

Expected Utility

# Finally, games...

- One-person games :(



# Lotteries

- To every action there is a consequence...



# Lotteries

- So far, in our modeling situations each action deterministically leads to a particular consequence

# Lotteries

- From now on, the correspondence between actions and consequences is **stochastic**
- The choice of an action is viewed as choosing a lottery where the prizes are the consequences



# Lotteries

- We will be interested in preferences and choices over the set of lotteries

(a)	Color	White	Red	Green	Yellow
	Chance %	90	6	1	3
	Prize \$	0	45	30	−15
(b)	Color	White	Red	Green	Yellow
	Chance %	90	7	1	2
	Prize \$	0	45	−10	−15

# Lotteries

- Let  $\mathbf{Z}$  be a set of consequences (prizes)
  - for now,  $\mathbf{Z}$  is finite
- A lottery is a probability measure on  $\mathbf{Z}$ 
  - a lottery  $\mathbf{p}$  is a function that assigns a nonnegative number  $\mathbf{p}(\mathbf{z})$  to each prize  $\mathbf{z}$ , where  $\sum_{\mathbf{z} \in \mathbf{Z}} \mathbf{p}(\mathbf{z}) = 1$
  - The number  $\mathbf{p}(\mathbf{z})$  is taken to be the objective probability of obtaining the prize  $\mathbf{z}$  given the lottery  $\mathbf{p}$

# Lotteries

$$p_a(0) = 0.9$$

		White	Red	Green	Yellow	
(a)	Color	White	Red	Green	Yellow	
	Chance %	90	6	1	3	$p_a$
	Prize \$	0	45	30	-15	$z_a$
		White	Red	Green	Yellow	
(b)	Color	White	Red	Green	Yellow	
	Chance %	90	7	1	2	$p_b$
	Prize \$	0	45	-10	-15	$z_b$



# Lotteries

- Which lottery do you prefer?

(a)



$$Z_a = \{\$35\}$$

$$p_a(\$35)=1$$

(b)



$$Z_b = \{\$20, \$80\}$$

$$p_b(\$20)=0.75$$

$$p_b(\$80)=0.25$$

OR

# Lotteries

- Which lottery do you prefer?

(a)



OR

(b)



# Lotteries

- Denote by  $[z]$  the degenerate lottery for which  $p(z) = 1$
- We will use the notation

$$\alpha x \oplus (1 - \alpha)y$$

to denote the lottery in which the prize  $x$  is realized with probability  $\alpha$  and the prize  $y$  with probability  $1 - \alpha$

# Lotteries

- Denote by  $L(\mathbf{Z})$  the (infinite) space containing all lotteries with prizes in  $\mathbf{Z}$
- A simplex in Euclidean space:

$$L(\mathbf{Z}) = \{x \in \mathbb{R}_+^{\mathbf{Z}} \mid \sum x_z = 1\}$$

where  $\mathbb{R}_+^{\mathbf{Z}}$  is the set of functions from  $\mathbf{Z}$  into  $\mathbb{R}_+$

# Preferences over Lotteries

- Let us think about examples of “sound” preferences over a space  $L(Z)$ 
  - What makes a lottery better than the other?

# Preferences over Lotteries

- Preference for uniformity
  - The decision maker prefers the lottery that is less disperse where dispersion is measured by
$$\sum_z (p(z) - 1/|Z|)^2$$
  - Example:
    - a lottery over my music collection (all songs have the same chance of being played)

# Preferences over Lotteries

- Preference for most likelihood
  - The decision maker prefers  $\mathbf{p}$  to  $\mathbf{q}$  if  $\max_z \mathbf{p}(\mathbf{z})$  is greater than  $\max_z \mathbf{q}(\mathbf{z})$
  - Example
    - Weather conditions before going out



# Preferences over Lotteries

- The size of the support
  - The decision maker evaluates each lottery by the number of prizes that can be realized with positive probability

$$\mathbf{supp}(p) = \{z | p(z) > 0\}$$

- He prefers a lottery  $\mathbf{p}$  over a lottery  $\mathbf{q}$  if

$$|\mathbf{supp}(p)| \leq |\mathbf{supp}(q)|$$

- Example
  - A multiple choice question in an exam



# Preferences over Lotteries

- These three examples are degenerate
- **Ignored** the consequences and were dependent on the probability vectors alone

# Preferences over Lotteries

- Increasing the probability of a “good” outcome
  - The set  $Z$  is partitioned into two disjoint sets  $G$  and  $B$  (good and bad), and between two lotteries the decision maker prefers the lottery that yields “good” prizes with higher probability
  - Example
    - Choosing a route from city A to city B

# Preferences over Lotteries

- The worst case
  - The decision maker evaluates lotteries by the worst possible case
  - He attaches a number  $v(z)$  to each prize  $z$  and
$$p \succeq q \text{ if } \min\{v(z) \mid p(z) > 0\} \geq \min\{v(z) \mid q(z) > 0\}$$
- Example:
  - "This criterion is often used in computer science, where one algorithm is preferred to another if...
  - it functions better in the worst case independently of the likelihood of the worst case occurring"

# Preferences over Lotteries

- Comparing the most likely prize
  - The decision maker considers the prize in each lottery that is most likely (breaking ties in some arbitrary way) and compares two lotteries according to a basic preference relation over  $\mathbf{Z}$
- Example
  - Selecting a career

# Preferences over Lotteries

- Lexicographic preferences:
  - The prizes are ordered  $\mathbf{z}_1, \dots, \mathbf{z}_K$ , and the lottery  $\mathbf{p}$  is preferred to  $\mathbf{q}$  if
    - $(p(\mathbf{z}_1), \dots, p(\mathbf{z}_K)) \geq_L (q(\mathbf{z}_1), \dots, q(\mathbf{z}_K))$
- Example
  - choosing a movie to watch
    - explosions  $\succ_1$  funny  $\succ_2$  romance  $\succ_3$  etc

# Preferences over Lotteries

- Expected utility:
  - A number  $v(\mathbf{z})$  is attached to each prize, and a lottery  $\mathbf{p}$  is evaluated according to its expected  $v$ , that is, according to  $\sum_{\mathbf{z}} \mathbf{p}(\mathbf{z}) v(\mathbf{z})$
  - Thus,



- Example
  - Games in a casino

# Preferences over Lotteries

- Examples could be even richer...

# Preferences over Lotteries

- The richness of examples calls for the classification of preference relations over lotteries
- Study of properties that these relations satisfy



# Preferences over Lotteries

- Formally state general principles (axioms) that may apply to preferences over the space of lotteries
  - Consistency requirement
  - Procedural aspect of decision making

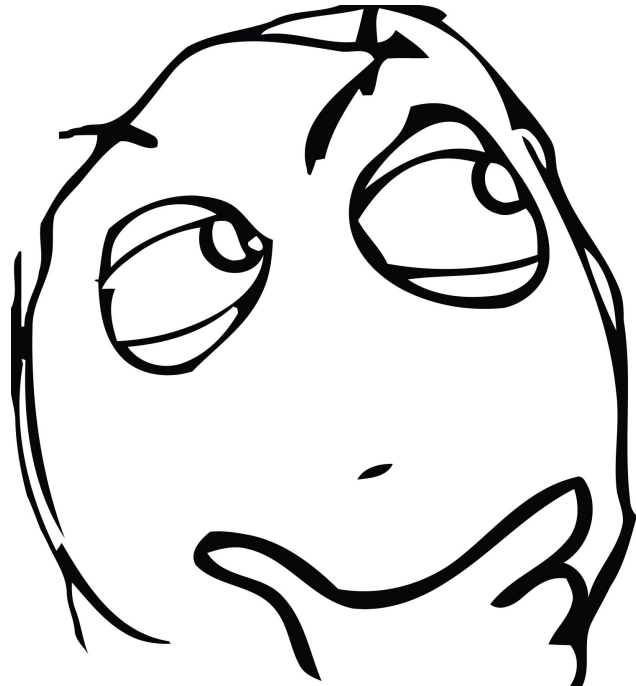


# Preferences over Lotteries

- A set of axioms characterizing a family of preferences is a justification for focusing on that specific family

# Preferences over Lotteries

- What are the desired properties of choice procedures over lotteries?



# von Neumann and Morgenstern Axiomatization

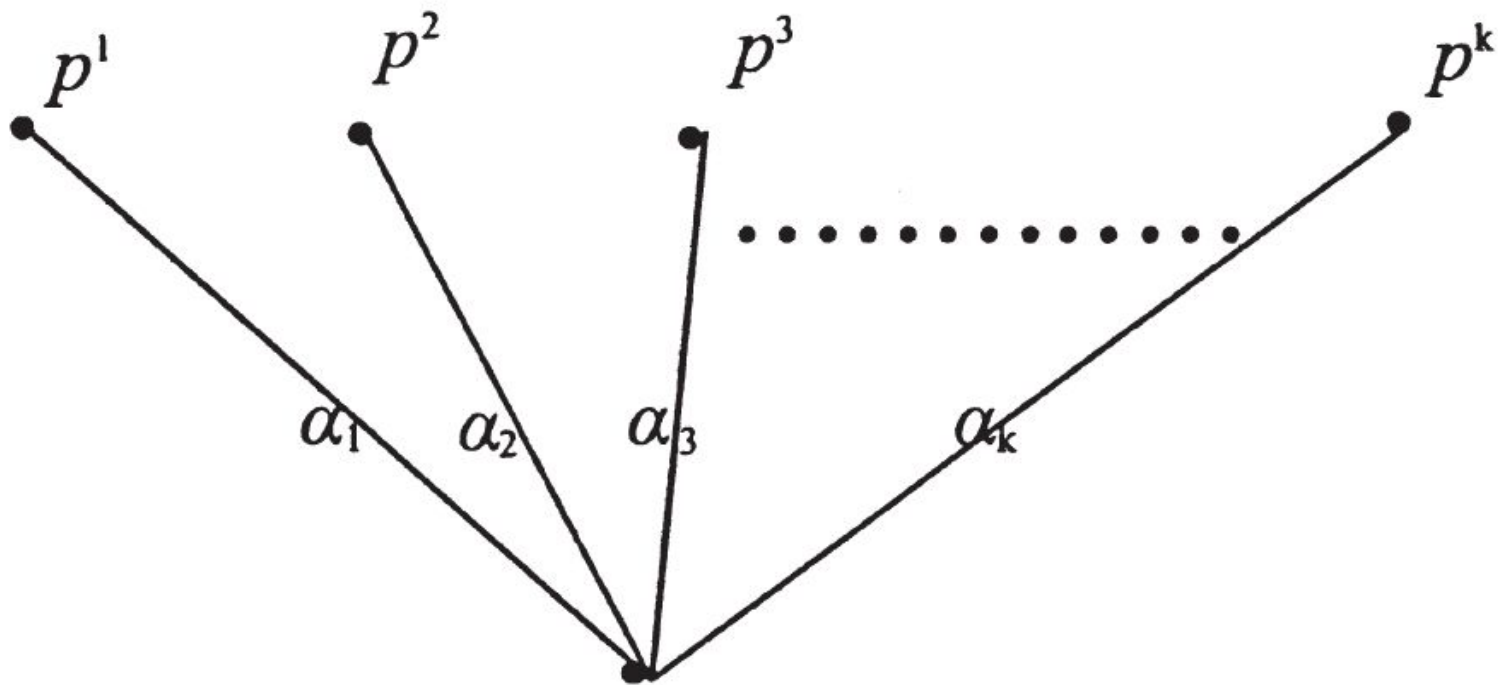
- Six axioms are usually presented
  - ordering of alternatives
  - reduction of compound lotteries
  - continuity
  - substitutability
  - transitivity
  - monotonicity

# An axiomatic treatment of utility

- **Assumption 1:** ordering of alternatives
  - The **preference relation**  $\succeq$  between two prizes  $\mathbf{z}_i$  and  $\mathbf{z}_j$  is transitive
  - Either  $\mathbf{z}_i \succeq \mathbf{z}_j$  or  $\mathbf{z}_j \succeq \mathbf{z}_i$
  - If  $\mathbf{z}_i \succeq \mathbf{z}_j$  and  $\mathbf{z}_j \succeq \mathbf{z}_k$ , then  $\mathbf{z}_i \succeq \mathbf{z}_k$
- Transitivity is not always seen in data!
  - specially when people are presented with paired comparisons

# Compound Lotteries

- $\oplus_{k=1}^K \alpha_k p^k$



# An axiomatic treatment of utility

- **Assumption 2:** reduction of compound lotteries

- A compound lottery

$$CL = \oplus_{k=1}^K \alpha_k p^k = (\alpha_1 p^1, \alpha_2 p^2, \dots, \alpha_K p^K)$$

- Any compound lottery is indifferent to a simple lottery with the same prize list

$$(\alpha_1 p^1, \alpha_2 p^2, \dots, \alpha_K p^K) \sim (p_1(z_1), p_2(z_2), \dots, p_r(z_r))$$

where

$$p_i(z_i) = \alpha_1 p^1(z_i) + \alpha_2 p^2(z_i) + \dots + \alpha_K p^K(z_i)$$

# An axiomatic treatment of utility

- **Assumption 3:** continuity
- Each prize  $z_i$  is indifferent to some lottery ticket involving just  $z_1$  (the best prize) and  $z_r$  (the worst)
- There exists a number  $p$  such that  $z_i$  is indifferent to  $[pz_1, (1-p)z_r]$
- And for  $Z = \{\$1, \$0.01, death\}$ ?



# An axiomatic treatment of utility

- **Assumption 4:** substitutibility
  - If  $z_i \sim z_j$ , then one may substitute the other in a lottery

# An axiomatic treatment of utility

- **Assumption 5:** transitivity
  - Preference and indifference among lotteries (or lottery tickets) are transitive relations

# An axiomatic treatment of utility

- From these five assumptions
  - it is possible to find for any lottery ticket one to which it is indifferent and which only involves  $\mathbf{z}_1$  and  $\mathbf{z}_r$

# An axiomatic treatment of utility

- **Assumption 6:** monotonicity
  - A lottery  $[p(z_1), (1-p)z_r]$  is preferred or indifferent to  $[p'(z_1), (1-p')z_r]$  if and only if...
  - ...  $p \geq p'$
  - Two lotteries involving only two prizes, I should prefer the one which renders the most preferred prize more probable
    - But is it always?



# von Neumann and Morgenstern Axiomatization

- Do these axioms make sense?
  - If we accept these six axioms, it turns out that we have no choice but to
    - accept the existence of single-dimensional utility functions
    - whose expected values agents want to maximize
  - If we do not want to reach this conclusion, we must therefore give up at least one of the axioms

# von Neumann and Morgenstern Axiomatization

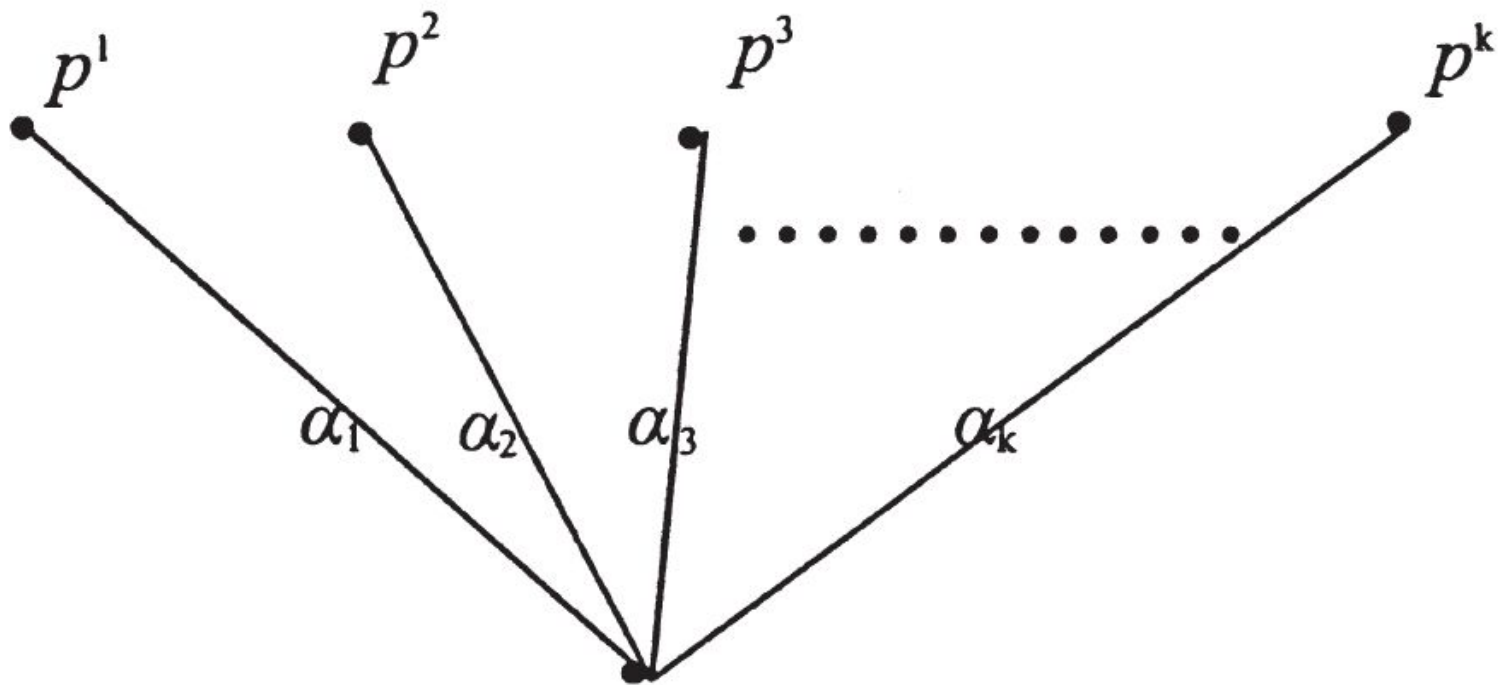
- Theorem (von Neumann and Morgenstern, 1944)
  - If  $\succsim$  satisfies **assumptions 1** through **6**, there are numbers  $u_i$  associated with  $z_i$  such that, for two lotteries  $p$  and  $q$ ,
  - $p_1(z_1) u_1 + \dots + p_r(z_r) u_r$  and  $q_1(z_1) u_1 + \dots + q_r(z_r) u_r$
  - reflect the preferences between the lotteries

# von Neumann and Morgenstern Axiomatization

- Two axioms may do the job
  - **I**: independence
  - **C**: continuity

# Compound Lotteries

- $\oplus_{k=1}^K \alpha_k p^k$





# An axiomatic treatment of utility

- **I**: independence
  - For any  $p, q, r \in L(Z)$
  - and any  $\alpha \in (0, 1)$ ,
  - $p \succeq q$  iff
  - $\alpha p \oplus (1 - \alpha)r \succeq \alpha q \oplus (1 - \alpha)r$
  - No correlation between lotteries

# An axiomatic treatment of utility

- **C**: continuity
  - If  $p \succ q$ , then there are neighborhoods  $B(p)$  of  $p$  and  $B(q)$  of  $q$  such that for all  $p' \in B(p)$  and  $q' \in B(q)$ ,  $p' \succ q'$
  - Alternatively, if  $p \succ q \succ r$ , then there exists  $\alpha \in (0, 1)$  such that:
    - $q \sim [\alpha p \oplus (1 - \alpha)r]$
  - Reality may be different...
    - if  $r$  is a lottery involving an extremely bad prize such as 'death'

# Preferences over Lotteries

- Preference for most likelihood
  - The decision maker prefers  $p$  to  $q$  if  $\max_z p(z)$  is greater than  $\max_z q(z)$
  - Example
    - Choosing which clothes I should wear

# An axiomatic treatment of utility

- Preferences for most likelihood
  - Satisfies **C** since the function  $\max\{p_1, \dots, p_K\}$  that represents it is continuous in probabilities
  - It does not satisfy **I** since, for example,
    - $[z_1] \sim [z_2]$ ,
    - $[z_1] = 1/2[z_1] \oplus 1/2[z_1] > 1/2[z_2] \oplus 1/2[z_1]$

# Preferences over Lotteries

- The worst case
  - The decision maker evaluates lotteries by the worst possible case
  - He attaches a number  $v(z)$  to each prize  $z$  and
$$p \succeq q \text{ if } \min\{v(z) \mid p(z) > 0\} \geq \min\{v(z) \mid q(z) > 0\}$$
- Example:
  - "This criterion is often used in computer science, where one algorithm is preferred to another if it functions better in the worst case independently of the likelihood of the worst case occurring"

# An axiomatic treatment of utility

- The worst case:
  - Does not satisfy **C**
  - In the two-prize case
  - $v(z_1) > v(z_2)$ ,  $[z_1] \succ 1/2[z_1] \oplus 1/2[z_2]$
  - In  $R_+^2$ , we can rewrite this as
  - $(1, 0) \succ (1/2, 1/2)$
  - Any neighborhood of  $(1, 0)$  contains lotteries that are not strictly preferred to  $(1/2, 1/2)$
  - Does not satisfy **I**
  - $[z_1] \succ [z_2]$  but  $1/2[z_1] \oplus 1/2[z_2] \sim 1/2[z_2] \oplus 1/2[z_2]$

# Preferences over Lotteries

- Expected utility:
  - A number  $v(\mathbf{z})$  is attached to each prize, and a lottery  $\mathbf{p}$  is evaluated according to its expected  $v$ , that is, according to  $\sum_{\mathbf{z}} \mathbf{p}(\mathbf{z}) v(\mathbf{z})$
  - Thus,



- Example
  - Games in a casino

# An axiomatic treatment of utility

- Expected utility:

$$\begin{aligned} U(\oplus_{k=1}^K \alpha_k p^k) &= \sum_{z \in Z} [\oplus_{k=1}^K \alpha_k p^k](z) v(z) = \sum_{z \in Z} \left[ \sum_{k=1}^K \alpha_k p^k(z) \right] v(z) \\ &= \sum_{k=1}^K \alpha_k \left[ \sum_{z \in Z} p^k(z) v(z) \right] = \sum_{k=1}^K \alpha_k U(p^k). \end{aligned}$$

- It is linear, so it satisfies **I**
- It is continuous in the probability vector, so it satisfies **C**



# Preferences over Lotteries

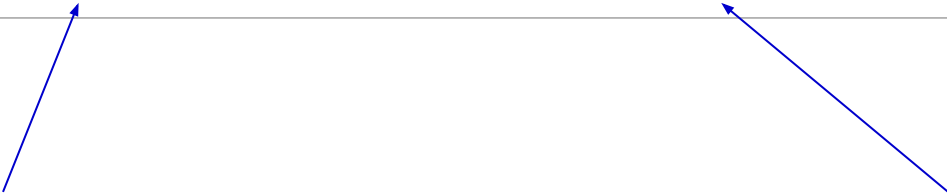
- Increasing the probability of a “good” outcome
  - The set **Z** is partitioned into two disjoint sets **G** and **B** (good and bad), and between two lotteries the decision maker prefers the lottery that yields “good” prizes with higher probability
  - Example
    - Choosing a route from city A to city B

# An axiomatic treatment of utility

- Increasing the probability of a “good” consequence
  - Satisfies **I** and **C**, since it can be represented by the expectation of **v** where **v(z) = 1** for **z ∈ G** and **v(z) = 0** for **z ∈ B**

# Utility Representation

- Theorem (vNM):
  - Let  $\succsim$  be a preference relation over  $L(\mathbf{Z})$  satisfying **I** and **C**
  - There are numbers  $(v(\mathbf{z}))_{\mathbf{z} \in \mathbf{Z}}$  such that



$U(\mathbf{p})$  is the utility of  
lottery  $\mathbf{p}$

$v(\mathbf{z})$  is the vNM utility,  
representing  $\succsim$  over  $\mathbf{Z}$

# Utility Representation

- Let  $\succsim$  be a preference relation over a set of lotteries
- If to each lottery  $p$  there is assigned a number  $U(p)$  such that  $U(p) \geq U(q)$  iff  $p \succsim q$ ,
- then there is a utility function  $U$  over  $L(Z)$
- When faced about uncertainty about which outcomes he will receive, the agent prefers outcomes that maximize  $U$

# Utility Representation

- If **I** and **C** (or the six axioms presented previously) are met, then there is a linear vNM utility function **v** over the prizes
- $v(z_1) = 1$ ,
- $v(z_i) = v_i$ , for  $1 < i < r$
- $v(z_r) = 0$ ,
- $U(p(z_1), \dots, p(z_r)) = p(z_1)v(z_1) + \dots + p(z_r)v(z_r)$

# Utility Representation

- Let  $\mathbf{U}(\mathbf{p})$  satisfy  $\succeq$  over  $\mathbf{L}(\mathbf{Z})$  from a linear vNM utility function  $\mathbf{v}(\mathbf{z})$
- For some  $\alpha > 0$  and  $\beta$ , we can make
- $w(\mathbf{z}) = \alpha \mathbf{v}(\mathbf{z}) + \beta$
- $W(\mathbf{p}) = \sum_{\mathbf{z} \in \mathbf{Z}} p(\mathbf{z}) w(\mathbf{z})$
- $W(\mathbf{p})$  will also satisfy  $\succeq$  over  $\mathbf{L}(\mathbf{Z})$

# Utility Representation

- What does this mean?
  - The absolute magnitudes of the utility function evaluated at different outcomes are unimportant
  - Instead, every positive affine transformation of a utility function yields another utility function for the same agent
    - In other words, if  $u(o)$  is a utility function for a given agent then  $u'(o) = \alpha u(o) + \beta$  is also a utility function for the same agent, as long as  $\alpha$  and  $\beta$  are constants and  $\alpha$  is positive

# Utility Representation

- Which lottery do you prefer?
  - $L = (0.25z_1, 0.25z_2, 0.25z_3, 0.25z_4)$
  - $L' = (0.15z_1, 0.50z_2, 0.15z_3, 0.20z_4)$
  - Suppose by continuity
    - $z_2 \sim z_2' = (0.6z_1, 0.4z_4)$
    - $z_3 \sim z_3' = (0.2z_1, 0.8z_4)$
  - $L = (0.25z_1, 0.25(0.6z_1, 0.4z_4), 0.25(0.2z_1, 0.8z_4), 0.25z_4)$
  - $L = (0.45z_1, 0.55z_4), L' = (0.48z_1, 0.52z_4)$
  - Thus,  $L' > L$



# Utility Representation

- Which lottery do you prefer?
  - $L = (0.25z_1, 0.25z_2, 0.25z_3, 0.25z_4)$
  - $L' = (0.15z_1, 0.50z_2, 0.15z_3, 0.20z_4)$
  - Suppose by continuity
    - $z_2 \sim z_2' = (0.6z_1, 0.4z_4)$
    - $z_3 \sim z_3' = (0.2z_1, 0.8z_4)$
  - $u'(x) = 2u(x) - 0.4$



# V or F

- A lottery  $\mathbf{p}$  is preferred to  $\mathbf{q}$  because the expected utility  $U(\mathbf{p})$  is greater than the expected utility  $U(\mathbf{q})$
- Fallacy!
  - The problem is the causal relationship, since the utility function is built upon the preferences and not the other way around

# V or F

- Suppose that  $A \succ B \succ C \succ D$  and that the vNM utilities of these prizes satisfy  $v(A) + v(D) = v(B) + v(C)$ , then  $(1/2B, 1/2C)$  should be preferred to  $(1/2A, 1/2D)$  because, although they have the same expected utility, the former has the smaller utility variance
- Fallacy!
  - Misinterpretation of utility: preferences precede the utility function

# V or F

- Suppose that  $A \succ B \succ C \succ D$  and that the vNM utility function has the property that  $v(A) - v(B) > v(C) - v(D)$ , then the change from  $B$  to  $A$  is more preferred than the change from  $D$  to  $C$
- Fallacy!
  - if  $D$  is “death”... :)

# Interpersonal comparison of utility

- Very hard to define a universal value of utility
  - It changes from person to person
  - a gamble of \$1 between a rich and a poor person
  - possible way to deal with it: normalize the maximum value to 1 and the lowest to 0