Bayesian Games

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Item location: Ft. Lauderdale
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Other examples

Very simple example

- **2** players
- Each draw a card from a deck
- The one who has the highest card wins everything
- Two actions available: to **bet** or to **check**
- **I**f **both bet**, the game is worth **$2**
- If **both check**, the game is worth **$1**
- If **one checks** and the **other bets**, the one who bets wins

Bayesian Games

- So far, all players know what game is being played
  - the number of players
  - the actions available for every player
  - the payoffs associated with each action vector
- Even in imperfect information games?

Bayesian Games

- Bayesian games allow us to represent players' uncertainties about the very game being played
- How?
  - This uncertainty is represented as a probability distribution over a set of possible games
    - Same players and actions
    - Different utility functions

Bayesian Games

- We make two assumptions

  1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs

  2. The beliefs of the different agents are posteriors, obtained by conditioning a common prior on individual private signals

Bayesian Games

- 1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs
  - One can imagine many other potential types of uncertainty that players might have about the game
    - how many players are involved, what actions are available to each player, etc
  - These other types of uncertainty can be reduced to uncertainty only about payoffs via problem reformulation

Bayesian Games

- Example: one player is uncertain about the number of actions available to the other players
  - We can reduce this uncertainty to uncertainty about payoffs by padding the game with irrelevant actions
  - E.g. a player does not know whether his opponent has only the two strategies ***L*** and ***R*** or also the third one ***C***

Bayesian Games

Let's change the game on the left

	L	R
U	$1, 1$	$1, 3$
D	$0, 5$	$1, 13$

	L	C	R
U	$1, 1$	$0, 2$	$1, 3$
D	$0, 5$	$2, 8$	$1, 13$

Bayesian Games

Newly added column is dominated

	L	C	R
U	$1, 1$	$0, -100$	$1, 3$
D	$0, 5$	$2, -100$	$1, 13$

	L	C	R
U	$1, 1$	$0, 2$	$1, 3$
D	$0, 5$	$2, 8$	$1, 13$

The uncertainty about the strategy space can be reduced to uncertainty about payoffs

Bayesian Games

- It is possible to reduce uncertainty about other aspects of the game to uncertainty about payoffs only
  - reductions have been shown for all the common forms of uncertainty

Bayesian Games

- 2. The beliefs of the different agents are posteriors, obtained by conditioning a common prior on individual private signals
  - common prior: everybody starts with the same beliefs of what is possible in the world
  - individual private signals: agents might get private information about what game is being played
  - From the signals, they update the priors to reach a posterior

Bayesian Games

- Still concerning Assumption #2
  - A bayesian game defines not only the uncertainties of agents about the game being played
  - but their beliefs about the beliefs of other agents about the game being played
  - and indeed an entire infinite hierarchy of nested beliefs (the so-called ***<u>epistemic type space</u>***)

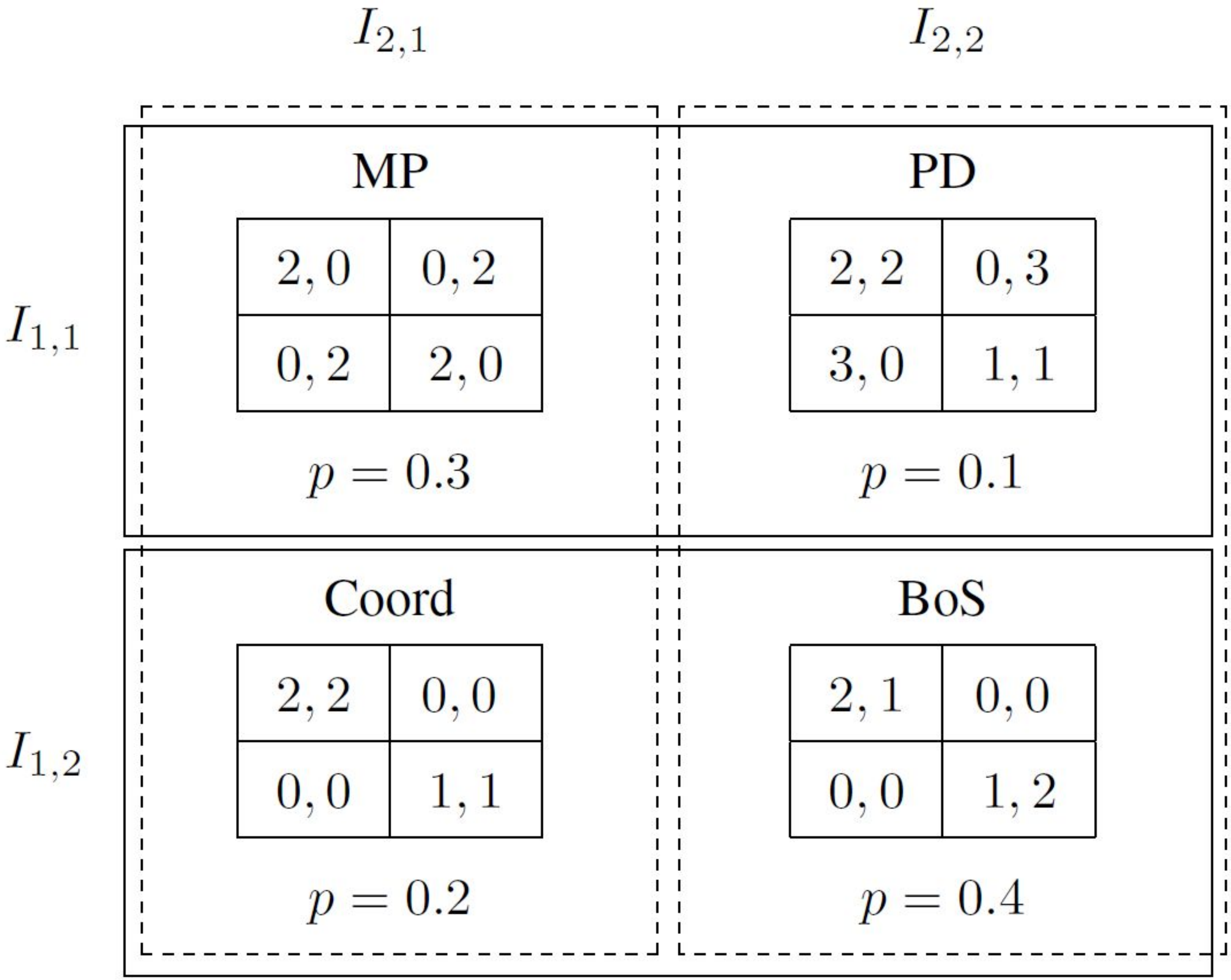
Definition

- There are several ways of presenting Bayesian games
- We will offer three different definitions
  - All three are equivalent, modulo some subtleties
  - Each formulation is useful in different settings and offers different intuition about the underlying structure of this family of games

Definition: Information Sets

- Under this definition, a Bayesian game consists of a set of games that differ only in:
  - their payoffs
  - a common prior defined over them
  - a partition structure over the games for each agent

Definition: Information Sets



Definition: Information Sets

- A Bayesian game is a tuple ***(N,G, P, I)*** where:
  - ***N*** is a set of agents
  - ***G*** is a set of games with ***N*** agents each such that if ***g, g' ∈ G*** then for each agent ***i ∈ N*** the strategy space in ***g*** is identical to the strategy space in ***g'***
  - ***P ∈ Π(G)*** is a common prior over games, where ***Π(G)*** is the set of all probability distributions over ***G***
  - $I = (I_1, \ldots, I_N)$ is a tuple of partitions of ***G***, one for each agent

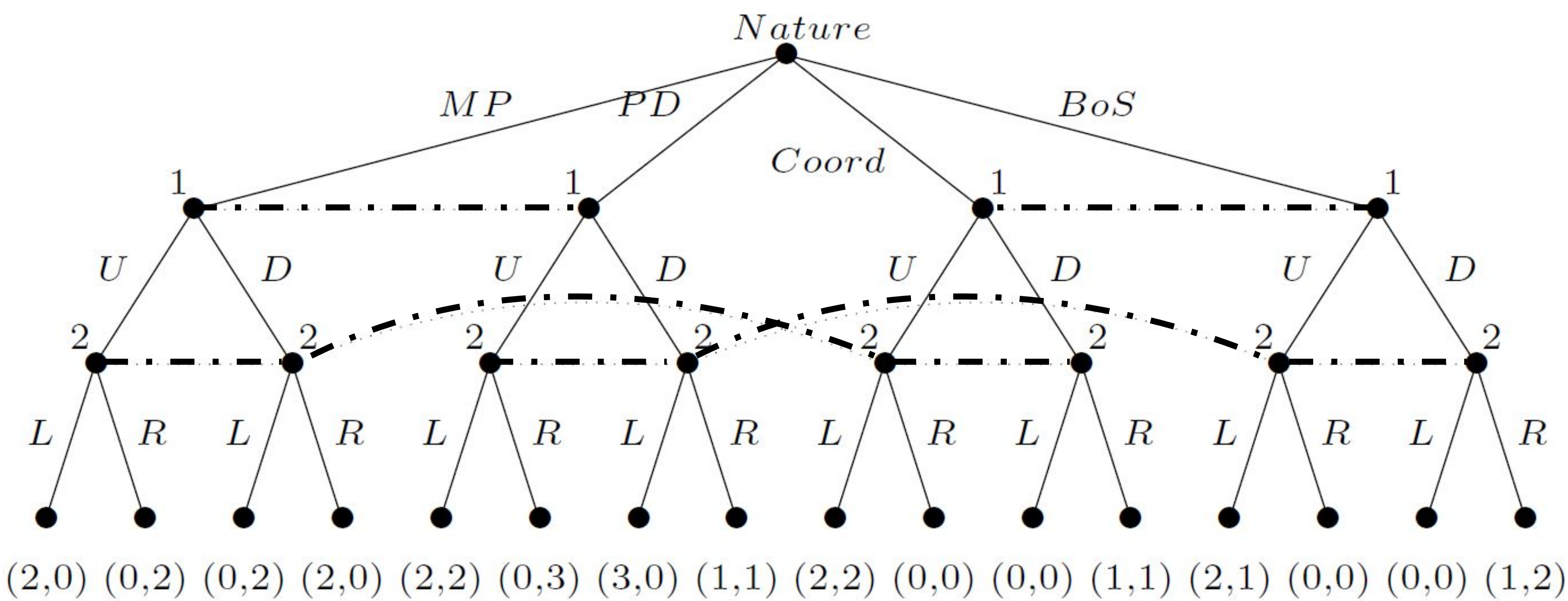
Definition: Extensive form with chance moves

- A special agent called Nature who makes probabilistic choices
- Nature makes all its choices at the beginning
- Nature does not have a utility function (or a constant one payoff), and has the unique strategy of randomizing in a commonly known way
- The agents receive individual signals about Nature’s choice, and these are captured by their information sets in a standard way

Definition: Extensive form with chance moves

- The agents have no additional information
  - The information sets capture the fact that agents make their choices without knowing the choices of others
- These chance moves of Nature require minor adjustments of existing definitions
  - replace payoffs by their expectations given Nature’s moves

Definition: Extensive form with chance moves



Definition: Extensive form with chance moves

- Can be initially more intuitive, but more cumbersome to work with
  - Use an extensive-form representation in a setting where players are unable to observe each others' moves
  - For this reason, we will not make further use of this definition
- One advantage: it extends very naturally to Bayesian games in which players move sequentially and do (at least sometimes) learn about previous players' moves

Definition: epistemic types

- Recall that a game may be defined by a set of players, actions, and utility functions
- In the first definition, agents are really only uncertain about the game's utility function
- Here we use the notion of an epistemic type, or simply a type, as a way of defining uncertainty directly over a game's utility function

Definition: epistemic types

- Definition
  - Bayesian game is a tuple (N, A, Θ, p, u) where:
    - N is a set of agents
    - $A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to player i
    - $\Theta = \Theta_1 \times \ldots \times \Theta_n$, where Θ_i is the type space of player i
    - $p : \Theta \to [0, 1]$ is a common prior over types
    - $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \to \mathbb{R}$ is the utility function for player i

Definition: epistemic types

- Definition
  - Bayesian game is a tuple (N, A, Θ, p, u) where:
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    - $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \rightarrow \mathbb{R}$ is the utility function for player i
- Assumption: all of the above is common knowledge among the players

Definition: epistemic types

- What is a type?
  - The type of agent encapsulates all the information possessed by the agent that is not common knowledge
- This is often quite simple...
  - e.g. the agent’s knowledge of his private payoff function
- ... but can also include his beliefs about other agents’ payoffs, about their beliefs about his own payoff, and any other higher-order beliefs

Definition: epistemic types

- For each of the agents we have two types, corresponding to his two information sets

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP $2,0$ | $0,2$ $0,2$ | $2,0$ $p = 0.3$	PD $2,2$ | $0,3$ $3,0$ | $1,1$ $p = 0.1$
$I_{1,2}$	Coord $2,2$ | $0,0$ $0,0$ | $1,1$ $p = 0.2$	BoS $2,1$ | $0,0$ $0,0$ | $1,2$ $p = 0.4$

MP

$2,0$	$0,2$
$0,2$	$2,0$

$p = 0.3$

PD

$2,2$	$0,3$
$3,0$	$1,1$

$p = 0.1$

Coord

$2,2$	$0,0$
$0,0$	$1,1$

$p = 0.2$

BoS

$2,1$	$0,0$
$0,0$	$1,2$

$p = 0.4$

Definition: epistemic types

A_1 = {U, D}
A_2 = {L, R}

$\Theta_1 = \{\theta_{1,1}, \theta_{1,2}\}$
$\Theta_2 = \{\theta_{2,1}, \theta_{2,2}\}$

joint distribution of these types:

$$p(\theta_{1,1}, \theta_{2,1}) = 0.3$$
$$p(\theta_{1,1}, \theta_{2,2}) = 0.1$$
$$p(\theta_{1,2}, \theta_{2,1}) = 0.2$$
$$p(\theta_{1,2}, \theta_{2,2}) = 0.4$$

$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP

2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

2, 2	0, 3
3, 0	1, 1

$p = 0.1$

$I_{1,2}$

Coord

2, 2	0, 0
0, 0	1, 1

$p = 0.2$

BoS

2, 1	0, 0
0, 0	1, 2

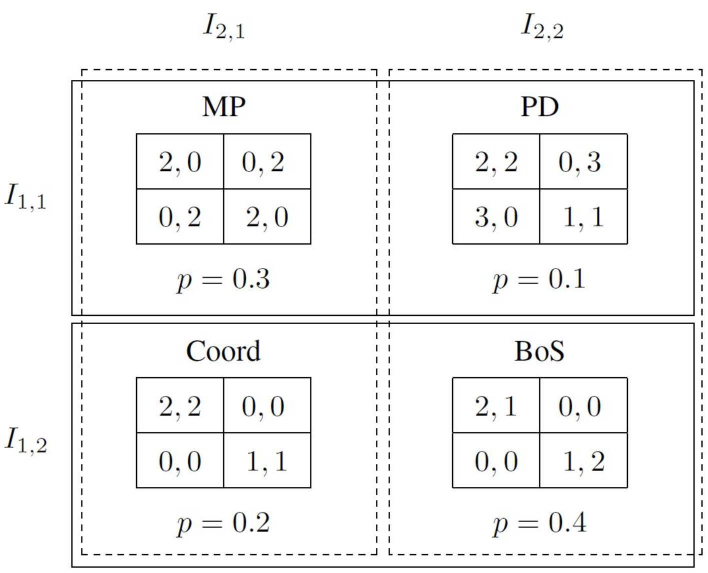
$p = 0.4$

the conditional probabilities for player ***1***:

$$p(\theta_{2,1} \mid \theta_{1,1}) = 3/4$$
$$p(\theta_{2,2} \mid \theta_{1,1}) = 1/4$$
$$p(\theta_{2,1} \mid \theta_{1,2}) = 1/3$$
$$p(\theta_{2,2} \mid \theta_{1,2}) = 2/3$$

Definition: epistemic types

- utility function for both players:



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Strategies and Equilibria

- How to reason about Bayesian games?
  - We will use the epistemic type definition
  - All of the concepts can also be expressed in terms of the other definitions

Strategies and Equilibria

- What is the agent's strategy space in a Bayesian game?
- In imperfect-information extensive-form games:
  - a pure strategy is a mapping from information sets to actions
- The definition is similar in Bayesian games:
  - a pure strategy $\boldsymbol{\alpha_i : \Theta_i \rightarrow A_i}$ is a mapping from every type agent ***i*** could have to the action he would play if he had that type

Strategies and Equilibria

- What is the agent's strategy space in a Bayesian game?
- Mixed strategies defined as probability distributions over pure strategies
- A mixed strategy for $\boldsymbol{i}$ is defined as $\boldsymbol{s_i} \in \boldsymbol{S_i}$, where $\boldsymbol{S_i}$ is the set of all $\boldsymbol{i}$'s mixed strategies
- $\boldsymbol{s_j(a_j \mid \theta_j)}$ to denote the probability under mixed strategy $\boldsymbol{s_j}$ that agent $\boldsymbol{j}$ plays action $\boldsymbol{a_j}$, given that $\boldsymbol{j}$'s type is $\boldsymbol{\theta_j}$

Other examples

Example

$Sheriff$

$Good$	$Shoot$	Not
$Shoot$	$-3, -1$	$-1, -2$
Not	$-2, -1$	$0, 0$

Bad	$Shoot$	Not
$Shoot$	$0, 0$	$2, -2$
Not	$-2, -1$	$-1, 1$

Strategies and Equilibria

- An environment with multiple sources of uncertainty
- What is the expected utility of an agent?
- Three meaningful notions of expected utility
  - *ex post*
    - computed based on all agents' actual types
  - *ex interim*
    - agent knows his own type but not the types of the other agents
  - *ex ante*
    - agent does not know anybody's type

Ex post expected utility

- Agent ***i***'s *ex post* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where the agents' strategies are given by ***s*** and the agent' types are given by ***θ***, is defined as

$$EU_i(s,\theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a,\theta)$$

- In this case, the only uncertainty concerns the other agents' mixed strategies

Ex interim expected utility

- Agent ***i***'s *ex interim* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where ***i***'s type is $\boldsymbol{\theta_i}$ and where the agents' strategies are given by the mixed-strategy profile ***s***, is defined as

$$EU_i(s,\theta_i) = \sum_{\boxed{\theta_{-i}\in\Theta_{-i}}} p(\theta_{-i}|\theta_i) \sum_{\boxed{a\in A}} \left(\prod_{j\in N} s_j(a_j|\theta_j) \right) \boxed{u_i(a,\theta_{-i},\theta_i)}$$

i must consider every assignment of types to the other agents $\boldsymbol{\theta_{-i}}$ and every pure action profile ***a*** in order to evaluate his utility function $\boldsymbol{u_i(a, \theta_i, \theta_{-i})}$

Ex interim expected utility

- Agent ***i***'s *ex interim* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where ***i***'s type is $\boldsymbol{\theta_i}$ and where the agents' strategies are given by the mixed-strategy profile ***s***, is defined as

$$EU_i(s,\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} \boxed{p(\theta_{-i}|\theta_i)} \sum_{a\in A} \boxed{\left(\prod_{j\in N} s_j(a_j|\theta_j)\right)} u_i(a,\theta_{-i},\theta_i)$$

Utility of ***i*** is weighted by
(1) the probability that the other players' types would be $\boldsymbol{\theta_{-i}}$ given that his own type is $\boldsymbol{\theta_i}$
(2) the probability that the pure action profile ***a*** would be realized given all players' mixed strategies and types

Ex interim expected utility

- Agent ***i***'s *ex interim* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where ***i***'s type is $\boldsymbol{\theta_i}$ and where the agents' strategies are given by the mixed-strategy profile ***s***, is defined as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) EU_i(s, (\theta_i, \theta_{-i}))$$

- Uncertainty over mixed strategies was already handled in the *ex post* case
- Thus, we can also write *ex interim* expected utility as a weighted sum of $\boldsymbol{EU_i(s, \theta)}$ terms

Ex ante expected utility

- Agent ***i***'s *ex ante* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where the agents' strategies are given by the mixed-strategy profile ***s***, is defined as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta)$$

or

ex post

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \boxed{EU_i(s, \theta)}$$

or

ex interim

Best response

- The <u>**set**</u> of agent ***i***'s best response to mixed-strategy profile $\boldsymbol{s}_{-i}$ are given by

what EU is this?

- $\boldsymbol{BR}_i$ is a set because there may be many strategies for ***i*** that yield the same expected utility

Best response

- It may seem odd that ***BR*** is calculated based on ***i***'s *ex ante* EU
- However, write $\boldsymbol{EU_i(s)}$ as 
- Observe that $\boldsymbol{EU_i(s_i', s_{-i}, \theta_i)}$ does not depend on strategies that ***i*** would play if his type were not $\boldsymbol{\theta_i}$
- We are in fact performing independent maximization of ***i***'s *ex interim* expected utilities conditioned on each type that he could have

Best response

- Intuitively speaking, if a certain action is best after the signal is received, it is also the best conditional plan devised ahead of time for what to do should that signal be received

Bayes–Nash equilibrium

- A <u>Bayes–Nash equilibrium</u> is a mixed strategy profile $\boldsymbol{s}$ that satisfies $\forall i \; s_i \in BR_i(s_{-i})$
- This is exactly the definition we gave for the Nash equilibrium
  - each agent plays a best response to the strategies of the other players
- <u>Difference</u>: built on top of the Bayesian game definitions of best response and expected utility

Bayes–Nash equilibrium

- Observe that we would not be able to define equilibrium in this way if an agent's strategies were not defined for every possible type
- In order for a given agent i to play a best response to the other agents $-i$, i must know what strategy each agent would play for each of his possible types
- Without this information, it would be impossible to evaluate the term $EU_i(s_i', s_{-i})$ in previous equations

Bayes–Nash equilibrium

- A BNE is a set of strategies, one for each type of player, such that no type has incentive to change his or her strategy given the beliefs about the types and what the other types are doing

Computing Equilibria

- Despite its similarity to the Nash equilibrium, the Bayes–Nash equilibrium may seem conceptually more complicated
- However, as we did with extensive-form games, we can construct a normal-form representation that corresponds to a given Bayesian game

Computing Equilibria

- The induced normal form for Bayesian games has an action for every pure strategy
  - The actions for an agent $\boldsymbol{i}$ are the distinct mappings from $\boldsymbol{\Theta_i}$ to $\boldsymbol{A_i}$
  - Each agent $\boldsymbol{i}$'s payoff given a pure-strategy profile $\boldsymbol{s}$ is his *ex ante* expected utility under $\boldsymbol{s}$

Computing Equilibria

- Then, the Bayes–Nash equilibria are precisely the Nash equilibria of its induced normal form
  - Nash's theorem applies directly to Bayesian games, i.e.,Bayes–Nash equilibria always exist

Example

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP $p = 0.3$	PD $p = 0.1$
$I_{1,2}$	Coord $p = 0.2$	BoS $p = 0.4$

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

$p = 0.1$

Coord

$2, 2$	$0, 0$
$0, 0$	$1, 1$

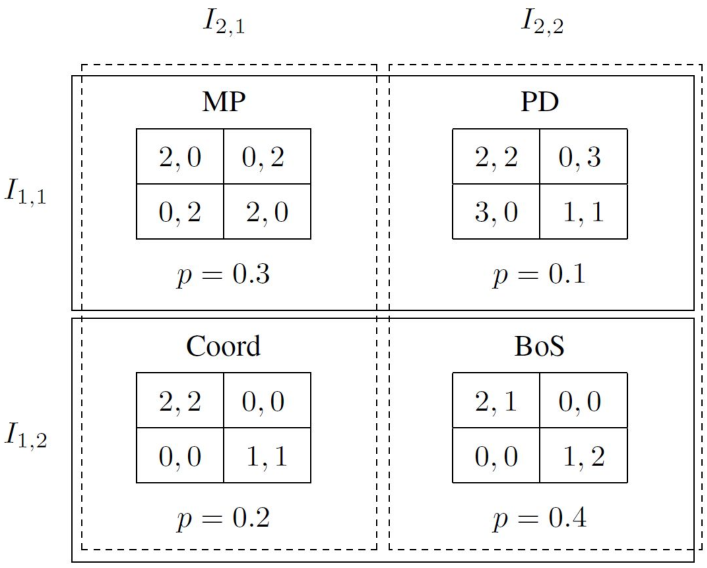
$p = 0.2$

BoS

$2, 1$	$0, 0$
$0, 0$	$1, 2$

$p = 0.4$

Example



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Computing Equilibria

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Computing Equilibria

- Each agent has four possible pure strategies
  - two types and two actions
- Player ***1***'s four strategies in the Bayesian game can be labeled ***UU***, ***UD***, ***DU***, and ***DD***
  - ***UU*** means that ***1*** chooses ***U*** regardless of his type
  - ***UD*** that he chooses ***U*** when he has type $\boldsymbol{\theta_{1,1}}$ and ***D*** when he has type $\boldsymbol{\theta_{1,2}}$, and so forth
- Similarly, we can denote the strategies of player ***2*** in the Bayesian game by ***RR***, ***RL***, ***LR***, and ***LL***

Computing Equilibria

- What if each agent has 2 actions and 3 types?
  - ***UUU***, ***UUD***, ***UDU***, ***UDD***, ***DUU***, ***DUD***, ***DDU***, ***DDD***
  - or $\boldsymbol{2^3}$ ($|A_i|$^$|\Theta_i|$) pure strategies in the induced normal form game

Computing Equilibria

- ***4 × 4*** normal-form game
  - these are the four strategies of the two agents
  - the payoffs are the expected payoffs in the individual games given the agents' common prior beliefs

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

Reminder: *ex ante* expected utility

- Agent ***i***'s *ex ante* expected utility in a Bayesian game ***(N, A, Θ, p, u)***, where the agents' strategies are given by the mixed-strategy profile ***s***, is defined as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta)$$

or

ex post

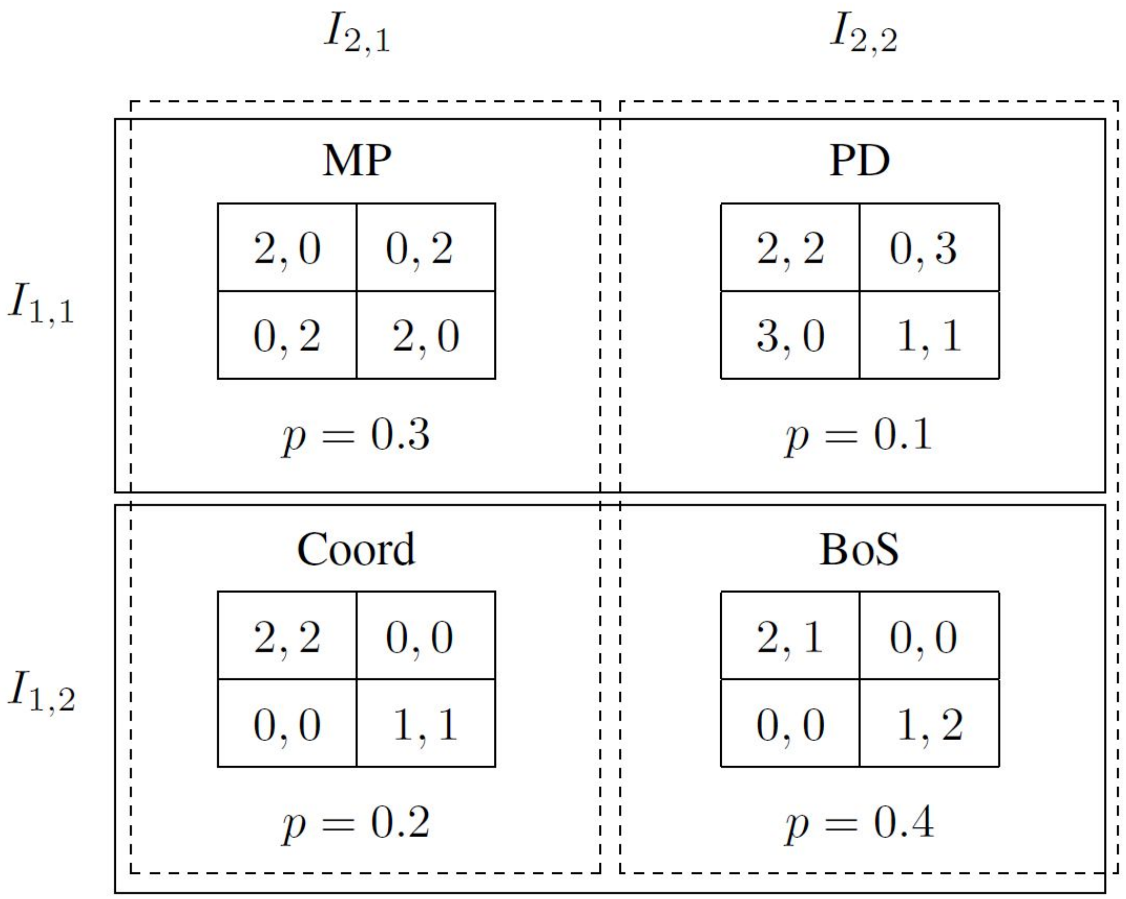
$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \boxed{EU_i(s, \theta)}$$

or

ex interim

Computing Equilibria

- For example, player ***2***'s *ex ante* expected utility under the strategy profile ***(UU,LL)*** is calculated as:



Computing Equilibria

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP $\begin{array}{|c|c|}\hline 2,0 & 0,2 \\ \hline 0,2 & 2,0 \\ \hline \end{array}$ $p = 0.3$	PD $\begin{array}{|c|c|}\hline 2,2 & 0,3 \\ \hline 3,0 & 1,1 \\ \hline \end{array}$ $p = 0.1$
$I_{1,2}$	Coord $\begin{array}{|c|c|}\hline 2,2 & 0,0 \\ \hline 0,0 & 1,1 \\ \hline \end{array}$ $p = 0.2$	BoS $\begin{array}{|c|c|}\hline 2,1 & 0,0 \\ \hline 0,0 & 1,2 \\ \hline \end{array}$ $p = 0.4$

$$
\begin{aligned}
u_2(UU, LL) &= \sum_{\theta \in \Theta} p(\theta) u_2(U, L, \theta) \\
&= p(\theta_{1,1}, \theta_{2,1}) u_2(U, L, \theta_{1,1}, \theta_{2,1}) + p(\theta_{1,1}, \theta_{2,2}) u_2(U, L, \theta_{1,1}, \theta_{2,2}) + \\
&\quad p(\theta_{1,2}, \theta_{2,1}) u_2(U, L, \theta_{1,2}, \theta_{2,1}) + p(\theta_{1,2}, \theta_{2,2}) u_2(U, L, \theta_{1,2}, \theta_{2,2}) \\
&= 0.3(0) + 0.1(2) + 0.2(2) + 0.4(1) = 1.
\end{aligned}
$$

Computing Equilibria

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

$$
\begin{aligned}
u_2(UU, LL) = &\sum_{\theta \in \Theta} p(\theta) u_2(U, L, \theta) \\
=& p(\theta_{1,1}, \theta_{2,1}) u_2(U, L, \theta_{1,1}, \theta_{2,1}) + p(\theta_{1,1}, \theta_{2,2}) u_2(U, L, \theta_{1,1}, \theta_{2,2}) + \\
& p(\theta_{1,2}, \theta_{2,1}) u_2(U, L, \theta_{1,2}, \theta_{2,1}) + p(\theta_{1,2}, \theta_{2,2}) u_2(U, L, \theta_{1,2}, \theta_{2,2}) \\
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Computing Equilibria

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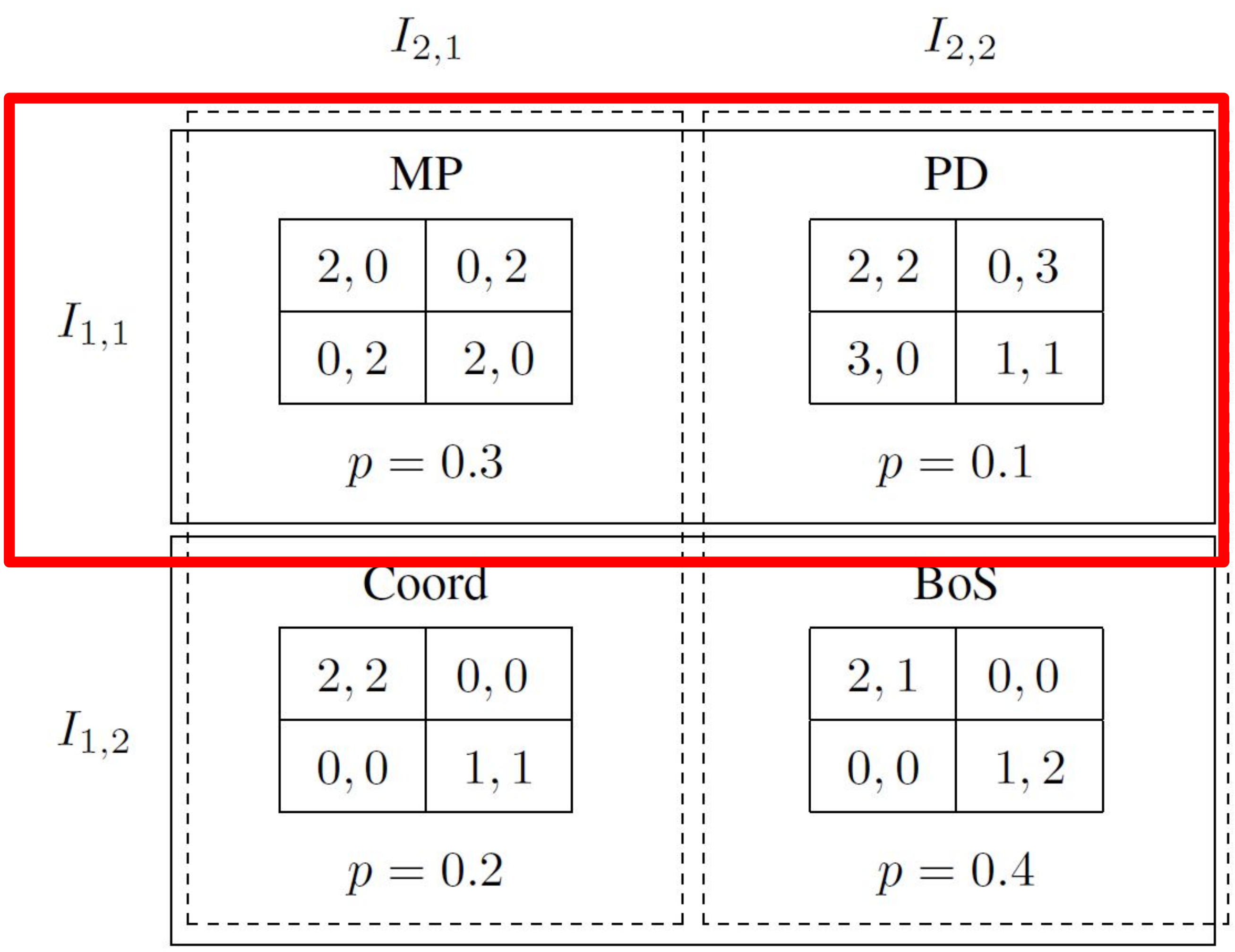
- Now the game may be analyzed straightforwardly
  - e.g.: player ***1***'s best response to ***RL*** is ***DU***

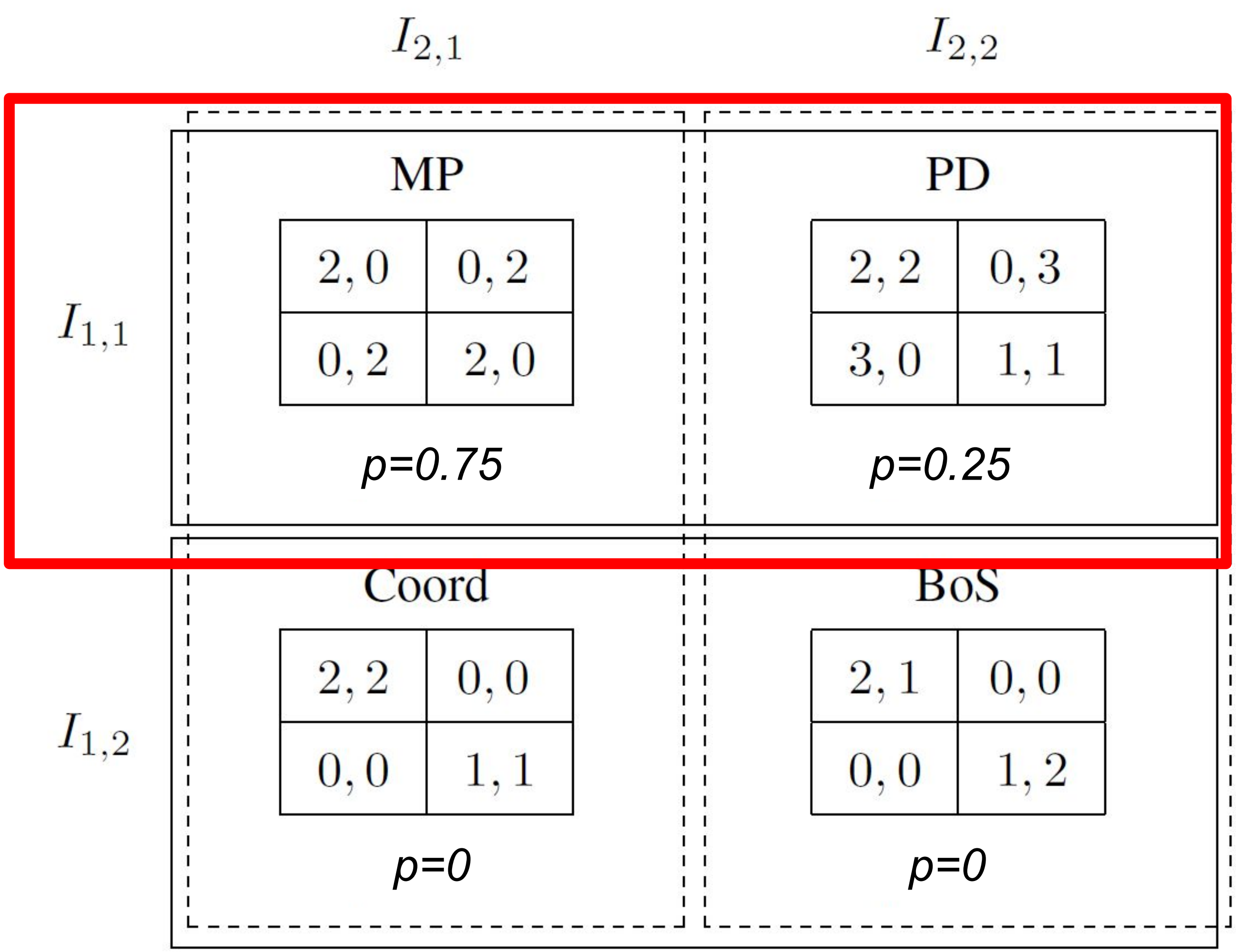
Computing Equilibria

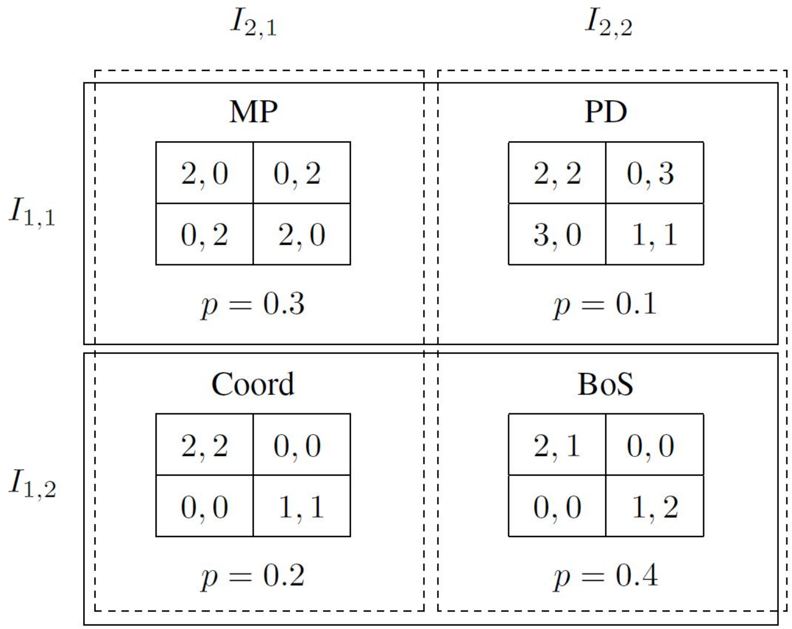
- Given a particular signal, the agent can compute the posterior probabilities and recompute the expected utility of any given strategy vector
- Thus in the previous example once the row agent gets the signal $\boldsymbol{\theta}_{1,1}$ he can update the expected payoffs and compute the new game

Computing Equilibria



Computing Equilibria





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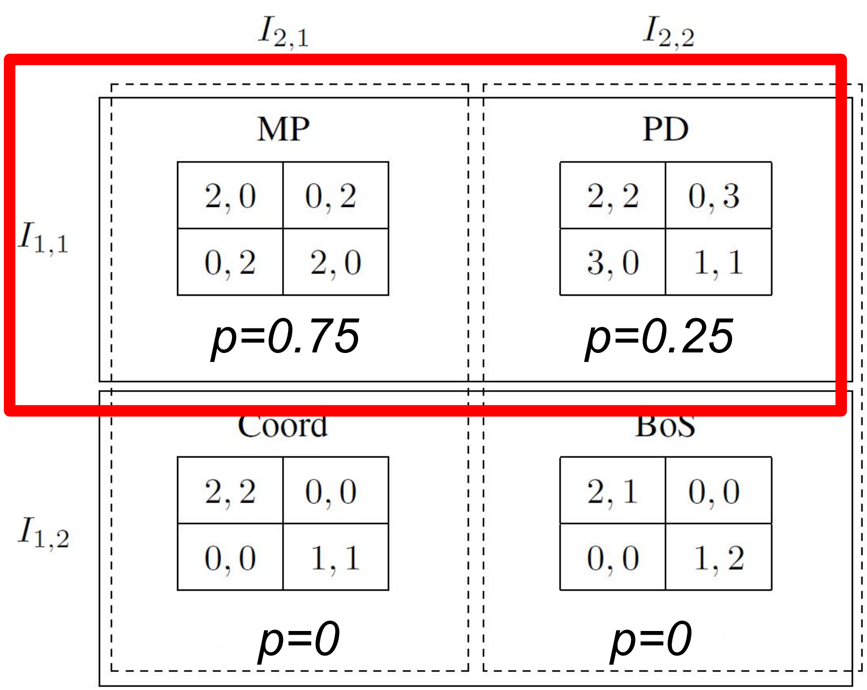

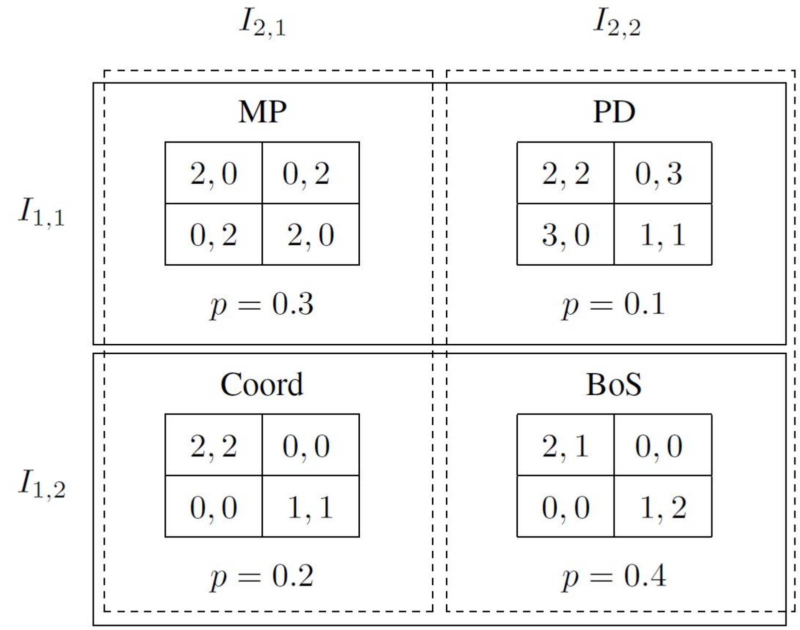

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
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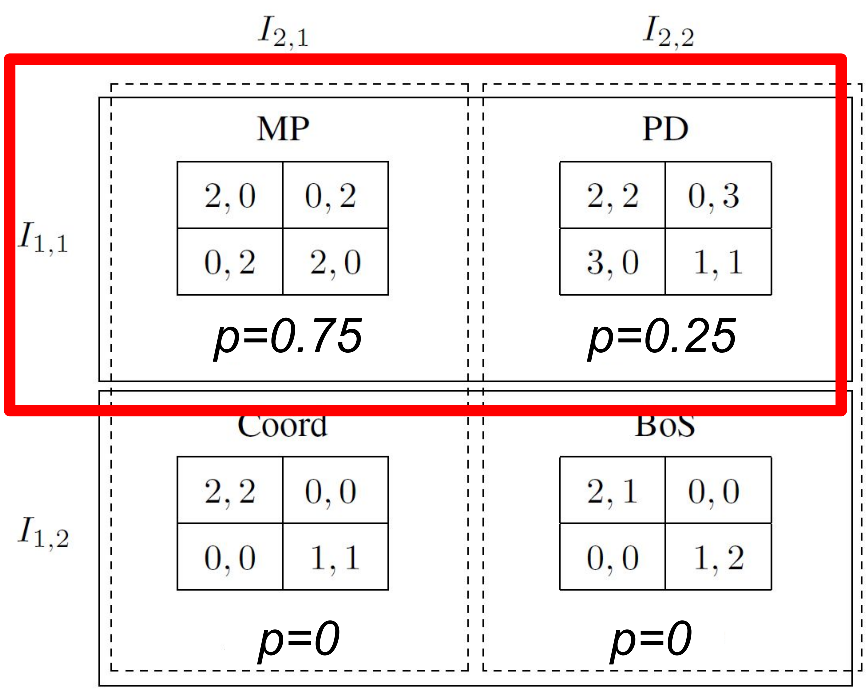

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Computing Equilibria

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Computing Equilibria

- ***DU*** is still a best response to ***RL***
  - But it won't always be the case
- The row player's payoffs are now independent of his choice of which action to take upon observing type $\boldsymbol{\theta_{1,2}}$
  - conditional on observing type $\boldsymbol{\theta_{1,1}}$ the player needs only to select a single action ***U*** or ***D***
  - this ***ex interim*** induced normal form could be written with four columns but only two rows

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
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	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Computing Equilibria

- Although we can use this matrix to find best responses for player ***1***, it turns out to be meaningless to analyze the Nash equilibria
- Why? :(

Computing Equilibria

- Although we can use this matrix to find best responses for player ***1***, it turns out to be meaningless to analyze the Nash equilibria
  - these expected payoffs are not common knowledge
  - if the column player were to condition on his signal, he would arrive at a different set of numbers
- It is only in the induced normal form in which the payoffs do not correspond to any *ex interim* assessment of any agent, that the Nash equilibria are meaningful

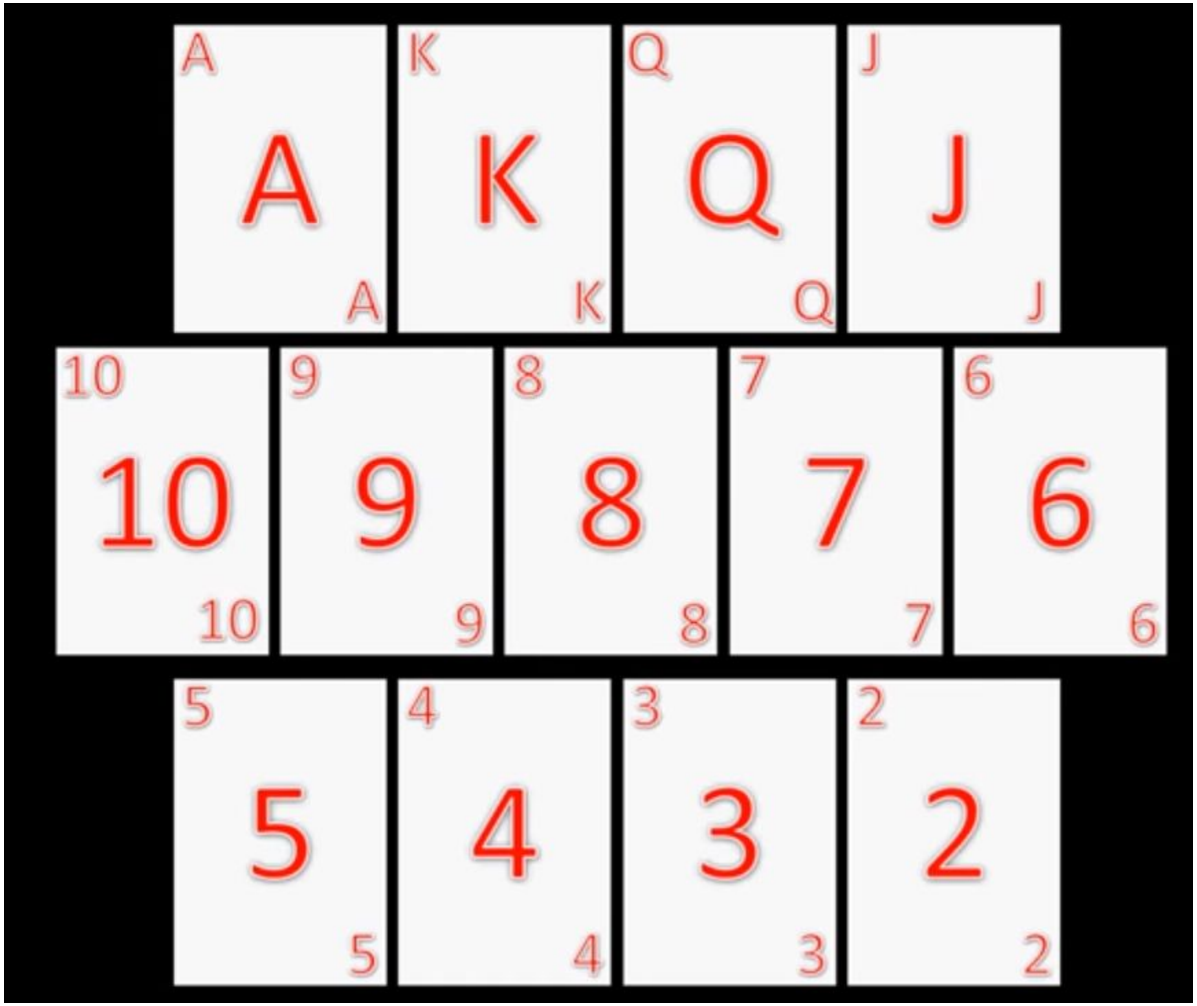
Computing Equilibria

- Expectmax algorithm
  - efficient for Bayesian games written using the “extensive form with chance moves” formulation
  - more details in the book

Dominance in Bayesian Games

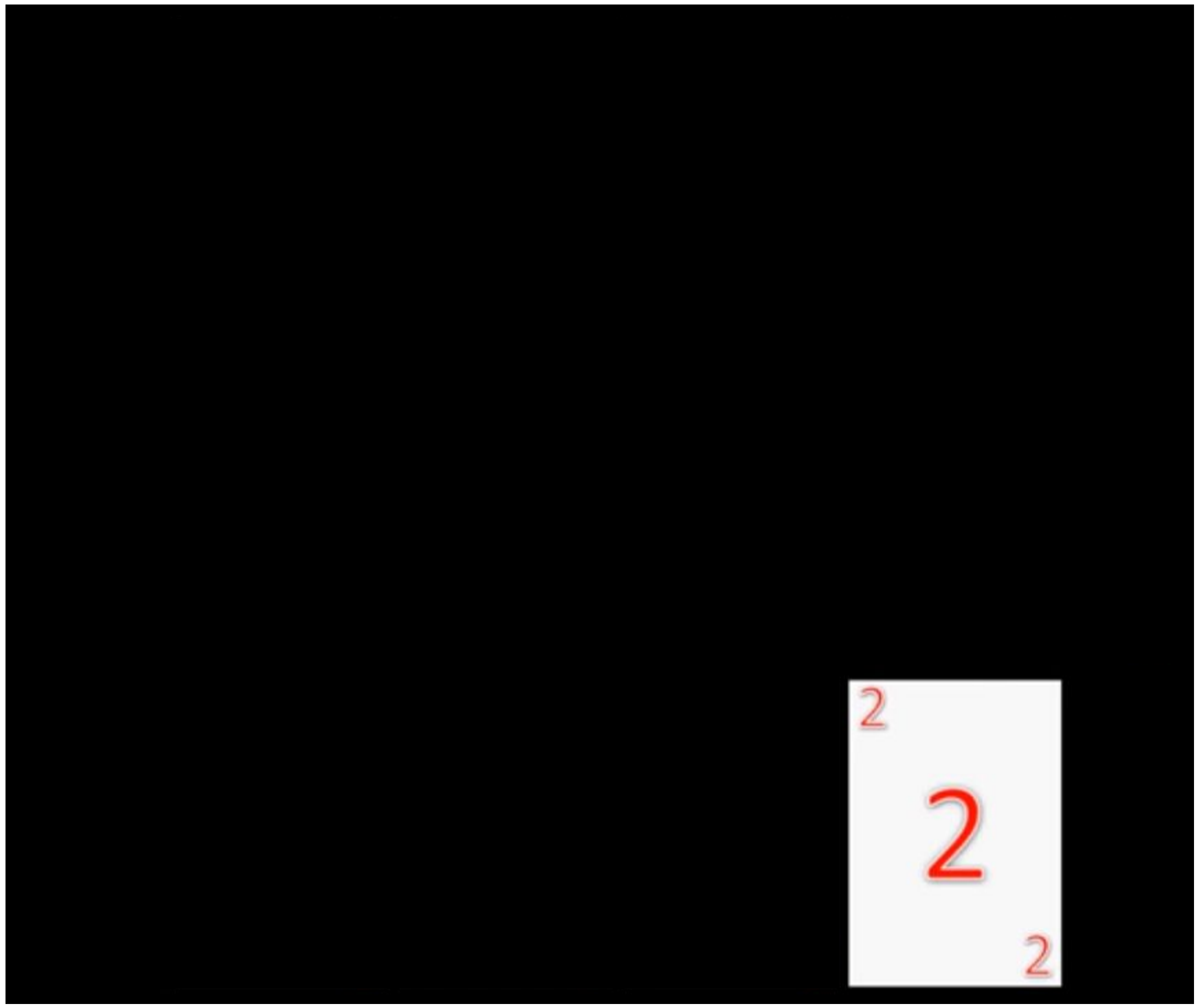
- Ex ante dominated strategy: a strategy for a **player** such that an alternative strategy for that **player** provides a greater payoff for that **player** regardless of all other players' strategies
- Interim dominated strategy: a strategy for a **type** such that an alternative strategy for that **type** provides a greater payoff for that **type** regardless of all other players' strategies

Dominance in Bayesian Games



ex ante dominance

Dominance in Bayesian Games



interim dominance

Dominance in Bayesian Games

PD = .2

	Left	Right
Up	3, 3	0, 4
Down	2, 1	1, 2

SH = .8

	Left	Right
Up	3, 3	0, 2
Down	2, 0	1, 1

Dominance in Bayesian Games

PD = .2

	Left	Right
Up	3, 3	0, 4
Down	2, 1	1, 2

interim dominance

Dominance in Bayesian Games

	LL	LR	RL	RR
Up	3, 3	.6, 2.2	2.4, 3.2	0, 2.4
Down	2, .2	1.2, 1	1.8, .4	1, 1.2

ex ante dominance

Dominance in Bayesian Games

- If a strategy is interim dominated, then it is ex ante dominated
- **The converse is not true**: if a strategy is ex ante dominated, there may or may not be any corresponding interim dominance

An example

Player 1 has one type. What Player 2 should do?

Is there any dominated strategy?

A = 0.5

	Left	Right
Up	0, 2	2, 5
Down	2, 2	0, 1

B = 0.5

	Left	Right
Up	0, 4	2, 5
Down	2, 4	0, 1

Claim: RL ex ante strictly dominates LR

Another example

Player 2 know his type. What is the solution of this game?

Are there any dominant strategies?

PD = 0.9

	Left	Right
Up	3, 3	0, 4
Down	2, 1	1, 2

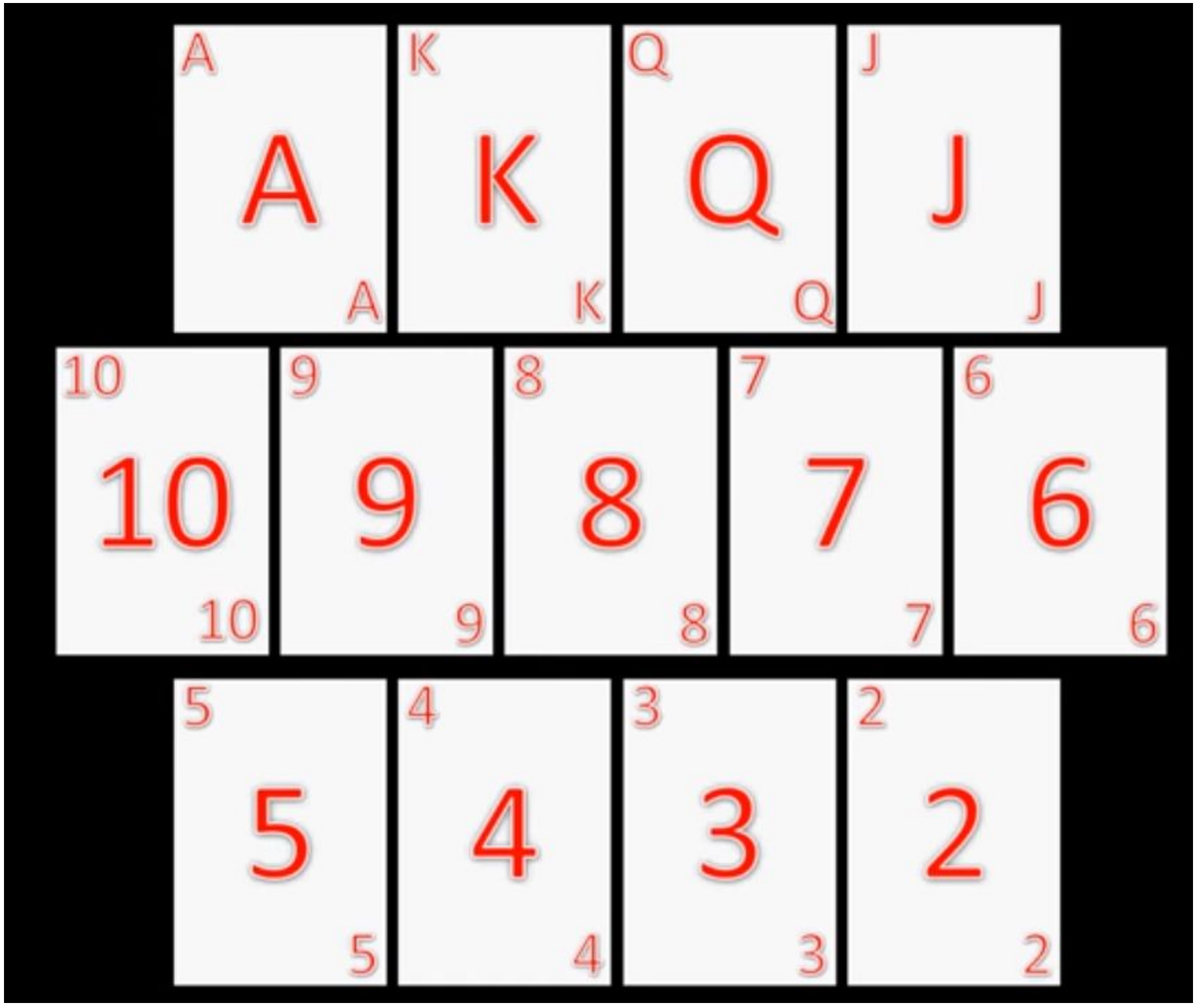
SH = 0.1

	Left	Right
Up	3, 3	0, 2
Down	2, 0	1, 1

Claim: Player 1 should play ‘Down’

Why Are There Antes in Poker?

- Two players, each will draw a card at random



Why Are There Antes in Poker?

- Two players, each will draw a card at random
- $\Theta_i = \{2, 3, ..., J, Q, K, A)$
- They will privately check the card and choose between two actions
  - bet 1$
  - fold
- If at least one player folds, each gets $0
- If both bet, the one with the highest card gets $1, the other loses $1

Why Are There Antes in Poker?

- What would you play?
- Naive strategy:
  - if I have a card higher than 8, I bet, else, I fold
- Why is this strategy naive?
  - because it is not the rational solution

Why Are There Antes in Poker?

- First observation:
  - impracticable to construct the matrix form of this game
    - Each player has 13 types
    - Each player has 2^13 pure strategies
    - Game matrix: 8192 x 8192

Why Are There Antes in Poker?

- What would you do if you have an Ace?
  - ***bet*** weakly dominates ***fold***

Why Are There Antes in Poker?

- What would you do if you have a 2?
  - fold
- What would you do if you have a 3?
  - fold, because if the other player has a 2, she will fold
- What would you do if you have a 4?
  - ...

Why Are There Antes in Poker?

- If there were not antes in poker, only the unbeatable type has incentive to bet

Ex post equilibrium

- Working with *ex post* utilities allows us to define an equilibrium concept that is stronger than the *Bayes–Nash equilibrium*
- Definition
  - An ***ex post equilibrium*** is a mixed-strategy profile ***s*** that satisfies

Ex post equilibrium

- This definition does not presume that each agent actually does know the others' types
- It says that no agent would ever want to deviate from his mixed strategy even if he knew the complete type vector $\boldsymbol{\theta}$
- This form of equilibrium is appealing because it is unaffected by perturbations in the type distribution $\boldsymbol{p(\theta)}$

Ex post equilibrium

- An *ex post* equilibrium does not ever require any agent to believe that the others have accurate beliefs about his own type distribution
  - a standard *Bayes–Nash* equilibrium can imply this requirement
- The *ex post* equilibrium is similar to equilibria in **dominant strategies**, which do not require any belief that other agents act rationally
  - Since many dominant strategy equilibria are also *ex post* equilibria, it is easy to believe that this relationship always holds

Ex post equilibrium

- Consider a two-player Bayesian game
  - each agent has two actions and two corresponding types: $\forall_{i \in N}$, $A_i = \Theta_i = \{H, L\}$
  - distributed uniformly: $\forall_{i \in N}$, $P(\theta_i = H) = 0.5$
  - the same utility function for each agent ***i***:

$$u_i(a, \theta) = \begin{cases} 10 & a_i = \theta_{-i} = \theta_i; \\ 2 & a_i = \theta_{-i} \neq \theta_i; \\ 0 & \text{otherwise.} \end{cases}$$

Ex post equilibrium

$$u_i(a,\theta) = \begin{cases} 10 & a_i = \theta_{-i} = \theta_i; \\ 2 & a_i = \theta_{-i} \neq \theta_i; \\ 0 & \text{otherwise.} \end{cases}$$

Ex post equilibrium

- What is the *ex interim* dominant strategy?
  - play $\boldsymbol{\theta}_i$
- What is the *ex post* best strategy?
  - play $\boldsymbol{\theta}_{-i}$
  - An equilibrium in these dominant strategies is not *ex post*

$$u_i(a,\theta) = \begin{cases} 10 & a_i = \theta_{-i} = \theta_i; \\ 2 & a_i = \theta_{-i} \neq \theta_i; \\ 0 & \text{otherwise.} \end{cases}$$

Ex post equilibrium

- Unfortunately, similar to equilibria in dominant strategies is that neither kind of equilibrium is guaranteed to exist :(

Recommended Lectures

- Game Theory 101 Course from William Spaniel
  - From (#64) to (#68)
  - starts at: https://www.youtube.com/watch?v=AzN-eV_Na10