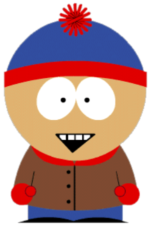
Class #7 – Part 1

Introduction no Noncooperative Game Theory: Games in Normal Form

Motivation

- Systems that include multiple autonomous entities with either diverging information or diverging interests, or both



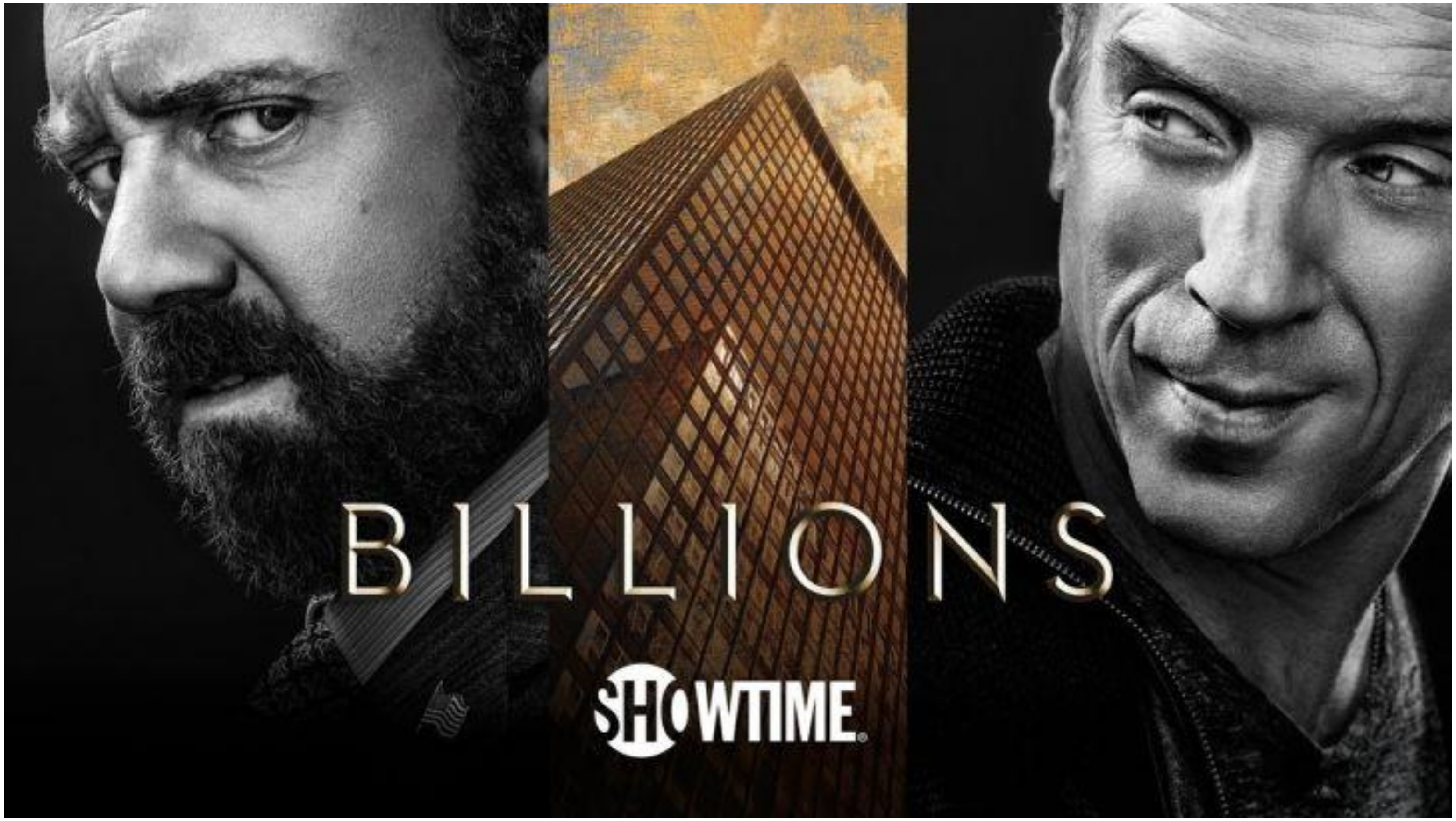
Games

- Which cell phone Cartman should pick?

Games

- And now?

Billions



[The Problem With Game Theory – The Philosophy of Billions - YouTube]
[Decisions Matter: Game Theory in Billions – Part I – Fan Fun with Damian Lewis]

Motivation

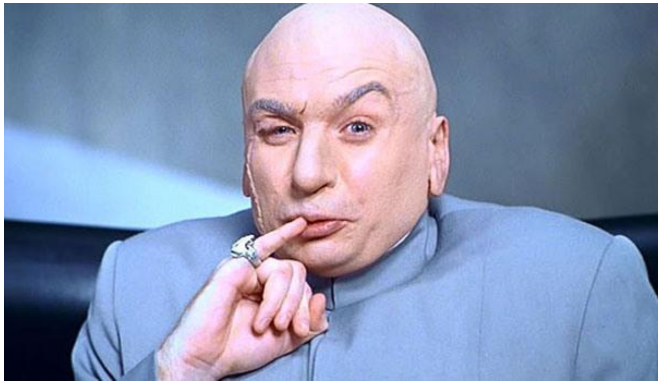
- Indeed, the **Internet** can be viewed as the ultimate platform for interaction among self-interested, distributed computational entities
  - Trading agents
  - “Interface agents” that facilitate the interaction between the user and various computational resources
  - Game-playing agents that assist (or replace) human players in a multiplayer game
  - Autonomous robots

Game Theory

- Outcome of a person's decision depends not just on her preferences, but also on the **choices made by others**
- Main question:
  - Which behaviors **tend to sustain** themselves when carried out in a larger population?

Self-Interested Agents

- What does it mean?
  - They want to cause harm to each other?
    - Not necessarily!
  - They care only about themselves?
    - Not necessarily!



Self-Interested Agents

- What does it mean?
  - Each agent has his own description of which states of the world he likes
    - which can include good things happening to other agents
  - and that he acts in an attempt to bring about these states of the world

Self-Interested Agents

- A utility function is a mapping from states of the world to real numbers
  - measures of an agent's level of happiness in the given states
- If uncertain about which state of the world he faces
  - expected value of his utility function with respect to the appropriate probability distribution over states

Self-Interested Agents

- The states can be thought of as the prizes in the context of lotteries

Example: friends and enemies

- Alice has three options: club (***c***), movie (***m***), watching a video at home (***h***)
- On her own, her utility for these three outcomes is ***100*** for ***c***, ***50*** for ***m*** and ***50*** for ***h***
- Alice also cares about Bob (<u>who she hates</u>) and Carol (<u>who she likes</u>)
- Bob is at the club ***60%*** of the time, and at the movies otherwise
- Carol is at the movies ***75%*** of the time, and at the club otherwise
- If Alice runs into Bob at the movies, she suffers disutility of ***40***; if she sees him at the club she suffers disutility of ***90***
- If Alice sees Carol, she enjoys whatever activity she's doing ***1.5*** times as much as she would have enjoyed it otherwise

Example: friends and enemies

- What should Alice do? Reminder: u(***home***) = 50

	60%	40%	
	$B = c$	$B = m$	
$C = c$	15	150	25%
$C = m$	10	100	75%

$A = c$

	60%	40%
	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

$A = m$

Alice chooses ***club***

Alice chooses ***movie***

Example: friends and enemies

- What should Alice do? Reminder: u(***home***) = 50

	60% $B = c$	40% $B = m$			60% $B = c$	40% $B = m$
$C = c$	15	150	25%	$C = c$	50	10
$C = m$	10	100	75%	$C = m$	75	15
	$A = c$				$A = m$	

Alice chooses ***club:***

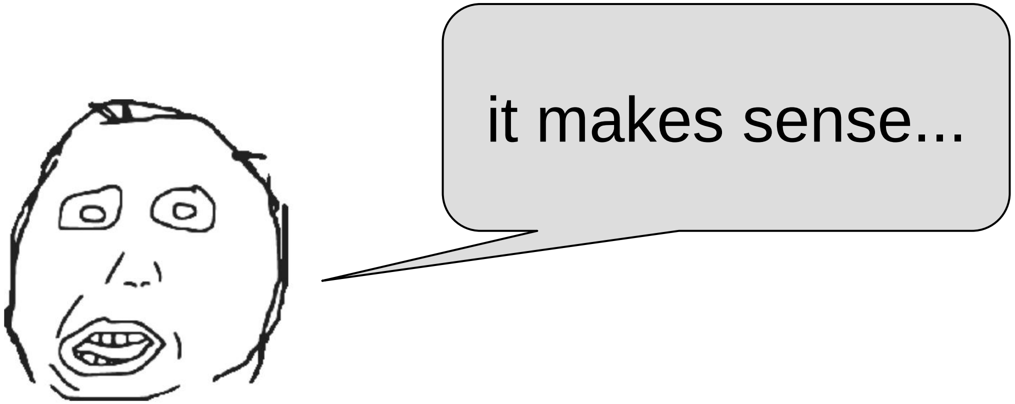
Eu(c) = 0.25(0.6 x 15 + 0.4 x 150) + 0.75(0.6 x 10 + 0.4 x 100) = 51.75

Alice chooses ***movies***

Eu(m) = 0.25(0.6 x 50 + 0.4 x 10) + 0.75(0.6 x 75 + 0.4 x 15) = 46.75

Example: friends and enemies

- What should Alice do? Reminder: u(***home***) = 50
  - Alice prefers to go to the ***club*** (though Bob is often there and Carol rarely is), and prefers staying ***home*** to going to the ***movies*** (though Bob is usually not at the ***movies*** and Carol almost always is)



Alice chooses ***club:***

Eu(c) = 0.25(0.6 x 15 + 0.4 x 150) + 0.75(0.6 x 10 + 0.4 x 100) = 51.75

Alice chooses ***movies***

Eu(m) = 0.25(0.6 x 50 + 0.4 x 10) + 0.75(0.6 x 75 + 0.4 x 15) = 46.75

What is a game?
Example 1

- Problem
  - You have an exam and a presentation tomorrow
  - You have time to prepare yourself for just one
  - The exam is individual and the presentation is together with a colleague
  - Which one should you pick?

What is a game?
Example 1

- Possible outcomes
  - Exam
    - If you study: 92
    - If you don't study: 80
  - Presentation
    - If both work: 100
    - If only one works: 92
    - If no one works: 84

  - The same outcomes are valid for your colleague

What is a game?
Example 1

- Possible outcomes (summary)
  - If you study and your colleague works on the presentation
    - (92 + 92) / 2 = **92**
  - If you both study
    - (92 + 84) / 2 = **88**
  - If you work on the presentation and your colleague studies
    - (80 + 92) / 2 = **86**
  - If you both work on the presentation
    - (80 + 100) / 2 = **90**

What is a game?

- A set of **players**
  - you and your partner
- A set of possible **strategies** for each player
  - to prepare for the presentation, or to study for the exam
- A set of **payoffs** for each player and for each joint choice of strategies (the more, the better)
  - the average grade

Games in Normal Form

- A (finite, n-person) normal-form game is a tuple ***(N,A,u)***, where:
  - ***N*** is a finite set of ***n*** players, indexed by ***i***
  - $A = A_1 \times \cdots \times A_n$, where A_i is a finite set of actions available to player ***i***
    – Each vector $a = (a_1, \ldots, a_n) \in A$ is called an action profile
  - $u = (u_1, \ldots, u_n)$ where $u_i : A \to \mathbb{R}$ is a real-valued utility (or payoff) function for player ***i***
    – $u : A^n \to \mathbb{R}^n$

What is a game?
Example 1

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

What is a game?

[SÍLVIO SANTOS-PROGRAMA SETE E MEIO-DUAS EDIÇÕES JUNTOS -2002-MP4 - YouTube]

What is a game?
Example 1

Payoff Matrix

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

What is a game?
Example 1

"Player 2" or "Column player"

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

"Player 1" or "Row player"

What is a game?
Example 1

Strategies $\boldsymbol{A_2}$ of Player 2

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

Strategies $\boldsymbol{A_1}$ of Player 1

What is a game?
Example 1

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

Payoffs of Player 1

What is a game?
Example 1

Payoffs of Player 2

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

What is a game?
Example 1

u(Exam, Presentation):
The outcome of the game if
P***1*** chooses '***Exam***' and P***2***
chooses '***Presentation***'

P**2** gets 86

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Figure 6.1: Exam or Presentation?

P***1*** gets 92

Reasoning about Behavior in a Game

- Everything that a player cares about is summarized in the player's payoffs
  - If a player is altruistic, the payoffs should reflect it

Reasoning about Behavior in a Game

- Each player knows everything about the structure of the game
  - her own list of possible strategies
  - who the other players are
  - the strategies available to them
  - what his or her payoff will be for any choice of strategies
- There are many studies on games with incomplete information

Reasoning about Behavior in a Game

- Rationality
  - each player wants to maximize her own payoff
  - each player actually succeeds in selecting the optimal strategy
    - Achieved usually with time and/or a lot of reasoning

Reasoning about Behavior in the Exam-or-Presentation Game

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Figure 6.1: Exam or Presentation?

Reasoning about Behavior in the Exam-or-Presentation Game

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Figure 6.1: Exam or Presentation?

if you knew your partner was going to study for the exam:
88 by also studying
86 by preparing for the presentation
Decision: you should **study for the exam**

Reasoning about Behavior in the Exam-or-Presentation Game

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

if you knew your partner was going to work on the presentation:
90 by also working
92 by studying
Decision: you should **study for the exam**

Reasoning about Behavior in the Exam-or-Presentation Game

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

No matter what your partner does, you should **study for the exam**

Strictly Dominant Strategy

- A strategy that is strictly better than all other options regardless of what the other player does
  - To study for the exam

Strictly Dominant Strategy

- It does not mean it will give the best payoff!

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$\boxed{90, 90}$	$86, \boxed{92}$
	Exam	$\boxed{92}, 86$	$88, 88$

Figure 6.1: Exam or Presentation?

Strictly Dominant Strategy

- It does not mean it will give the best payoff!

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

- Even if we agree to work on the presentation together, I have a clear incentive to deviate
  - $92 > 90$

The Prisoner's Dilemma

- Proposed by Flood and Dresher while working at RAND in 1950

Merrill Flood

Melvin Dresher

The Prisoner's Dilemma

- In June 1949, Flood wanted to buy a used Buick from a RAND employee who was moving back East

The Prisoner's Dilemma

- Honor among thieves

	Mr. Big sticks to agreement	**Mr. Big cheats**
You stick to agreement	Deal goes through: you get money, Mr. Big gets diamond	You get nothing, Mr. Big walks away with diamond and money
You cheat	You walk away with money and diamond, Mr. Big gets nothing	A lot of trouble for nothing: you keep diamond, Mr. Big keeps money

The Prisoner's Dilemma

- The Flood Dresher's experiment

	JW's Strategy 1 [Defect]	**JW's Strategy 2 [Cooperate]**
AA's Strategy 1 [Cooperate]	−1¢, 2¢	1/2¢, 1¢
AA's Strategy 2 [Defect]	0, 1/2¢	1¢, −1¢

Armen Alchian (AA) vs. John D. Williams (JW)

Armen Alchian vs. John D. Williams

Game	AA	JW	AA's comments	JW's comments

The Prisoner's Dilemma

- **Two suspects** are being interrogated in separate rooms.
- The police strongly suspect that these two individuals are responsible for a robbery, but there is not enough evidence to convict either of them of the robbery. However, they both resisted arrest and **can be charged with that lesser crime, which would carry a one-year sentence.**
- Each of the suspects is told the following story. **If you confess, and your partner doesn't confess, then you will be released** and your partner will be charged with the crime. Your confession will be sufficient to convict him of the robbery and **he will be sent to prison for 10 years**. **If you both confess,** then we don't need either of you to testify against the other, and you will **both be convicted of the robbery**. (Although in this case your sentence will be **4 years** only because of your guilty plea.) **If neither of you confesses**, then we can't convict either of you of the robbery, so we will **charge each of you with resisting arrest (1 year)**. Your partner is being offered the same deal.
- Do you want to confess?

The Prisoner's Dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

Figure 6.2: Prisoner's Dilemma

The Prisoner's Dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$\boxed{-10, 0}$
	C	$0, -10$	$\boxed{-4, -4}$

Figure 6.2: Prisoner's Dilemma

If P2 confesses:

- if P1 does not confess: -10
- if P1 confesses: -4
- **Decision: to confess**

The Prisoner's Dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

Figure 6.2: Prisoner's Dilemma

If P2 does not confess:

- if P1 does not confess: -1
- if P1 confesses: 0
- **Decision: to confess**

The Prisoner's Dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$\boxed{-4, -4}$

Figure 6.2: Prisoner's Dilemma

The strategy "to confess" strictly dominates the strategy "to not confess" for both players

The Prisoner's Dilemma

- It serves as a highly streamlined depiction of the difficulty in establishing cooperation in the face of individual self-interest

- Are there other real-world situations?
  - Performance enhancing drugs in sports
  - Arms race between opposing nations
  - etc

The Prisoner's Dilemma

		Athlete 2	
		Don't Use Drugs	*Use Drugs*
Athlete 1	*Don't Use Drugs*	3, 3	1, 4
	Use Drugs	4, 1	2, 2

Figure 6.3: Performance-Enhancing Drugs

The Prisoner's Dilemma

- It only manifests itself when the conditions are right
  - Making the exam easier, i.e., 100 if you study, 96 if you do not

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	98, 98	94, 96
	Exam	96, 94	92, 92

Let's go back to school...

- You and your partner are preparing slides for a joint project presentation
- You can't reach your partner by phone and need to start working on the slides now
- You have to decide whether to prepare your half of the slides in PowerPoint or in Apple's Keynote software
  - it will be much easier to merge your slides together with your partner's if you use the same software

Common payoff games (pure coordination games)

- A ***common payoff game*** is a game in which for all action profiles $\boldsymbol{a} \in A_1 \times \cdot \cdot \cdot \times A_n$ and any pair of agents $\boldsymbol{i}$, $\boldsymbol{j}$, it is the case that $u_i(a) = u_j(a)$

Common payoff games (pure coordination games)

- Suppose that both you and your project partner each prefer Keynote to PowerPoint

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

Figure 6.8: Unbalanced Coordination Game

Common payoff games (pure coordination games)

- Suppose that both you do not care about which software you use

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

Figure 6.7: Coordination Game

Common payoff games
(pure coordination games)

- Examples?

Common payoff games (pure coordination games)

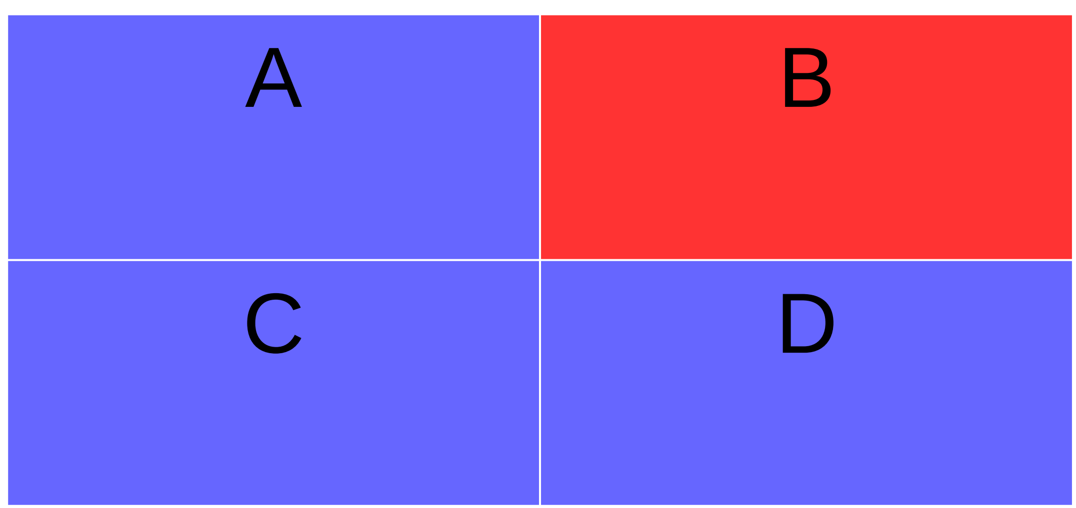
- Examples
  - two manufacturing companies that work together extensively need to decide whether to configure their machinery in metric units of measurement or English units of measurement
  - two platoons in the same army need to decide whether to attack an enemy's left flank or right flank
  - two people trying to find each other in a crowded mall need to decide whether to wait at the north end of the mall or at the south end
- Both choices can be fine, provided that both participants make the same choice

Common payoff games (pure coordination games)

- Two people unable to communicate with each other are asked to select a square in a four-square panel
- If and only if both select the same one, they will each receive a prize

Common payoff games (pure coordination games)

- Select a square in the panel
- If and only if both select the same one, they will each receive a prize



Schelling's Focal Point

- "Focal point[s] for each person's expectation of what the other expects him to expect to be expected to do"

Schelling's Focal Point

- "Focal point[s] for each person’s expectation of what the other expects him to expect to be expected to do"

Schelling's Focal Point

- It is a solution that people will tend to use in the absence of communication, because it seems natural, special, or relevant to them
- Such points are highly useful in negotiations, because we cannot completely trust our negotiating partners' words

Constant-sum games (zero-sum games)

- A two-player normal-form game is constant sum if there exists a constant $\boldsymbol{c}$ such that for each strategy profile $\boldsymbol{a} \in \boldsymbol{A_1} \times \boldsymbol{A_2}$ it is the case that $\boldsymbol{u_1(a) + u_2(a) = c}$
- For convenience, assume that $\boldsymbol{c=0}$, that is, that we have a zero-sum game
  - Does it make any difference?

Constant-sum games (zero-sum games)

- Matching pennies

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Constant-sum games (zero-sum games)

- Rock, paper, scissors

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Hybrid games

- In general, games can include elements of both <u>coordination</u> and <u>competition</u>
  - Prisoner’s Dilemma?

Hybrid games

- Battle of the sexes
  - Husband and wife wish to go to the movies, and they can select among two movies:
  - “Lethal Weapon (LW)” and “Wondrous Love (WL)”
  - They much prefer to go together rather than to separate movies, but while the wife (player 1) prefers LW, the husband (player 2) prefers WL

Hybrid games

- Battle of the sexes

		Husband	
		LW	WL
Wife	LW	2, 1	0, 0
	WL	0, 0	1, 2

Variants on the Basic Coordination Game

- Suppose that two people are out hunting
- If they work together, they can catch a stag (which would be the highest-payoff outcome)

- but on their own each can catch a hare

Variants on the Basic Coordination Game

		Hunter 2	
		Hunt Stag	*Hunt Hare*
Hunter 1	*Hunt Stag*	$4, 4$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

Figure 6.10: Stag Hunt

Variants on the Basic Coordination Game

- Similar to the Unbalanced Coordination Game
- Except that if the two players **miscoordinate**:
  - the one who was trying for the higher-payoff outcome gets penalized more than the one who was trying for the lower-payoff outcome

		Hunter 2	
		Hunt Stag	*Hunt Hare*
Hunter 1	*Hunt Stag*	$4, 4$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

Figure 6.10: Stag Hunt

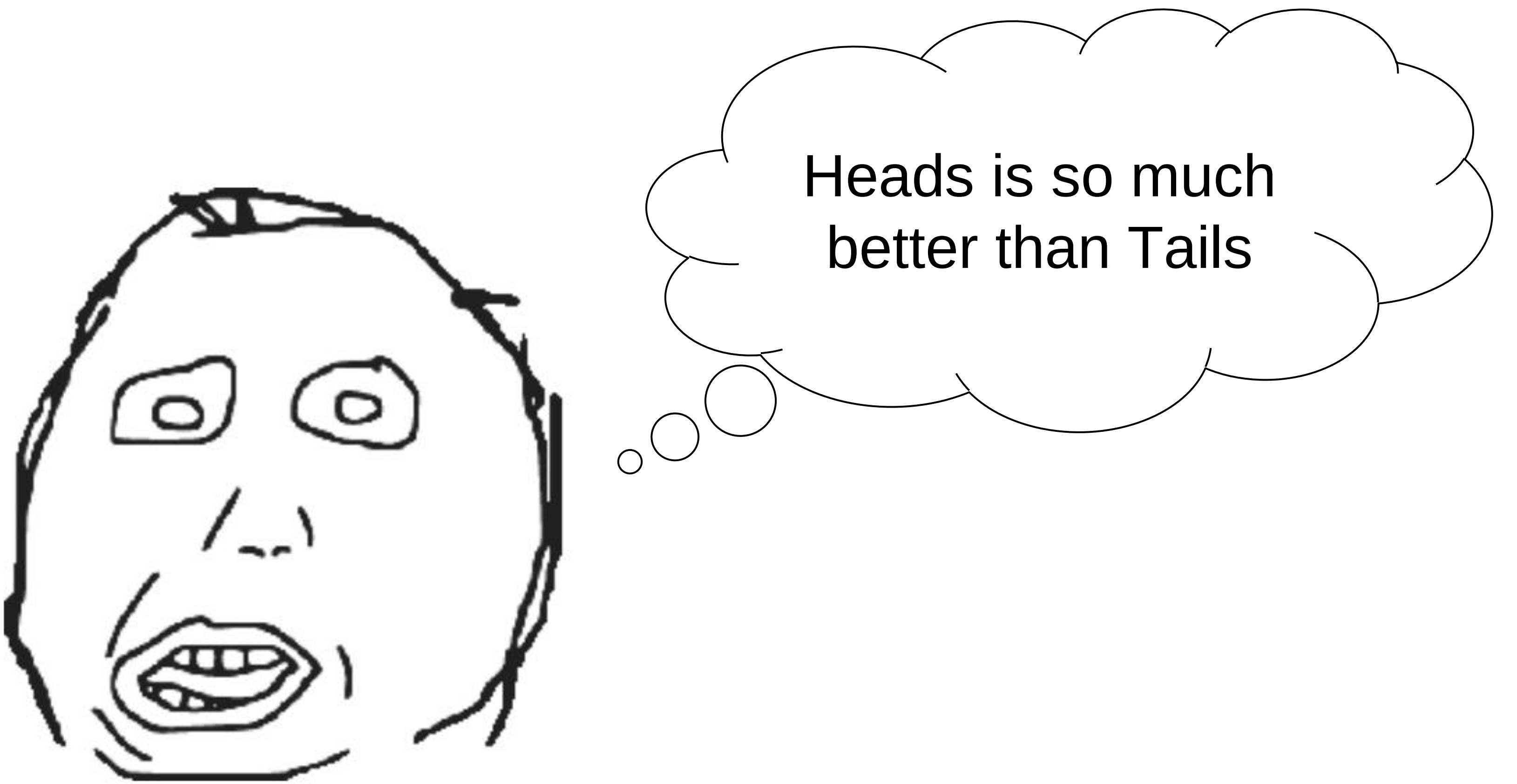
Constant-sum games (zero-sum games)

- Which action should I play?

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Strategies in Normal Form Games

- It would be a pretty bad idea to play any deterministic strategy in matching pennies



Strategies in Normal Form Games

- **Idea**: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability

Strategies in Normal Form Games

- **Idea**: confuse the opponent by playing randomly
- for any set $\boldsymbol{X}$ let $\prod(\boldsymbol{X})$ be the set of all probability distributions over $\boldsymbol{X}$
- Let the set of all strategies for $\boldsymbol{i}$ be $\boldsymbol{S_i = \prod(A_i)}$
- Let the set of all strategy profiles be $\boldsymbol{S = S_1 \times \ldots \times S_n}$

Strategies in Normal Form Games

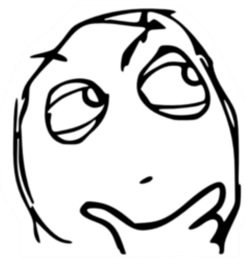
- What is your payoff if all the players follow mixed strategy profile ***s* ∈ *S***?
  - We can't just read this number from the game matrix anymore
    - we won't always end up in the same cell
- We use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

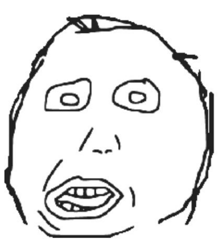
Strategies in Normal Form Games

- Matching pennies



$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$



	$s_2(H) = 0.4$ Heads	$s_2(T) = 0.6$ Tails
$s_1(H) = 0.8$ Heads	0.32 $1, -1$	0.48 $-1, 1$
Tails $s_1(T) = 0.2$	0.08 $-1, 1$	0.12 $1, -1$

s = (s$_1$*, s*$_2$*)*

s = ((0.8,0.2), (0.4,0.6))

Pr(a|s) = $\prod_{j \in N} s_j(a_j)$

Pr((H,H) |s) = 0.8 x 0.4
Pr((H,T) |s) = 0.8 x 0.6
Pr((T,H) |s) = 0.2 x 0.4
Pr((T,T) |s) = 0.2 x 0.6

$u_1(s) = \Sigma_{a \in A} u_1(a) Pr(a|s)$
$u_1(s) = 0.32(1) + 0.48(-1) + 0.08(-1) + 0.12(1)$
$u_1(s) = -0.12$
$u_2(s) = +0.12$

Solving a Game

- What is an optimal strategy for a given agent?
  - The best strategy depends on the choices of others

Solving a Game

- Solution concepts
  - Certain subsets of outcomes that are interesting in one sense or another
    - Pareto optimality
    - Nash equilibrium

Analyzing a Game

- Can some outcomes of a game be said to be better than others?

	L	R
U	1, 1000	0,0
D	2,1	1, 0

Analyzing a Game

- Can some outcomes of a game be said to be better than others?
- We have no way of saying that one agent's interests are more important than another's
  - E.g., multiply player ***1***’s payoffs by 1,000 to change the sum of agents’ utilities
- Intuition: we don't know what currency has been used to express each agent's payoffs

Analyzing a Game

- Are there situations where we can still prefer one outcome to another?

Pareto Domination

- Idea: sometimes, one outcome ***o*** is at least as good <u>for every agent</u> as another outcome ***o'***, and there is some agent who strictly prefers ***o*** to ***o'***
- In this case, it seems reasonable to say that ***o*** is better than ***o'***
- We say that ***o*** Pareto-dominates ***o'***

Pareto Domination

- Strategy profile $\boldsymbol{s}$ Pareto dominates strategy profile $\boldsymbol{s'}$ if for all $\boldsymbol{i \in N}$, $\boldsymbol{u_i(s) \geq u_i(s')}$, and there exists some $\boldsymbol{j \in N}$ for which $\boldsymbol{u_j(s) > u_j(s')}$
- Observe that we define Pareto domination over <u>strategy profiles</u>, not just <u>action profiles</u>

Pareto Optimality

- An <u>outcome</u> ***o*** is Pareto-optimal if there is no other <u>outcome</u> ***o'*** that Pareto-dominates it
- or
- A <u>strategy profile</u> ***s*** is Pareto-optimal if there does not exist another <u>strategy profile</u> $s' \in S$ that Pareto dominates ***s***
- Can a game have more than one Pareto-optimal outcome?
- Does every game have at least one Pareto-optimal outcome?

Pareto Optimality

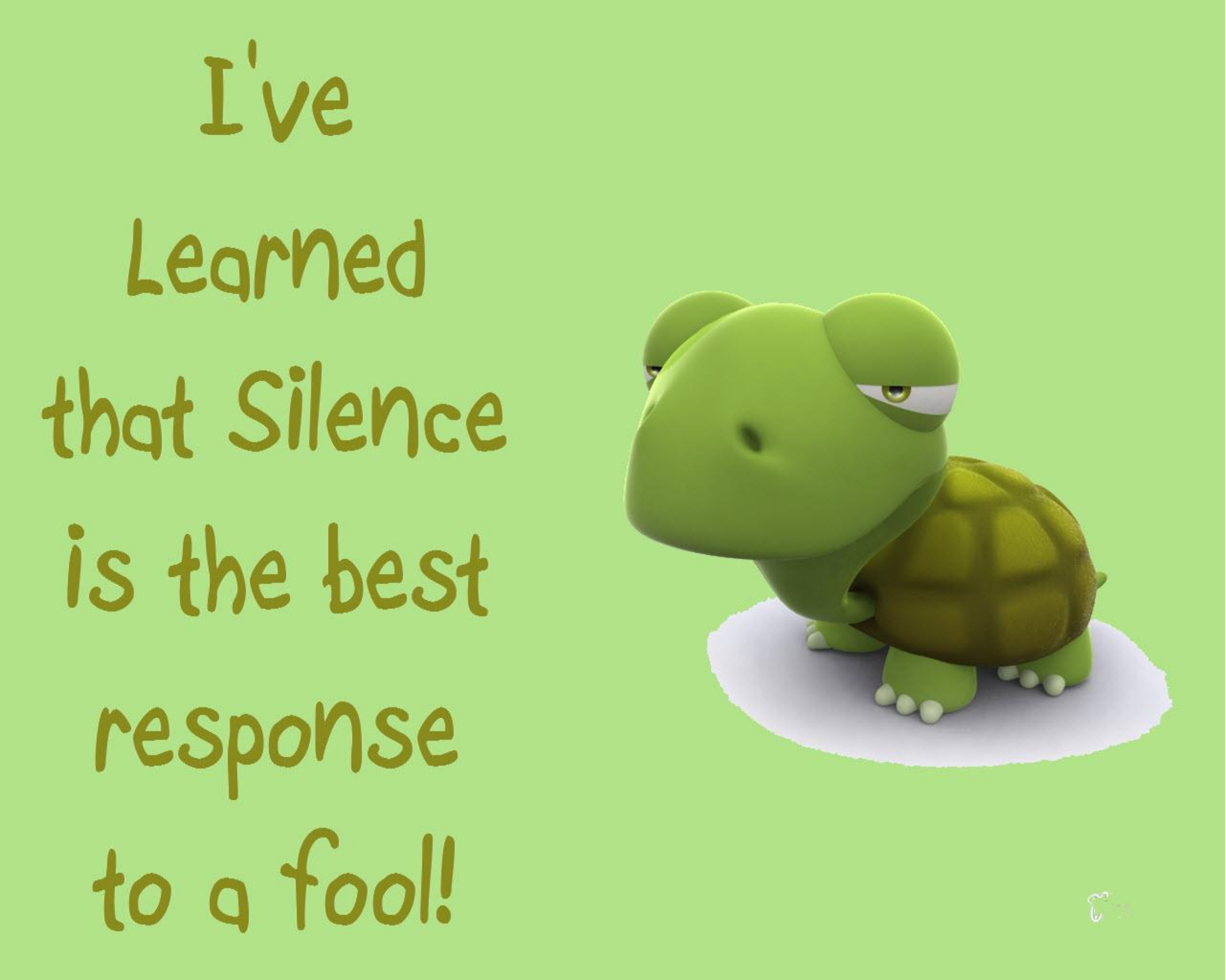
		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Figure 6.1: Exam or Presentation?

		Husband: LW	Husband: WL
Wife	LW	$2, 1$	$0, 0$
	WL	$0, 0$	$1, 2$

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Best Response



Best Response

- **Best response**
  - it is the best choice of one player, given a belief about what the other player will do
  - in the Exam-or-Presentation Game, we determined your best choice in response to each possible choice of your partner

Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Player ***i***'s best response to the strategy profile $\boldsymbol{s_{-i}}$ is a mixed strategy $\boldsymbol{s_i^* \in S_i}$ such that
- $\boldsymbol{u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})}$ for all strategies $\boldsymbol{s_i \in S_i}$
- Best responses are unique?
- $\boldsymbol{s_i^* \in BR(s_{-i})}$ ***iff*** $\boldsymbol{\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})}$

Best Response

- Best responses are unique?
  - If player ***2***'s strategy is $s_2 = (0.5,0.5)$, what is your best response?
  - $s_1 = (0.5,0.5)$, $u_1(s) = 0.25(1+0+0+1) = 0.5$
  - $s_1 = (0.75,0.25)$, $u_1(s) = 0.5$
  - $s_1 = (0.38,0.62)$, $u_1(s) = 0.5$

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Best Response

- Best responses are unique?
  - It is unique when there is a unique best response that is a pure strategy
    - $s_2 = (1,0)$
    - $s_2 = (0.8,0.2)$

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Best Response

- Best response is not a solution concept
  - Does not identify an interesting set of outcomes
- However, we can leverage the idea of best response to define what is arguably the most central notion in non-cooperative game theory

Nash Equilibrium

- Imagine a setting where no agent knows anything about what the others will do
- What can we say about which actions will occur?
  - Idea: look for stable action profiles

An Example: A Three-Client Game

- Suppose there are two firms that each hope to do business **with one** of three large clients, A, B, and C
- Each firm has three possible strategies: whether to approach A, B, **or** C

An Example: A Three-Client Game

- If the two firms approach the same client, then the client will give half its business to each
- Firm 1 is too small to attract business on its own, so if it approaches one client while Firm 2 approaches a different one, then Firm 1 gets a payoff of 0
- If Firm 2 approaches client B or C on its own, it will get their full business

An Example: A Three-Client Game

- However, A is a larger client, and will only do business with the firms if both approach A
- Because A is a larger client, doing business with it is worth 8 (and hence 4 to each firm if it's split), while doing business with B or C is worth 2 (and hence 1 to each firm if it's split)

An Example: A Three-Client Game

- If the two firms approach the same client, then the client will give half its business to each
- **Firm 1** is too small to attract business on its own, so if it approaches one client while **Firm 2** approaches a different one, then **Firm 1** gets a payoff of **0**
- If **Firm 2** approaches client **B** or **C** on its own, it will get their full business
- However, **A** is a larger client, and will only do business with the firms if both approach **A**
- Because **A** is a larger client, doing business with it is worth **8** (and hence **4** to each firm if it's split), while doing business with **B** or **C** is worth **2** (and hence **1** to each firm if it's split)

An Example: A Three-Client Game

		Firm 2		
		A	B	C
	A			
Firm 1	B			
	C			

Figure 6.6: Three-Client Game

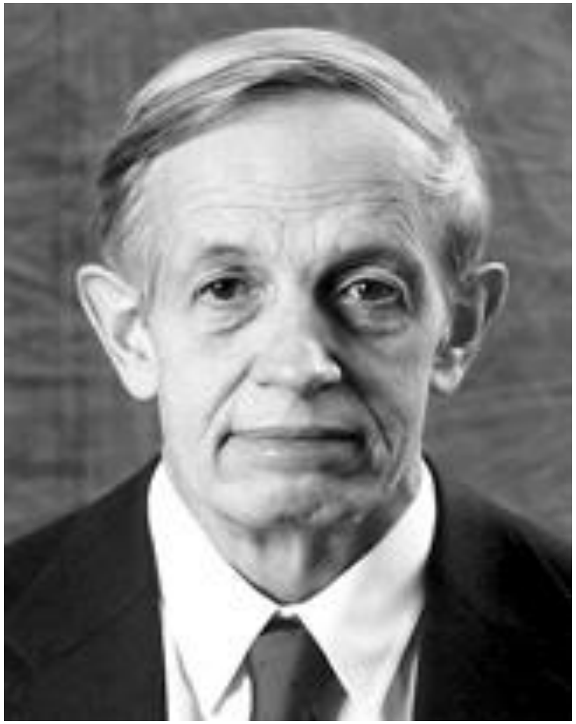
An Example: A Three-Client Game

		Firm 2 A	Firm 2 B	Firm 2 C
Firm 1	A	$4, 4$	$0, 2$	$0, 2$
	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

Figure 6.6: Three-Client Game

Defining Nash Equilibrium

- John Nash proposed a simple but powerful principle for reasoning about behavior in general games
- Even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other



Defining Nash Equilibrium

- A pair of strategies $(\boldsymbol{s_i}; \boldsymbol{s_j})$ is a **Nash equilibrium** if $\boldsymbol{s_i}$ is a best response to $\boldsymbol{s_j}$, and $\boldsymbol{s_j}$ is a best response to $\boldsymbol{s_i}$
- If the players choose strategies that are best responses to each other, then no player has an incentive to deviate to an alternative strategy
- The system is in a kind of equilibrium state, with no force pushing it toward a different outcome

Defining Nash Equilibrium

		Firm 2		
		A	B	C
Firm 1	A	$4,4$	$0,2$	$0,2$
	B	$0,0$	$1,1$	$0,2$
	C	$0,0$	$0,2$	$1,1$

Figure 6.6: Three-Client Game

Defining Nash Equilibrium

		Firm 2		
		A	B	C
	A	$4, 4$	$0, 2$	$0, 2$
Firm 1	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

Figure 6.6: Three-Client Game

(A; A) is a Nash Equilibrium

Pure Strategy Nash Equilibrium

- A strategy profile $\boldsymbol{a = (a_1, \ldots, a_n)}$ is a pure strategy Nash equilibrium if, <u>for all agents</u> $\boldsymbol{i}$, $\boldsymbol{a_i}$ is a best response to $\boldsymbol{a_{-i}}$
- $\boldsymbol{a = (a_1, \ldots, a_n)}$ is a pure strategy Nash equilibrium iff $\boldsymbol{\forall i,\ a_i \in BR(a_{-i})}$
- Intuitively, a Nash equilibrium is a stable strategy profile: no agent would want to change his strategy if he knew what strategies the other agents were following

Pure Strategy Nash Equilibrium

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

	Left	Right
Left	1	0
Right	0	1

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

	Heads	Tails
Heads	1	-1
Tails	-1	1

Pure Strategy Nash Equilibrium

- How to find?
  - To **check all pairs of strategies**, and ask for each one of them whether the individual strategies are best responses to each other
  - To **compute each player's best response(s)** to each strategy of the other player, and then find strategies that are mutual best responses

Nash Equilibrium

- What is your payoff if all the players follow a mixed strategy profile $\boldsymbol{s} \in \boldsymbol{S}$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Nash Equilibrium

- We use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Nash Equilibrium

- A strategy profile $\boldsymbol{s = (s_1, \ldots, s_n)}$ is a Nash equilibrium if, <u>for all agents</u> $\boldsymbol{i}$, $\boldsymbol{s_i}$ is a best response to $\boldsymbol{s_{-i}}$
- Intuitively, a Nash equilibrium is a stable strategy profile: no agent would want to change his strategy if he knew what strategies the other agents were following

Nash Equilibrium

- A strategy profile $\boldsymbol{s = (s_1, \ldots, s_n)}$ is a strict Nash equilibrium if, for all agents $\boldsymbol{i}$ and for all strategies $\boldsymbol{s_i' \neq s_i}$, $\boldsymbol{u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})}$
- A strategy profile $\boldsymbol{s = (s_1, \ldots, s_n)}$ is a weak Nash equilibrium if, for all agents $\boldsymbol{i}$ and for all strategies $\boldsymbol{s_i' \neq s_i}$, $\boldsymbol{u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})}$ and $\boldsymbol{s}$ is not a <u>strict Nash equilibrium</u>

Nash Equilibrium

- Theorem (Nash, 1951)
  - Every game with a finite number of players and action profiles has **at least one** Nash equilibrium
    - with mixed strategies

Fun Game

	L	R
T	$80, 40$	$40, 80$
B	$40, 80$	$80, 40$

Fun Game

	L	R
T	$320, 40$	$40, 80$
B	$40, 80$	$80, 40$

Fun Game

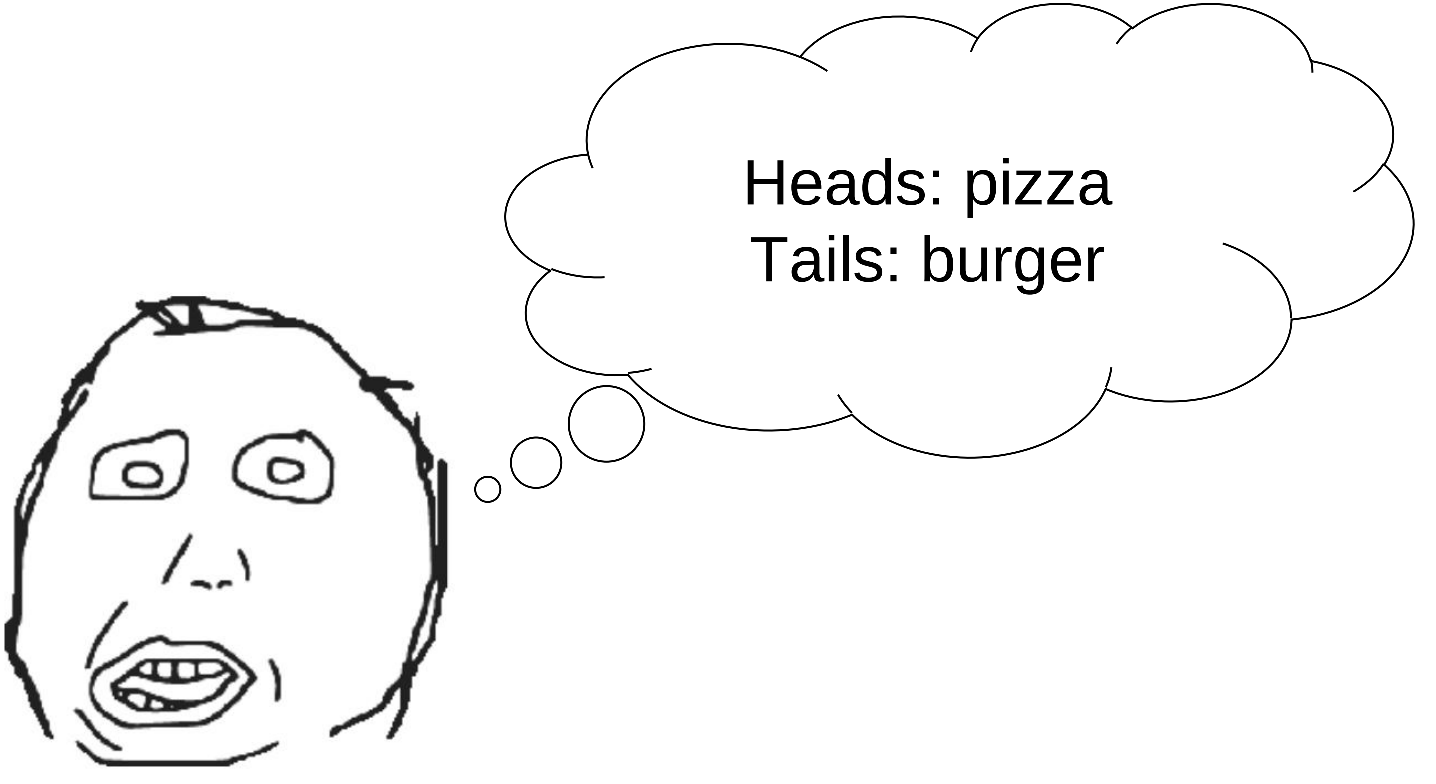
	L	R
T	$44, 40$	$40, 80$
B	$40, 80$	$80, 40$

Fun Game

- What does row player do in equilibrium of this game?
  - row player randomizes 50-50 all the time
  - that's what it takes to make column player indifferent
- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time

Mixed strategies

- Some counter-intuitive features...
  - Do people really play them?



Mixed strategies

- Ignacio Palacios-Heurta (2003) “Professionals Play Minimax” Review of Economic Studies, Volume 70, pp 395-415
- 1417 Penalty kicks from FIFA games: Spain, England, Italy...
- Ignacio considers left, center, or right choices, and considered which leg the kicker used
- I will keep it down to left and right choices ignoring kicker’s leg

Mixed strategies

- $u_i(s)$ = empirical probability of winning

$Kicker/Goalie$	$Left$	$Right$
$Left$	$.58, .42$	$.95, .05$
$Right$	$.93, .07$	$.70, .30$

Mixed strategies

- Let's solve the game...
  - $0.58q + 0.95(1\text{-}q) = 0.93q + 0.70(1\text{-}q)$
  - $q = 0.42$
  - $0.42p + 0.07(1\text{-}p) = 0.05p + 0.30(1\text{-}p)$
  - $p = 0.38$

		q	1-q
	Kicker/Goalie	*Left*	*Right*
p	*Left*	.58, .42	.95, .05
1-p	*Right*	.93, .07	.70, .30

Mixed strategies

- Let's solve the game...
  - q = 0.42, p = 0.38
- Do players randomize accordingly?

		q	1-q
	Kicker/Goalie	*Left*	*Right*
p	*Left*	.58, .42	.95, .05
1-p	*Right*	.93, .07	.70, .30

Mixed strategies

	Goalie Left	*Goalie Right*	*Kicker Left*	*Kicker Right*
Nash Freq.	.42	.58	.38	.62

Mixed strategies

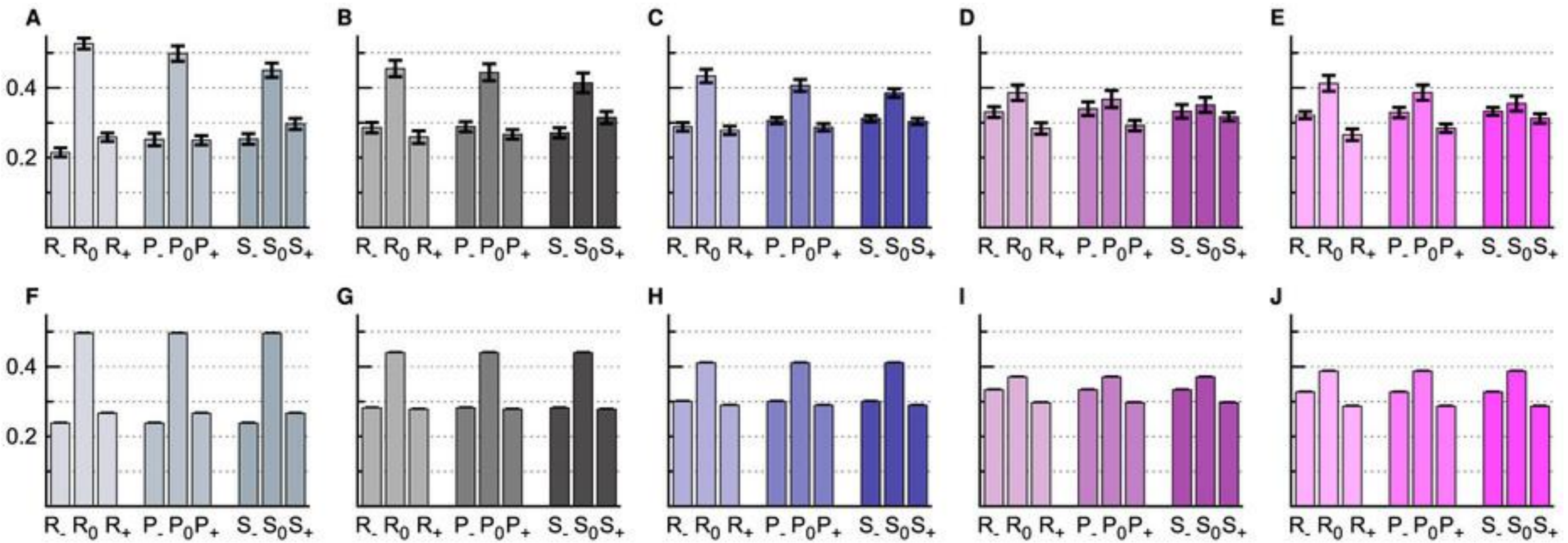
	Goalie Left	*Goalie Right*	*Kicker Left*	*Kicker Right*
Nash Freq.	.42	.58	.38	.62
Actual Freq.	.42	.58	.40	.60

Mixed strategies

- Do players randomize well over time?
  - Pretty well...
- How about under pressure? Do they become predictable?
  - Statisticians/Game Theorists are being hired to keep watch this
- Other sports
  - Tennis players and serves, M. Walker and J. Wooders, American Economic Review (2001) “Minimax Play and Wimbeldon” volume 91, pp 1521-1538

Mixed strategies

- Social cycling and conditional responses in the **Rock-Paper-Scissors game** Zhijian Wang, Bin Xu & Hai-Jun Zhou Scientific Reports 4, Article number: 5830 (2014)



Mixed strategies

- Important note about mixed strategies:
  - since players are able to receive any linear combination of their utility values, then all values must be defined very carefully
  - should respect preference relations over lotteries!