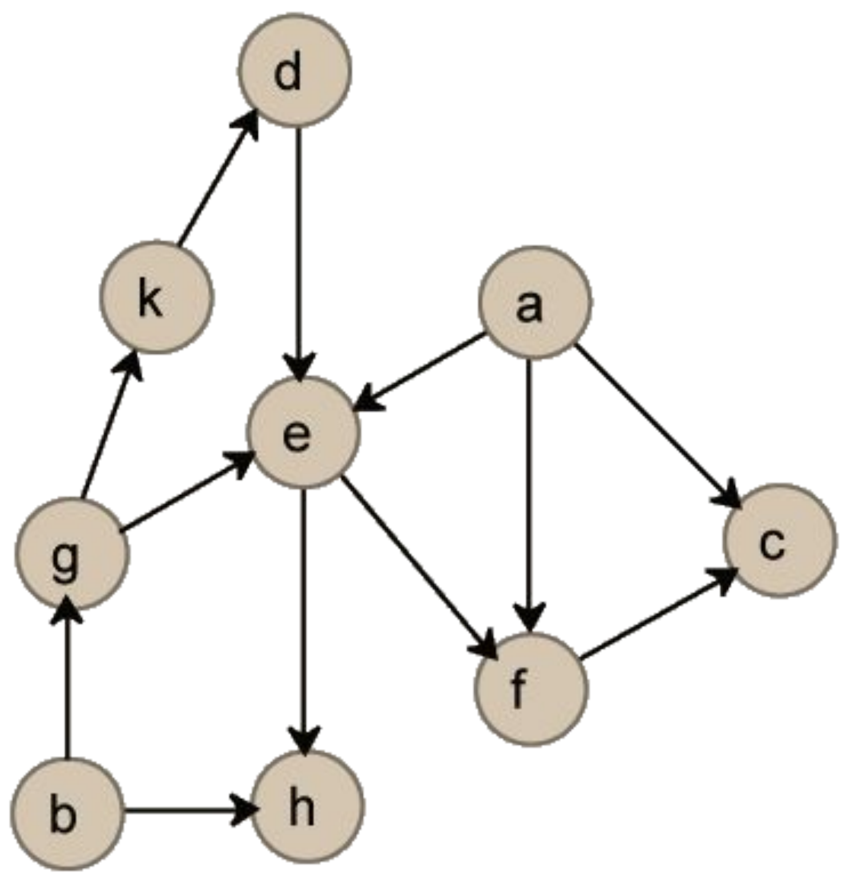
Utility

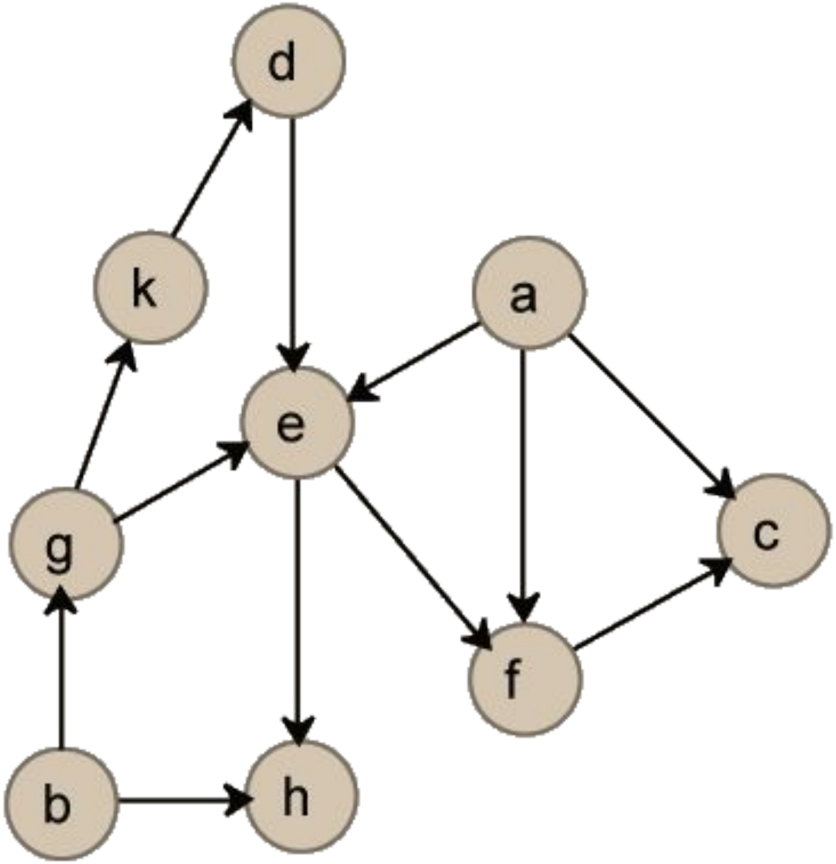
Reminder: preferences

- DAG (Directed Acyclic Graph)
  - There is a direct edge from ***i*** to ***j*** if $j \succ i$



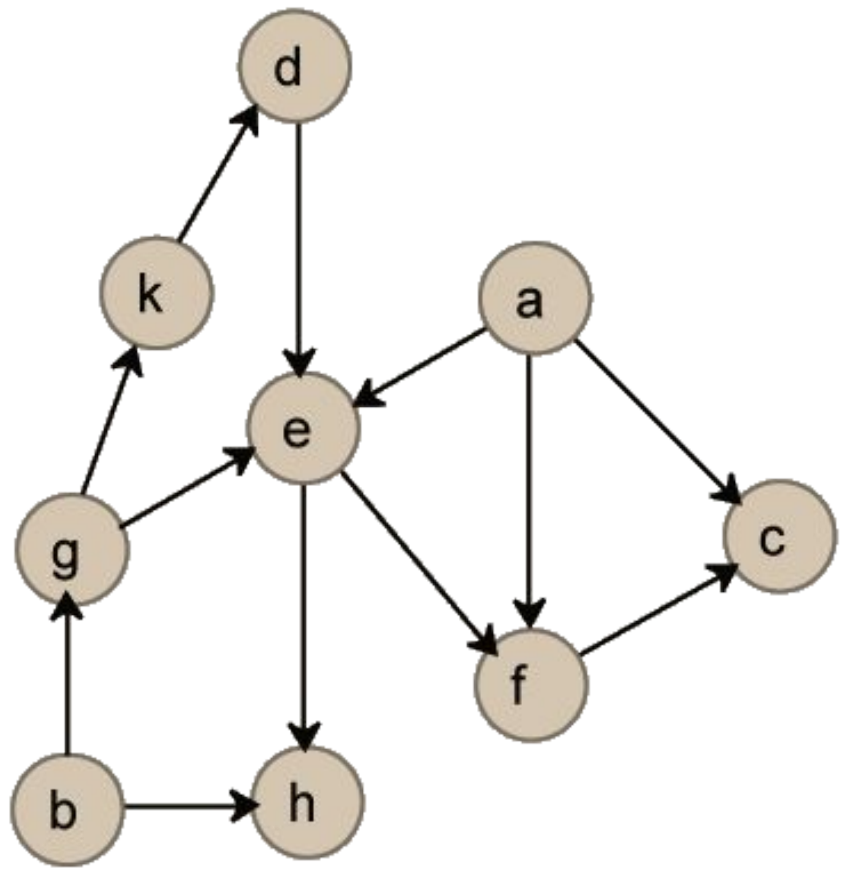
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** which are the most preferred nodes?



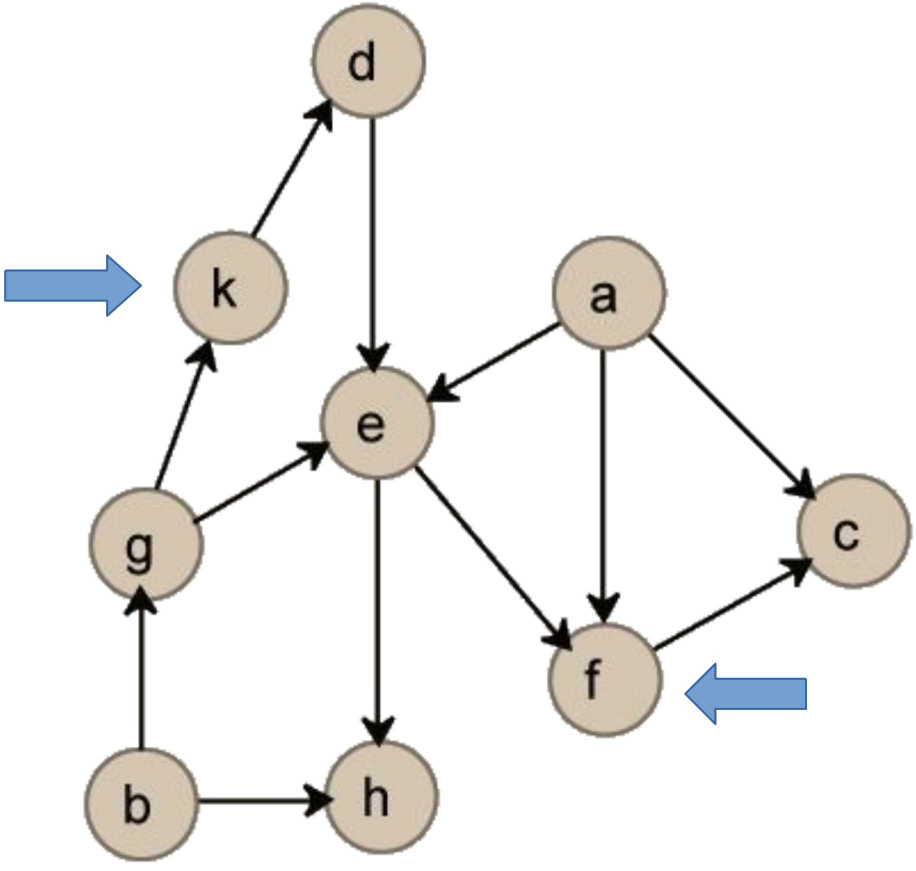
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** which are the least preferred nodes?



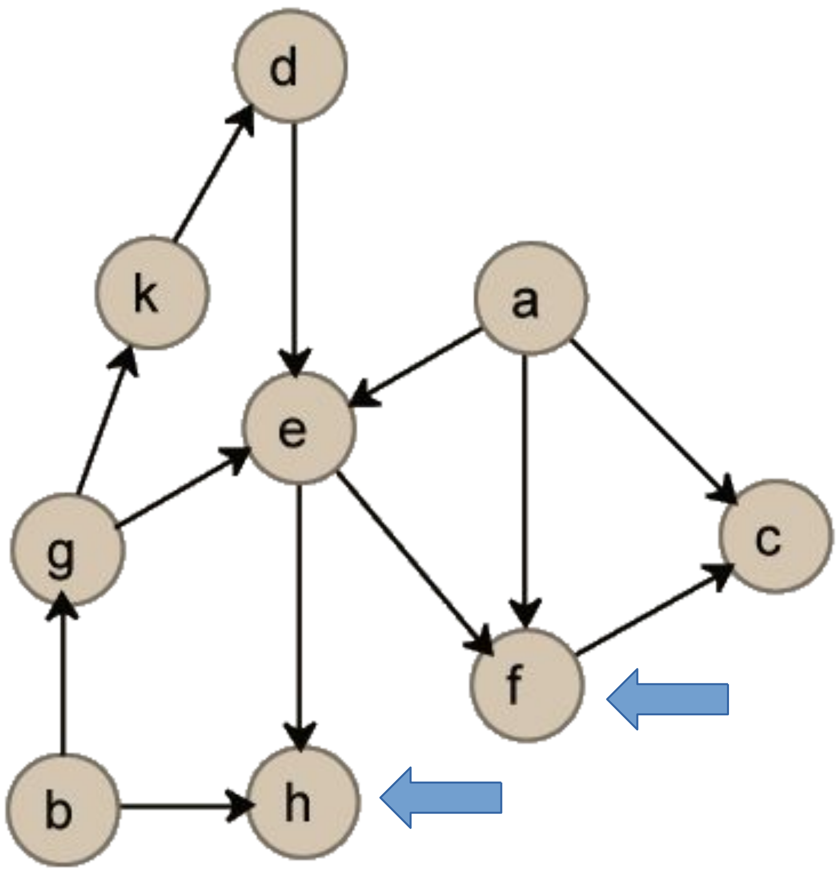
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** is ***k*** preferred to ***f***?



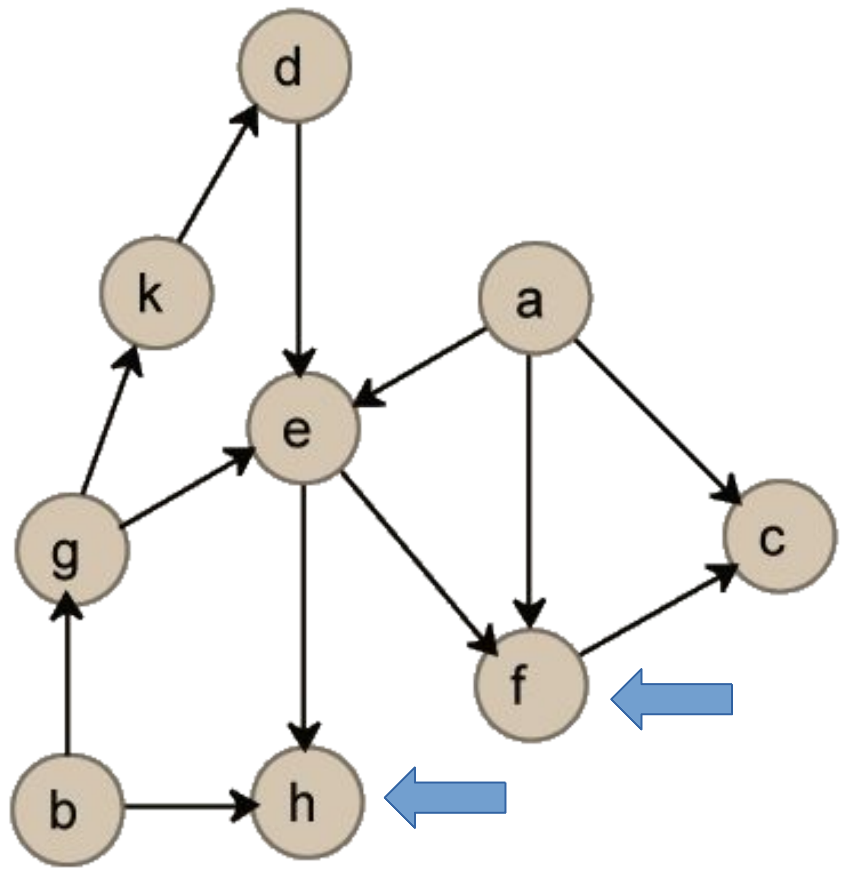
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** is ***h*** preferred to ***f?***



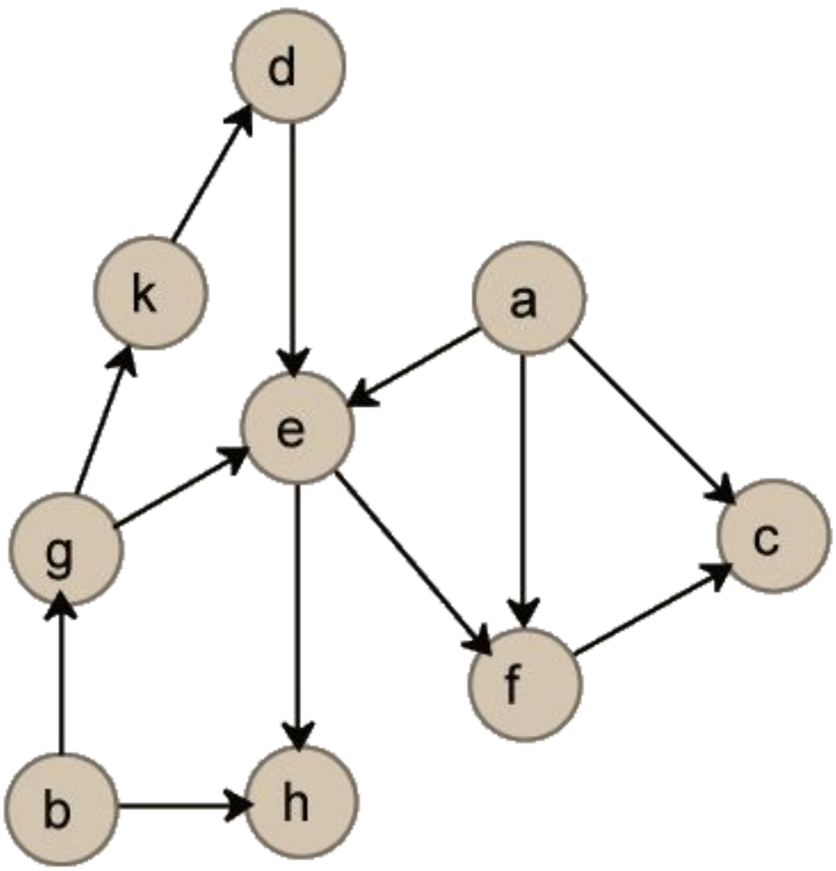
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** how can I identify two indifferent nodes?



Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if ***j*** $\succ$ ***i***
  - ***Q:*** Is this preference relation transitive?



The Concept of Utility Representation

- If the number of alternatives is **small**
  - preference relation can be an ordered list from best to worst

The Concept of Utility Representation

- In some cases, the alternatives are grouped into a <u>small number of categories</u>
  - we describe the preferences on X by specifying the preferences on the set of categories

The Concept of Utility Representation

- “I prefer the fastest car”
- “I prefer the taller basketball player”
- “I prefer the more expensive present”
- “I prefer a teacher who gives higher grades”

The Concept of Utility Representation

- They can naturally be specified by
  - $\mathbf{x} \succsim \mathbf{y}$ if $\mathbf{V(x)} \geq \mathbf{V(y)}$ (or $\mathbf{V(x)} \leq \mathbf{V(y)}$), where $\boldsymbol{V : X} \rightarrow \mathbb{R}$
- For example, the preferences stated by "I prefer the taller basketball player" can be expressed formally by
  - **X** is the set of all conceivable basketball players, and **V(x)** is the height of player **x**

The Concept of Utility Representation

- Note that the statement $\boldsymbol{x} \succsim \boldsymbol{y}$ if $\boldsymbol{V(x) \geq V(y)}$ always defines a preference relation because…

  … the relation **≥** on $\mathbb{R}$ satisfies <u>completeness</u> and <u>transitivity</u>

The Concept of Utility Representation

- We say that the function $\boldsymbol{U : X \rightarrow \mathbb{R}}$ represents the preference $\succsim$ if for all ***x*** and ***y*** $\in$ ***X, x*** $\succsim$ ***y*** iff $\boldsymbol{U(x) \geq U(y)}$
- If the function ***U*** represents the preference relation $\succsim$, we refer to it as a ***utility function***, and we say that $\succsim$ has a utility representation

The Concept of Utility Representation

- Claim:

  If $\boldsymbol{U}$ represents $\succsim$, then for any <u>strictly increasing</u> function $\boldsymbol{f : \mathbb{R} \to \mathbb{R}}$, the function $\boldsymbol{V(x) = f(U(x))}$ …

  … represents $\succsim$ as well

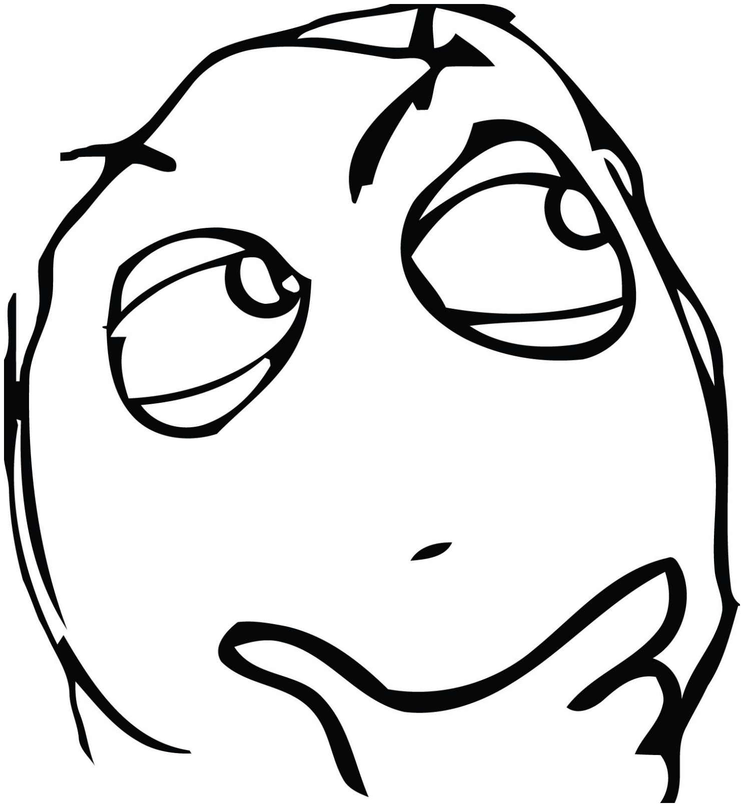
- Proof:
  - $\boldsymbol{a \succsim b}$
  - ***iff*** $\boldsymbol{U(a) \geq U(b)}$ (since $\boldsymbol{U}$ represents $\succsim$)
  - ***iff*** $\boldsymbol{f(U(a)) \geq f(U(b))}$ (since $\boldsymbol{f}$ is strictly increasing)
  - ***iff*** $\boldsymbol{V(a) \geq V(b)}$

The Concept of Utility Representation

- Claim:
  - If $\boldsymbol{U}$ represents $\succsim$, then for any <u>strictly increasing</u> function $\boldsymbol{f} : \mathbb{R} \rightarrow \mathbb{R}$, the function $\boldsymbol{V(x) = f(U(x))}$ represents $\succsim$ as well
- What does this mean?
  - Various forms of utility functions may exist to represent a preference relation

Existence of a Utility Representation

- If any preference relation could be represented by a ***utility function***, then…

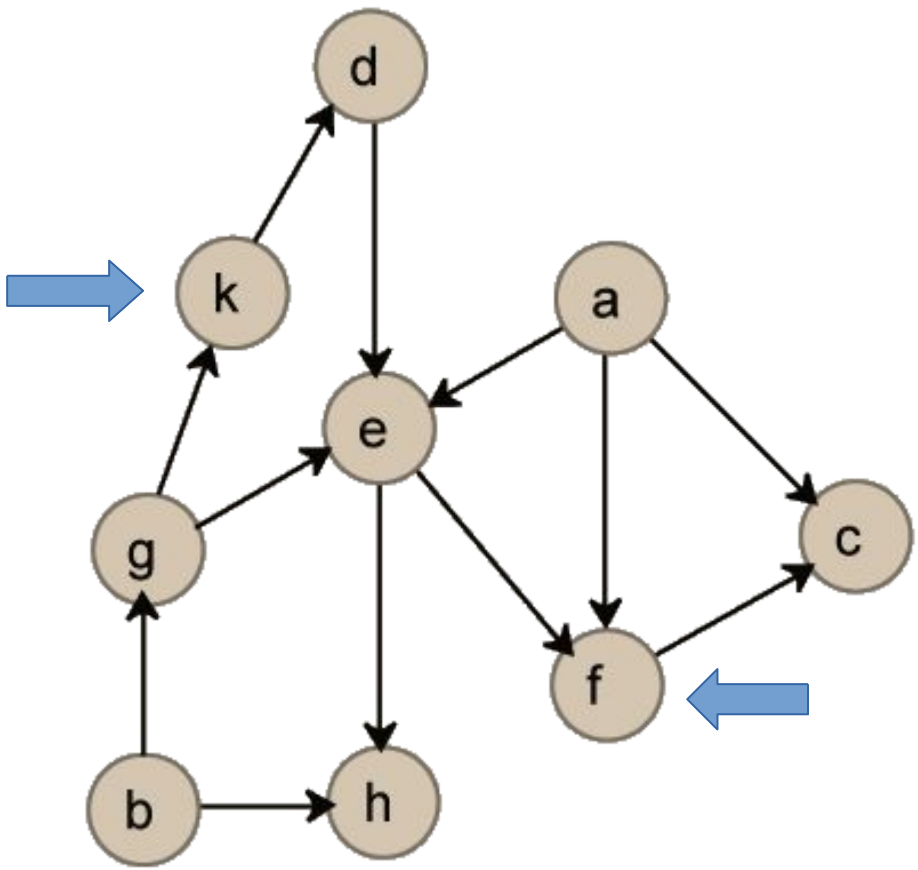


Existence of a Utility Representation

- If any preference relation could be represented by a ***utility function***, then it would "grant a license" to use utility functions rather than preference relations with no loss of generality
- Why is this important?

Reminder: preferences

- Easier to compare two items
  - ***Q:*** is ***k*** preferred to ***f***?



Existence of a Utility Representation

- Possibility of numerical representations carrying additional meanings
  - Ex:
  - ***a*** is preferred to ***b*** more than ***c*** is preferred to ***d***

Existence of a Utility Representation

- Under what assumptions do utility representations exist?

Existence of a Utility Representation

- Lemma:
  - In any finite set $\boldsymbol{A} \subseteq \boldsymbol{X}$, there is a minimal element (similarly, there is also a maximal element)
- That is, there is an element that is less preferred to any other element
- Which property guarantees that?

Existence of a Utility Representation

- Lemma:
  - In any <u>finite</u> set $\boldsymbol{A} \subseteq \boldsymbol{X}$, there is a minimal element (similarly, there is also a maximal element)
- Proof:
  - By induction on the size of $\boldsymbol{A}$
  - If $\boldsymbol{A}$ is a singleton, then by completeness its only element is minimal
  - For the inductive step, let $\boldsymbol{A}$ be of cardinality $\boldsymbol{n+1}$ and let $\boldsymbol{x} \in \boldsymbol{A}$. The set $\boldsymbol{A - \{x\}}$ is of cardinality $\boldsymbol{n}$ and by the inductive assumption has a minimal element denoted by $\boldsymbol{y}$
  - If $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{y}$ is minimal in $\boldsymbol{A}$
  - If $\boldsymbol{y} \succsim \boldsymbol{x}$, then by ***<u>transitivity</u>*** $\boldsymbol{z} \succsim \boldsymbol{x}$ for all $\boldsymbol{z} \in \boldsymbol{A - \{x\}}$, and thus $\boldsymbol{x}$ is minimal

Reminder

- Recall that ***X*** is ***countable*** and ***infinite*** if there is a ***one-to-one function*** from ***X*** onto the natural numbers
- It is possible to specify an enumeration of all its members
  - $\{x_n\}_{n=1,2,...}$

Existence of a Utility Representation

- **Claim:**
  - If ***X*** is countable, then any preference relation on ***X*** has a utility representation with a range ***(−1, 1)***

Existence of a Utility Representation

- Proof:
  - Let $\{x_n\}$ be an enumeration of all elements in X
  - Set $U(x_1) = 0$
  - Assume that you have completed the definition of the values $U(x_1), . . . ,U(x_{n-1})$ so that $x_k \succsim x_l$ *iff* $U(x_k) \geq U(x_l)$
  - If x_n is indifferent to x_k for some $k < n$, then assign $U(x_n) = U(x_k)$
  - If not, by transitivity, all numbers in the nonempty set $\{U(x_k)|\ x_k \prec x_n\} \cup \{-1\}$ are below all numbers in the nonempty set $\{U(x_k)|\ x_k \succ x_n\} \cup \{1\}$
  - Choose $U(x_n)$ to be between the two sets
  - This guarantees that for any $k < n$ we have $x_n \succsim x_k$ *iff* $U(x_n) \geq U(x_k)$
  - Thus, the function defined on $\{x_1, . . . , x_n\}$ represents the preferences on those elements
  - To complete the proof that U represents $\succsim$, take any two elements, x and $y \in X$. For some k and l we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ *iff* $U(x_k) \geq U(x_l)$

Existence of a Utility Representation

- Let's put that in practice…

Which bundle do you prefer?

Bundle #1

3 chocolates
6 snacks
4 sodas

Bundle #2

2 chocolates
2 snacks
10 sodas

Bundle #3

1 beer
1 snack
1 chocolate

Lexicographic Preferences

Bundle #1

3 chocolates
6 snacks
4 sodas

Bundle #2

2 chocolates
2 snacks
10 sodas

Bundle #3

1 beer
1 snack
1 chocolate

Lexicographic Preferences

- Let $(\succsim_k)_{k=1,\ldots,K}$ be a ***K-tuple*** of preferences over the set X
- The lexicographic preferences induced by those preferences are defined by $x \succsim_L y$ if
  - (1) there is k^* such that for all $k < k^*$ we have $x \sim_k y$ and $x \succ_{k*} y$ or
  - (2) $x \sim_k y$ for all k
- The lower the k, the more relevant it is

Lexicographic Preferences

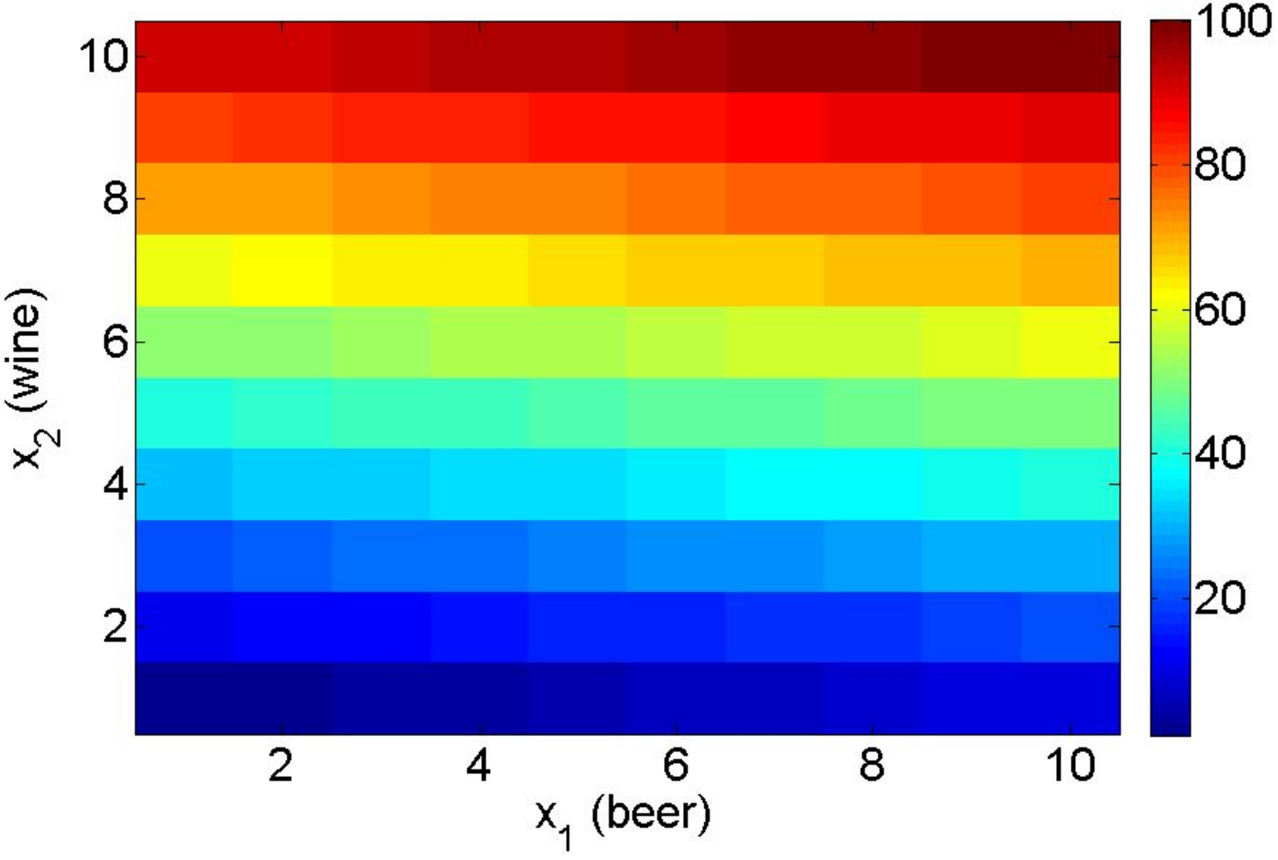
- Example:
  - Let $\boldsymbol{X}$ be the unit square, that is, $\boldsymbol{X = [0, 1] \times [0, 1]}$
  - Let $\boldsymbol{x} \succsim_{\boldsymbol{k}} \boldsymbol{y}$ if $\boldsymbol{x_k \geq y_k}$
  - The lexicographic preferences $\succsim_{\boldsymbol{L}}$ induced from $\succsim_{\boldsymbol{1}}$ and $\succsim_{\boldsymbol{2}}$ are:
    - $\boldsymbol{(a_1, a_2) \succsim_L (b_1, b_2)}$ if $\boldsymbol{a_1 > b_1}$ or both $\boldsymbol{a_1 = b_1}$ and $\boldsymbol{a_2 \geq b_2}$

Lexicographic Preferences

- **Claim:**
  - The lexicographic preference relation $\succsim_L$ on $[0, 1] \times [0, 1]$, induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ ($k = 1, 2$), **does not have** a utility representation

Lexicographic Preferences

- Why it cannot be represented by a utility function?



$(x_1,x_2) \succsim (y_1,y_2)$ if
$(x_2 > y_2)$ or
$(x_2 = y_2$ and ***$x_1 > y_1)$***

Lexicographic Preferences

- Proof:

Theorem: $\mathbb{R}$ is uncountable

Theorem: every subset of $\mathbb{Q}$ is countable

Corollary: there is no one-to-one function $f : \mathbb{R} \rightarrow \mathbb{Q}$

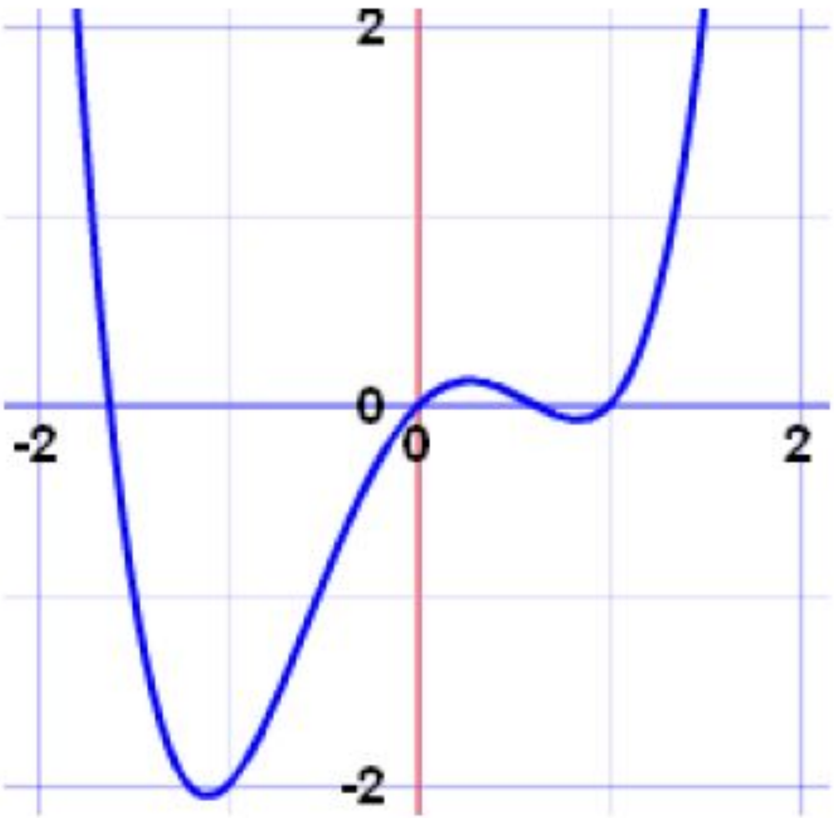
Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$

Lexicographic Preferences

- Proof:
  - Assume by contradiction that the function $u : X \rightarrow \mathbb{R}$ represents $\succsim_L$
  - For any $a \in [0, 1]$, $(a, 1) \succ_L (a, 0)$ and, therefore, $u(a, 1) > u(a, 0)$
  - Let $q(a)$ be a ***rational number*** in the nonempty interval $I_a = (u(a, 0), u(a, 1))$
  - The function q is a function from $[0, 1]$ into the set of ***rational numbers*** $\mathbb{Q}$
  - It is a one-to-one function since if $b > a$, then $(b, 0) \succ_L (a, 1)$, $u(b, 0) > u(a, 1)$, and, therefore, the intervals I_a and I_b are disjoint
    - For all b, there is a rational $q(b)$ number
  - Thus, $q(a) \neq q(b)$
  - But the cardinality of the rational numbers is lower than that of the continuum, a contradiction

Reminder: continuous functions

- Is this function continuous?



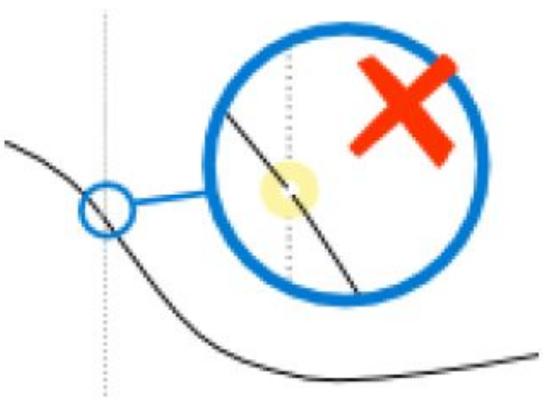
Reminder: continuous functions

- A function is continuous when its graph is a single unbroken curve…

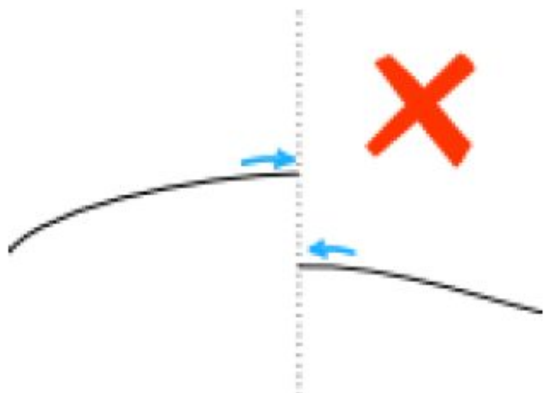
- … that you could draw without lifting your pen from the paper

Reminder: continuous functions

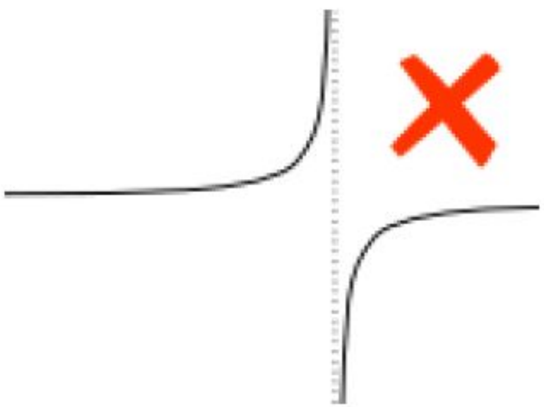
- Examples:



Not Continuous
(hole)



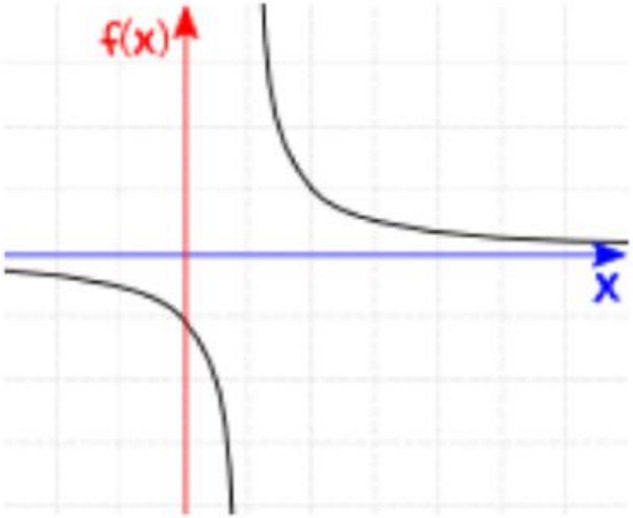
Not Continuous
(jump)



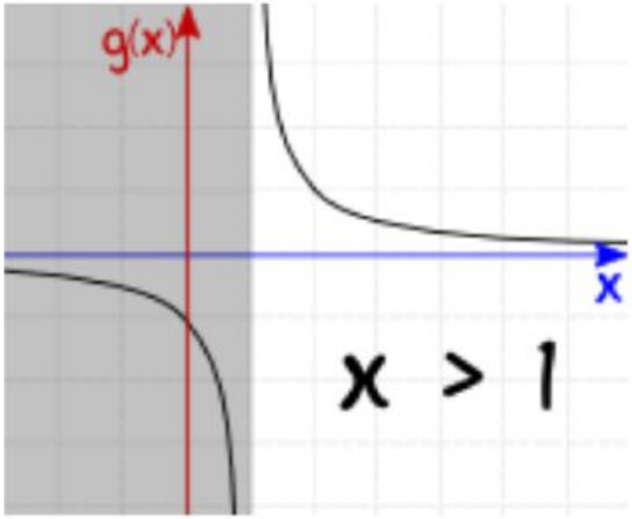
Not Continuous
(vertical asymptote)

Reminder: continuous functions

- Examples:



f(x) = 1/(x-1)
over all **Real Numbers**
NOT continuous



g(x) = 1/(x-1) for **x>1**
Continuous

Reminder: continuous functions

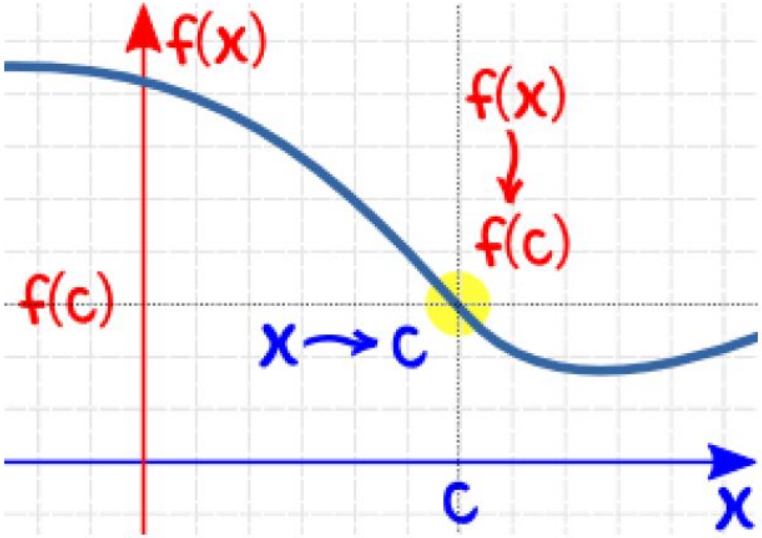
- Formal definition:
  - A function ***f*** is <u>continuous</u> when, for every value ***c*** in its Domain:
    - ***f(c)*** is defined, and

$$\lim_{x \to c} f(x) = f(c)$$

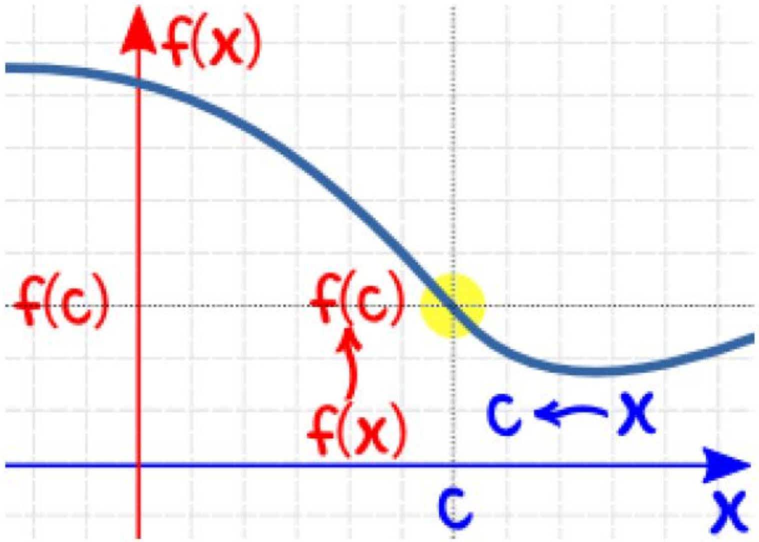
    - the limit of ***f(x)*** as ***x*** approaches ***c*** equals ***f(c)***
    - “as ***x*** gets closer and closer to ***c***, ***f(x)*** gets closer and closer to ***f(c)***”

Reminder: continuous functions

as x approaches c (from left)
then f(x) approaches f(c)



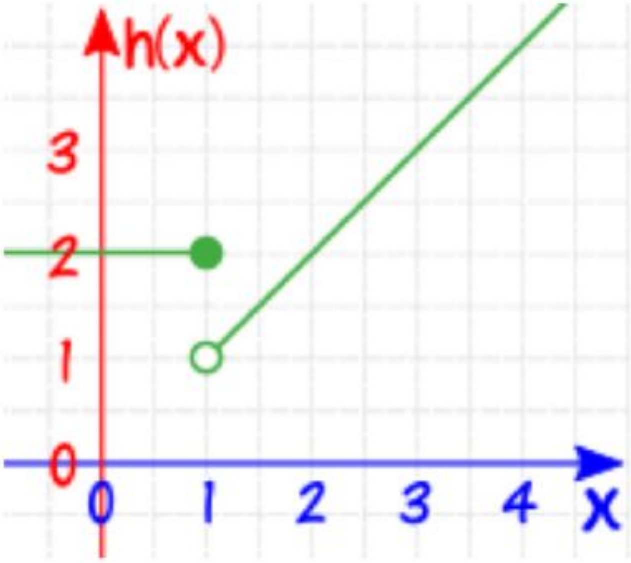
AND
as x approaches c (from right)
then f(x) approaches f(c)



Reminder: continuous functions

$$h(x) = \begin{cases} 2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$$

which looks like:



- It is defined at ***x=1***, since ***h(1) = 2***
- But you cannot say what the <u>limit</u> is at ***x=1***
  - from the left: ***2***
  - from the right: ***1***

Continuity of Preferences

- Why is it important to talk about this?

Continuity of Preferences

- In economics, the set ***X*** is often an ***infinite*** subset of a Euclidean space
- In $\mathbb{R}^1$
  - Ex: gold
- In $\mathbb{R}^2$
  - Ex: (salary, vacation time per year)
- In $\mathbb{R}^3$
  - Ex: (coffee, bread, milk)

Continuity of Preferences

- In economics, the set ***X*** is often an ***infinite*** subset of a Euclidean space
- Is there a utility representation in such a case?

Continuity of Preferences

- Which one do you prefer?

12 free months

100k miles

OR

120

20 boosters

Continuity of Preferences

- Which one do you prefer?

12 free months

OR

99.8k miles

120

19 boosters

Continuity of Preferences

- Which one do you prefer?

12 free months

100k miles

OR

119

20 boosters

Continuity of Preferences

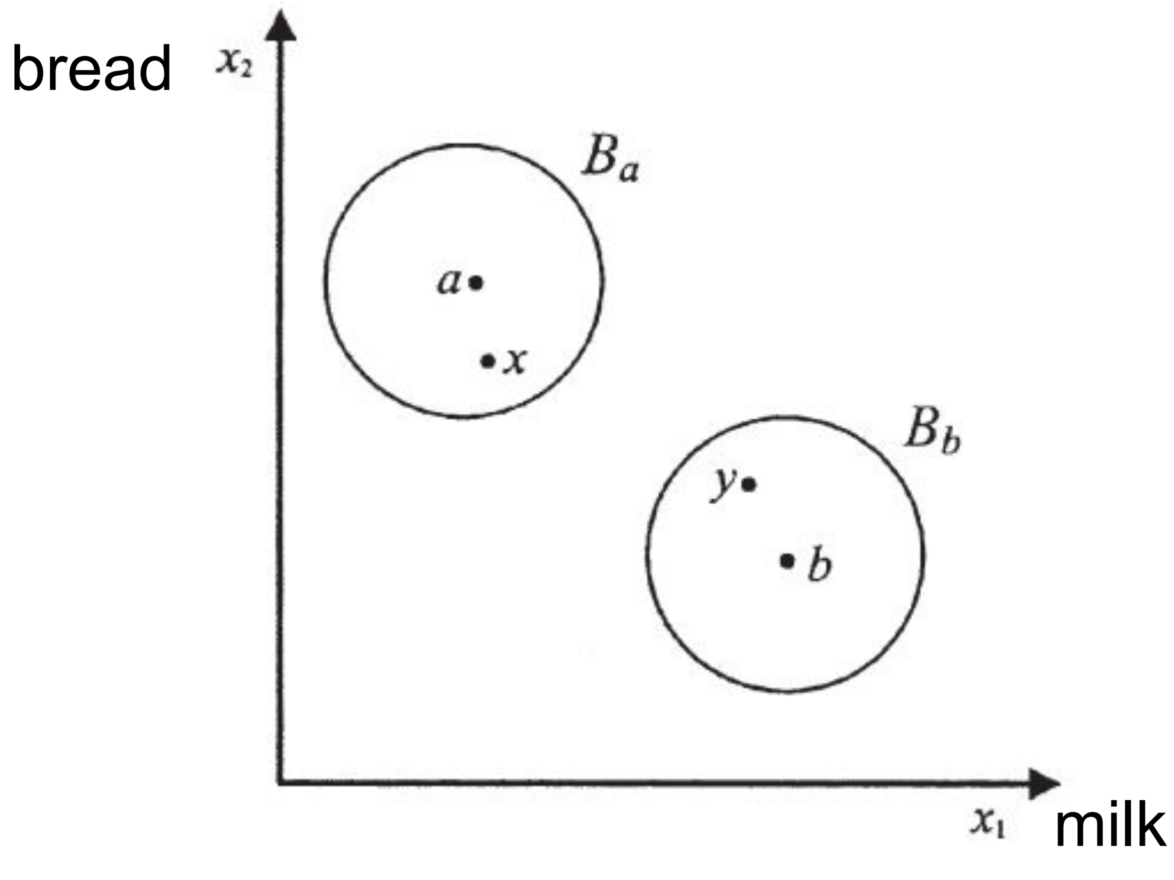
- The basic intuition, captured by the notion of a ***continuous preference relation***, is that if ***a*** is preferred to ***b***, then "small" deviations from ***a*** or from ***b*** will not reverse the ordering

Continuity of Preferences

- **Definition C1**
  - A preference relation $\succsim$ on $\boldsymbol{X}$ is <u>continuous</u> if whenever $\boldsymbol{a} \succ \boldsymbol{b}$ **<u>there are</u>** balls (neighborhoods in the relevant topology) $\boldsymbol{B_a}$ and $\boldsymbol{B_b}$ around $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, such that <u>for all</u> $\boldsymbol{x} \in \boldsymbol{B_a}$ and $\boldsymbol{y} \in \boldsymbol{B_b}$, $\boldsymbol{x} \succ \boldsymbol{y}$

Continuity of Preferences

- **Definition C1**



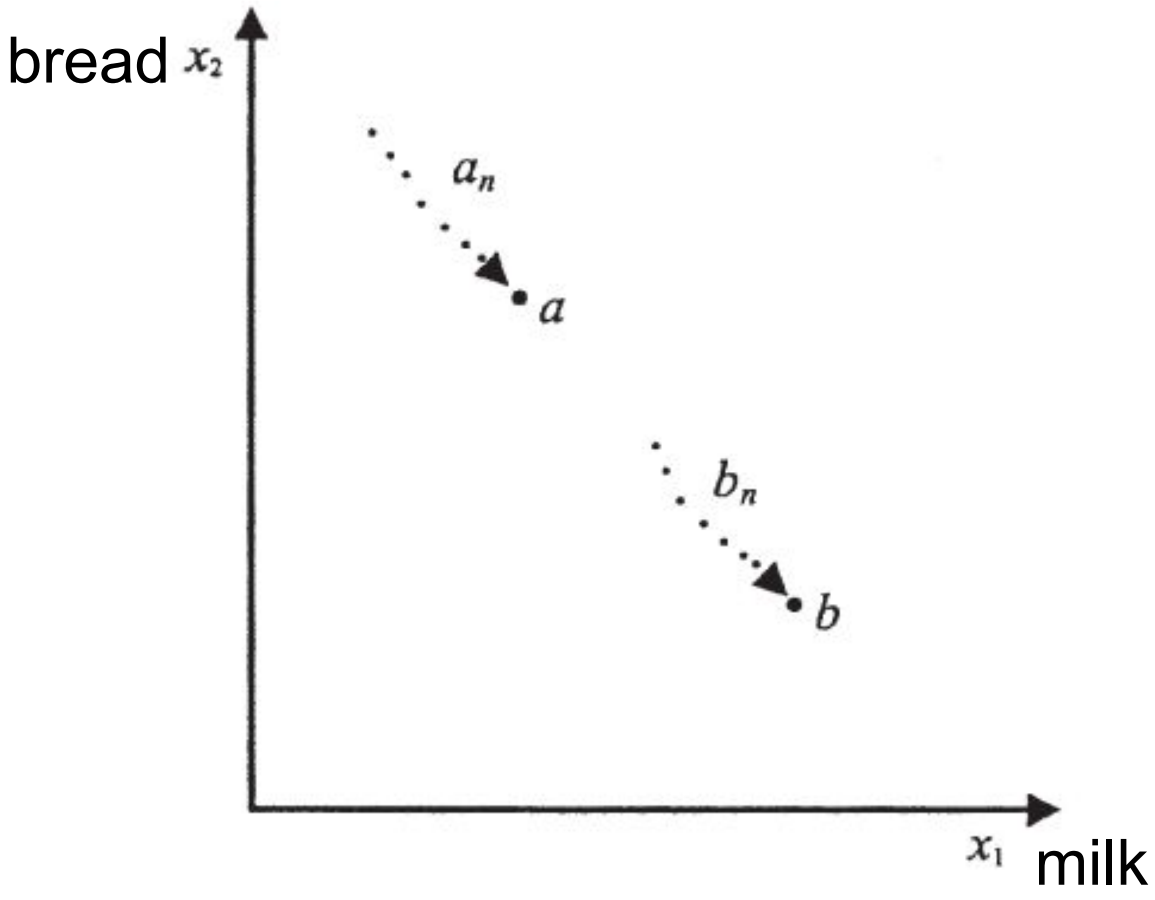
C1

Continuity of Preferences

- **Definition C2**
  - A preference relation $\succsim$ on $\boldsymbol{X}$ is continuous if the graph of $\succsim$ (i.e., the set $\boldsymbol{\{(x, y) | x \succsim y\} \subseteq X \times X}$) is a closed set (with the product topology)
  - That is, if $\boldsymbol{\{(a_n, b_n)\}}$ is a sequence of pairs of elements in $\boldsymbol{X}$ satisfying $\boldsymbol{a_n \succsim b_n}$ <u>for all</u> $\boldsymbol{n}$ and $\boldsymbol{a_n \rightarrow a}$ and $\boldsymbol{b_n \rightarrow b}$, then $\boldsymbol{a \succsim b}$

Continuity of Preferences

- **Definition C2**



C2

Continuity of Preferences

- Claim:
  - The preference relation $\succsim$ on $\boldsymbol{X}$ satisfies ***C1*** if and only if it satisfies ***C2***

Continuity of Preferences

- Proof: (if)
  - Assume that $\succsim$ on $\boldsymbol{X}$ is continuous according to ***C1***. Let $\{(a_n, b_n)\}$ be a sequence of pairs satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$
  - If it is not true that $a \succsim b$ (i.e., $b \succ a$), then there exist two balls B_a and B_b around a and b, respectively, such that for all $y \in B_b$ and $x \in B_a$, $y \succ x$
  - There is an N large enough such that for all $n > N$, both $b_n \in B_b$ and $a_n \in B_a$
  - Therefore, for all $n > N$, we have $b_n \succ a_n$, which is a contradiction

Continuity of Preferences

- Proof: (only if)
  - Assume that $\succsim$ is continuous according to ***C2***
  - Let $\boldsymbol{a} \succ \boldsymbol{b}$
  - Assume by contradiction that for all $\boldsymbol{n}$ there exist $\boldsymbol{a_n} \in \boldsymbol{Ball(a, 1/n)}$ and $\boldsymbol{b_n} \in \boldsymbol{Ball(b, 1/n)}$ such that $\boldsymbol{b_n} \succsim \boldsymbol{a_n}$
  - The sequence $\boldsymbol{(b_n, a_n)}$ converges to $\boldsymbol{(b, a)}$
  - By the second definition, $\boldsymbol{(b, a)}$ is within the graph of $\succsim$, that is, $\boldsymbol{b} \succsim \boldsymbol{a}$, which is a contradiction

Continuity of Preferences

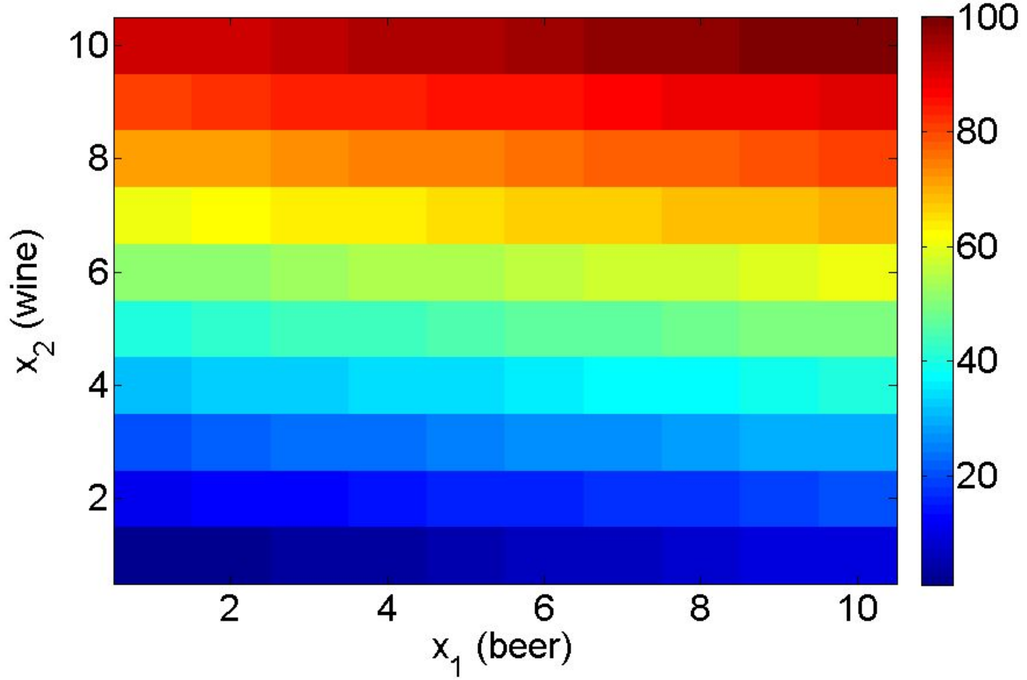
- Remark #1
  - $\succsim$ on $\boldsymbol{X}$ is represented by a continuous function $\boldsymbol{U}$, then $\succsim$ is continuous
  - To see this, note that if $\boldsymbol{a} \succ \boldsymbol{b}$, then $\boldsymbol{U(a) > U(b)}$
  - Let $\boldsymbol{\varepsilon = (U(a) - U(b))/2}$
  - By the continuity of $\boldsymbol{U}$, there is a $\boldsymbol{\delta > 0}$ such that for all $\boldsymbol{x}$ distanced less than $\boldsymbol{\delta}$ from $\boldsymbol{a}$, $\boldsymbol{U(x) > U(a) - \varepsilon}$, and for all $\boldsymbol{y}$ distanced less than $\boldsymbol{\delta}$ from $\boldsymbol{b}$, $\boldsymbol{U(y) < U(b) + \varepsilon}$
  - Thus, for $\boldsymbol{x}$ and $\boldsymbol{y}$ within the balls of radius $\boldsymbol{\delta}$ around $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, $\boldsymbol{x \succ y}$

Continuity of Preferences

- Remark #2
  - The lexicographic preferences that were used in the counterexample to the existence of a utility representation are not continuous
  - Why?
  - This is because ***(1, 1)*** $\succ$ ***(1, 0)***, but in any ball around ***(1, 1)*** there are points inferior to ***(1, 0)***

Continuity of Preferences

- Remark #2
  - The lexicographic preferences that were used in the counterexample to the existence of a utility representation are not continuous



Continuity of Preferences

- Remark #3
  - Note that the second definition of continuity can be applied to any binary relation over a topological space, not just to a preference relation
  - For example, the relation = on the real numbers ($\mathbb{R}$) is continuous, whereas the relation ≠ is not
    - Why?

Debreu's theorem

- Continuous preferences have a continuous utility representation
- Proof in the book

Reminder

- Claim:
  - If $\boldsymbol{U}$ represents $\succsim$, then for any <u>strictly increasing</u> function $\boldsymbol{f : \mathbb{R} \rightarrow \mathbb{R}}$, the function $\boldsymbol{V(x) = f(U(x))}$ represents $\succsim$ as well
- Proof:
  - $\boldsymbol{a \succsim b}$
  - ***iff*** $\boldsymbol{U(a) \geq U(b)}$ (since $\boldsymbol{U}$ represents $\succsim$)
  - ***iff*** $\boldsymbol{f(U(a)) \geq f(U(b))}$ (since $\boldsymbol{f}$ is strictly increasing)
  - ***iff*** $\boldsymbol{V(a) \geq V(b)}$

Problem

- a) Is the statement "if both ***U*** and ***V*** represent $\succsim$, then there is a strictly monotonic function $\boldsymbol{f : \mathbb{R} \rightarrow \mathbb{R}}$ such that ***V(x) = f(U(x))***" correct?
- FALSE

Let $X = \Re$ and preferences be represented by the utility functions

$$V(x) = x \qquad \text{and} \qquad U(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0. \end{cases}$$

The only increasing function $f : \Re \rightarrow \Re$ that satisfies $V(x) = f(U(x))$ is

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } 0 < x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

Problem

- b) Can a continuous preference relation be represented by a discontinuous utility function?
- True: The preferences ($\boldsymbol{x} \succsim \boldsymbol{y}$ if $\boldsymbol{x} \geq \boldsymbol{y}$) is represented by $\boldsymbol{U}$ in (a) are continuous, though $\boldsymbol{U}$ is discontinuous

Problem

- c) Show that in the case of $X = \mathbb{R}$, the preference relation that is represented by the discontinuous utility function ***u(x) = floor(x)*** is not a continuous relation
  - floor(x): the largest integer ***n*** such that $\boldsymbol{x \geq n}$