180)*** is the only equilibrium, for all ***R > 1***
- What happens?
  - with ***R = 5*** most people choose ***295-300***
  - with ***R = 180*** most people choose ***180***

ε-Nash Equilibrium

- Fix $\varepsilon > 0$
- A strategy profile $s = (s_1, \ldots, s_n)$ is an **ε-Nash equilibrium** if, for all agents i and for all strategies $s_i' \neq s_i$,
- $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \varepsilon$

Traveler's Dilemma

- What is the minimum value of ε that would give an ε-Nash equilibrium and would maximize their expected payoffs?
  - Maximum value both players can get
    - ***300***
  - If one deviates, her utility can be
    - $299 + R$
  - Definition of ε-Nash
    - $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \varepsilon$
  - Then…
    - $300 \geq 299 + R - \varepsilon$
    - $\varepsilon \geq R-1 \rightarrow \varepsilon = R-1$

ε-Nash Equilibrium

- What are the Nash equilibria?
- What are the ε-Nash equilibria?

	L	R
U	$1, 1$	$0, 0$
D	$1 + \frac{\epsilon}{2}, 1$	$500, 500$

ε-Nash Equilibrium

- Neither player's payoff under the ***ε***-Nash equilibrium is within ***ε*** of his payoff in a Nash equilibrium
  - In general both players' payoffs under an ***ε***-Nash equilibrium can be arbitrarily less than in any Nash equilibrium
  - The problem is that the requirement that player ***1*** cannot gain more than ***ε*** by deviating from the ***ε***-Nash equilibrium strategy profile of ***(U,L)*** does not imply that player ***2*** would not be able to gain more than ***ε*** by best responding to player ***1***'s deviation

ε-Nash Equilibrium

- Second, some ***ε***-Nash equilibria might be very unlikely to arise in play
  - Although player ***1*** might not care about a gain of ***ε***, he might reason that the fact that ***D*** dominates ***U*** would lead player ***2*** to expect him to play ***D***, and that player ***2*** would thus play ***R*** in response

ε-Nash Equilibrium

- Advantage
  - Some games may have only irrational, i.e. not exact, Nash equilibrium points [Nash, 1951]