Lista 1

Teoria dos Jogos em Computação

Professor: Pedro O.S. Vaz de Melo

Utility

1. Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values.

R: Suppose there is a countable set $\boldsymbol{X} = \mathbb{Q}$. The utility function **U: X → ℤ** cannot represent the preference relation of the set $\boldsymbol{X}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. It means that for any arbitrary real numbers **x** < **y**, there are **infinitely many** rational numbers **z** such that **x < z < y.** Nevertheless, an utility function that returns integer values **U(x) < U(z) < U(y)** is discrete (not dense), meaning that there are **finitely many** integers U(z) to this preference relation.

2. A farmer wants to dig a well in a square field. The preferences of the farmer on the possible locations are lexicographic, i.e:

If $x_1 < x_2$ then $(x_1,y_1) \succ (x_2,y_2)$ for all y_1,y_2.

If $x_1 = x_2 = x$, then $(x,y_1) \succ (x,y_2)$ iff $y_1 < y_2$.

First, assume that the field has dimensions [0,1000] × [0,1000] and construct a linear utility function that represents this relation. Second, construct a utility function assuming that the field has dimensions [0,∞] × [0,∞]. For both cases, assume that the well location must have **integer** coordinates.

R: First, the utility function $\boldsymbol{U(x,y) = 1000 - x - \frac{y}{2000}}$ represents a lexicographic preference relation in the [0,1000] x [0,1000] assuming only **integer** coordinates (**x, y** ∈ ℤ). This preference can be described by an utility function until it is bounded, because there are pairs of terms A, B that construct the function U(x, y) = f(x, y) = f(x) + f(y) = -Ax -By iff the greatest change in y (1000) is less than the tinniest change in x (1).

So, the second set does not have an utility function to represent the same lexicographic preferences because for any $x_1 < x_2$ there is a pair of values y_1, y_2 such that it is impossible to stablish coefficients A and B to satisfy the first preference.

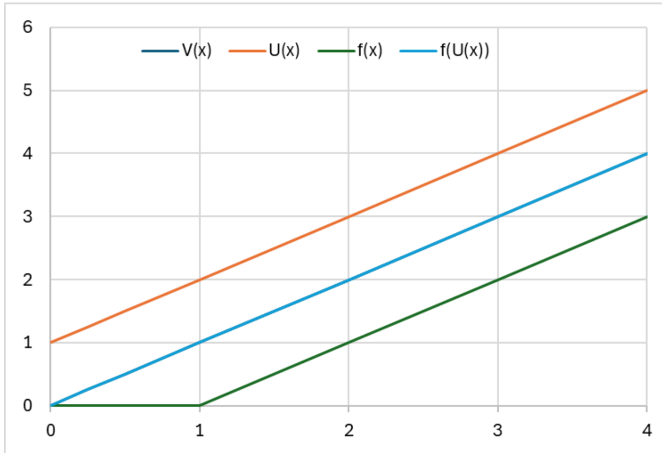
3. Is the statement "if both U and V represent $\succsim$, then there is a strictly monotonic function f: R→R such that $V(x) = f(U(x))$" correct?

Tip: consider $V(x) = x$ and $U(x) = \begin{cases} x & if \quad x \le 0 \\ x+1 & if \quad x > 0 \end{cases}$

R: The assumption is incorrect. The only function f(x) that satisfies V(x) = f(U(x)) is the function below:

$$\boldsymbol{f(x) = \begin{cases} x, & x \le 0 \\ 0, & 0 < x \le 1 \\ x - 1, & x > 1 \end{cases}}$$

The function f(x) is not strictly monotonic, because in the interval of (0, 1] it is stationary, as shown in the graph below:



4. Can a continuous preference relation be represented by a discontinuous utility function?

R: Yes. The utility function presented in 3) is discontinuous at x = 0 but it represents the preference relation **$x \succcurlyeq y$ iff $x \geq y$.**

5. Show that in the case of $X = \mathrm{R}$, the preference relation that is represented by the discontinuous utility function $u(x) = floor(x)$ is not a continuous relation. $floor(x)$: the largest integer n such that $x \geq n$

R: Yes. The utility function presented in 3) is discontinuous at x = 0 but it represents the preference relation **$x \succcurlyeq y$ iff $x \geq y$.**

Choice

6. The following are descriptions of decision–making procedures. Discuss whether the procedures can be described in the framework of the choice model discussed in this course and whether they are compatible with the "rational man" paradigm. In other words, can I construct a utility function $u(x)$ based solely on the set of alternatives $x \in X$ according with these procedures? Explain why (e.g. with an example).

a. The decision maker chooses an alternative in order to maximize another person's suffering.

R: Assuming that the utility function is described by the suffering caused by the alternatives in the set X, $U(x_1) > U(x_2)$ iff the suffering caused by alternative x_1 is greater than x_2. This assumption is complete and transitive, so it can be applied to the rational man.

b. The decision maker asks his two children to rank the alternatives and then chooses the alternative that is the best on average (you can use your own definition of "best on average").

R: If a numerical value is linked to the ranking of the children, then an average can be calculated to represent the utility function of "best on average". Example:

	X = {a, b, c}		
	Kid 1	Kid 2	
Position	Choices	Choices	Value
1º	a	c	3
2º	c	b	2
3º	b	a	1
Avg			
a	4		
b	3		
c	5		

In this function, a weight value is stablished to each ranking (1º to 3º), then the "Avg" utility function is defined by the weighted value to each alternative of the children based on their rankings. So, the decision-maker can define that **c>a because Avg(c) > Avg(a)**

c. The decision maker has an ideal point in mind and chooses the alternative that is closest to it.

R: If the decision maker is searching for an apartment, and the ideal one is the closest to his current savings (suppose that is 500K). Then, he can make a list of the options and its prices to stablish an utility function $U(x) = 1 - (x/500-1)^2$. This function stablishes a higher utility to the prices with lower distances to the target (500). In a general form, distances functions can be used by the decision maker to use this criterion.

d. The decision maker looks for the alternative that appears most often in a list of alternatives.

R: It's not possible to determine an utility function U(x) because, in that case, the function would need to account for the other alternatives on the set X to stablish an utility for the specific alternative x. Example:

X = {a, b, c}

List 1 = {a, a, b, c, a}. In that case, the decision maker choses "a"

List 2 = {b, b, b, c, a} In that case, the decision maker choses "b"

Different choices are made in the same set, making impossible to stablish an utility function U(x) depending only on the x alternative.

e. The decision maker has an ordering in mind and always chooses the median element.

R: In the same way as presented above, in that case any utility function would need to account for the other elements in X to stablish an utility to the specific alternative x in the set, and this dependency makes impossible to use this criterion for a decision maker.

7. Consider the following choice procedure: a decision maker has a strict ordering $\succsim$ over the set X and assigns to each $x \in X$ a natural number *class*(x) to be interpreted as the "class" of x. Given a choice problem A, he chooses the best element in A from those belonging to the most common class in A (i.e., the class that appears in A most often). If there is more than one most common class, he picks the best element from the members of A that belong to a most common class with the highest class number.

a. Is this procedure consistent with the "rational man" paradigm?

R: No. the "most common" choice is dependent on the other elements in the set to stablish the utility of the alternative x in the set X, making it unfeasible to the decision maker.

b. Define the relation: xPy if x is chosen from $\{x,y\}$. Show that the relation P is a strict ordering (complete, asymmetric, and transitive).

R: Completeness: for any two alternatives x and y, the decision maker choses one of them (in that case, x is chosen).

Asymmetric: The relation P determines that xPy, so yPx does not occur.

Transitivity: If xPy and yPz, then xPz. It means that, in the set of alternatives {x, y, z} when class (x) = class (y) = class (z), the relation xPy determines that x is chosen, even though there is a new set {x, z} with class(x) = class (z).

Expected Utility

8. Which lottery do you prefer? $L = (0.25z_1, 0.25z_2, 0.25z_3, 0.25z_4)$ OR

$L' = (0.15z_1, 0.50z_2, 0.15z_3, 0.20z_4)$

Suppose, by continuity: $z_2 \sim z_2' = (0.6z_1, 0.4z_4)$ and $z_3 \sim z_3' = (0.2z_1, 0.8z_4)$

R: In that case, z_2 and z'_2 can be substituted to $(0.6z_1, 0.4z_4)$ and z_3, z'_3 can be substituted to $(0.2z_1, 0.8z_4)$. It means that the lottery L can be written as:

$L = (0.25z_1, 0.25(0.6z_1, 0.4z_4), 0.25(0.2z_1, 0.8z_4), 0.25z_4) \rightarrow L = (0.45z_1, 0.55z_4)$

$L' = (0.15z_1, 0.5(0.6z_1, 0.4z_4), 0.15(0.2z_1, 0.8z_4), 0.2z_4) \rightarrow L' = (0.48z_1, 0.52z_4)$

If $U(z_1) \succ U(z_4)$, then $L' \succ L$

9. **T** or **F.** Justify or give a counterexample.

a. A lottery p is preferred to q because the expected utility $U(p)$ is greater than the expected utility $U(q)$.

R: F. The preference relationship is what comes first and what guides the construction of the utility function U(x).

b. Suppose that $A \succ B \succ C \succ D$ and that the vNM utilities of these prizes satisfy $v(A) + v(D) = v(B) + v(C)$, then $(\frac{1}{2}B, \frac{1}{2}C)$ should be preferred to $(\frac{1}{2}A, \frac{1}{2}D)$ because, although they have the same expected utility, the former has the smaller utility variance.

R: F. In that case, the decision maker is indifferent to those lotteries, because the expected utilities are the same expected utility.

c. Suppose that $A \succ B \succ C \succ D$ and that the vNM utility function has the property that $v(A) - v(B) \succ v(C) - v(D)$, then the change from B to A is more preferred than the change from D to C.

R: F. In that case, the decision maker cannot make his decision upon utility differences between alternatives, but with respect to entire lotteries and expected utilities. An counterexample for that is the set of prizes:

A: Stay Healthy, B: Get moderately ill, C: Loose an arm, D: Die

In that case, the utility gain from not dying is way more relevant than the utility gain from getting ill to staying healthy. It means that vNM needs to account for all prizes and its utilities and not only their differences.

10. Verify whether each of the following preference relations over lotteries satisfy (or not) von Neumann and Morgenstern axioms (*I* and *C*). Consult the book "Lecture Notes in Microeconomic Theory by Ariel Rubinstein", Pages 95 and 96, for more details.

a. The worst case (the decision maker evaluates lotteries by the worst possible case).

R: Violates I – The worst case ignores any change in compound lotteries, breaking independency. It looks only for the worst case scenario.

Violates C – The worst case scenario is not continuous, because it looks only for the minimum value, and small changes in the lotteries can change the decision because it's bounded by the previous min value.

b. Increasing the probability of a "good" consequence.

R: Violates I – Since there is no correlation between lotteries, the decision maker cannot choose a lottery which the expected utilities are the same but the max probability of a prize is greater than the other one.

Risk Aversion

11. Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete (t = 1,2,...) and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that:

a. Adam likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.

b. Adam is **impatient**. He would be delighted to increase his consumption on day t from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later $t + 1$.

c. In any day in which he does not have an apple, he prefers to get 1 apple immediately in exchange for 2 apples tomorrow.

d. Adam expects to live for 120 years.

Show that if (poor) Adam is offered a stream of 2 apples starting in day 4 for the rest of his expected life, he would be willing to exchange that offer for 1 apple right away. Tips:

-(b) means that <u>one single apple</u> is promised to Adam on day $t + 1$

-initial stream offered to Adam can be represented by (0,0,0,2,2,...,2,2)

-evolve it due to Adam's preferences

t	Initial	First	Second	Third	...	120th
1	0	0	0	1	...	2
2	0	0	1	0	...	0
3	0	1	0	1	...	0
4	2	0	1	0	...	0
...	2	2	0	1	...	0
120	2	2	2	0	...	0

R: As the iteration occurs, Adam keeps shortening the number of days of 2 apples promised

and receiving the first one earlier, until he prefers receiving two immediately.

12. Given the pairs of lotteries in tables 1 and 2, in each case, which one do you prefer? Explain considering the *First-Order Stochastic Domination* concept.

Table 1: (a) or (b)?

chance %	90	6	1	3 -
(a) prize $	0	45	30	15
chance %	90	7	1 -	2 -
(b) prize $	0	45	10	15

Table 2: (c) or (d)?

chance %	40	35	15	10
(c) prize $	0	10	50	200
chance %	40	35	15	10
(d) prize $	0	25	40	180

R: In both cases the first option is chosen due to the intermediate prizes and its probabilities. For the second table, for example, the first option is preferred because there is a probability of winning at least 50 reais of 0.25 against 0.01

Ao menos (-15)	Ao menos -10	Ao menos 0	Ao menos 30	Ao menos 45			
1	0.97	0.97	0.07	0.06	**A)**		
1	0.98	0.97	0.07	0.07			
Ao menos 0	Ao menos 10	Ao menos 25	Ao menos 40	Ao menos 50	Ao menos 180	Ao menos 200	
1	0.6	0.25	0.25	0.25	0.01	0.01	**A)**
1	0.6	0.6	0.25	0.01	0.01	0	

13. A gambling house charges $15 for the lottery below.

lottery	**$0**	**$36**	**$64**
p	0.50	0.30	0.20

Will a person, whose utility function over money is $u(x) = \frac{5}{4}\sqrt{x}$, pay to to play *p*? Justify your answer.

R: The expected utility of the lottery is 0.5*0 + 0.3*(1.25*6)+0.2*(1.25*8) = 4.25. Since the expected utility is smaller than the value charged by the gambling house, the decision maker does not have motive to play p, because in the long-term (playing it successive times) the gambling house will exploit him.