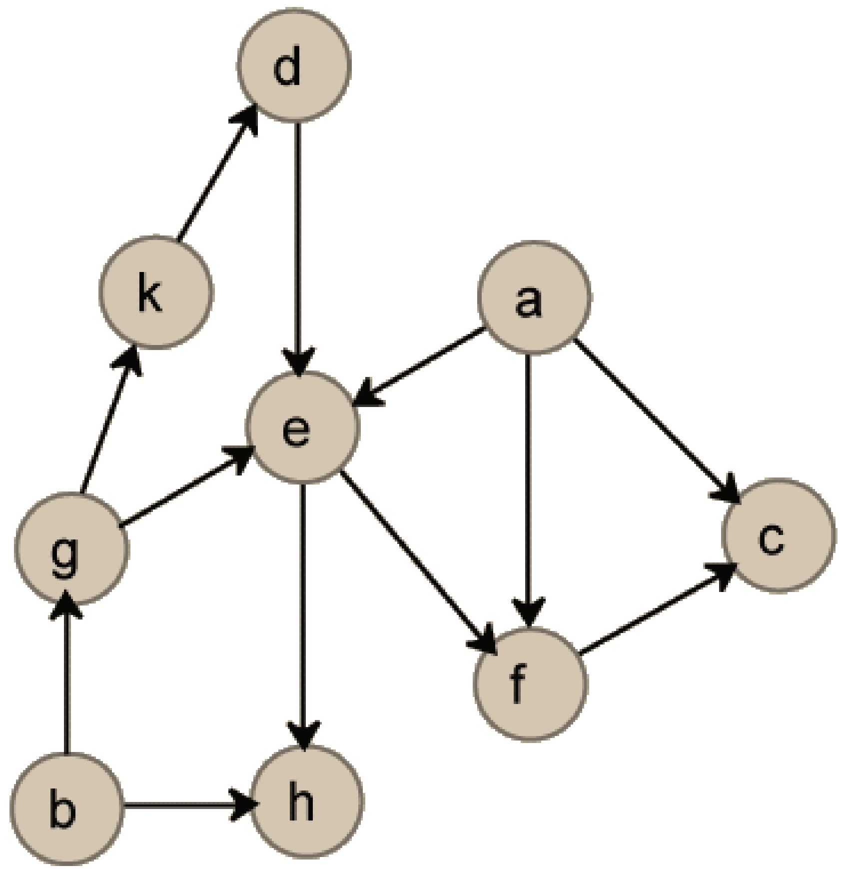
Utility

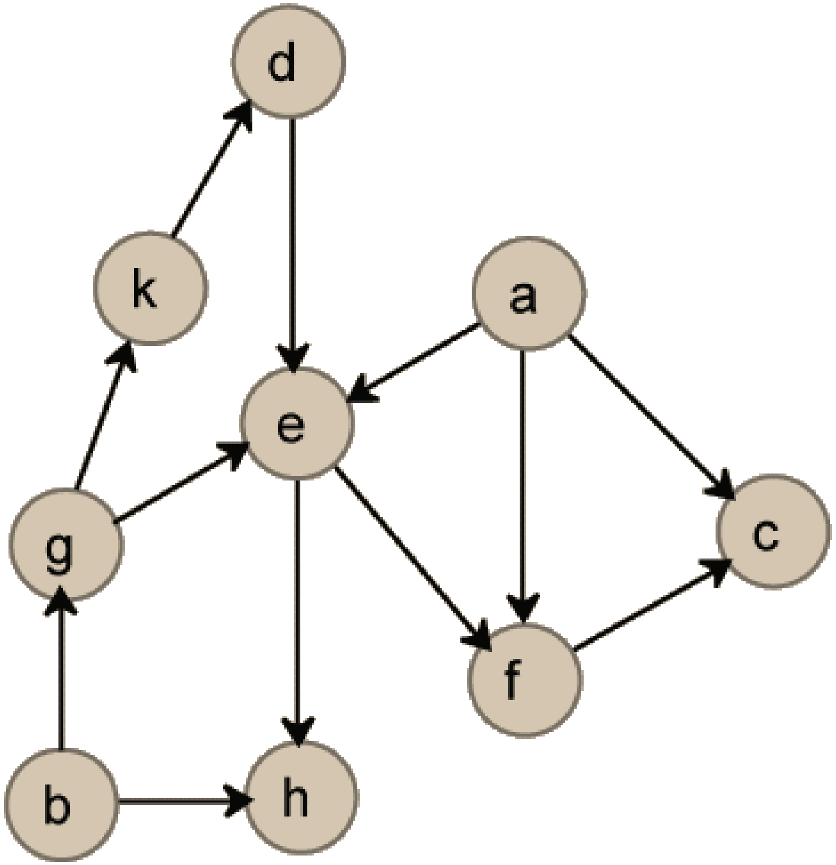
Reminder: preferences

- DAG (Directed Acyclic Graph)
  - There is a direct edge from $\boldsymbol{i}$ to $\boldsymbol{j}$ if $\boldsymbol{j} \succ \boldsymbol{i}$



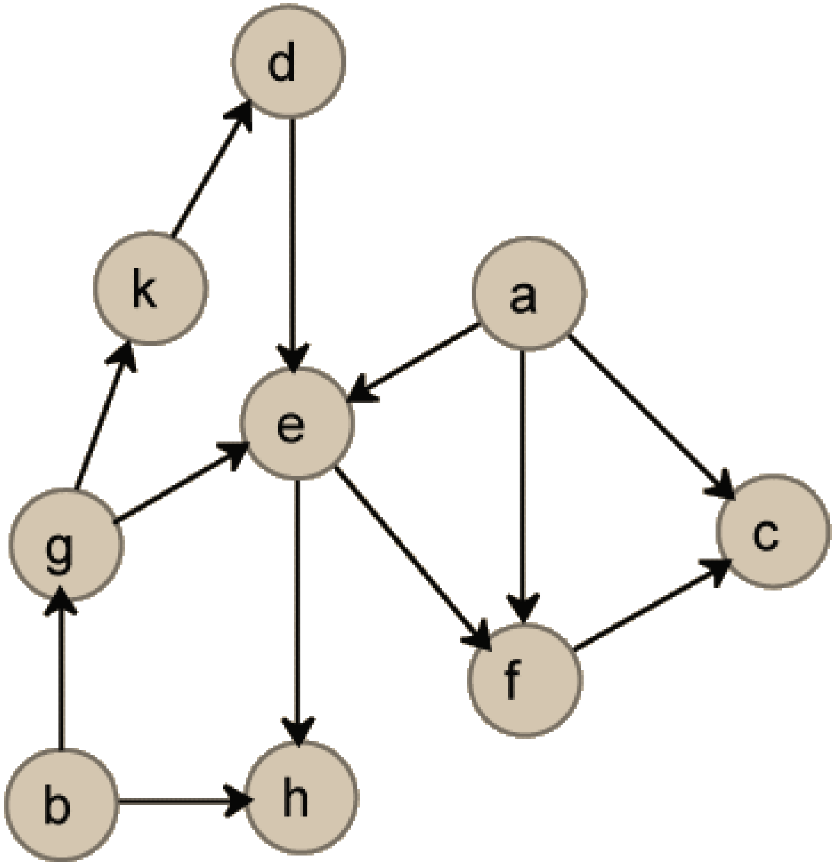
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** which are the most preferred nodes?



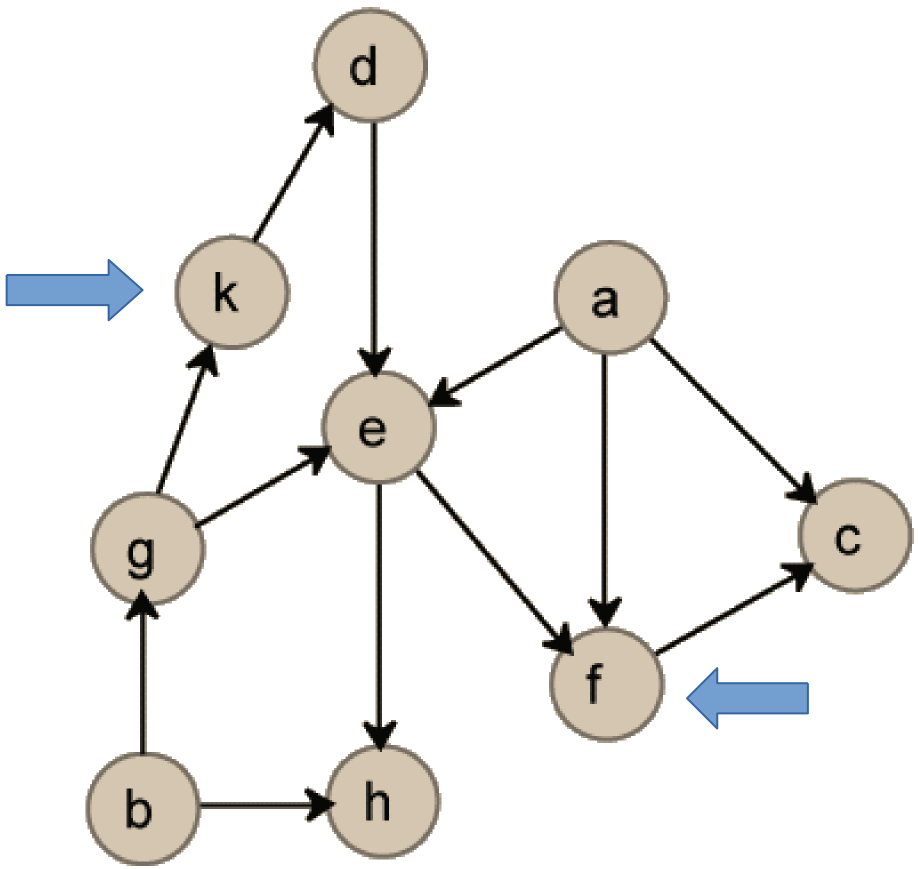
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** which are the least preferred nodes?



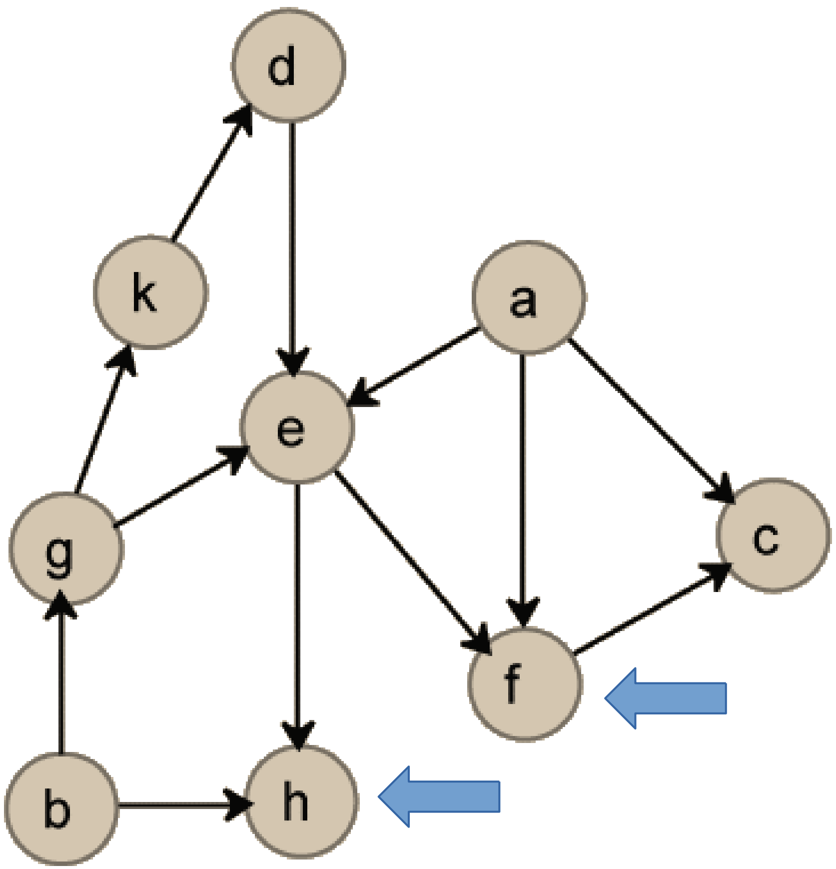
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** is ***k*** preferred to ***f***?



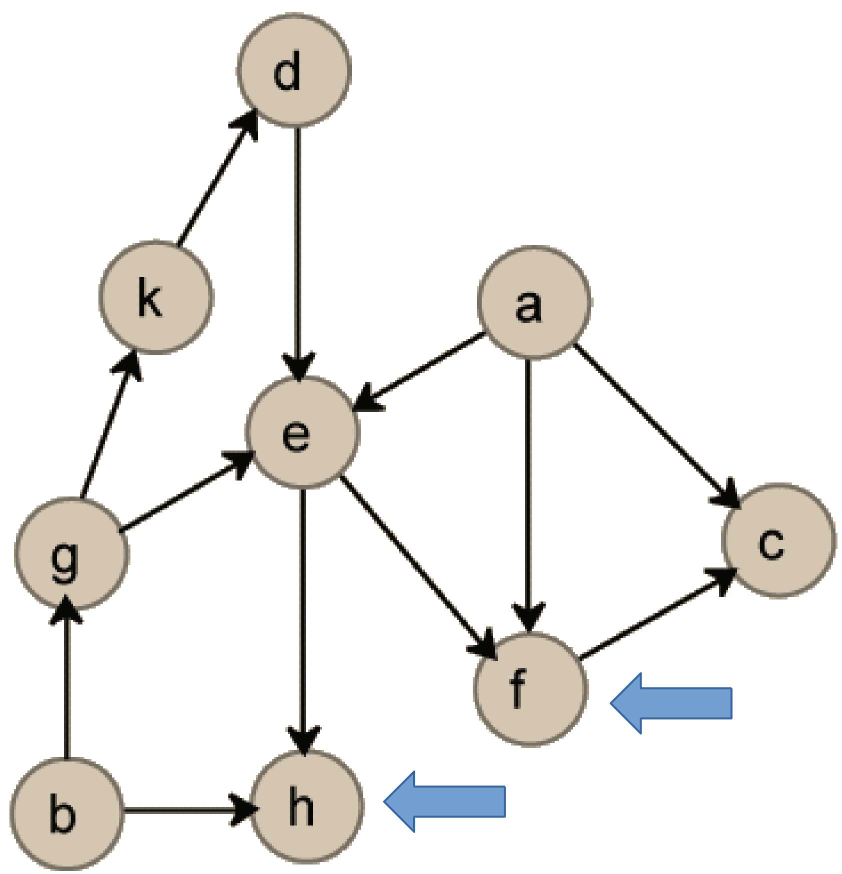
Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** is ***h*** preferred to ***f?***



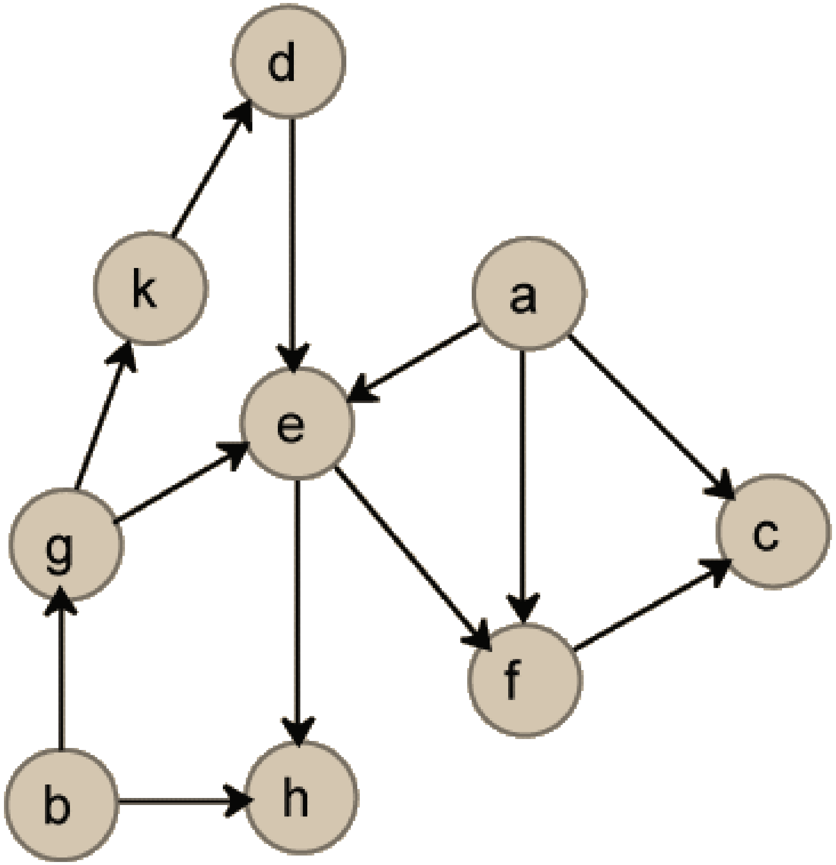
Reminder: preferences

- There is a direct edge from $\boldsymbol{i}$ to $\boldsymbol{j}$ if $\boldsymbol{j} \succ \boldsymbol{i}$
  - ***Q:*** how can I identify two indifferent nodes?



Reminder: preferences

- There is a direct edge from ***i*** to ***j*** if $j \succ i$
  - ***Q:*** Is this preference relation transitive?



The Concept of Utility Representation

- If the number of alternatives is **small**
  - preference relation can be an ordered list from best to worst

The Concept of Utility Representation

- In some cases, the alternatives are grouped into a small number of categories
  - we describe the preferences on ***X*** by specifying the preferences on the set of categories

The Concept of Utility Representation

- “I prefer the fastest car”
- “I prefer the taller basketball player”
- “I prefer the more expensive present”
- “I prefer a teacher who gives higher grades”

The Concept of Utility Representation

- They can naturally be specified by
  - **x** $\succsim$ **y** if **V(x) ≥ V(y)** (or **V(x) ≤ V(y)**), where $\boldsymbol{V : X} \rightarrow \mathbb{R}$
- For example, the preferences stated by “I prefer the taller basketball player” can be expressed formally by
  - **X** is the set of all conceivable basketball players, and **V(x)** is the height of player **x**

The Concept of Utility Representation

- Note that the statement $\boldsymbol{x} \succsim \boldsymbol{y}$ if $\boldsymbol{V(x) \geq V(y)}$ always defines a preference relation because…

  … the relation $\geq$ on $\mathbb{R}$ satisfies completeness and transitivity

The Concept of Utility Representation

- We say that the function $U : X \to \mathbb{R}$ represents the preference $\succsim$ if for all x and $y \in X$, $x \succsim y$ iff $U(x) \geq U(y)$
- If the function U represents the preference relation $\succsim$, we refer to it as a ***utility function***, and we say that $\succsim$ has a utility representation

The Concept of Utility Representation

- Claim:

  If ***U*** represents $\succsim$, then for any <u>strictly increasing</u> function $f : \mathbb{R} \rightarrow \mathbb{R}$, the function $V(x) = f(U(x))$ …

  … represents $\succsim$ as well

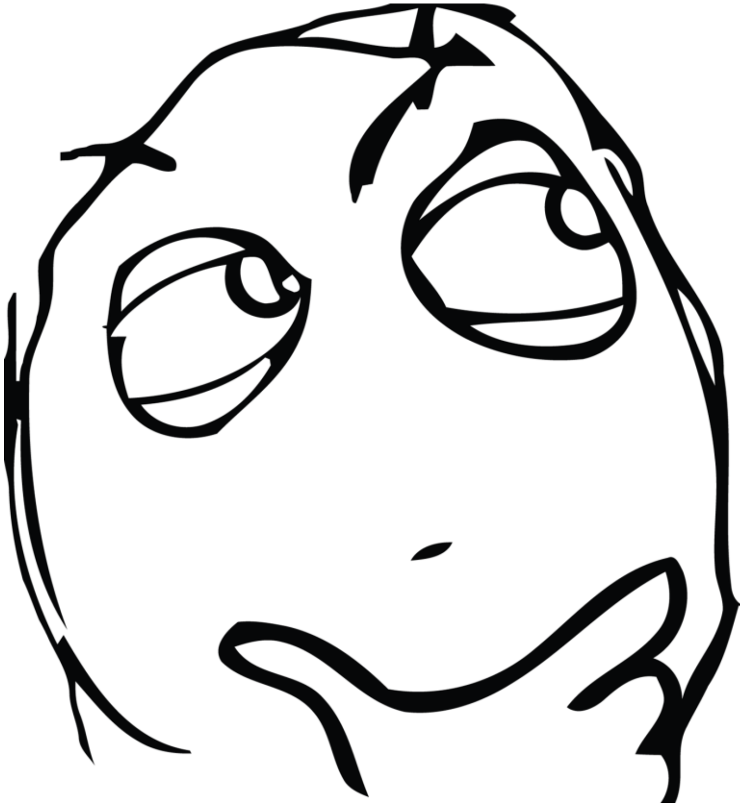
- Proof:
  - $a \succsim b$
  - ***iff*** $U(a) \geq U(b)$ (since ***U*** represents $\succsim$)
  - ***iff*** $f(U(a)) \geq f(U(b))$ (since ***f*** is strictly increasing)
  - ***iff*** $V(a) \geq V(b)$

The Concept of Utility Representation

- Claim:
  - If $\boldsymbol{U}$ represents $\succsim$, then for any <u>strictly increasing</u> function $\boldsymbol{f : \mathbb{R} \to \mathbb{R}}$, the function $\boldsymbol{V(x) = f(U(x))}$ represents $\succsim$ as well
- What does this mean?
  - Various forms of utility functions may exist to represent a preference relation

Existence of a Utility Representation

- If any preference relation could be represented by a ***<u>utility function</u>***, then…

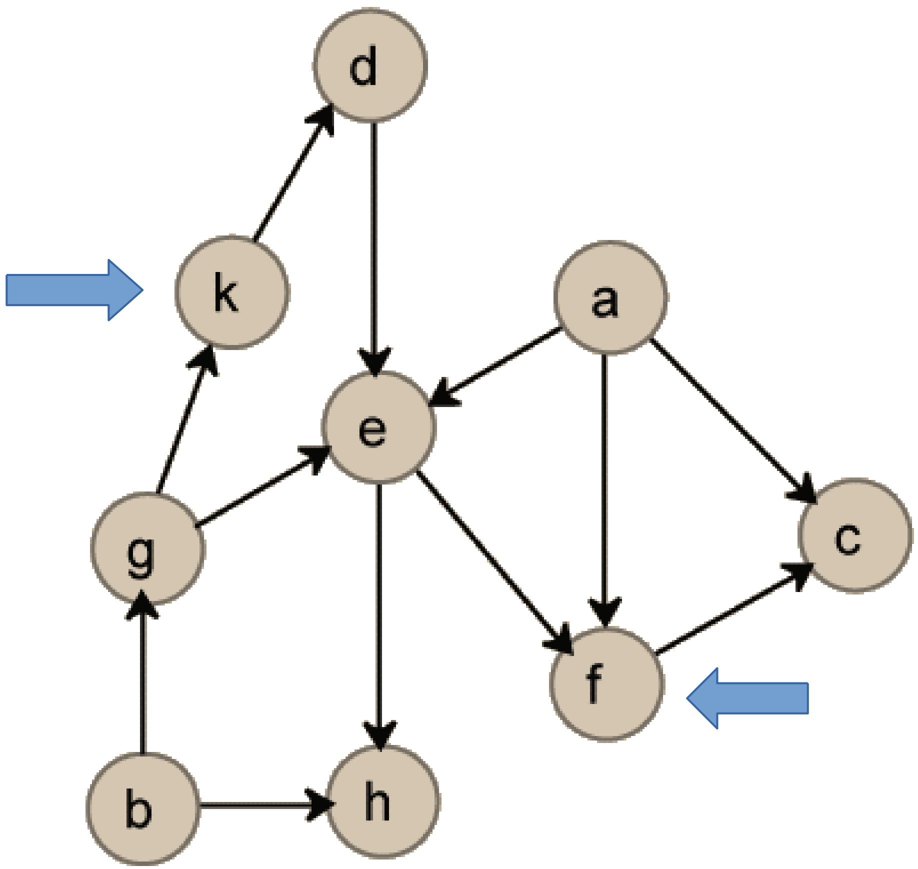


Existence of a Utility Representation

- If any preference relation could be represented by a ***utility function***, then it would “grant a license” to use utility functions rather than preference relations with no loss of generality
- Why is this important?

Reminder: preferences

- Easier to compare two items
  - ***Q:*** is ***k*** preferred to ***f***?



Existence of a Utility Representation

- Possibility of numerical representations carrying additional meanings
  - Ex:
  - ***a*** is preferred to ***b*** more than ***c*** is preferred to ***d***

Existence of a Utility Representation

- Under what assumptions do utility representations exist?

Existence of a Utility Representation

- Lemma:
  - In any finite set $\boldsymbol{A} \subseteq \boldsymbol{X}$, there is a minimal element (similarly, there is also a maximal element)
- That is, there is an element that is less preferred to any other element
- Which property guarantees that?

Existence of a Utility Representation

- Lemma:
  - In any finite set $A \subseteq X$, there is a minimal element (similarly, there is also a maximal element)
- Proof:
  - By induction on the size of A
  - If A is a singleton, then by completeness its only element is minimal
  - For the inductive step, let A be of cardinality $n+1$ and let $x \in A$. The set $A - \{x\}$ is of cardinality n and by the inductive assumption has a minimal element denoted by y
  - If $x \succsim y$, then y is minimal in A
  - If $y \succsim x$, then by ***transitivity*** $z \succsim x$ for all $z \in A - \{x\}$, and thus x is minimal

Reminder

- Recall that ***X*** is ***<u>countable</u>*** and ***<u>infinite</u>*** if there is a ***<u>one-to-one function</u>*** from ***X*** onto the natural numbers
- It is possible to specify an enumeration of all its members
  - $\{x_n\}_{n=1,2,...}$

Existence of a Utility Representation

- **Claim:**
  - If X is countable, then any preference relation on X has a utility representation with a range $(-1, 1)$

Existence of a Utility Representation

- Proof:
  - Let $\{x_n\}$ be an enumeration of all elements in X
  - Set $U(x_1) = 0$
  - Assume that you have completed the definition of the values $U(x_1), \ldots, U(x_{n-1})$ so that $x_k \succsim x_l$ *iff* $U(x_k) \geq U(x_l)$
  - If x_n is indifferent to x_k for some $k < n$, then assign $U(x_n) = U(x_k)$
  - If not, by transitivity, all numbers in the nonempty set $\{U(x_k) | x_k \prec x_n\} \cup \{-1\}$ are below all numbers in the nonempty set $\{U(x_k) | x_k \succ x_n\} \cup \{1\}$
  - Choose $U(x_n)$ to be between the two sets
  - This guarantees that for any $k < n$ we have $x_n \succsim x_k$ *iff* $U(x_n) \geq U(x_k)$
  - Thus, the function defined on $\{x_1, \ldots, x_n\}$ represents the preferences on those elements
  - To complete the proof that U represents $\succsim$, take any two elements, x and $y \in X$. For some k and l we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ *iff* $U(x_k) \geq U(x_l)$

Existence of a Utility Representation

- Let's put that in practice…

Which bundle do you prefer?

Bundle #1

3 chocolates
6 snacks
4 sodas

Bundle #2

2 chocolates
2 snacks
10 sodas

Bundle #3

1 beer
1 snack
1 chocolate

Lexicographic Preferences

Bundle #1

3 chocolates
6 snacks
4 sodas

Bundle #2

2 chocolates
2 snacks
10 sodas

Bundle #3

1 beer
1 snack
1 chocolate

Lexicographic Preferences

- Let $(\succsim_k)_{k=1,\ldots,K}$ be a ***K-tuple*** of preferences over the set X
- The lexicographic preferences induced by those preferences are defined by $x \succsim_L y$ if
  - (1) there is k^* such that for all $k < k^*$ we have $x \sim_k y$ and $x \succ_{k*} y$ or
  - (2) $x \sim_k y$ for all k
- The lower the k, the more relevant it is

Lexicographic Preferences

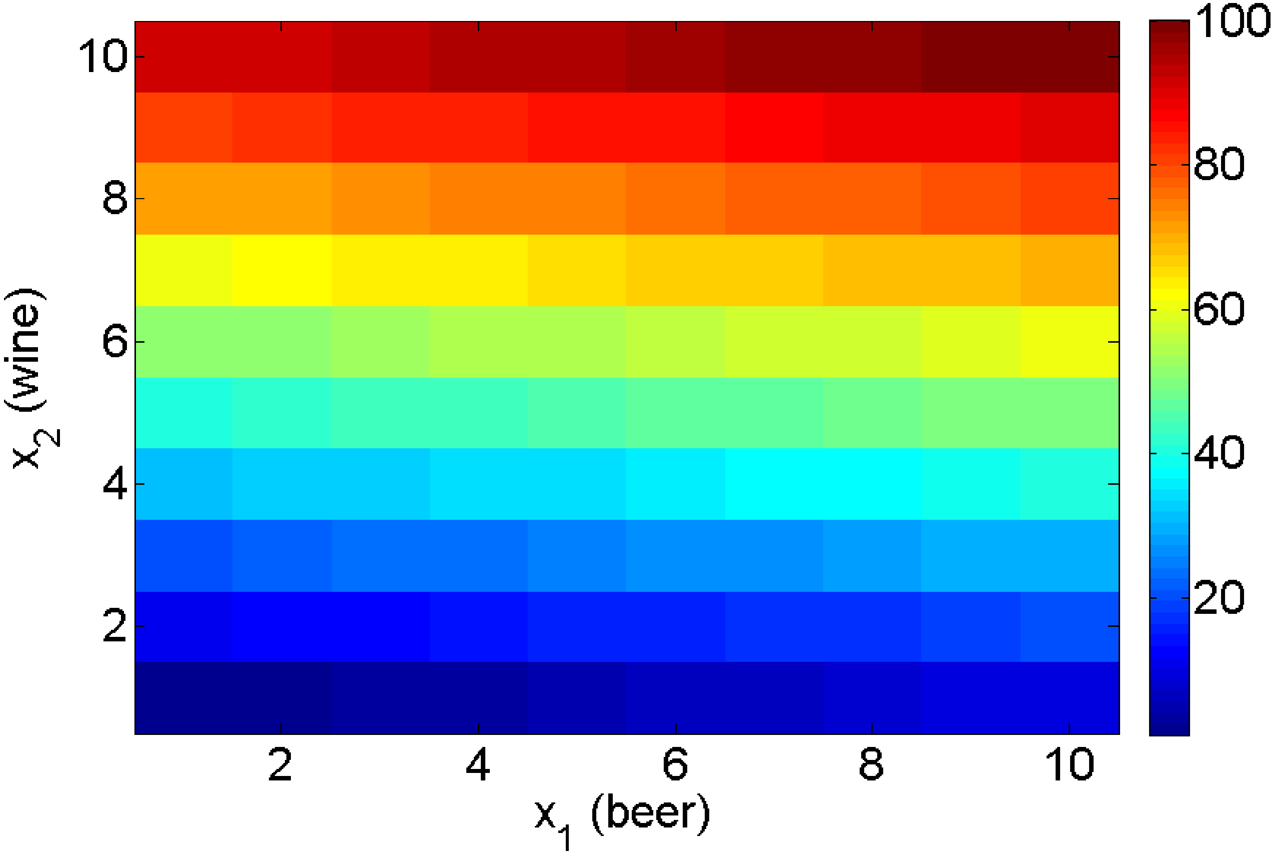
- Example:
  - Let $\boldsymbol{X}$ be the unit square, that is, $\boldsymbol{X = [0, 1] \times [0, 1]}$
  - Let $\boldsymbol{x \succsim_k y}$ if $\boldsymbol{x_k \geq y_k}$
  - The lexicographic preferences $\succsim_L$ induced from $\succsim_1$ and $\succsim_2$ are:
    - $\boldsymbol{(a_1, a_2) \succsim_L (b_1, b_2)}$ if $\boldsymbol{a_1 > b_1}$ or both $\boldsymbol{a_1 = b_1}$ and $\boldsymbol{a_2 \geq b_2}$

Lexicographic Preferences

- **Claim:**
  - The lexicographic preference relation $\succsim_L$ on $[0, 1] \times [0, 1]$, induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ ($k = 1, 2$), **does not have** a utility representation

Lexicographic Preferences

- Why it cannot be represented by a utility function?



$(x_1,x_2) \succsim (y_1,y_2)$ if
$(x_2 > y_2)$ or
$(x_2 = y_2$ and ***$x_1 > y_1)$***

Lexicographic Preferences

- Proof:

Theorem: $\mathbb{R}$ is uncountable

Theorem: every subset of $\mathbb{Q}$ is countable

Corollary: there is no one-to-one function f : $\mathbb{R} \rightarrow \mathbb{Q}$

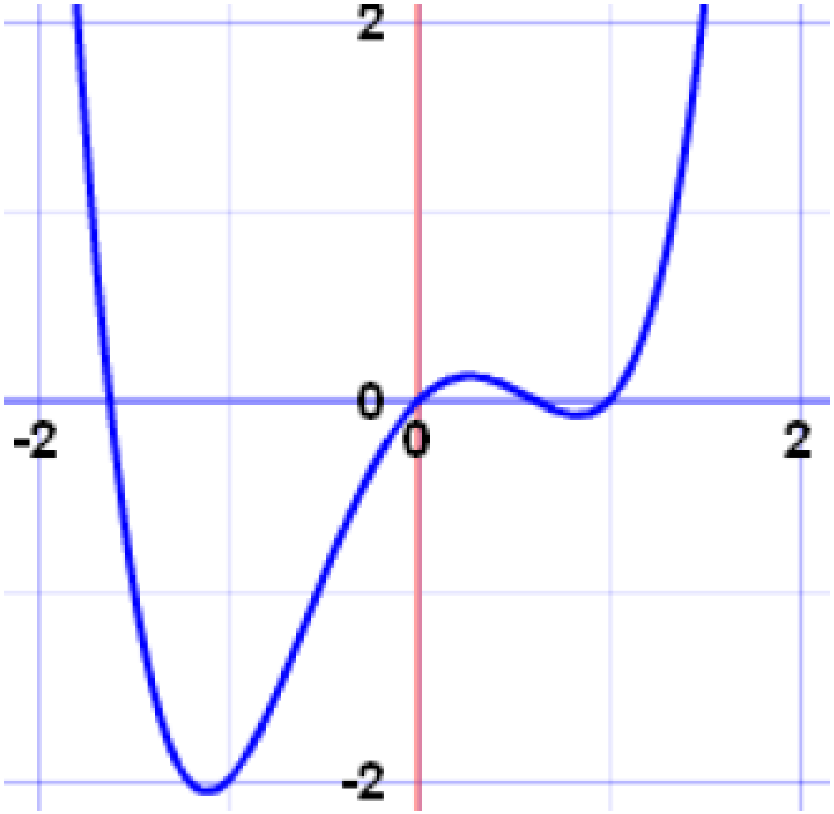
Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$

Lexicographic Preferences

- Proof:
  - Assume by contradiction that the function $\boldsymbol{u : X \rightarrow \mathbb{R}}$ represents $\succsim_L$
  - For any $\boldsymbol{a \in [0, 1], (a, 1) \succ_L (a, 0)}$ and, therefore, $\boldsymbol{u(a, 1) > u(a, 0)}$
  - Let $\boldsymbol{q(a)}$ be a ***rational number*** in the nonempty interval $\boldsymbol{I_a = (u(a, 0), u(a, 1))}$
  - The function $\boldsymbol{q}$ is a function from $\boldsymbol{[0, 1]}$ into the set of ***rational numbers*** $\mathbb{Q}$
  - It is a one-to-one function since if $\boldsymbol{b > a}$, then $\boldsymbol{(b, 0) \succ_L (a, 1)}$, $\boldsymbol{u(b, 0) > u(a, 1)}$, and, therefore, the intervals $\boldsymbol{I_a}$ and $\boldsymbol{I_b}$ are disjoint
    - For all $\boldsymbol{b}$, there is a rational $\boldsymbol{q(b)}$ number
  - Thus, $\boldsymbol{q(a) \neq q(b)}$
  - But the cardinality of the rational numbers is lower than that of the continuum, a contradiction

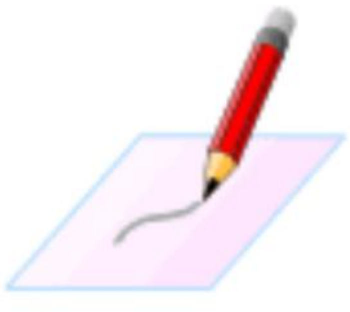
Reminder: continuous functions

- Is this function continuous?



Reminder: continuous functions

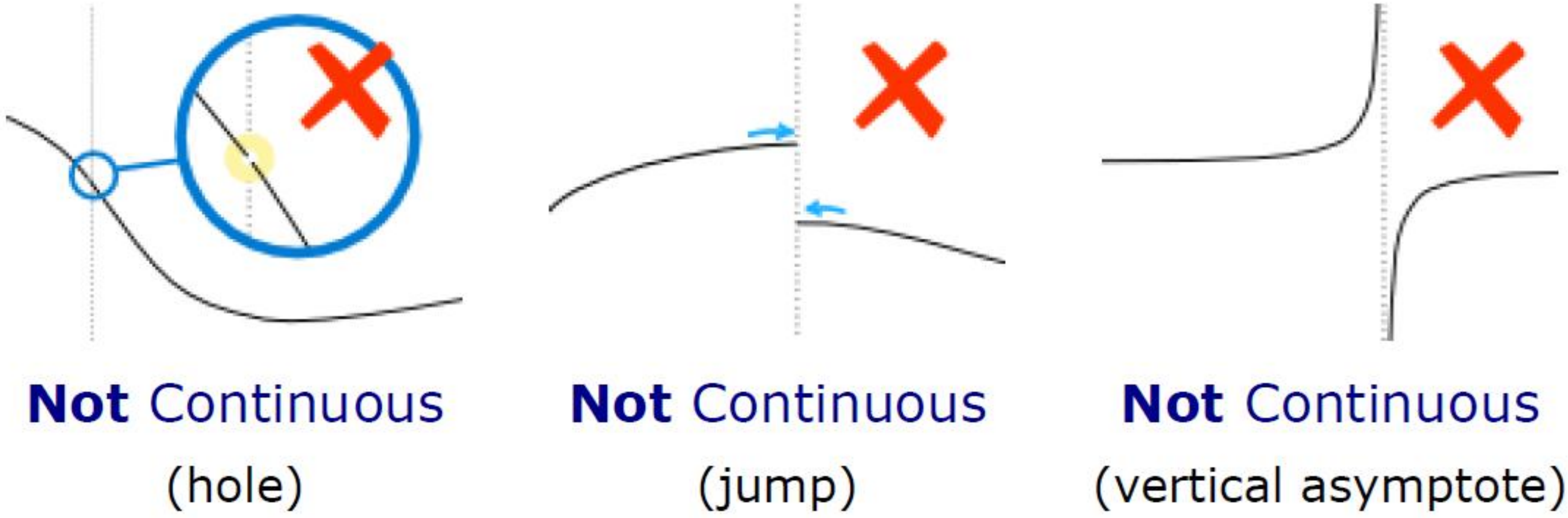
- A function is continuous when its graph is a single unbroken curve…



- … that you could draw without lifting your pen from the paper

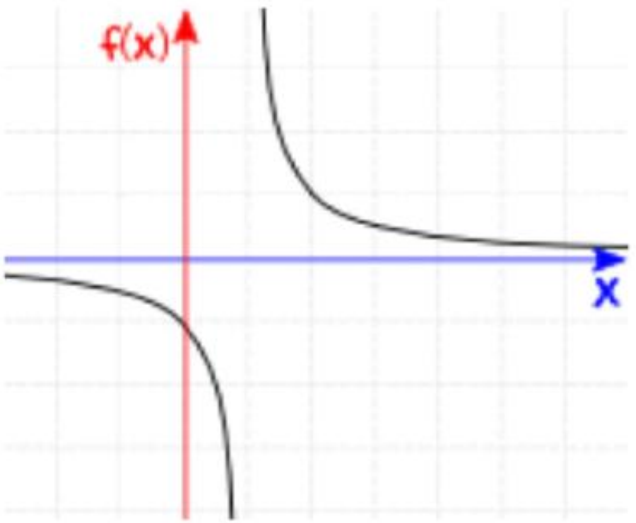
Reminder: continuous functions

- Examples:



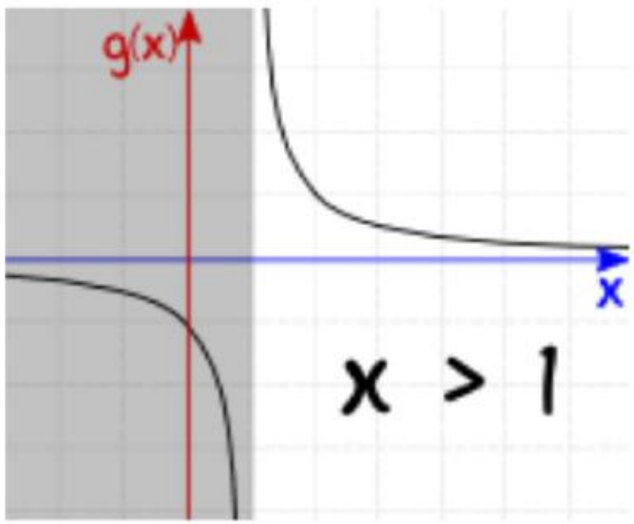
Reminder: continuous functions

- Examples:



f(x) = 1/(x-1)
over all **Real Numbers**

NOT continuous



g(x) = 1/(x-1) for **x>1**

Continuous

Reminder: continuous functions

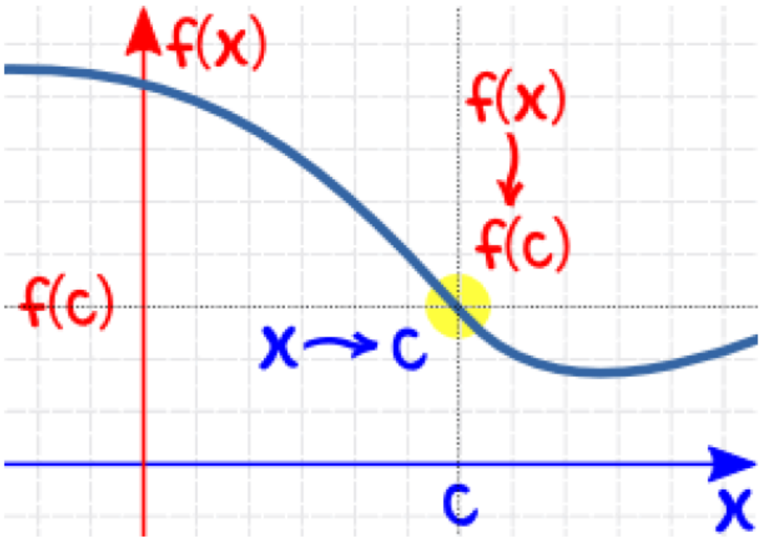
- Formal definition:
  - A function ***f*** is <u>continuous</u> when, for every value ***c*** in its Domain:
    - ***f(c)*** is defined, and

$$\lim_{x \to c} f(x) = f(c)$$

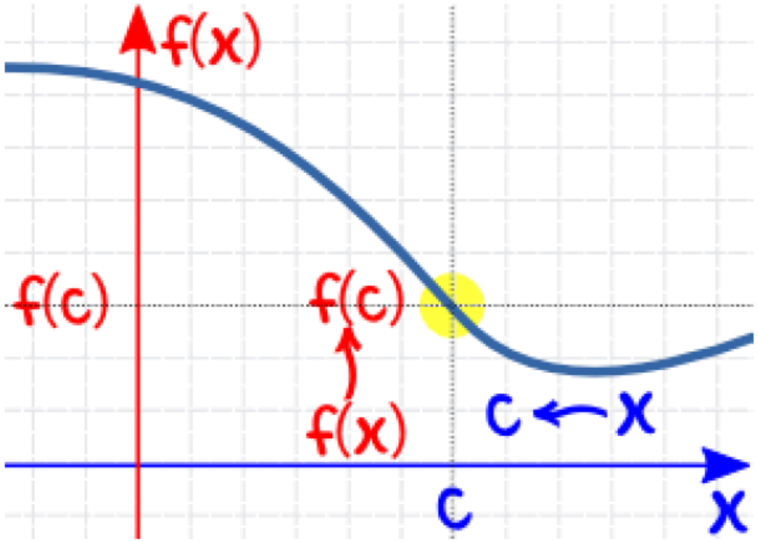
    - the limit of ***f(x)*** as ***x*** approaches ***c*** equals ***f(c)***
    - "as ***x*** gets closer and closer to ***c***, ***f(x)*** gets closer and closer to ***f(c)***"

Reminder: continuous functions

as x approaches c (from left)
then f(x) approaches f(c)



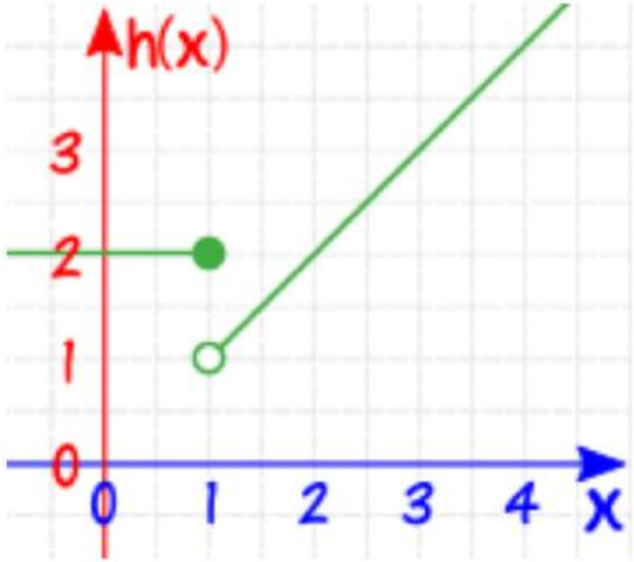
AND
as x approaches c (from right)
then f(x) approaches f(c)



Reminder: continuous functions

$$h(x) = \begin{cases} 2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$$

which looks like:



- It is defined at ***x=1***, since ***h(1) = 2***
- But you cannot say what the limit is at ***x=1***
  - from the left: ***2***
  - from the right: ***1***

Continuity of Preferences

- Why is it important to talk about this?

Continuity of Preferences

- In economics, the set ***X*** is often an ***<u>infinite</u>*** subset of a Euclidean space
- In $\mathbb{R}^1$
  - Ex: gold
- In $\mathbb{R}^2$
  - Ex: (salary, vacation time per year)
- In $\mathbb{R}^3$
  - Ex: (coffee, bread, milk)

Continuity of Preferences

- In economics, the set ***X*** is often an ***infinite*** subset of a Euclidean space
- Is there a utility representation in such a case?

Continuity of Preferences

- Which one do you prefer?

12 free months

100k miles

OR

120

20 boosters

Continuity of Preferences

- Which one do you prefer?

12 free months

OR

99.8k miles

120

19 boosters

Continuity of Preferences

- Which one do you prefer?

12 free months

100k miles

OR

119

20 boosters

Continuity of Preferences

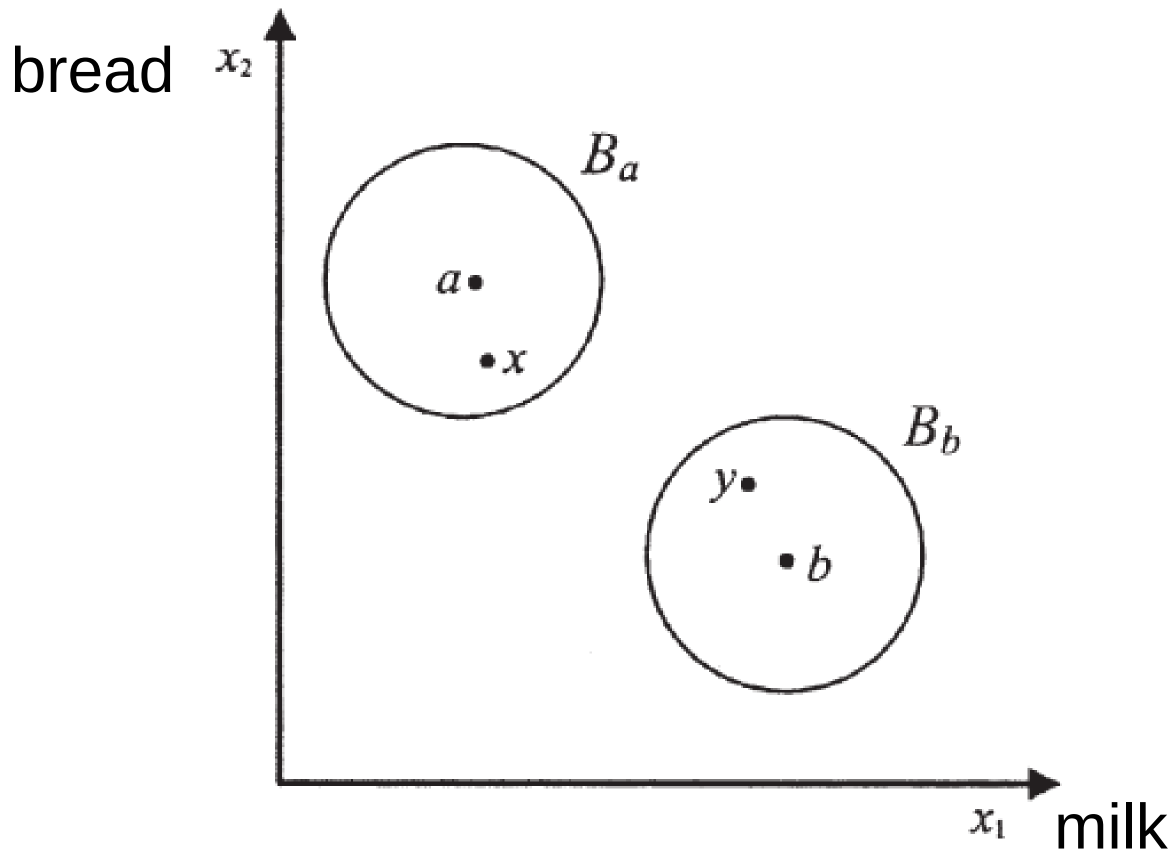
- The basic intuition, captured by the notion of a ***continuous preference relation***, is that if ***a*** is preferred to ***b***, then "small" deviations from ***a*** or from ***b*** will not reverse the ordering

Continuity of Preferences

- **Definition C1**
  - A preference relation $\succsim$ on $\boldsymbol{X}$ is <u>continuous</u> if whenever $\boldsymbol{a} \succ \boldsymbol{b}$ **<u>there are</u>** balls (neighborhoods in the relevant topology) $\boldsymbol{B_a}$ and $\boldsymbol{B_b}$ around $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, such that <u>for all</u> $\boldsymbol{x} \in \boldsymbol{B_a}$ and $\boldsymbol{y} \in \boldsymbol{B_b}$, $\boldsymbol{x} \succ \boldsymbol{y}$

Continuity of Preferences

- **Definition C1**



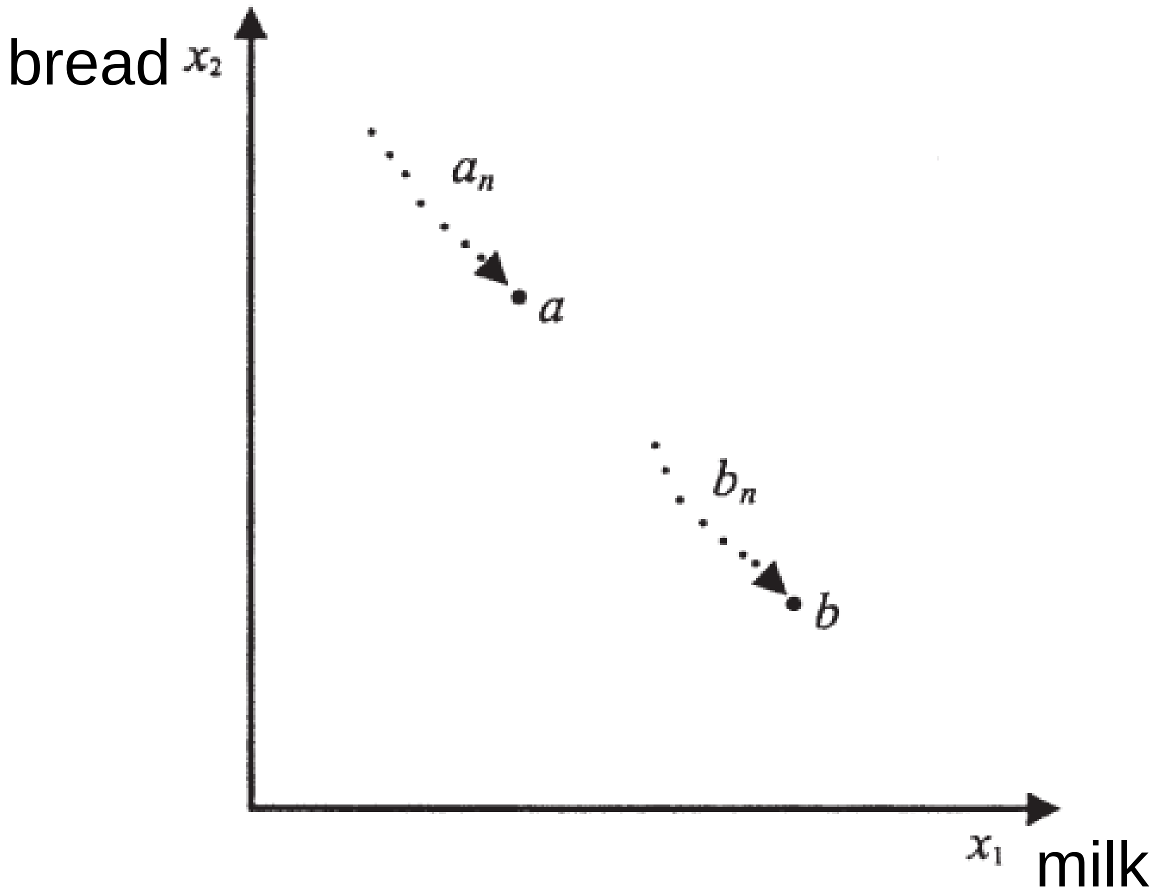
C1

Continuity of Preferences

- **Definition C2**
  - A preference relation $\succsim$ on $\boldsymbol{X}$ is continuous if the graph of $\succsim$ (i.e., the set $\boldsymbol{\{(x, y)|x \succsim y\}} \subseteq \boldsymbol{X \times X}$) is a closed set (with the product topology)
  - That is, if $\boldsymbol{\{(a_n, b_n)\}}$ is a sequence of pairs of elements in $\boldsymbol{X}$ satisfying $\boldsymbol{a_n} \succsim \boldsymbol{b_n}$ <u>for all</u> $\boldsymbol{n}$ and $\boldsymbol{a_n} \to \boldsymbol{a}$ and $\boldsymbol{b_n} \to \boldsymbol{b}$, then $\boldsymbol{a} \succsim \boldsymbol{b}$

Continuity of Preferences

- **Definition C2**



C2

Continuity of Preferences

- Claim:
  - The preference relation $\succsim$ on ***X*** satisfies ***C1*** if and only if it satisfies ***C2***

Continuity of Preferences

- Proof: (if)
  - Assume that $\succsim$ on $\boldsymbol{X}$ is continuous according to ***C1***. Let $\boldsymbol{\{(a_n, b_n)\}}$ be a sequence of pairs satisfying $\boldsymbol{a_n} \succsim \boldsymbol{b_n}$ for all $\boldsymbol{n}$ and $\boldsymbol{a_n} \to \boldsymbol{a}$ and $\boldsymbol{b_n} \to \boldsymbol{b}$
  - If it is not true that $\boldsymbol{a} \succsim \boldsymbol{b}$ (i.e., $\boldsymbol{b} \succ \boldsymbol{a}$), then there exist two balls $\boldsymbol{B_a}$ and $\boldsymbol{B_b}$ around $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, such that for all $\boldsymbol{y} \in \boldsymbol{B_b}$ and $\boldsymbol{x} \in \boldsymbol{B_a}$, $\boldsymbol{y} \succ \boldsymbol{x}$
  - There is an $\boldsymbol{N}$ large enough such that for all $\boldsymbol{n > N}$, both $\boldsymbol{b_n} \in \boldsymbol{B_b}$ and $\boldsymbol{a_n} \in \boldsymbol{B_a}$
  - Therefore, for all $\boldsymbol{n > N}$, we have $\boldsymbol{b_n} \succ \boldsymbol{a_n}$, which is a contradiction

Continuity of Preferences

- Proof: (only if)
  - Assume that $\succsim$ is continuous according to ***C2***
  - Let $\boldsymbol{a} \succ \boldsymbol{b}$
  - Assume by contradiction that for all $\boldsymbol{n}$ there exist $\boldsymbol{a_n} \in \boldsymbol{Ball(a, 1/n)}$ and $\boldsymbol{b_n} \in \boldsymbol{Ball(b, 1/n)}$ such that $\boldsymbol{b_n} \succsim \boldsymbol{a_n}$
  - The sequence $\boldsymbol{(b_n, a_n)}$ converges to $\boldsymbol{(b, a)}$
  - By the second definition, $\boldsymbol{(b, a)}$ is within the graph of $\succsim$, that is, $\boldsymbol{b} \succsim \boldsymbol{a}$, which is a contradiction

Continuity of Preferences

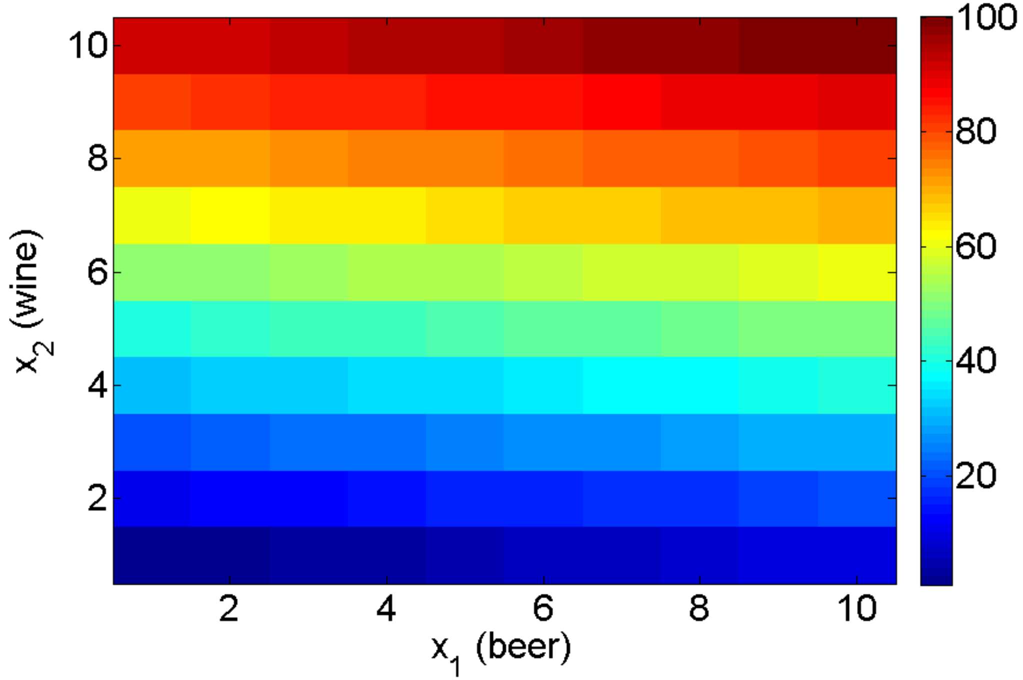
- Remark #1
  - $\succsim$ on $\boldsymbol{X}$ is represented by a continuous function $\boldsymbol{U}$, then $\succsim$ is continuous
  - To see this, note that if $\boldsymbol{a \succ b}$, then $\boldsymbol{U(a) > U(b)}$
  - Let $\boldsymbol{\varepsilon = (U(a) - U(b))/2}$
  - By the continuity of $\boldsymbol{U}$, there is a $\boldsymbol{\delta > 0}$ such that for all $\boldsymbol{x}$ distanced less than $\boldsymbol{\delta}$ from $\boldsymbol{a}$, $\boldsymbol{U(x) > U(a) - \varepsilon}$, and for all $\boldsymbol{y}$ distanced less than $\boldsymbol{\delta}$ from $\boldsymbol{b}$, $\boldsymbol{U(y) < U(b) + \varepsilon}$
  - Thus, for $\boldsymbol{x}$ and $\boldsymbol{y}$ within the balls of radius $\boldsymbol{\delta}$ around $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, $\boldsymbol{x \succ y}$

Continuity of Preferences

- Remark #2
  - The lexicographic preferences that were used in the counterexample to the existence of a utility representation are not continuous
  - Why?
  - This is because $\boldsymbol{(1, 1) \succ (1, 0)}$, but in any ball around $\boldsymbol{(1, 1)}$ there are points inferior to $\boldsymbol{(1, 0)}$

Continuity of Preferences

- Remark #2
  - The lexicographic preferences that were used in the counterexample to the existence of a utility representation are not continuous



Debreu's theorem

- Continuous preferences have a continuous utility representation
- Proof in the book