Lecture Notes in Microeconomics

(The Economic Agent)
2nd Edition

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Solution Manual

for January 2015 version

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The document contains the solutions to all the questions that appear in the January 2015 version of the book. Be aware that the problem sets in the book will be modified in the future.

I would be grateful if you alert me to any mistakes you find in the manual.

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Problem Set 1 - Preferences

Problem 1.

Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$. Show that the set (of sets!) $\{I(x) | x \in X\}$ is a partition of X, that is,

(i) For all x and y, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.

(ii) For every $x \in X$, there is $y \in X$ such that $x \in I(y)$.

(i) Note that the indifference relation is symmetric and transitive.

Let $x, y \in X$, and assume that $I(x) \cap I(y) \neq \emptyset$, which means that there exists $z \in I(x) \cap I(y)$. Let a be any element in one of the sets, let us say $I(x)$. It means that $a \sim x$. But $z \sim x$ and $z \sim y$ since $z \in I(x) \cap I(y)$. By symmetry of the indifference relation, $x \sim z$. By transitivity of $\sim$, it follows that $a \sim y$, and thus $a \in I(y)$.

(ii) Let $x \in X$. The completeness of $\succsim$ means that $x \sim x$. Consequently, $x \in I(x)$.

Problem 2.

Kreps (1990) introduces another formal definition for preferences. His primitive is a binary relation P interpreted as "strictly preferred". He requires P to satisfy:

Asymmetry: For no x,y do we have both xPy and yPx.

Negative Transitivity (NT): $\forall x,y,z \in X$, if xPy, then either xPz or zPy (or both).

Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

We will closely follow the proof presented in the lecture notes. The following steps are required:

a. Construct an interpretation-preserving function T that maps a binary relation P satisfying asymmetry and negative-transitivity into preference relations.

Consider the following candidate function T, which maps any relation P into a binary relation defined by

$xT(P)y$ if not yPx.

Note that T preserves interpretation: If "y is not strictly preferred to x" according to Kreps' formalization, then "x is at least as good as y".

b. Prove that $T(P)$ is a preference relation.

Completeness of $T(P)$: Kreps' asymmetry property of P says that for any x and y in X, either not xPy or not yPx. Thus either $xT(P)y$ or $yT(P)x$.

Transitivity: Let $x,y,z \in X$ be such that $xT(P)y$ and $yT(P)z$. If not $xT(P)z$, then zPx. By NT, either yPx or zPy. Thus either not $xT(P)y$ or not $yT(P)z$, a contradiction.

c. Prove that T is one-to-one.

Let P_1 and P_2 be two different relations satisfying Kreps' properties. Then there is a pair x,y such that xP_1y and not xP_2y (or the opposite), and thus not $yT(P_1)x$ and $yT(P_2)x$. Thus, $T(P_1) \neq T(P_2)$, implying that T is one to one.

d. Prove that T maps onto all preference relations.

Let $\succsim$ be a preference relation. Define P, by:

xPy if not $y \succsim x$.

P preserves Kreps' properties:

Asymmetry: Since $\succsim$ is complete, then we never have both xPy and yPx.

NT: Let $x,y,z \in X$ be such that xPy, and thus not $y \succsim x$. Therefore it is not true that both $y \succsim z$ and $z \succsim x$. Therefore either zPy or xPz.

Finally, note that $T(P) = \succsim$.

Problem 3.

Let Z be a finite set and let X be the set of all nonempty subsets of Z. Let $\succsim$ be a preference relation on X (not Z). An element $\mathbf{A} \in \mathbf{X}$ is interpreted as a "menu", i.e. "the option to choose an alternative from the set $\mathbf{A}$". Consider the following two properties of preference relations on X:

1. **If $A \succsim B$ and C is a set disjoint to both A and B, then $A \cup C \succsim B \cup C$, and if $A \succ B$ and C is a set disjoint to both A and B, then $A \cup C \succ B \cup C$.**
2. **If $x \in Z$ and $\{x\} \succ \{y\}\ \forall y \in A$, then $A \cup \{x\} \succ A$, and if $x \in Z$ and $\{y\} \succ \{x\}\ \forall y \in A$, then $A \succ A \cup \{x\}$.**

a. Discuss the plausibility of the properties.

Consider an appealing interpretation of the formal model: The elements in Z are the alternatives which might be chosen at the end of a decision process, a set A is a set of candidates to be considered seriously in the second stage. If we have in mind that the economic agent is certain about his preferences in the later stage then (1) is problematic: if the best element in menu A is better than the best in menu B, and menu C includes an even better element, then $A \succ B$ but $A \cup C \sim B \cup C$, violating (1). Also, if any element of menu A is strictly better than $z \in Z$, then $A \succ \{z\}$ but $A \sim A \cup \{z\}$, violating (2). The properties make more sense if the decision maker has in mind a tiny possibility that he will err in his choice or that there is a possibility that an alternative which he chooses will not be feasible at the end.

b. Provide an example of a preference relation that:

(**1**) **Satisfies both properties**.

The relation $\succsim$ defined by $A \succsim B$ if $|A| \geq |B|$ satisfies (1) and (2) in a degenerate way (since for all x and y we have $\{x\} \sim \{y\}$).

A "better" class of examples (including the previous one): Let X be divided to two sets G and B. Define a preference relation by the utility function $u(A) = |A \cap G| - |A \cap B|$. Clearly it satisfies both properties.

(**2**) **Satisfies the first but not the second property**.

Let $z^* \in Z$. Define $\succsim$ over X whereby $A \succ B \Leftrightarrow z^* \in A, z^* \notin B$ and $A \sim B$ otherwise.

(1) Let $A, B, C \in X$ be such that $A \succsim B$ and C is disjoint to both A and B. If $A \succ B$, then $z^* \in A$ and $z^* \notin B, C$, which implies that $A \cup C \succ B \cup C$. If $A \sim B$, then either z^* is a member of both sets $A \cup C$ and $B \cup C$ or of none. In both cases $A \cup C \sim B \cup C$.

(not 2) Let $A = \{z^*\}$ and $y \neq z^*$. Then $A \succ \{y\}$ but $A \sim A \cup \{y\}$ violating the second part of (2).

More generally, attach to each element x a non-negative number $v(x)$ and define a preference relation by a utility function $U(A) = \Sigma_{a \in A} v(a)$. Then the preference relation satisfies (1) but not (2).

(**3**) **Satisfies the second but not the first property**.

Let $\succsim^*$ be a preference relation on Z. Define $\succsim$ by $A \succsim B$ if

(a) the $\succsim^*$ –best element in A is strictly better than the $\succsim^*$ –best element in B, or

(b) the agent is indifferent between the best elements but the $\succsim^*$ –worst element in A is weakly better than the $\succsim^*$ –worst element in B.

(not 1) Let $a \succ^* b \succ^* c \succ^* d$. Then $\{b\} \succ \{c\}$, but $\{b\} \cup \{a,d\} \sim \{c\} \cup \{a,d\}$.

(2) Let $A \in X$ and $z \in Z$. If z is strictly $\succsim^*$ –better (strictly $\succsim^*$ –worse) then all $a \in A$, then $A \cup \{z\} \succ A \quad (A \succ A \cup \{z\})$.

c. **Show that if there are** $x,y,z \in Z$ **such that** $\{x\} \succ \{y\} \succ \{z\}$, **then there is no preference relation satisfying both properties**.

Assume $\succsim$ satisfies (1) and (2), with $\{x\} \succ \{y\} \succ \{z\}$ for some $x,y,z \in Z$. From (2), $\{x\} \succ \{x,y\}$ and $\{y,z\} \succ \{z\}$. Applying (1) to the above, $\{x,z\} \succ \{x,y,z\}$ and $\{x,y,z\} \succ \{x,z\}$, a contradiction.

Problem 4.

Let $\succ$ be an asymmetric binary relation on a finite set X that does not have cycles. **Show** (**by induction on the size of X**) **that $\succ$ can be extended to a complete ordering** (**i**.**e**., **a complete**, **asymmetric**, **and transitive binary relation**).

Note that if a set A is finite and $\succ$ is an acyclic relation on A (there are no cycles), then there must exist an $x \in A$ such that there is no $y \in A$ such that $y \succ x$.

Since X is finite, then there exists an $x_1 \in X$ such that there is no $y \in X$ such that $y \succ x_1$. Define $x_1 \succ^* y$ for all such $y \in X - \{x_1\}$. Again, there exists an $x_2 \in X - \{x_1\}$ such that there is no $y \in X - \{x_1\}$ such that $y \succ x_2$. Define $x_2 \succ^* y$ for all such y, and so on. By induction we can define $\succ^*$ for all $x \in X$.

By construction, the relation $\succ^*$ is complete, asymmetric, extends $\succ$ and transitive: let $x_i \succ^* x_j$ and $x_j \succ^* x_h$. Then $i < j$ and $j < h$ and therefore, $x_i \succ^* x_h$.

Problem 5.

You have read an article in a "prestigious" journal about a decision maker (DM) whose mental attitude towards elements in a finite set X is represented by a binary relation $\succ$, which is a-symmetric and transitive but not necessarily complete. The incompleteness is the result of an assumption that a DM is sometimes unable to compare between alternatives.

Another, presumingly stronger, assumption made in the article is that the DM uses the following procedure: he has n criteria in mind, each represented by an ordering (a-symmetric, transitive and complete) $\succ_i$ $(i = 1, \ldots, n)$. The DM decides that $x \succ y$ if and only if $x \succ_i y$ for every i.

1. Verify that the relation $\succ$ generated by this procedure is a-symmetric and transitive. Try to convince a reader of the paper that this is an attractive assumption by giving a "real life" example in which it is "reasonable" to assume that a DM uses such a procedure in order to compare between alternatives.

$\succ$ is a-symmetric: If $x \succ y$ then by definition, $x \succ_i y$ for every i. Since $\succ_i$ are a-syemmtric, $y \not\succ_i x$ for all i, and by definition also $y \not\succ x$.

$\succ$ is transitive: Let $x \succ y$ and $y \succ z$. By definition, $x \succ_i y$ and $y \succ_i z$ for every i. Since $\succ_i$ are transitive, also $x \succ_i z$ for all i, and by definition $x \succ z$.

An example: A parent who considers destinations for a family vacation who ranks the different destinations according to the orderings of his children: he prefers A to B iff all his children prefer A to B.

It can be claimed that the additional assumption regarding the procedure that generates $\succ$ is not a "serious" one since given any asymmetric and transitive relation, $\succ$, one can find a set of complete orderings $\succ_1, \ldots, \succ_n$ such that $x \succ y$ iff $x \succ_i y$ for every i.

2. Demonstrate this claim for the relation on the set $X = \{a, b, c\}$ according to which only $a \succ b$ and the comparison between [b and c] and [a and c] are not determined.

Let $a \succ_1 b \succ_1 c$ and $c \succ_2 a \succ_2 b$. The two relations agree only on $a \succ_i b$.

3. (Main part of the question) Prove this claim for the general case.

Guidance (for c): given an asymmetric and transitive relation $\succ$ on an arbitrary X, define a set of complete orderings $\{\succ_i\}$ and prove that $x \succ y$ iff for every i, $x \succ_i y$.

First, note that if X is a finite set and P is a asymmetric and transitive relation on X then P does not have any cycles and thus P can be extended to a complete ordering of X (see problem 4).

Let Λ be the set of all complete orderings which extends $\succ$. We will see that $a \succ b$ if and only if $a \succ_i b$ for all $\succ_i \in \Lambda$:

(i) If $a \succ b$, then $a \succ_i b$ for all i since any $\succ_i \in \Lambda$ is an extension of $\succ$.

(ii) If not $a \succ b$, then let $\succ^*$ be the relation $\succ$ extended to include also $b \succ^* a$. The relation $\succ^*$ does not have cycles: if there is a cycle $x_1 \succ^* \ldots \succ^* x_n = x_1$ then

(a) if for some i we have $x_i = b \succ^* a = x_{i+1}$ then since $a = x_{i+1} \succ^* x_{i+2} \ldots \succ^* x_n = x_1 \succ^* \ldots \succ^* x_i = b$ by transitivity $a \succ b$ contradicting the assumption.

(b) otherise, by thranstivity $x_1 \succ x_2$ but also $x_2 \succ x_1$ conradicting asymettry.

Thus, $\succ^*$ can be extended to a complete ordering $\succ'$ which will be an extension of $\succ$ as well. Hence, there is an extension $\succ' \in \Lambda$ for which not $a \succ' b$.

Problem 6.

Listen to the illusion called the Shepard Scale. (**You can find it on the internet**. **Currently**, **it is available at http**://**www**.**youtube**.**com**/**watch**?**v**=**boJD\gTLavA**
and http://**en**.**wikipedia**.**org**/**wiki**/**Shepard_tone**.)

Can you think of any economic analogies?

The Shepard Scale consists of three separate scales that play the same tone at different octaves. As the notes ascend, one scale drops its pitch an octave, a change the listener does not notice because the other two scales continue to ascend monotonically, which "covers up" the drop. Several notes later, a second scale drops its pitch an octave, and so on. Thus, the Shepard Scale sounds as if it perpetually ascends, even though the same finite set of notes are repeated. See http://en.wikipedia.org/wiki/Shepard_tone for explanation.

The phenomenon is a reminder of an example due to Fishburn and LaValle (1988):

Our illustration for decision under uncertainty assumes that states are the faces of a standard die with probability 1/6 for each state. Two acts with results that depend on the up face after one roll are f_1 and f_2:

	1	2	3	4	5	6
f_1	\$1000	\$ 500	\$600	\$700	\$800	\$900
f_2	\$ 900	\$1000	\$500	\$600	\$700	\$800

Many of us prefer the lottery f_2 to the lottery f_1. However, one can easily construct the other 4 lotteries f_3, f_4, f_5, f_6, (increase the \$500 to \$1000 and reduce the other prizes by \$100), such that we would prefer f_3 to f_2,f_4 to f_3, ... and f_1 to f_6.

See Peter C. Fishburn and Irving H. LaValle (1988). "Context-Dependent Choice with Nonlinear and Nontransitive Preferences", Econometrica, Vol. 56, 1221-1239. Stable URL: http://www.jstor.org/stable/1911365

Problem Set Two – Utility

Problem 1.

The purpose of this problem is to make sure that you fully understand the basic concepts of utility representation and continuous preferences. **Prove or disprove the following**:

a. **Is the statement "if both U and V represent $\succsim$, then there is a strictly monotonic function $f : \Re \to \Re$ such that $V(x) = f(U(x))$" correct**?

False: Let $X = \Re$ and preferences be represented by the utility functions

$$V(x) = x \qquad \text{and} \qquad U(x) = \begin{cases} x & \text{if } x \le 0 \\ x+1 & \text{if } x > 0. \end{cases}$$

The only increasing function $f : \Re \to \Re$ that satisfies $V(x) = f(U(x))$ is

$$f(x) = \begin{cases} x & \text{if } x \le 0 \\ 0 & \text{if } 0 < x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

which is not *strictly* increasing.

b. **Can a continuous preference relation be represented by a discontinuous utility function**?

True: The preferences ($x \succsim y$ if $x \ge y$) is represented by U in (a) are continuous, though U is discontinuous.

c. **Show that in the case of $X = \Re$, the preference relation that is represented by the discontinuous utility function $u(x) = [x]$ (the largest integer n such that $x \ge n$) is not a continuous relation**.

$1 \succ 1/2$, but $1 - \epsilon \sim 1/2$ for $\epsilon > 0$ small, violating C1.

d. **Show that the two definitions of a continuous preference relation**, **C1 and C2**, **are equivalent to**

Definition C3: $\forall x \in X$, **the upper and lower contours** $\{y \mid y \succsim x\}$ **and** $\{z \mid x \succsim z\}$ **are closed sets in** X.

Definition C4: $\forall x \in X$, **the sets** $\{y \mid y \succ x\}$ **and** $\{z \mid x \succ z\}$ **are open sets in** X.

(C3 $\Leftrightarrow$ C4) By completeness, the sets $\{y \mid x \succ y\}$ and $\{y \mid y \succ x\}$ are the complementary to $\{y \mid y \succsim x\}$ and $\{y \mid x \succsim y\}$ correspondingly. Thus the formers are open sets iff the later are closed sets.

(C1 $\Rightarrow$ C4) Let $x \in X$ and $a \in \{y \mid y \succ x\}$. By C1, there exists an $\epsilon > 0$ such that $Ball(a, \varepsilon) \subseteq \{y \mid y \succ x\}$, ($Ball(a, \varepsilon)$ is the set of points in X that are less than ϵ distance from a). Thus $\{y \mid y \succ x\}$ is open. The argument for $\{z \mid x \succ z\}$ open is analogous.

(C4 $\Rightarrow$ C1) Let us use the notation $B \succ x$ to mean that $y \succ x$ for all $y \in B$.

Let $x \succ y$. Assume first that there exists a $z \in X$ such that $x \succ z \succ y$. By C4, there exist $\epsilon_1, \epsilon_2 > 0$ such that $Ball(x, \epsilon_1) \succ z$ and $z \succ Ball(y, \epsilon_2)$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. By transitivity, every point in $Ball(x, \varepsilon)$ is strictly better than every point in $Ball(y, \varepsilon)$.

Next, assume that there does not exist a $z \in X$ such that $x \succ z \succ y$. As above, by C4 there exists an $\epsilon > 0$ such that $Ball(x, \varepsilon) \succ y$ and $x \succ Ball(y, \varepsilon)$. Since there is no z such that $x \succ z \succ y$, then $Ball(x, \varepsilon) \succsim x$ and $y \succsim Ball(y, \varepsilon)$, and thus by transitivity, every point in $Ball(x, \varepsilon)$ is strictly better than every point in $Ball(y, \varepsilon)$.

Problem 2.

Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values.

Let $X = \mathbb{N}$, which is countable. Define preferences to be such that

$$1 \succ 3 \succ 5 \succ \ldots \succ 2 \succ 4 \succ \ldots.$$

By contradiction, assume that there exists a utility function $u : X \to \mathbb{Z}$ that represents $\succsim$. Then $u(1) = N$ and $u(2) = n$ for some $n, N \in \mathbb{Z}$. But there are an infinite number of odd numbers, implying that u maps to an infinite number of integers between n and N, a contradiction.

Problem 3.

Let $\succsim$ be continuous preferences on a set $X \subseteq \Re^n$which contains the interval connecting the points x and z. Show that if $y \in X$ and $x \succsim y \succsim z$, then there is a point m on the interval connecting x and z such that $y \sim m$.

Construct inductively the sequence $\{(x^n, z^n, m^n)\}$ as follows: Start with define $x^0 = x$, $z^0 = z$ and the midpoint $m^0 = 1/2x^0 + 1/2z^0$.

If $m^n \sim y$ then we found the point we look for. Otherwise, $m^n \succ y$ or $y \succ m^n$.

If $m^n \succ y$ let $x^{n+1} = m^n$ and $z^{n+1} = z^n$.

If $y \succ m^n$ let $x^{n+1} = x^n$ and $z^{n+1} = m^n$.

In any case define $m^{n+1} = 1/2x^{n+1} + 1/2z^{n+1}$.

If none of the points $m^n \sim y$ then $x^n \succ y \succ z^n$ for all n. Both sequences (x^n) and (z^n) converge to some m^* on the interval between x and z.

Since $\succsim$ is continuous, then $m^* \succsim y$ and $y \succsim m^*$, and thus $m^* \sim y$.

Another possible proof: the interval between x and z is a connected set. The two sets $\{a|a \succ y\}$ and $\{a|y \succ a\}$ are disjoint by definition and open by the continuity of $\succsim$. Two disjoint open sets cannot cover a connected set and therefore there is at least one point on this interval such that $y \sim m$.

Problem 4.

Consider the sequence of preference relations $(\succsim^n)_{n=1,2,\ldots}$, defined on $\Re^2_+$ where $\succsim^n$ is represented by the utility function $u_n(x_1,x_2) = x_1^n + x_2^n$. We will say that the sequence $\succsim^n$ converges to the preferences $\succsim^*$ if for every x and y such that $x \succ^* y$, there is an N such that for every $n > N$ we have $x \succ^n y$. Show that the sequence of preference relations $\succsim^n$ converges to the preferences $\succsim^*$ which are represented by the function $max\{x_1,x_2\}$.

Let $x \succ^* y$. Since $\max\{x_1,x_2\} > \max\{y_1,y_2\}$, then there exists an $\epsilon > 0$ such that $\max\{x_1,x_2\} > (1+\epsilon)\max\{y_1,y_2\}$. Consequently, for n large enough, $[\max\{x_1,x_2\}]^n > 2[\max\{y_1,y_2\}]^n$. But $x_1^n + x_2^n \geq [\max\{x_1,x_2\}]^n$ and $2\max\{y_1,y_2\}^n \geq y_1^n + y_2^n$, and thus $x \succ^n y$ for n large enough.

Problem 5.

Let X be a finite set and let $(\succsim, \succ\succ)$ be a pair where $\succsim$ is a preference relation and $\succ\succ$ is a transitive sub-relation of $\succ$ (by sub-relation, we mean $x \succ\succ y$ implies $x \succ y$). We can think about the pair as representing the responses to the questionnaire A where $A(x,y)$ is the question:

How do you compare x and y? Tick one of the following five options:

☐ **I very much prefer x over y** $(x \succ\succ y)$

☐ **I prefer x over y** $(x \succ y)$

☐ **I am indifferent** (I)

☐ **I prefer y over x** $(y \succ x)$

☐ **I very much prefer y over x** $(y \succ\succ x)$

Assume that the pair satisfies extended transitivity: If $x \succ\succ y$ and $y \succsim z$, or if $x \succsim y$ and $y \succ\succ z$ then $x \succ\succ z$. We say that a pair $(\succsim, \succ\succ)$ is represented by a function u if

$u(x) = u(y)$ **iff** $x \sim y$,

$u(x) - u(y) > 0$ **iff** $x \succ y$, **and**

$u(x) - u(y) > 1$ **iff** $x \succ\succ y$.

Show that every extended preference $(\succsim, \succ\succ)$ can be represented by a function u.

Denote $A \succ B$ if $a \succ b$ for all $a \in A$ and $b \in B$. Let $X_1, X_2, ..., X_K$ be the $\succsim$ indifference sets such that $X_K \succ X_{K-1} \succ \ldots \succ X_1$. Define first $u(X_1) = 0$.

Let us define $u(X_k)$ for $k > 1$.

(1) if $X_k \succ\succ X_{k-1}$, then $u(X_k) = u(X_{k-1}) + 2$

(2) if X_k is not $\succ\succ$ even of X_1, then $u(X_k) \in (u(X_{k-1}), 1)$

(3) otherwise, there exists a maximal $m(k)$ such that $X_k \succ\succ X_{m(k)}$. Define $u(X_k)$ such that $u(X_k) > u(X_{k-1})$ and $1 + u(X_{m(k)+1}) > u(X_k) > u(X_{m(k)}) + 1$.

Clearly, $x \sim y$ iff $u(x) = u(y)$

Also, if $x \succ y$ then $u(x) > u(y)$, since we picked $u(X_k)$ as an increasing sequence.

Finally, if $x \succ\succ y$, $x \in X_k$ and $y \in X_m$ then $m(k) \geq m$ and $u(x) > u(X_{m(k)}) + 1 \geq u(y) + 1$.

Problem 6.

The following is a typical example of a utility representation theorem: **Let** $X = \Re^2_+$.
Assume that a preference relation $\succsim$ **satisfies the following three properties**:

ADD: $(a_1, a_2) \succsim (b_1, b_2)$ **implies that** $(a_1 + t, a_2 + s) \succsim (b_1 + t, b_2 + s)\ \forall s, t$.

SMON: **If** $a_1 \geq b_1$ **and** $a_2 \geq b_2$, **then** $(a_1, a_2) \succsim (b_1, b_2)$. **In addition**, **if either** $a_1 > b_1$ **or** $a_2 > b_2$ **then** $(a_1, a_2) \succ (b_1, b_2)$.

CON: **Continuity**.

a. **Show that if** $\succsim$ **has a linear representation** (**that is**, $\succsim$ **are represented by a utility function** $u(x_1, x_2) = \alpha x_1 + \beta x_2$ **with** $\alpha, \beta > 0$), **then** $\succsim$ **satisfies ADD**, **SMON**, **CON**.

ADD: Let $s, t \in \Re$ and $x, y \in X$ be such that $x \succsim y$. Note that $(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow \alpha x_1 + \beta x_2 \geq \alpha y_1 + \beta y_2 \Leftrightarrow \alpha(x_1 + t) + \beta(x_2 + s) \geq \alpha(y_1 + t) + \beta(y_2 + s) \Leftrightarrow u(x_1 + t, x_2 + s) \geq u(y_1 + t, y_2 + s) \Leftrightarrow (x_1 + t, x_2 + s) \succsim (y_1 + t, y_2 + s)$.

SMON: Let $x, y \in X$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$ with at least one strict inequality. Since $\alpha, \beta > 0$, then $\alpha x_1 + \beta x_2 > \alpha y_1 + \beta y_2$, which implies that $(x_1, x_2) \succ (y_1, y_2)$.

CON: $u(x_1, x_2)$ is continuous, and thus $\succsim$ is continuous.

b. **Show that for any pair of the three properties there is a preference relation that does not satisfy the third property**.

Satisfies only ADD, SMON: Lexicographic preferences satisfy *ADD* and *SMON*, but are not continuous (see the lecture notes).

Satisfies only ADD, CON: The preferences represented by $u(x_1, x_2) = x_1 - x_2$ satisfy *ADD* and *CON*, but not *SMON* since $(1,1) \succ (1,2)$.

Satisfies only MON, CON: Preferences represented by $u(x_1, x_2) = x_1^2 + x_2^2$ satisfy *SMON* and *CON*, but not *ADD* since $(3,0) \succ (2,1)$ and $(3,3) \prec (2,4)$.

c. **Show that if** $\succsim$ **satisfies the three properties**, **then it has a linear representation**.

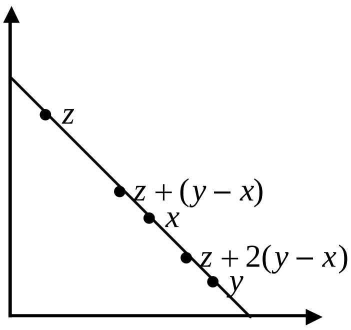
Assume first that x and y are two different points such that $x \sim y$. Then:

(i) $(x + y)/2 \sim y$. Otherwise, $(x + y)/2 \succ y$ would imply that
$x = \frac{x+y}{2} + \frac{x-y}{2} \succ y + \frac{x-y}{2} = \frac{x+y}{2} \succ y$ by ADD, a contradiction.

(ii) $z = (1 - \alpha)x + \alpha y \sim x$ for $\alpha \in [0,1]$. Define $\{(x^n, y^n)\}$ inductively as follows: let $x^0 = x$, $y^0 = y$. Let $m^0 = (x^0 + y^0)/2$.

Assume z belongs to $[x^n, y^n]$ and its length is $1/2^n$ the length of $[x, y]$. The point z belongs to at least one of the intervals $[x^n, m^n]$ and $[m^n, y^n]$. Define $[x^{n+1}, y^{n+1}]$ to be one of those intervals which contains z. Now, all $x^n \sim x$ for all n. The sequence $x^n \to z$, therefore by continuity $z \sim x$.

(iii) Let z be on the line which connects x and y, $z \sim x$. Without loss of generality, assume that z is closer to x. There is n such that $w = z + n(y - x)$ is between x and y. By ADD if $a - x = b - y$ (that is a-b=x-y) then $a \sim b$. Thus by transitivity $z \sim w \sim x$.



By SMON there is an $\varepsilon > 0$ such that $a = (x_1 + \varepsilon, x_2) \succ x \succ (x_1, x_2 - \varepsilon) = b$. By question 3, there exists y (different than x) on the interval which connects a and b such that $x \sim y$. Thus, every point is on a difference line which is a line. The indifference lines must be parallel since otherwise we will get a contradiction to *ADD*.

d. **Characterize the preference relations which satisfy ADD**, **SMON and an additional property MUL**:

$$(a_1, a_2) \succsim (b_1, b_2) \text{ implies that } (\lambda a_1, \lambda a_2) \succsim (\lambda b_1, \lambda b_2) \text{ for any } \lambda \geq 0.$$

Define $s = sup\{x | (0,1) \succ (x,0)\}$ (by SMON the set is not empty).

Case (1): $s = \infty$ or $s = 0$: the preferences must be lexicographic with priority for the second or first components, respectively.

Assume $s = \infty$.

If $a_2 > b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $(a_1, a_2 - b_2) \succ (b_1, 0)$ (by ADD) iff $(a_1/(a_2 - b_2), 1) \succ (b_1/(a_2 - b_2), 0)$ (by MUL), which is always true (by $s = \infty$).

If $a_2 = b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $a_1 > b_1$ (by SMON).

Thus, we have a lexicographic relation with priority for the second component.

If $s = 0$ then it follows that $s = sup\{y | (1,0) \succ (0,y)\} = \infty$ and the preferences must be lexicographic with priority for the first component.

Case (2): $\infty > s > 0$

Let (a_1, a_2) and (b_1, b_2) be two vectors with $a_1 \leq b_1$. (a_1, a_2) relates to (b_1, b_2) as $(0, a_2 - b_2)$ relates to $(b_1 - a_1, 0)$ (by ADD) and thus as $((b_1 - a_1)/(a_2 - b_2), 0)$ relates to $(0, 1)$. This relation is determined by the comparison of $(b_1 - a_1)/(a_2 - b_2)$ to s, which is equivalent to the comparison of $a_1 + sa_2$ and $b_1 + sb_2$.

Therefore, if $(0, 1) \sim (s, 0)$ then $x_1 + sx_2$ represents the preferences. If $(0, 1) \succ (s, 0)$ or $(0, 1) \prec (s, 0)$ then the preferences are lexicographic with the first priority to $x_1 + sx_2$ and the second to x_2 or x_1 accordingly.

Problem 7.

Utility is a numerical representation of preferences. One can think about the numerical representation of other abstract concepts. Here, you will try to come up with a possible numerical representation of the concept "approximately the same" (see Luce (1956) and Rubinstein (1988)). For simplicity, let $X = [0,1]$. Consider the following six properties of S:....

c. Let S be a binary relation that satisfies the above six properties and let $\epsilon > 0$. Show that there is a strictly increasing and continuous function $H : X \to \Re$ such that $aSb \Leftrightarrow |H(a) - H(b)| \leq \epsilon$.

Note the definitions of $m(x)$ and $M(x)$ in the question.

Define $\{x_n\}$ by $x_0 = 0$, $x_1 = M(0)$, $x_2 = M(x_1) = M(M(0))$ and so on. By S6, $\{x_n\}$ is increasing and bounded above by 1, and thus $\{x_n\}$ converges to $x^* \leq 1$. By S5, there exists an N such that $x_{N-1}Sx^*$, and thus $x^* \leq M(x_{N-1}) = x_N$. Since x^* is the upper bound of $\{x_n\}$, then $x^* = 1$ by S6. Define N to be the smallest integer such that $x_N = 1$, and thus $0 = x_0 <\ldots< x_N = 1$.

Lemma 1: If $a \in [x_n, x_{n+1}]$, where $1 \leq n \leq N-1$, then $m(a) \in [x_{n-1}, x_n]$.

Proof: Since x_nSx_{n+1}, then x_nSa by S4, and thus $m(a) \leq x_n$. Moreover, $x_{n-1} \leq m(a)$, as otherwise $m(a) < x_{n-1}$ and $M(x_{n-1}) = x_n \leq M(m(a))$, violating the assumption that M increasing.

Lemma 2: $m(a)$ is strictly increasing and continuous on $(x_1, 1]$.

Proof: $m(a) > 0$ if $a > x_1$, as otherwise $aS0$, and thus $M(0) \geq a > x_1$, a contradiction. By S6, the lemma is proved.

Define

$$H(a) = \begin{cases} \frac{\epsilon}{x_1}a & \text{if } a \leq x_1 \\ H(m(a)) + \epsilon & \text{if } a > x_1. \end{cases}$$

H is clearly continuous and strictly increasing on $[0, x_1]$, with $H(x_1) = \epsilon$.

If $a \in (x_1, x_2]$, then $H(a) = \epsilon[m(a)/x_1 + 1]$ since $m(a) \in [0, x_1]$ by Lemma 1. Thus H is strictly increasing and continuous on $(x_1, x_2]$ by Lemma 2. Since $m(x_1) = 0$, then $H(x) \to \epsilon$ as $x \to x_1$ from the right, and thus H in continuous and strictly increasing on $[0, x_2]$, with $H(x_2) = 2\epsilon$.

More generally, if $a \in (x_n, x_{n+1}]$, where $n \leq N-1$, then $m(a) \in [x_{n-1}, x_n]$, $m(m(a)) \in [x_{n-2}, x_{n-1}]$ and so on by Lemma 1. Therefore $H(a) = \epsilon[m(\ldots m(a)\ldots)/x_1 + n]$, which is strictly increasing and continuous by Lemma 2, where $m(\ldots m(a)\ldots)$ applies m inductively n times. Since $H(x) \to n\epsilon$ as $x \to x_n$ from the right, then H is strictly increasing and continuous on $[0, x_{n+1}]$.

Let $a, b \in [0,1]$ where $a < b$. If $b \leq x_1$, then H represents S by S4. Otherwise, aSb iff $H(m(b)) \leq H(a) < H(b)$ iff $|H(b) - H(a)| \leq \epsilon$, where the first iff follows from aSb iff $m(b) \leq a < b$ and H strictly increasing, and the second iff follows from $H(b) = H(m(b)) + \epsilon$.

Problem Set 3 – Choice

Problem 1.

The following are descriptions of decision making procedures. **Discuss whether the procedures can be described in the framework of the choice model discussed in this lecture and whether they are compatible with the** "**rational man**" **paradigm**.

a. **The DM chooses an alternative in order to maximize another person**'**s suffering**.

Assuming that the relation "the other person suffers more from x than he does from y" is complete and transitive, the DM is maximizing a well-defined preference relation.

b. **The DM asks his two children to rank the alternatives and then chooses the alternative that is the best** "**on average**".

The question is, of course, what does the expression "on average" mean. If the DM ranks all alternatives in X and uses the ranking to attach the number to each alternative a in any set A (independently of A), then the DM's behavior is consistent with rationality. But if the score of an alternative is recalculated for every choice set then his behavior may be inconsistent with the rational man paradigm. For example, assume that one child ranks the alternatives a, d, e, b, c and the other as b, c, a, d, e. Then, the element a is chosen from the set $\{a, b, c, d, e\}$ while b is chosen from $\{a, b, c\}$.

c. **The DM has an ideal point in mind and chooses the alternative that is closest to it**.

Let x be the ideal point and $d(a, b)$ the distance function between $a, b \in X$. The behavior is rationalized by the preferences represented by $u(a) = -d(a, x)$.

d. **The DM looks for the alternative that appears most often in the choice set**.

A choice function C is not well-defined. The DM's behavior is different when faced with the group of elements (a, a, b) than when faced with the group (a, b, b), even though in both cases he chooses from the set $\{a, b\}$.

e. **The DM has an ordering in mind and always chooses the median element**.

C violates condition α. Assume that the order of the grand set $X = \{a, b, c, d, e\}$ is alphabetical. Then, $C(\{a, b, c, d, e\}) = \{c\}$ but $C(\{a, b, c\}) = \{b\}$.

Problem 2.

A choice correspondence C satisfiies the path independence property if for every set A and a partition of A into A_1 and A_2 ($A_1, A_2 \neq \emptyset$, $A = A_1 \cup A_2$ **and** $A_1 \cap A_2 = \emptyset$) **we have** $C(A) = C((C(A_1) \cup C(A_2))$. (**Of course this definition applies also for choice functions**).

a. **Show that the rational decision maker satisfies this property**.

Let A_1, A_2 **be a partition of** A. **Let** $x \in C_{\succsim}(A)$ **and** $x \in A_i$. **Then** $x \in C_{\succsim}(A_i)$ **and** $x \in C_{\succsim}(C_{\succsim}(A_1) \cup C_{\succsim}(A_2))$.

b. **Find examples of choice procedures that do not satisfy this property**.

(i) The "second best" procedure. If $x \succ y \succ z$, then $C(\{x,y,z\}) = \{y\}$ while $C(\{C(\{x,y\}), C(\{z\})\}) = \{z\}$.

(ii) Let X be partitioned into Y and Z and let $\succ$ be an ordering on X. Let $C(A)$ be the $\succ$ -minimal element if all alternatives in A are in Y and the $\succ$ -maximal alternative otherwise. If $Y = \{a,b\}$, $Z = \{c,d\}$ and $a \succ b \succ c \succ d$, then $\{C(\{a,b,c,d\}) = \{a\}$ but $C(\{C(\{a,b\}), C(\{c,d\})\}) = \{b\}$.

c. **Show that if a choice function satisfies path independence**, **then it is consistent with rationality**.

We will show that Condition α is satisfied. Let $A \subset B \subseteq X$ be such that $C(B) \in A$. By path independence, $C(B) = C(\{C(A), C(B \setminus A)\}$. Since $C(B)$ is in A, then it is not in $B \setminus A$. Therefore $C(B)$ is identical to $C(A)$.

d. **Find an example of a multi-valued choice function satisfying path independence which cannot be rationalized**.

Let $a \succ b \succ c \succ d$ and $C(A)$ be a set which contains the best and worst elements in A. C satisfies path independence (verify) but violates WA since $a \in C(\{a,b,c,d\})$ and $c \in C(\{a,b,c\})$ but $c \notin C(\{a,b,c,d\})$.

Problem 3.

Let X be a finite set. Check whether the following three choice correspondences satisfy WA:

1) $C(A) = \{x \in A \mid$ **the number of** $y \in X$ **for which** $V(x) \geq V(y)$ **is at least** $|X|/2\}$ **and if this set is empty, then** $C(A) = A$.

By C, a satisfactory element is one which is in the upper half of the elements in the grand set.

C satisfies WA. Define:

$$G = \{x \in X \mid V(x) \geq V(y) \text{ for at least } |X|/2 \text{ alternatives } y \in X\}.$$

Let $x,y \in A \cap B$ such that $x \in C(A)$ and $y \in C(B)$.

If $x \in G$, then $x \in C(B)$.

If $x \notin G$, then there are no elements of A in G and thus $y \notin G$. Since $y \in C(B)$, then $C(B) = B$ and thus $x \in C(B)$ as well.

Alternatively: define $u(x) = 1$ if $x \in G$ and 0 otherwise. Clearly, $C_u = C$.

2) $D(A) = \{x \in A \mid$ **the number of** $y \in A$ **for which** $V(x) \geq V(y)$ **is at least** $|A|/2\}$.

By D, a satisfactory element is one in the upper half of the elements of the choice set.

It is not necessarily consistent with the rational man paradigm:

Let $V(a) > V(b) > V(c) > V(d) > V(e)$. Then, $c \in C(\{a,b,c,d,e\})$ and $a \in C(\{a,b,c\})$, but $c \notin C(\{a,b,c\})$.

3) $E(A) = \{x \in A \mid x \succsim_1 y$ **for every** $y \in A$ **or** $x \succsim_2 y$ **for every** $y \in A\}$, **where** $\succsim_1$ **and** $\succsim_2$ **are two orderings over** X.

By E, a satisfactory element is one which is optimal according to one of the two criteria. It is not necessarily consistent with the rational man paradigm. Let $x \succ_1 y \succ_1 z$ and $y \succ_2 z \succ_2 x$. Then, $z \in C(\{x,z\})$ and $x \in C(\{x,y,z\})$, but $z \notin C(\{x,y,z\})$.

Problem 4.

Consider the following choice procedure: A decision maker has a strict ordering $\succsim$ over the set X and assigns to each $x \in X$ a natural number $class(x)$ to be interpreted as the "class" of x. Given a choice problem A, he chooses the best element in A from those belonging to the most common class in A (i.e., the class that appears in A most often). If there is more than one most common class, he picks the best element from the members of A that belong to a most common class with the highest class number.

a. Is this procedure consistent with the "rational man" paradigm?

No. Let $a \succ b \succ c \succ d \succ e$, $class(a) = class(b) = class(c) = 1$ and $class(d) = class(e) = 2$. $C(\{a,b,c,d,e\}) = a$ but $C(\{a,d,e\}) = d$, thus violating α.

b. Define the relation $\mathbf{xPy}$ if $\mathbf{x}$ is chosen from $\{\mathbf{x},\mathbf{y}\}$. Show that the relation P is a strict ordering (complete, asymmetric and transitive).

By definition, P is complete and asymmetric. We will see that it is also transitive. That is, if xPy and yPz, then xPz.

If xPy and yPz, then $[class(x) > class(y)$ or $class(x) = class(y)$ and $x \succsim y]$, and $[class(y) > class(z)$ or $class(y) = class(z)$ and $y \succsim z]$. If either $class(x) > class(y)$ or $class(y) > class(z)$, then $class(x) > class(z)$ and $C(\{x,z\}) = \{x\}$. Otherwise, $class(x) = class(z)$ and $x \succsim z$ and thus $x \in C(\{x,z\})$.

Alternatively: Note that P is identical to the lexicographic preferences with first priority given to class and second priority to the relation $\succsim$.

Problem 5.

Consider the following two choice procedures. Explain each procedure and try to persuade a skeptic that they "make sense". Determine For each of them whether they are consistent with the "rational man" model.

a. The primitives of the procedure are two numerical (one-to-one) functions u **and** v **defined on** X **and a number** v^*. **For any given choice problem** A, **let** $a^* \in A$ **be the maximizer of** u **over** A **and let** $b^* \in A$ **be the maximizer of** v **over** A. **The decision maker chooses** a^* **if** $v(a^*) \geq v^*$ **and** b^* **if** $v(a^*) < v^*$.

One interpretation of this procedure is that the DM actually wants to maximize v but pretends to maximize u. If the maximization of u yields a result which is too bad for him, he abandons the pretense and maximizes v. The procedure may fail condition α. For example,

Element	$u(\bullet)$	$v(\bullet)$
x	3	1
y	2	2
z	1	3

and let $v^* = 2$.

Then $C(\{x,y,z\}) = z$ but $C(\{y,z\}) = y$.

b. The primitives of the procedure are two numerical (one-to-one) functions u **and** v **defined on** X **and a number** u^*. **For any given choice problem** A, **the decision maker chooses the element** $a^* \in A$ **that maximizes** u **if** $u(a^*) \geq u^*$ **and the element** $b^* \in A$ **that maximizes** v **if** $u(a^*) < u^*$.

In this case, the DM, cares about the value of u only if it is at least u^*. Otherwise, he cares about v. The DM behaves as if he is maximizing lexicographic preferences with first priority given to the function $u'(x)$, which receives the value $u(x)$ if $u(x) \geq u^*$ and $u^* - 1$ otherwise, and second priority to $v(x)$.

Problem 6.

The standard economic model assumes that choice is made from a set. Let us construct a model where the choice is assumed to be made from a list (note that the list $< a, b >$ is distinct from $< a, a, b >$ and $< b, a >$). Let X be a finite grand set. A list is a nonempty finite vector of elements in X. In this problem, consider a choice function C to be a function that assigns a single element from $\{a_1, \ldots, a_k\}$ to each vector $L = \langle a_1, \ldots, a_k \rangle$. Let $\langle L_1, \ldots, L_m \rangle$ be the concatenation of the m lists $L_1, \ldots, L_m$ (note that if the length of L_i is k_i, the length of the concatenation is $\sum_{i=1,\ldots,m} k_i$). We say that L' extends the list L if there is a list M such that $L' = \langle L, M \rangle$.

We say that a choice function C satisfies property I if for all $L_1, \ldots, L_m$, $C(\langle L_1, \ldots, L_m \rangle) = C(\langle C(L_1), \ldots, C(L_m) \rangle)$.

a. Interpret Property I. Give two examples of choice functions that satisfy I and two examples that do not.

Property I is analogous to path independence.

Two choice functions that satisfy I:

(i) Choose the first alternative in L.

(ii) Choose the first alternative in L that is "at least as good as" some $\tilde{x} \in X$ and choose the last element in L if there is no such alternative.

Two choice functions that violate I:

(i) Choose the second alternative in L.

(ii) Choose the last alternative such that the alteratives from the start of the sequence up to that alternative are in ascending order.

b. Define formally the following two properties of a choice function:

Order Invariance OI: A change in the order of the elements in the list does not alter the choice.

Let $L = \langle a_1, \ldots, a_K \rangle$. A permutation of L is a list $L^\pi = \langle a_{\pi(1)}, \ldots, a_{\pi(K)} \rangle$, where π is a permutation of $\{1, \ldots, K\}$. C satisfies OI if $C(L) = C(L^\pi)$ for every permutation π.

Duplication Invariance DI: Deleting an element that appears elsewhere in the list does not change the choice.

C satisfies DI if $C(L) = C(L')$ whenever (i) $L = \langle \langle L_1 \rangle, x, \langle L_2 \rangle \rangle$, $L' = \langle \langle L_1 \rangle, \langle L_2 \rangle \rangle$ and x appears in either L_1 or L_2, or (ii) $L = \langle x, \langle L' \rangle \rangle$ and x appears in L', or (iii) $L = \langle \langle L' \rangle, x \rangle$ and x appears in L'.

Show that Duplication Invariance implies Order Invariance.

Let $L^{\pi} = \langle a_{\pi(1)}, \ldots, a_{\pi(K)} \rangle$ be a permutation of $L = \langle a_1, \ldots, a_K \rangle$. By using DI K times, both $C(L) = C(\langle L, L^{\pi} \rangle)$ and $C(L^{\pi}) = C(\langle L, L^{\pi} \rangle)$.

c. **Characterize the choice functions that satisfy Duplication Invariance and Property** I.

Claim: Let C be a choice function over the *lists* of X. If C satisfies DI and I, then there exists a rationalizable choice function $\overline{C}$ over the *sets* of X such that $C(L) = \overline{C}(\{L\})$, where $\{L\}$ is the *set of elements* in L.

Proof: Let K, L be two lists such that $\{K\} = \{L\}$. By DI, which implies OI as well, choice is preserved when the duplicate alternatives in both lists are removed and the resulting lists are reshuffled so that the remaining alternatives appear in the same order. Thus, $C(K) = C(L)$ and thus $\overline{C}$ is well-defined and single-valued.

By Problem 2(d), showing that $\overline{C}$ satisfies path independence is sufficient for rationalizability. For any *set* $S \subseteq X$, define $\langle S \rangle$ to be some *list* of elements in S. Let $A, B \subseteq X$ be disjoint. Then:

$$\begin{aligned}
\overline{C}(A \cup B) &= C(\langle \langle A \rangle, \langle B \rangle \rangle) && \text{by def. of } \overline{C} \\
&= C(\langle C(\langle A \rangle), C(\langle B \rangle) \rangle) && \text{by property } I \\
&= C(\langle \overline{C}(A), \overline{C}(B) \rangle) && \text{by def. of } \overline{C} \\
&= \overline{C}(\{\overline{C}(A), \overline{C}(B)\}) && \text{by def. of } \overline{C}
\end{aligned}$$

Assume now that at the back of the decision maker's mind there is a value function u **defined on the set** X (**such that** $u(x) \neq u(y)$ **for all** $x \neq y$). **For any choice function** C, **define** $v_C(L) = u(C(L))$. **We say that** C **accommodates a longer list if whenever** L' **extends** L, $v_C(L') \geq v_C(L)$ **and there is a pair of lists L' and L**, **such that L' extends L and** $v_C(L') > v_C(L)$.

d. **Give two interesting examples of choice functions that accommodate a longer list**.

(i) Choose the u-maximal element in L.

(ii) Choose the second u-best alternative in L.

e. **Give two interesting examples of choice functions which satisfy property** I **but do not accommodate a longer list**.

(i) Choose the first alternative in L that yields at least utility $\tilde{u}$ and choose the last alternative in L if $u(x) < \tilde{u}$ for all $x \in L$.

(ii) Choose the first element in L.

Problem 7.

Let X be a finite set. We say that a choice function c is lexicographically rational if there exists a profile of preference relations $\{\succ_a\}_{a \in X}$ (not necessarily distinct) and an ordering O over X such that for every set $A \subset X$, $c(A)$ is the $\succ_a$ -maximal element in A, where a is the O -maximal element in A.

A decision maker who follows this procedure is attracted by the most notable element in the set (as described by O). If a is that element, he applies the ordering $\succ_a$ and chooses the $\succ_a$ -best element in the set.

We say that C satisfies the reference point property if for every set A, there exists $a \in A$ such that if $a \in A'' \subset A' \subset A$ and $C(A') \in A''$, then $C(A'') = C(A')$.

a. Show that a choice function C is lexicographically rational if and only if it satisfies the reference point property.

($\Rightarrow$) Assume C is lex. rational. For every set A we'll show that the $O-maximal$ element $a \in A$ satisfies the requirement of the reference point property. Note that for any $A' \subset A$ containing a, a is still the $O-maximal$ element. Thus, for all subsets of A containing a, $C(A')$ is determined by $\succ_a$. Thus condition α is satisfied for all subsets of A containing a and the reference point property holds.

($\Leftarrow$) Assume C satisfies the reference point property. We build the representation recursively. Consider the set X. By the reference point property there exists an element a_1 such that for all subsets of X that contain a_1, condition α holds. Define $Y = \{x \in X | C(\{x, a_1\} = x\}$. For any $x, y \in Y$, $x \neq y$, define $x \succ_{a_1} y$ if $x = C(\{a_1, x, y\})$ (including the case $y = a_1$), and for any $x \notin Y$ define $a_1 \succ_{a_1} x$. Extend $\succ_{a_1}$ such that the elements in $X \backslash Y$ are ordered arbitrarily. This preference relation, $\succ_{a_1}$, is well defined by condition α, and it rationalizes the choices of C whenever a_1 is available. Now consider the set $X \setminus \{a_1\}$, and repeat the procedure. We'll find an a_2 and $\succ_{a_2}$ such that $\succ_{a_2}$ represents all choices of C whenever a_2 is available in the subsets of $X \setminus \{a_1\}$. For completeness, assume that for any $i \neq 1$, $a_i \succ_{a_2} a_1$. Also construct the $O-preference$ such that $a_1 \succ_O a_2$. By induction we can complete the $O-preference$, and $\succ_a$ for every a in X.

b. Try to come up with a procedure satisfying the reference point axiom which is not stated explicitly in the language of the lexicographical rational choice function.

Choosing the second best alternative:

For every set A, the best alternative a satisfies the requirement of the reference point property: let $a \in A'' \subset A' \subset A$ and let $C(A')$ be the second best alternative in A'. If

$C(A') \in A''$ then $C(A')$ is also the second best alternative in A'' and thus $C(A'') = C(A')$.
In order to decsribe this procedure as lexicographically rational, assume the DM chooses the second best alternative according to some ordering $\succ$ and define:

1. the ordering O to be the original ordering $\succ$
2. $\succ_a$ to be an ordering indentical to $\succ$ except that a is moved to be the worst alternative.

Given a set A the DM is attracted by the $O-$maximal alternative a, which is the best alternative, but applies an ordering $\succ_a$ in which a is last. Therefore, the best alternative in A according to $\succ_a$ is in fact the second best alternative according to $\succ$.

Problem 8.

Consider a decision maker who has in mind a set of rationales and an asymmetric complete relation over a finite set X. Given $A \subset X$, he chooses the best alternative that he can rationalize.

Formally, **we say that a choice function C is rationalized if there is an asymmetric complete relation $\succ$ (not necessarily transitive!) and a set of partial orderings (asymmetric and transitive) $\{\succ_k\}_{k=1...K}$ (called rationales) such that $C(A)$ is the $\succ$-maximal alternative from among those alternatives found to be maximal in A by at least one rationale (given a binary relation $\succ$ we say that x is $\succ$-maximal in A if $x \succ y$ for all $y \in A$). Assume that the relations are such that the procedure always leads to a solution**.

We say that a choice function C satisfies The Weak Weak Axiom of Revealed Preference (WWARP) if for all $\{x,y\} \subset B_1 \subset B_2$ ($x \neq y$) and $C\{x,y\} = C(B_2) = x$, then $C(B_1) \neq y$.

a. **Show that a choice function satisfies WWARP if and only if it is rationalized**. **For the proof**, **construct rationales**, **one for each choice problem**.

($\Leftarrow$) Let us see first that the axiom is satisfied by any rationalized choice function: If x is chosen from B_2 then has a rationale in B_2 (I.e. there is a rationale $\succ_k$ such that x is the $\succ_k$-maximal in B_2). Thus, it has a rationale also in B_1. If y were chosen from B_1, then it has a rationale in B_1 as well. Since y is chosen from B_1 it must be that $y \succ x$. For $\{x,y\}$ both x and y have rationales and thus y would have been chosen from $\{x,y\}$, a contradiction.

($\Rightarrow$) Let C be a choice function satisfying WWARP. For every set B, define $x \succ_B y$ iff $x = C(B)$ and $y \in B$. Obviously, this rationale is a very partial ordering. As to the top preferences $\succ$, they are elicited by the choice from the two-element sets: $x \succ y$ if $C(\{x,y\}) = x$.

To see that those definitions "work", assume $C(B_1) = y$ but there is rationale $\succ_{B_2}$ and a x which is $\succ_{B_2}$-maximal in B_1 such that $x \succ y$. It must be that $B_1 \subset B_2$ and $C(B_2) = x$. By definition of $\succ$ also $C(\{x,y\}) = x$. A contradiction to $WWARP$.

b. **What do you think about the axiomatization**?

There might be other ways in which people's choices satisfy WWARP. Axiomatizing such a choice with partial orderings for each subset might be representing the choice procedure in a much more complex way than actual.

Consider the "warm-glow" procedure: The decision maker has two complete orderings in mind: one moral $\succsim_M$ and one selfish $\succsim_S$. He chooses the most moral alternative m as long as he doesn't "lose" too much by not choosing the most selfish alternative. Formally, for every alternative s there is some alternative $l(s)$ such that if the most selfish alternative is s then he is willing to choose m as long as $m \succsim_S l(s)$. If $l(s) \succ_S m$, he chooses s.

The function l satisfies (i) $s \succsim_S l(s)$ and (ii) $s \succsim_S s'$ implies $l(s) \succsim_S l(s')$.

c. Show that WWARP is satisfied by this procedure.

Assume in contradiction that WWARP is violated, i.e. there exists $\{x,y\} \subset B_1 \subset B_2$ ($x \neq y$) and $C(\{x,y\}) = C(B_2) = x$, and $C(B_1) = y$.

Assume x is the moral maximal in B_2. Clearly x is also the moral maximal in B_1 and this implies that y is the selfish maximal in B_1. Since x is chosen in B_2, it must be that $x \succeq_s l(s_{B_2})$, where s_{B_2} is the selfish maximal in B_2. But $B_1 \subset B_2$ implies $s_{B_2} \succeq_s s_{B_1} = y$. By monotonicity of $l(s)$, $x \succeq_s l(y)$ which contradicts that $C(B_1) = y$.

Assume that x is the selfish maximal in B_2. Clearly x is also the selfish maximal in B_1, hence y is the moral maximal in B_1. This implies that in $\{x,y\}$, x is the selfish maximal and y is the moral maximal. Since x is chosen in $\{x,y\}$, it must be that $y \not\succeq_s l(x)$. But since x is also selfish maximal in B_1 this contradicts that $C(B_1) = y$.

d. Show directly that the warm-glow procedure is rationalized (in the sense of the definition in this problem).

There are two rationales, the selfish and the moral orderings. The final relation $\succ$ is the moral ordering, that is $x \succ y$ if $x \succ_m y$. However, if $l(y) \succ_s x$, then it is reversed, that is $y \succ x$.

To see that this works, given any set we choose the moral and selfish maximal, m and s. Then we apply the final ordering. Note that if $m \succeq_s l(s)$ then the ordering says $m \succ s$. Otherwise, $s \succ m$ as desired.

Problem Set 4 – Consumer Preferences

Problem 1.

Consider the preference relations on the interval $[0,1]$ **which are continuous**. **What can you say about those preferences which are also strictly convex**?

We will show that a continuous preference relation $\succsim$ on $X = [0,1]$ is strictly convex iff there exists a point x^* such that $b \succ a$ for all $a < b \leq x^*$ or all $x^* \geq b > a$.

(a) Let $\succsim$ be continuous and strictly convex. Since the preferences are continuous and X is compact there exists a unique $x^* \in X$ that maximizes the preferences (see Lecture 5). Let $0 \leq a < b \leq x^*$. By definition $a \precsim x^*$ and $b = \alpha a + (1-\alpha)x^*$ for some $\alpha \in [0,1)$ and thus, by strict convexity $a \prec b$. The case, for two points in $[x^*, 1]$ is analogous .

(b) Assuming that the preferences are increasing in $[0, x^*]$ and decreasing in $[x^*, 1]$, we will show that they satisfy strict convexity. Let $\alpha \in (0,1)$ and $a, b \in X$ be such that $a \neq b$ and $a \succsim b$. It must be that $\alpha a + (1-\alpha)b$ is either between a and x^* or between b and x^*. If $\alpha a + (1-\alpha)b$ is between b and x^*, then $\alpha a + (1-\alpha)b \succ b$. If it is between a and x^*, then $\alpha a + (1-\alpha)b \succ a \succsim b$.

Problem 2.

Show that if the preferences $\succsim$ satisfy continuity and monotonicity, then the function $u(x)$ defined by $x \sim (u(x), \ldots, u(x))$ is continuous.

Let x be a point in X. By definition $u(x) \geq 0$. We need to show that for any $\epsilon > 0$ there exists δ such that $|u(x) - u(y)| < \epsilon$ for any $y \in Ball(x, \delta)$.

If $u(x) - \epsilon \geq 0$, then by monotonicity, $x \succ (u(x) - \epsilon, \ldots, u(x) - \epsilon)$. By continuity, there exists $\delta_1 > 0$ such that $u(y) > u(x) - \epsilon$ for $y \in Ball(x, \delta_1)$.

If $u(x) - \epsilon < 0$, then for any $\delta_1 > 0$, $u(y) > u(x) - \epsilon$ for $y \in Ball(x, \delta_1)$.

Similarly, by monotonicity, $(u(x) + \epsilon, \ldots, u(x) + \epsilon) \succ x$ and thus $(u(x) + \epsilon, \ldots, u(x) + \epsilon) \succ Ball(x, \delta_2)$ for some $\delta_2 > 0$ by continuity. Therefore, $u(x) + \epsilon > u(y)$ for $y \in Ball(x, \delta_2)$.

Define $\delta = \min\{\delta_1, \delta_2\}$. Then, $|u(x) - u(y)| < \epsilon$ for any $y \in Ball(x, \delta)$.

Problem 3.

In a world with two commodities, **consider the following condition**: **The preference relation** $\succsim$ **satisfies Convexity 4 if for all** x **and** $\epsilon > 0$

$$(x_1, x_2) \sim (x_1 - \epsilon, x_2 + \delta_1) \sim (x_1 - 2\epsilon, x_2 + \delta_1 + \delta_2) \textit{ implies } \delta_2 \geq \delta_1.$$

Interpret Convexity 4 and show that for strong monotonic and continuous preferences, **it is equivalent to the convexity of the preference relation**.

Interpretation: If after an x_1 is reduced by ϵ, the consumer must be compensated with δ units of good 2 in order to remain indifferent to x, then he must be compensated with *at least* 2δ units of good 2 if his consumption of x_1 is decreased by 2ϵ.

Convexity 1 $\Rightarrow$ **Convexity 4**: Let $(x_1, x_2) \sim (x_1 - \epsilon, x_2 + \delta_1) \sim (x_1 - 2\epsilon, x_2 + \delta_1 + \delta_2)$. By convexity 1,

$$\begin{aligned}(x_1 - \epsilon, x_2 + \frac{\delta_1 + \delta_2}{2}) &= \frac{1}{2}(x_1, x_2) + \frac{1}{2}(x_1 - 2\epsilon, x_2 + \delta_1 + \delta_2) \\ &\succsim (x_1 - \epsilon, x_2 + \delta_1).\end{aligned}$$

Then, $(\delta_1 + \delta_2)/2 \geq \delta_1$ by monotonicity and thus $\delta_2 \geq \delta_1$.

Convexity 4 $\Rightarrow$ **Convexity 1**: First, we show that if $x \sim y$, then $(x + y)/2 \succsim y$. If $x \neq y$ then by strong monotonicity we can WLOG assume $x_1 > y_1$ and $y_2 > x_2$. Define $\Delta > 0$ by $\Delta = (y_2 - x_2)/2$ and $\epsilon = (x_1 - y_1)/2$. By strong monotonicity

$$\begin{aligned}(x_1 - \epsilon, x_2 + 2\Delta) = (\frac{x_1 + y_1}{2}, y_2) &\succ y \sim x \\ &\succ (\frac{x_1 + y_1}{2}, x_2) = (x_1 - \epsilon, x_2).\end{aligned}$$

By continuity, there exists $\delta > 0$ such that

$$(x_1, x_2) \sim (x_1 - \epsilon, x_2 + \delta) \sim y = (x_1 - 2\epsilon, x_2 + 2\Delta).$$

By Convexity 4, $2\Delta - \delta \geq \delta$ and thus $\Delta \geq \delta$. By monotonicity,

$$\frac{x + y}{2} = (x_1 - \epsilon, x_2 + \Delta) \succsim (x_1 - \epsilon, x_2 + \delta) \sim y.$$

Now if $x \succsim y$, then there exists z on the interval which connects 0 and x, such that $z_k \leq x_k$ for all k and $z \sim y$. Then, by monotonicity and the previous result, $(x + y)/2 \succsim (z + y)/2 \succsim y$. The rest follows from the following Lemma:

Lemma: If $\succsim$ are continuous preferences, then $\succsim$ are convex iff [$x \succsim y$ implies $(x + y)/2 \succsim y$] for all $x, y \in X$.

Proof: Assume $x \succsim y$ and $z = \alpha x + (1 - \alpha)y$ for $\alpha \in [0, 1]$. We will show that $z \succsim y$. Construct a sequence $\{(x^n, y^n)\}$ such that both $x^n, y^n \succsim y$ and z between x^n and y^n. Define $x^0 = x$, $y^0 = y$. Continue inductively. Let $m^n = (x^n + y^n)/2$. Then, m^n is at least as good as either x^n or y^n and the above argument and transitivity imply that it is at least as good as y. Define:

$x^{n+1} = m^n$ and $y^{n+1} = y^n$ if z lies between y^n and m^n, and
$x^{n+1} = x^n$ and $y^{n+1} = m^n$ otherwise.

Thus, $x^{n+1}, y^{n+1} \succsim y$ and z is between x^{n+1} and y^{n+1}. Since $y^n, x^n \to z$, $z \succsim y$ by continuity.

Problem 4.

Complete the proof (for all K) of the claim that any continuous preference relation satisfying strong monotonicity quasi-linearity in all commodities can be represented by a utility function of the form $\sum_{k=1}^{K} \alpha_k x_k$, where $\alpha_k > 0$ for all k.

Proof by induction on K: We have already proved this for $K = 1$ and 2.

Let $\succsim$ be a preference relation satisfying the problem's assumptions. Consider the preferences restricted to the set of all vectors of the type $(0, x_2, \ldots, x_K)$. The preferences satisfy Continuity, Strong Monotonicity and Quasi-Linearity in goods $2, .., K$. By the induction hypothesis, there is a vector of positive numbers $(\alpha_k)_{k=2,..,K}$ such that $(0, x_2, \ldots, x_K) \sim (0, \sum_{k=2}^{K} \alpha_k x_k, 0, \ldots, 0)$.

By quasi-linearity in good 1, $(x_1, x_2, \ldots, x_K) \succsim (y_1, y_2, \ldots, y_K)$ iff $(x_1, \sum_{k=2}^{K} \alpha_k x_k, 0, \ldots, 0) \succsim (y_1, \sum_{k=2}^{K} \alpha_k y_k, 0, \ldots, 0)$.

The relation over all vectors of the type $(x_1, x_2, 0, \ldots, 0)$ satisfies the three properties in the first two dimensions. Thus, there exists $\beta_1, \beta_2 > 0$ such that $(x_1, x_2, 0, \ldots, 0) \sim (\beta_1 x_1 + \beta_2 x_{2,} 0, , 0, \ldots, 0)$ and thus $x \sim (\beta_1 x_1 + \sum_{k=2}^{K} \beta_2 \alpha_k x_k, 0 \ldots, 0)$ and by strong monotonicity in the first good, the preferences have a linear utility representation.

Problem 5.

Show that for any consumer's preference relation $\succsim$ satisfying continuity, monotonicity, strong monotonicity with respect to commodity 1 and quasi-linearity with respect to commodity 1, there exists a number $v(x)$ such that $x \sim (v(x), 0, \ldots, 0)$ for every vector x.

Since $\succsim$ satisfies continuity and monotonicity every bundle is indifferent to a bundle on the main diagonal. Thus, it is sufficient to show the claim for bundles on the main diagonal.

Let $e = (1, \ldots, 1)$ and define

$$T = \{\alpha \in \Re_+ \mid \alpha e \succ (x_1, 0, \ldots, 0) \text{ for all } x_1 \in \Re_+\}.$$

We will see that $T = \emptyset$. Assume that $T \neq \emptyset$. Let $\gamma = \inf T$. There are two cases:

Case 1: $\gamma \in T$. Then $\gamma > 0$ and by strict monotonicity of commodity 1, $(1+\gamma, \gamma, \ldots, \gamma) \succ \gamma e$. By continuity, there exists $\epsilon > 0$ such that

$$(1+\gamma, \gamma - \epsilon, \ldots, \gamma - \epsilon) \succ \gamma e \succ (x_1, 0, \ldots, 0)$$

for all x_1.

Since $\gamma - \epsilon < \inf T$, there exists an x_1^* such that $(x_1^*, 0, \ldots, 0) \succsim (\gamma - \epsilon, \gamma - \epsilon, \ldots, \gamma - \epsilon)$ and by quasi-linearity in commodity 1,

$$(x_1^* + 1 + \epsilon, 0, \ldots, 0) \succsim (1+\gamma, \gamma - \epsilon, \ldots, \gamma - \epsilon), \text{ a contradiction.}$$

Case 2: $\gamma \notin T$. Then $(\beta, 0, \ldots, 0) \succsim \gamma e$ for some β. By strong monotonicity of commodity 1, $(\beta + 1, 0, \ldots, 0) \succ \gamma e$. By continuity, there is an $\epsilon > 0$ such that $(\beta + 1, 0, \ldots, 0) \succ (\gamma + \epsilon)e$, which contradicts $\gamma = \inf T$.

Thus, $T = \emptyset$ and for any bundle on the main diagonal, αe, there exists a bundle $(x_1, 0, \ldots, 0)$ such that $(x_1, 0, \ldots, 0) \succsim \alpha e \succsim (0, \ldots, 0)$. By continuity there exists a number $v(\alpha e)$ such that $(v(\alpha e), 0, \ldots, 0) \sim \alpha e$.

Problem 6.

We say that a preference relation satisfies separability if it can be represented by an additive utility function, that is, a function of the type $u(x) = \sum_k v_k(x_k)$.

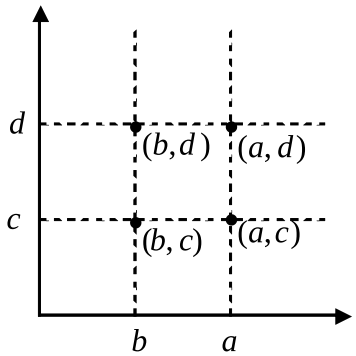
a. **Show that such preferences satisfy condition S**: **for any subset of commodities** J, **and for any bundles** a,b,c,d, **we have**

$$(a_J, c_{-J}) \succsim (b_J, c_{-J}) \Leftrightarrow (a_J, d_{-J}) \succsim (b_J, d_{-J})$$

where (x_J, y_{-J}) **is the vector that takes the components of** x **for any** $k \in J$ **and takes the components of** y **for any** $k \notin J$.

$$\begin{aligned}(a_J, c_{-J}) \succsim (b_J, c_{-J}) &\Leftrightarrow \sum_{k\in J} v_k(a_k) + \sum_{i\notin J} v_i(c_i) \geq \sum_{k\in J} v_k(b_k) + \sum_{i\notin J} v_i(c_i)\\ &\Leftrightarrow \sum_{k\in J} v_k(a_k) + \sum_{i\notin J} v_i(d_i) \geq \sum_{k\in J} v_k(b_k) + \sum_{i\notin J} v_i(d_i)\\ &\Leftrightarrow (a_J, d_{-J}) \succsim (b_J, d_{-J}).\end{aligned}$$

Graphically, if two bundles lie on the same horizontal line and $(a,c) \succsim (b,c)$, then a change of c to d will preserve the preference relation, that is $(a,d) \succsim (b,d)$.



b. **Show that for** $K = 2$ **such preferences satisfy the** $Hexagon - condition$: **If** $(a,b) \succsim (c,d)$ **and** $(c,e) \succsim (f,b)$ **then** $(a,e) \succsim (f,d)$.

$v_1(a) + v_2(b) \geq v_1(c) + v_2(d)$ and

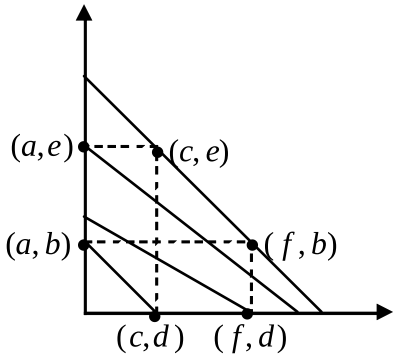
$v_1(c) + v_2(e) \geq v_1(f) + v_2(b)$ implies

$v_1(a) + v_2(e) \geq v_1(f) + v_2(d)$.

c. **Give an example of a continuous preference relation which satisfies condition**

S and does not satisfy separability.

Consider any preference relation with linear indifference curves as depicted:



Such preferences violate the Hexagon Condition.

Problem 7.

a. **Show that the preferences represented by the utility function** $\min\{x_1,\ldots,x_K\}$ **are not differentiable**.

Let $x^* = (a,\ldots,a)$ and let $v(x^*)$ be a candidate set of subjective values. Without loss of generality, let $v_1(x^*) > 0$. Then, $(1,0,0,\ldots,0)\cdot v(x^*) > 0$ but $(a+\epsilon,a,a,\ldots,a) \sim x^*$ for all ϵ, and thus $(+1,0,0,\ldots,0) \notin D(x^*)$, a contradiction.

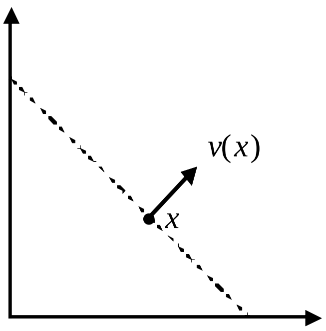
b. **Check the differentiability of the lexicographic preferences in** $\Re^2$.

Lexicographic preferences are not differentiable. Let $x \in \Re^2$ and assume that $v(x)$ is a vector of subjective values. Since $x+\epsilon(0,1) \succ x$ for all $\epsilon > 0$, then $(0,1)$ is an improving direction and $v_2(x) > 0$. Then, for small $\delta > 0$, $(-\delta,1)\cdot v(x) > 0$. However, $(-\delta,1)$ is not an improving direction, a contradiction.

c. **Assume that** $\succsim$ **is monotonic**, **convex and differentiable such that for every** x, **we have** (*) $D(x) = \{d \mid (x+d) \succ x\}$. **What can you say about** $\succsim$?

We will show that the indifference curves are linear.

By differentiability and (*) there exists $v(x)$ such that $d\cdot v(x) > 0$ iff $x+d \succ x$. Graphically, any point above the dotted line is strictly better than x:



We will show that for any $z \in X$ on the dotted line (that is $zv(x) = xv(x)$), we have $x \sim z$.

First let us see that (**) any $z \in X$ on the dotted line satisfies $x \succsim z$. If $z \succ x$, then by (*) $(z-x) \in D(x)$ and by differentiability $(z-x)\cdot v(x) > 0$ but as $zv(x) = xv(x)$, a contradiction. Note that it must be that $v(z) = v(x)$, since otherwise there would be a point on $\{y \mid yv(z) = zv(z)\}$such that $yv(x) > xv(x)$ but by (*) $y \succ x$ though $x \succsim z \succsim y$. Thus $(x-z)\cdot v(z) = 0$ and by (**) (applied to z) $z \succsim x$.

d. **Assume that $\succsim$ is a monotonic**, **convex and differentiable preference relation**. **Let** $E(x) = \{d \in \Re^K \mid$ **there exists** $\epsilon^* > 0$ **such that** $x + \epsilon d \prec x$ **for all** $\epsilon \le \epsilon^*\}$. **Show that** $\{-d \mid d \in D(x)\} \subseteq E(x)$ **but not necessarily** $\{-d \mid d \in D(x)\} = E(x)$.

We first show that $\{-d \mid d \in D(x)\} \subseteq E(x)$. By contradiction, let $d \in D(x)$ be such that $-d \notin E(x)$. WLOG $x + d \succ x$ and $x - d \succsim x$. By definition of $D(x)$, $d \cdot v(x) > 0$ and $e \cdot v(x) > 0$ for some e with $e_k \le d_k$ with at least one strict inequality. For $\epsilon > 0$ small enough $x + \epsilon e \succ x$. By convexity any convex combination of $x + \epsilon e$ and $x - d$ is at least as good as x but the segment contains points which by monotonicity are at least as bad as x.

Let $\succsim$ be represented by $u(x) = x_1 x_2$. Since u is quasi-concave, has continuous partial derivatives and satisfies $u_i(x) > 0$. Thus, the relation $\succsim$ is convex, monotonic and differentiable. Let $d = (1,-1)$ and note that $-d \in E(2,2)$ but $d \notin D(2,2)$.

e. **Consider the consumer's preferences in a world with two commodities defined by**:

$$\mathbf{u}(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 \le 1 \\ 1 + 2x_1 + x_2 & \text{if } x_1 + x_2 > 1 \end{cases}.$$

Show that these preferences are not continuous but nevertheless are differentiable according to our definition.

If $x_1 + x_2 \le 1$, then differentiability holds for $v(x) = (1,1)$ and if $x_1 > 1$, then differentiability holds for $v(x) = (2,1)$. The preferences are not continuous, since $(0,2) \succ (1,0)$, but $(0,2) \prec (1+\epsilon, 0)$ for $\epsilon > 0$.

Problem Set 5– Demand: Consumer Choice

Problem 1.

Show that if a consumer has a homothetic preference relation, then his demand function is homogeneous of degree one in w.

Let $\lambda > 0$ and $y \in B(p,\lambda w)$. Then $y/\lambda \in B(p,w)$. Since $x(p,w) \succsim y/\lambda$ and preferences are homothetic, then $\lambda x(p,w) \succsim y$, and thus, $\lambda x(p,w)$ is the best element in $B(p,\lambda w)$, that is $x(p,\lambda w) = \lambda x(p,w)$.

Problem 2.

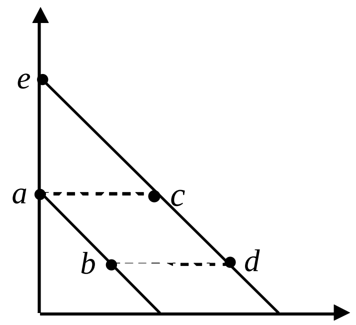
Consider a consumer in a world with $K = 2$, **who has a preference relation that is monotonic**, **continuous**, **strictly convex**, **and quasi-linear in the first commodity**. **How does the demand for the first commodity change with** w?

Claim: For any p either there is no w such that $x_1(p,w) > 0$ or there exists an $w^* \geq 0$ such that if $w \leq w^*$, then the consumer does not consume the first commodity, and if $w \geq w^*$, then the first commodity absorbs all changes in wealth, that is $x_1(p,w) = \frac{w-w^*}{p_1}$.

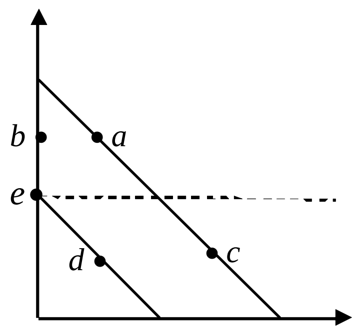
Proof: Normalize $p_1 = 1$. First, we show if $x_1(p,w) = 0$, then $x_1(p,w') = 0$ for $w' \leq w$.

Let $w' \leq w$. Denote $a = (0, w'/p_2)$, $b \in B(p,w')$ such that $pb = w'$, $c = (a_1 + (w - w'), a_2)$, $d = (b_1 + (w - w'), b_2)$ and $e = (0, w/p_2)$.

Note that $pd = w$ and c is between e and d. Since $e \succsim d$ then $c \succsim d$. By the quasi-linearity $a \succsim b$. Therefore $x(p,w') = a = (0, w'/p_2)$.



Now, if there is w such that $x_1(p,w) > 0$ then by continuity of the demand function $w^* = \max\{w \mid x_1(p,w) = 0\}$ exists. We need to show that $x(p,w) = (w - w^*, w^*/p_2)$ for all $w \geq w^*$. If $x_2(p,w) > w^*/p_2$, for some $w > w^*$, let $a = x(p,w)$ and by the quasi linearity in the first commodity $b = (0, a_2) \succsim y$ for all $y \in B(p, p_2 x_2(p,w))$ contradicting the definition of w^* (see graph). If $x_2(p,w) < w^*/p_2$, for some $w > w^*$, then $c = x(p,w)$ and
$d = (x_1(p,w) - (w - w^*), x_2(p,w)) \succsim e = (0, w^*/p_2)$, a contradiction to the definition of w^*.



Problem 3.

Define a Demand Correspondence, $X(p,w) : \Re^{K+1}_{++} \to \Re^{K}_{+}$, **to be the** **set** **of all solutions to the consumer's problem in** $B(p,w)$.

a. **Calculate** $X(p,w)$ **for the case of** $K = 2$ **and preferences represented by** $x_1 + x_2$.

If preferences are represented by $x_1 + x_2$, then for $p_1 \neq p_2$ the consumer buys the cheapest good; otherwise he is indifferent between all bundles on the budget line. Therefore,

$$X(p,w) = \left\{ \begin{array}{ll} (0, w/p_2) & if \;\; p_2 < p_1 \\ \{(\alpha, w/p_2 - \alpha) | \alpha \in [0, w/p_1]\} & if \;\; p_2 = p_1 \\ (w/p_1, 0) & if \;\; p_1 < p_2 \end{array} \right\}$$

b. **Let** $\succsim$ **be a continuous preference relation** (**not necessarily convex**). **Show that** $X(p,w)$ **is upper semi-continuous**.

(**A correspondence** $F : A \to B$ **is said to be upper semi-continuous if for every converging sequence** $a^n \in A$ **with** $\lim a^n \in A$, **and for every converging sequence** $b^n \in B$ **such that** $\lim b^n$ **exists and** $b^n \in F(a^n)$, **it holds that** $\lim b^n \in F(\lim a^n)$.)

Let $(p^n, w^n) \to (p,w)$, with $\{(p^n, w^n)\}, (p,w) \in \Re^{K+1}_{++}$, be a converging sequence of prices and wealth vectors. Let $x^n \to x$, $x^n \in \Re^K_+$, be a converging sequence of optimal bundles in $B(p^n, w^n)$, i.e. $x^n \in X(p^n, w^n)$.

Clearly, $p^n x^n \leq w^n$ for every n and therefore $px = w$, i.e. $x \in B(p,w)$. Assume, in contradiction, that $x \notin X(p,w)$. Thus, there exists a $y \in B(p,w)$ such that $y \succ x$. By continuity, there exists an $\epsilon > 0$ such that $Ball(y,\varepsilon) \succ Ball(x,\varepsilon)$ and therefore there exists a bundle $z \in Ball(y,\varepsilon)$ such that $pz < w$ and $z \succ x$. Since $p^n \to p$ and $w^n \to w$, $p^n z \leq w^n$ for large enough n. Since $x^n \to x$, $x^n \in Ball(x,\varepsilon)$ for large enough n and thus $z \succ x^n$, a contradiction to x^n being optimal in $B(p^n, w^n)$.

Problem 4.

Determine whether the following consumer behavior patterns are fully rationalized (assume K=2):

a. **The consumer consumes up to quantity 1 of commodity 1 and spends his excess wealth on commodity 2**.

Yes. $x(p,w)$ is rationalized by the monotonic preferences represented by

$$u(x) = \begin{cases} x_1 & \text{if } x_1 < 1 \\ 1 + x_2 & \text{if } x_1 \geq 1 \end{cases}$$

b. **The consumer chooses the bundle** (x_1, x_2) **which satisfies** $\frac{x_1}{x_2} = \frac{p_1}{p_2}$ **and costs** w. (**Does the utility function** $u(x) = x_1^2 + x_2^2$ **rationalize the consumer's behavior**?)

No. The behavior violates the WA. $x((2,1),5) = (2,1)$ and $x((1,2),5) = (1,2)$. Both bundles are affordable in both budget sets.

The function $u(x) = x_1^2 + x_2^2$ does not rationalize $x(p,w)$ since a consumer maximizing u would allocate all wealth to the cheapest good. The "first order condition" approach is not appropriate because preferences represented by u are not convex.

Problem 5.

In this question, we consider a consumer who behaves differently from the classic consumer we talked about in the lecture. Once again we consider a world with K commodities. The consumer's choice will be from budget sets. The consumer has in mind a preference relation that satisfies continuity, monotonicity, and strict convexity; for simplicity, assume it is represented by a utility function u.

The consumer maximizes utility up to utility level u^0. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least u^0.

a. Formulate the consumer's problem.

$$\begin{array}{lll} \max_{x \in B(p,w)} u(x) & & \text{if } \max_{x \in B(p,w)} u(x) < u^0, \text{ and} \\ \max_{x \in B(p,w)} x_1 & \text{s.t. } u(x) \geq u^0 & \text{if } \max_{x \in B(p,w)} u(x) \geq u^0. \end{array}$$

b. Show that the consumer's procedure yields a unique bundle.

If $\max_{x \in B(p,w)} u(x) < u^0$, then the consumer acts as in the standard framework. $x(p,w)$ exists because preferences are continuous and is unique because preferences are strictly convex.

If $\max_{x \in B(p,w)} u(x) \geq u^0$, define $\tilde{B} = \{x \in B(p,w) \mid u(x) \geq u^0\}$, which is compact, and convex (by strict convexity). Then $\max_{x \in \tilde{B}} x_1$ exists. If both y and z are solutions then by the strict convexity $u((y+z)/2) > u_0$ and thus there is a vector x such that $u(x) > u^0$ and $x_1 > y_1$ contradicting the optimality of y in $\tilde{B}$.

c. Is this demand procedure rationalizable?

Yes. The procedure is rationalized by

$$v(x) = \begin{cases} u(x) & \text{if } u(x) < u^0 \\ u^0 + x_1 & \text{if } u(x) \geq u^0. \end{cases}$$

d. Does the demand function satisfy Walras Law?

Yes. Preferences are monotonic.

e. **Show that in the domain of (p,w) for which there is a feasible bundle yielding utility of at least u^0 the consumer's demand function for commodity 1 is decreasing in p_1 and increasing in w.**

In both cases the budget set is enlarging and the consumer could obtain more x_1 and preserve u^0.

f. **Is the demand function continuous**?

Yes.

Let $z(p,w)$ be the solution of $\max_{x\in B(p,w)} u(x)$. Note that in this case $z(p,w)$ is not necessarily the consumer's demand $x(p,w)$.

First, we show that demand is continuous in prices. Let $\{p^n\}$ converge to p. If $u(z(p,w)) < u^0$ then for n large enough $max_{x\in B(p^n,w)} u(x) < u^0$ and the demand is $z(p^n,w)$ converges to $z(p,w)$ which is the demand in (p,w).

Assume $u(z(p,w)) \geq u^0$. Let $m = \inf_{i,n} p_i^n > 0$, the infimum of the commodity prices. Then $x(p^n,w) \in [0,w/m]^K$ for all n, and thus WLOG we can assume that $x(p^n,w)$ converges to a bundle y. By contradiction, assume that $y \neq x(p,w)$.

If $u(y) < u^0$, then by continuity, there exists an $\epsilon > 0$ such that $Ball(x(p,w),\varepsilon) \succ Ball(y,\varepsilon)$. Then, there exists a $a \in Ball(x(p,w),\varepsilon)$ such that $pa < w$ and $a \succ y$. For n large, $p^n a \leq w$ and $a \succ x(p^n,w)$, a contradiction.

If $u(y) \geq u^0$, then $x_1(p,w) > y_1$. Let $a = \frac{1}{2}x(p,w) + \frac{1}{2}y$. Then $a_1 > y_1$, $a \in B(p,w)$ and $u(a) > u^0$ by strict convexity. By continuity, there exists an $\epsilon > 0$ small such that $a_1 - \epsilon > y_1 + \epsilon$, $p \cdot (a - \epsilon e_1) < w$ and $u(a - \epsilon e_1) \geq u^0$. Thus for n large, $B(p_n,w)$ contains $a - \epsilon e_1$ which yields utility larger than u^0 and quantity larger than $x_1(p^n,w)$, a contradiction.

Now, let $\{(p^n,w^n)\}$ converge to (p,w). Since $x(p,w)$ is homogeneous of degree zero, then

$$x(p^n,w^n) = x(\frac{p^n}{w^n},1).$$

Since demand is continuous in p, then

$$x(\frac{p^n}{w^n},1) \to x(\frac{p}{w},1) = x(p,w),$$

where the equality follows from $x(p,w)$ being homogeneous of degree zero.

Problem 6.

It's a common practice in economics to view aggregate demand as being derived from the behavior of a "representative consumer". Give two examples of "well-behaved" consumer preference relations that can induce average behavior that is not consistent with maximization by a "representative consumer". (That is, construct two "consumers", 1 and 2, who choose the bundles x^1 and x^2 out of the budget set A and the bundles y^1 and y^2 out of the budget set B so that the choice of the bundle $\frac{x^1+x^2}{2}$ from A and of the bundle $\frac{y^1+y^2}{2}$ from B is inconsistent with the model of the rational consumer).

Let $(p^A, w^A) = ((1,2), 8)$, $(p^B, w^B) = ((2,1), 8)$ and

$$u_1(x) = \begin{cases} x_1 & \text{if } x_1 < 4 \\ 4 + x_2 & \text{if } x_1 \geq 4 \end{cases} \qquad u_2(x) = \begin{cases} x_2 & \text{if } x_2 < 4 \\ 4 + x_1 & \text{if } x_2 \geq 4 \end{cases}$$

The demands of the two agents in A will be $(4,2)$ and $(0,4)$ and thus $\overline{x^A}(p,w) = (2,3)$. Similarly, $\overline{x^B}(p,w) = (3,2)$. Both average bundles are interior in A and in B. Thus, we the average demand violates the weak axiom.

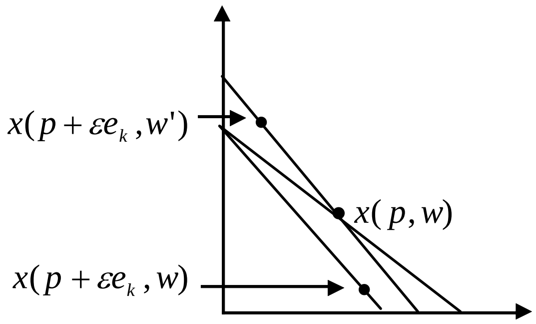
Problem 7.

A commodity k is Giffen if the demand for the k'th good is increasing in p_k. A commodity k is inferior if the demand for the commodity decreases with wealth. **Show that if there is a vector (p, w) such that the demand for the k'th commodity is rising after its price has increased**, **then there is a vector (p', w') such that the demand of the k'th commodity is falling after the income has increased** (**Giffen implies inferior**).

Let e_k be the vector with the k'th component being 1 and all other components being 0. We have $x_k(p + \epsilon e_k, w) > x_k(p, w)$. Let $w' \geq w$ be the "compensating" wealth level, that is $[p + \epsilon e_k] \cdot x(p, w) = w'$. Thus, $x(p + \epsilon e_k, w') \succsim x(p, w)$. By definition,

$px(p + \epsilon e_k, w') + \epsilon x_k(p + \epsilon e_k, w') = [p + \epsilon e_k] \cdot x(p + \epsilon e_k, w') \leq w' = w + \epsilon x_k(p, w)$.

If $x_k(p + \epsilon e_k, w') > x_k(p, w)$ then $px(p + \epsilon e_k, w') < w$ contradicting the optimality of $x(p, w)$ in $B(p, w)$.



Problem Set 6 – More Economic Agents: a Consumer Choosing Budget Sets, a Dual Consumer and a Producer

Problem 1.

Imagine that you are reading a paper in which the author uses the indirect utility function $v(p_1,p_2,w) = w/p_1 + w/p_2$. You suspect that the author's conclusions in the paper are the outcome of the"fact" that the function v is inconsistent with the model of the rational consumer. Take the following steps to make sure that this is not the case:

a. Use Roy's Identity to derive the demand function.

$$x_i(p,w) = -\frac{\partial v(p,w)/\partial p_i}{\partial v(p,w)/\partial w} = -\frac{-w/p_i^2}{(p_1+p_2)/p_1p_2} = \frac{wp_j}{p_i(p_1+p_2)}.$$

b. Show that if demand is derived from a smooth utility function, then the indifference curve at the point (x_1,x_2) has the slope $-\sqrt{x_2}/\sqrt{x_1}$.

By part (a), $x_i(p,w) > 0$ for $i = 1,2$. Note that $x_2(p,w) = (\frac{p_1}{p_2})^2 x_1(p,w)$.

If u is quasi-concave, then

$$\frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = \frac{p_1}{p_2} = \sqrt{\frac{x_2}{x_1}}$$

c. Construct a utility function with the property that the ratio of the partial derivatives at the bundle (x_1,x_2) is $\sqrt{x_2}/\sqrt{x_1}$.

$u(x) = (\sqrt{x_1} + \sqrt{x_2})$.

d. Calculate the indirect utility function derived from this utility function. Do you arrive at the original $v(p_1,p_2,w)$? If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c)?

The indirect utility function derived from u is

$u(x(p,w)) = (\sqrt{wp_1/p_2(p_1+p_2)} + \sqrt{wp_2/p_1(p_1+p_2)}) = \sqrt{w(p_1+p_2)}/\sqrt{p_1p_2}$.

The function $u^2(x)$ represents the same preference relation and $u^2(x(p,w)) = v(p,w)$.

Problem 2.

Show that if the preferences are monotonic, **continuous and strictly convex**, **then the Hicksian demand function** $h(p,z)$ **is continuous**.

Let $\{(p^n,z^n)\}$ converge to (p,z). Define $\bar{z}_k = \sup\{z_k^n\}$, $m = \inf\{p_k^n\}$ and $M = \sup\{p_k^n\}$. The consumer does not need more than $M\sum_k \bar{z}_k$ to obtain any z^n. Thus, $h_l(p^n,z^n) \leq M\sum_k \bar{z}_k/m$ for all l. Thus, WLOG we can assume that $h(p^n,z^n)$ converges to some bundle h^*.

By contradiction, assume that $h^* \neq h(p,z)$. By the continuity of the preferences $h^* \succsim z$, and thus (assuming that $h(p,z)$ is uniquely defined) $ph^* > ph(p,z)$. There exists an $\epsilon > 0$ such that $ph^* > p[h(p,z) + (\epsilon,..,\epsilon)]$ and by monotonicity of the preferences $h(p,z) + (\epsilon,..,\epsilon) \succ z$. Then, for n large enough, $p^n h(p^n,z^n) > p^n[h(p,z) + \epsilon e]$ and $h(p,z) + \epsilon e \succ z^n$, a contradiction.

Problem 3.

One way to compare budget sets is by using the indirect preferences which involves comparing $\mathbf{x}(p,w)$ and $x(p',w)$. Following are two other approaches to making such a comparison.

Define:

$$CV(p,p',w) = w - e(p',z) = e(p,z) - e(p',z)$$

where $z = x(p,w)$. **This is the answer to the question**: **What is the change in wealth that would be equivalent**, **from the perspective of** (p,w), **to the change in price vector from** p **to** p'?

Define:

$$EV(p,p',w) = e(p,z') - w = e(p,z') - e(p',z')$$

where $z' = x(p',w)$. **This is the answer to the question**: **What is the change in wealth that would be equivalent**, **from the perspective of** (p',w), **to the change in price vector from** p **to** p'?

Now, **answer the following questions regarding a consumer in a two**-**commodity world with a utility function** u:

a. **For the case of the preferences represented by** $u(x_1,x_2) = x_1 + x_2$, **calculate the two consumer surplus measures**.

$$CV(p,p',w) = w - \frac{w\min\{p_1',p_2'\}}{\min\{p_1,p_2\}} = w\frac{\min\{p_1,p_2\} - \min\{p_1',p_2'\}}{\min\{p_1,p_2\}}$$

$$EV(p,p',w) = \frac{w\min\{p_1,p_2\}}{\min\{p_1',p_2'\}} - w = w\frac{\min\{p_1,p_2\} - \min\{p_1',p_2'\}}{\min\{p_1',p_2'\}}$$

b. **Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity**.

Assume further that the first good is a normal good (**the demand is increasing with wealth**). **What is the relation of the two measures to the** "**area below the demand function**" (**which is a standard third definition of consumer surplus**)?

Let $b = p_1'' < p_1' = a$ and let A denote the area under the demand curve for commodity 1 between a and b.

Let $u' = v((a,p_2),w)$ and $u'' = v((b,p_2),w)$.

Let $h_1(p,u)$ denote the hicksian demand $h_1(p,z)$ when $u(z) = u$.
If commodity 1 is a normal good, then $h_1(p,u)$ is increasing in u, and thus

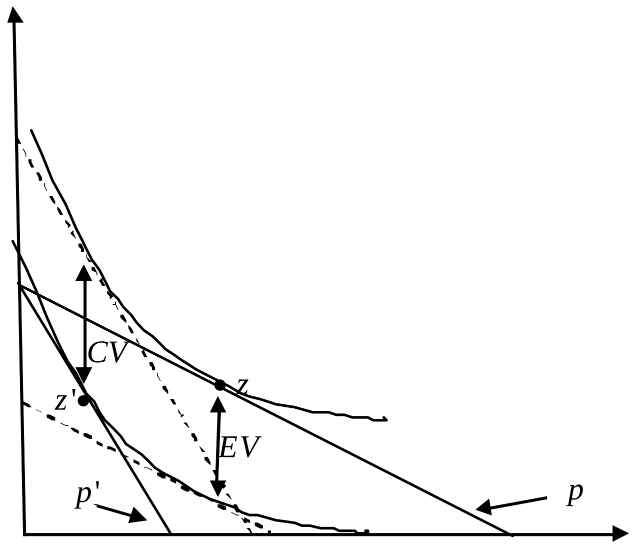
$$h_1((t,p_2),u') \leq h_1((t,p_2),v((t,p_2),w)) \quad \leq h_1((t,p_2),u'') \quad \text{for } t \in [a,b].$$

Recall that by the Dual Roy's identity, $\frac{\partial e((t,p_2),u)}{\partial p_1} = h_1((t,p_2),u)$ By integrating,

$$\begin{aligned} CV(p',p'',w) &= \int_a^b \frac{\partial e((t,p_2),u')}{\partial p_1}dt = \int_a^b h_1((t,p_2),u')dt \\ &\leq \int_a^b h_1((t,p_2),v((t,p_2),w))dt = \int_a^b x_1((t,p_2),w))dt = A \\ &\leq \int_a^b h_1((t,p_2),u'')dt = \int_a^b \frac{\partial e((t,p_2),u'')}{\partial p_1}dt = EV(p',p'',w). \end{aligned}$$

c. **Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities**.

Recall that by Question 4 of PS5, the consumer's demand adjusts to a change in wealth by adjusting consumption of the quasi-linear good (in this case commodity 2) and the consumption of commodity 1 is constant. Consider the following diagram where CV and EV are drawn. Since the indifference curves are parallel horizontally, the two measures are identical.



Problem 4.

a. **Verify that you know the envelope theorem**, **which states conditions under which the following is correct**:

Consider a maximization problem $\max_x\{u(x,\alpha_1,,..,\alpha_n) \mid g(x,\alpha_1,,..,\alpha_n) = 0\}$. **Let** $V(\alpha_1,,..,\alpha_n)$ **be the value of the maximization**. **Then** $\frac{\partial V}{\partial \alpha_i}(a_1,,..,a_n) = \frac{\partial(u-\lambda g)}{\partial \alpha_i}(x^*(a_1,,..,\alpha_n),a_1,,..,\alpha_n)$ **where** $x^*(a_1,,..,\alpha_n)$ **is the solution to the maximization problem and** λ **is the Lagrange multiplier associated with the solution of the maximization problem**.

b. **Derive the Roy's identity from the envelope theorem** (**hint**: **show that in this context** $\frac{\partial V/\partial \alpha_i}{\partial V/\partial \alpha_j}(a_1,,..,a_n) = \frac{\partial g/\partial \alpha_i}{\partial g/\partial \alpha_j}(x^*(a_1,,..,\alpha_n),a_1,,..,\alpha_n)$).

In the context of the consumer's problem $(a_1,,..,\alpha_n)$ are $(p_1,\ldots,p_k,w)$, $u(x,p_1,\ldots,p_k,w) = u(x)$ and $g(x,p_1,\ldots,p_k,w) = w - px$. Note that the prices and wealth do not effect the function u. The utility depends on the prices and wealth only through their effect on the bundle x. Denote $p = (p_1,,..,p_k)$.

The envelope theorem states that:

$$\partial V/\partial p_i(p,w) = \partial u/\partial p_i(x^*(p,w),p,w) - \lambda\partial g/\partial p_i(x^*(p,w),p,w)$$

$$\partial V/\partial w(p,w) = \partial u/\partial w(x^*(p,w),p,w) - \lambda\partial g/\partial w(x^*(p,w),p,w)$$

Becasue $\partial u/\partial p_i = \partial u/\partial w = 0$, we obtain

$$\partial V/\partial p_i(p,w) = -\lambda\partial g/\partial p_i(x^*(p,w),p,w) \text{ and}$$

$$\partial V/\partial w(p,w) = -\lambda\partial g/\partial w(x^*(p,w),p,w).$$

By taking ratios and canceling out λ:

$$\frac{\partial V/\partial p_i}{\partial V/\partial w}(p,w) = \frac{\partial g/\partial p_i}{\partial g/\partial w}(x^*(p,w),p,w).$$

Recall that $g(x,p,w) = w - px$, and thus $\partial g/\partial p_i(x^*(p,w),p,w) = -x_i^*$ and $\partial g/\partial w(x^*(p,w),p,w) = 1$. Plugging this we get $\frac{-\partial v/\partial p_i}{\partial v/\partial w}(p,w) = x_i^*$ which is exactly Roy's identity.

c. **What makes it is easy to prove Roy's identity without using the envelope theorem**?

The fact that the utility does not depend directly on the prices (only through the bundle x).

Problem 5.

Assume that technology Z and the production function f describe the same producer who produces commodity K using inputs $1, \ldots, K-1$. Show that Z is a convex set if and only if f is a concave function.

Z is convex

iff $(-v, y), (-v', y') \in Z$, $\lambda \in [0,1]$ implies that $(-\lambda v - (1-\lambda)v', \lambda y + (1-\lambda)y') \in Z$

iff $y \leq f(v)$, $y' \leq f(v')$, $\lambda \in [0,1]$ implies that $\lambda y + (1-\lambda)y' \leq f(\lambda v + (1-\lambda)v')$

iff $\lambda f(v) + (1-\lambda)f(v') \leq f(\lambda v + (1-\lambda)v')$

iff f is concave.

Problem 6.

Consider a producer who uses L inputs to produce $K-L$ outputs. Denote by w the price vector of the L inputs. Let $a_k(w,y)$ be the demand for the k'th input when the price vector is w and the output vector he wishes to produce is y. Show the following:

a. $C(\lambda w,y) = \lambda C(w,y)$.

$C(\lambda w,y) = \min_{\{a\,|\,(-a,y)\in Z\}} \lambda wa = \lambda \min_{\{a\,|\,(-a,y)\in Z\}} wa = \lambda C(w,y)$.

b. C **is nondecreasing in any input price** w_k.

Assume $w_l' \geq w_l$ for all l.

$C(w',y) = w'a(w',y) \geq wa(w',y) \geq wa(w,y) = C(w,y)$.

c. C **is concave in** w.

Let w, w' be input prices, $w'' = \lambda w + (1-\lambda)w'$ for $\lambda \in [0,1]$. Then

$$\begin{aligned} C(w'',y) &= [\lambda w + (1-\lambda)w']a(w'',y) = \lambda wa(w'',y) + (1-\lambda)w'a(w'',y) \\ &\geq \lambda C(w,y) + (1-\lambda)C(w',y). \end{aligned}$$

d. **Shepherd's lemma**: **If** C **is differentiable**, $\frac{\partial C}{\partial w_k}(w^*,y) = a_k(w^*,y)$ **(the** k**th input commodity)**.

Fix y. C is now a function of w. For every w, $C(w,y) \leq wa(w^*,y)$. $C(w^*,y) = w^*a(w^*,y)$. Thus $\{(w,c) \mid c = wa(w^*,y)\}$ is tangent to the graph of the function $C(w,y)$ at $(w^*, C(w^*,y))$. Since C is differentiable $\nabla C(w^*,y) = a(w^*,y)$.

e. **If** C **is twice continuously differentiable**, **then for any two commodities** j **and** k, $\partial a_j/\partial w_k(w,y) = \partial a_k/\partial w_j(w,y)$.

By Shapherd's Lemma and Young's Theorem (mixed partial derivatives are equal):

$$\frac{\partial a_j(w,y)}{\partial w_k} = \frac{\partial^2 C(w,y)}{\partial w_k \partial w_j} = \frac{\partial^2 C(w,y)}{\partial w_j \partial w_k} = \frac{\partial a_k(w,y)}{\partial w_j}.$$

Problem 7.

Consider a firm producing one commodity using L inputs, which maximizes production subject to the constraint of achieving a level of profit ρ (and does not produce at all if he cannot). Show that under reasonable assumptions:

a. **The firm's problem has a unique solution for every price vector**.

Denote the production function by $y = f(a)$. For a given a vector of inputs $a = (a_1, \dots a_L)$, and price vector $p = (p_{a_1}, \dots, p_{a_L}, p_y)$, the profit function is $\pi(a,p) = p_y y - p_a a$.

Let $D(p) = \{a \in \Re^L_+ \mid p_y f(a) - p_a a \geq \rho\}$. The firm solves $\max_{a \in D(p)} f(a)$. Let $a(p)$ be the firm's input demand and $y(p) = f(a(p))$ be the firm's optimal output (if $D(p) = \phi$ then $a(p) = y(p) = 0$).

If the production technology is strictly convex ($f(a)$ strictly concave, decreasing returns to scale), bounded from above, and all prices are strictly positive then we have a unique solution to our problem.

Assume for contradiction that there are two solutions, a and a'. It must be that $f(a) = f(a')$ and $\pi(a,p) = \pi(a',p) = \rho$, otherwise we would be able to increase production. Now look at a convex combination of these two points. For $\lambda \in (0,1)$, due to strict concavity of $f(a)$, $f(\lambda a + (1-\lambda)a') > f(a) = f(a')$, and therefore $\pi(\lambda a + (1-\lambda)a', p) > \rho$ (since $p_a a = p_a a' = p_a(\lambda a + (1-\lambda)a')$ and $p_y f(a) = p_y f(a') < p_y f((\lambda a + (1-\lambda)a'))$). But this means $\lambda a + (1-\lambda)a'$ is better than a and a'.

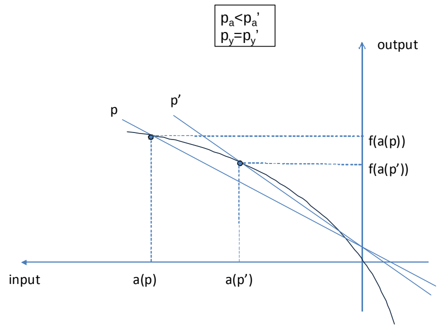
b. **The firm's supply function satisfies monotonicity in prices**.

Consider increasing the output price: let p and p' be price vectors s.t. $p'_y > p_y$ and $p'_a = p_a$. For all a if $p_y f(a) - p_a a \geq \rho$ then $p'_y f(a) - p'_a a > \rho$ and for $\epsilon > 0$ small enough also $p'_y f(a + (\epsilon, \dots, \epsilon)) - p'_a(a + (\epsilon, \dots, \epsilon)) > \rho$.

Therefore, $f(a(p', \rho)) \geq f(a(p, \rho) + (\varepsilon, \dots, \varepsilon)) > f(a(p, \rho))$ (assuming that *f* is increasing).

Now consider increasing the input price: let p and p' be price vectors s.t. $p'_y = p_y$ and $p'_a > p_a$. Similarly, For all a if $p'_y f(a) - p'_a a \geq \rho$ then $p_y f(a) - p_a a > \rho$ and for $\epsilon > 0$ small enough also $p_y f(a + (\epsilon, \dots, \epsilon)) - p_a(a + (\epsilon, \dots, \epsilon)) > \rho$.

Therefore, $f(a(p, \rho)) \geq f(a(p', \rho) + (\varepsilon, \dots, \varepsilon)) > f(a(p', \rho))$ (assuming that *f* is increasing).



c. The firm's supply function satisfies continuity in prices when $\rho = 0$:

Assume that f is continuous and strictly concave. We will show that $a(p)$ and $y(p)$ are continuous:

Let $\{p^n\}$ converge to p. Define $\overline{p_y} = \sup\{p_y^n\}$ and $\underline{p_a} = \inf\{a_L^n\}$, and thus $D(p^n) \subseteq D(\underline{p_a}, \ldots, \underline{p_a}, \overline{p_y})$ for all n. Then

$$a(p^n) \in \{a \in \Re_+^L \mid \overline{p_y} f(a(\underline{p_a}, \ldots, \underline{p_a}, \overline{p_y})) - (\underline{p_a}, \ldots, \underline{p_a}) \cdot a \geq \rho\}$$

since $\overline{p_y} f(a(\underline{p_a}, \ldots, \underline{p_a}, \overline{p_y})) \geq p_y^n f(a^n)$ and $(\underline{p_a}, \ldots, \underline{p_a}) \leq p_a^n$ for all n. If $\{a(p^n)\}$ does not converge to $a(p)$ then there is a subsequence that converges to some $a^* \neq a(p)$. Since f is continuous, $p_y f(a^*) - p_a a^* \geq \rho$. Thus, by strict concavity, $f(a^*) < f(a(p))$.

Now (i) $f((a^* + a(p))/2) > f(a^*)$ and (ii) $p_y f(a^*) \geq p_a a^* + \rho$ and $p_y f(a(p)) \geq p_a a(p) + \rho$ and by the strict concavity of f,

$p_y f((a^* + a(p))/2) > (p_y f(a^*) + p_y f(a(p)))/2 \geq p_a((a^* + a(p)/2) + \rho$.

Therefore we found a' such that $f(a') > f(a^*)$ and $p_y f(a') - p_a a' > \rho$. Therefore, for n large enough, $p_y^n f(a') - p_a^n a' \geq \rho$ and $f(a') > f(a^n)$, a contradiction.

d. The firm's supply function is monotonic in ρ.

Intuitively, as ρ increases, keeping the prices the same, we shift the price line with same slope up. Due to convexity of the production technology this will lead to less production.

Formally, assume $\rho' > \rho$. For all a if $p_y f(a) - p_a a \geq \rho'$ then $p_y f(a) - p_a a > \rho$ and for $\epsilon > 0$ small enough also $p_y f(a + (\epsilon, \ldots, \epsilon)) - p_a(a + (\epsilon, \ldots, \epsilon)) > \rho$. Therefore, $f(a(p, \rho)) \geq f(a(p, \rho') + (\varepsilon, \ldots, \varepsilon)) > f(a(p, \rho'))$.

Problem 8.

It is usually assumed that the cost function C is convex in the output vector. Much of the research on production has been aimed at investigating conditions under which convexity is induced from more primitive assumptions about the production process. Convexity often fails when the product is related to the gathering of information or data processing.

Consider, for example, a firm conducting a telephone survey immediately following a TV program. Its goal is to collect information about as many viewers as possible within 4 units of time. The wage paid to each worker is w (even when he is idle). In one unit of time, a worker can talk to one respondent or be involved in the transfer of information to or from exactly one colleague. At the end of the 4 units of time, the collected information must be in the hands of one colleague (who will announce the results).

Define the firm's product, calculate the cost function and examine its convexity.

The firm's product is units of information.

Denote the agents by 1,...,n. Let $i \to j$ stands for i transfers information to j and let $\to j$ stands for j collects information from a viewer. Denote a procedure by a sequence of square brackets, each stands for one period and contains the transfers of information during that period. Agent 1 will be the agent who announces the result.

$n = 1$. Since there is only one agent, he can use the four units of time only for collecting information from four viewers. Thus, the maximum number of responses is 4. An optimal procedure:

$[\to 1][\to 1][\to 1][\to 1]$

2. The last unit of time has to be used to transfer information from some agent, 2, to agent 1. Therefore, both agents can collect information only for the first three units of time. Thus, the maximum number of responses is 6.

An optimal procedure:

$[\to 1, \to 2][\to 1, \to 2][\to 1, \to 2][2 \to 1]$

3. Again, the last unit of time has to be used to transfer information from one agent, let us say 2, to agent 1. However, one of these two agents has to get the information from agent 3 one period earlier. Thus, there are two agents who are free to collect information for two periods and one agent who is free for three periods. The total number of collected responses is thus, bounded by 7.

An optimal procedure:

$[\to 1, \to 2, \to 3][\to 1, \to 2, \to 3][\to 1, 3 \to 2][2 \to 1]$

4. If the firm employs 4 or more agents it can collect at most 8 responses.

Again, the last unit of time has to be used to transfer information from one agent, let us say 2, to 1. It is sufficient to show that each of them cannot hold more than 4 units of information after three periods. To see it note that after two periods each agent can hold not more than 2 units and thus, after three periods he will have 3 units if he makes a call himself, or 4 units if he gets the information collected earlier by another agent.

An optimal procedure:

$[\to 1, \to 2, \to 3, \to 4][\to 1, \to 2, \to 3, \to 4][3 \to 1, 4 \to 2][2 \to 1]$

Let y be the "output", the number of responses the center collected and $C(y,w)$ be the minimum cost of producing y. Then,

y	0	$1,2,3,4$	$5,6$	7	8
$C(y,w)$	0	w	$2w$	$3w$	$4w$

Obviously, C is not convex: $C(2,w) = w > w/2 = .5C(0,w) + .5C(4,w)$.

Problem 9.

An event that could have occurred with probability 0.5 either did or did not occur. A firm must provide a report in the form of "the event occurred" or "the event did not occur". The quality of the report (the firm's product), denoted by q, is the probability that the report is correct. Each of k experts (input) prepares an independent recommendation which is correct with probability $1 > p > 0.5$. The firm bases its report on the k recommendations in order to maximize q.

a. Calculate the production function $q = f(k)$ for (at least) $k = 1, 2, 3, \ldots$

Experts k	$f(k)$
0	0.5
1	p
2	$p^2 + p(1-p) = p$
3	$p^3 + 3p^2(1-p) > p$

b. We say that a "discrete" production function is concave if the sequence of marginal product is nonincreasing. Is the firm's production function concave?

No. Marginal product is positive from 0 to 1, zero from 1 to 2 and then positive from 2 to 3.

Assume that the firm will get a prize of M if its report is actually correct. Assume that the wage of each worker is w.

c. Explain why it is true that if f is concave, the firm chooses k^* so that the k^*th worker is the last one for whom marginal revenue exceeds the cost of a single worker.

The firm's profits if it employs k workers are: $Mf(k) - kw$. If f is concave, then for any worker $k < k^*$, the firm's marginal profit $M[f(k+1) - f(k)] - w$ is positive, whereas for any worker $k > k^*$, the firm's marginal profit is negative.

d. Is this conclusion true in our case?

No. Since the marginal revenue of the second expert is 0, while it is possible that it is optimal for the firm to hire 3 experts.

Problem 10.

An economic agent is both a producer and a consumer. **He has** a_0 **units of good** 1. **He can use some of** a_0 **to produce commodity** 2. **His production function** f **satisfies monotonicity**, **continuity**, **strict concavity**. **His preferences satisfy monotonicity**, **continuity and convexity**. **Given he uses** a **units of commodity** 1 **in production he is able to consume the bundle** $(a_0 - a, f(a))$ **for** $a \leq a_0$. **The agent has in his** "**mind**" **three** "**centers**":

***The pricing center declares a price vector** (p_1, p_2).

***The production center takes the price vector as given and operates according to one of the following two rules**

Rule 1: **maximizing profits**, $p_2 f(a) - p_1 a$.

Rule 2: **maximizing production subject to the constraint of not making any losses**,, **i**.**e**. $p_2 f(a) - p_1 a \geq 0$.

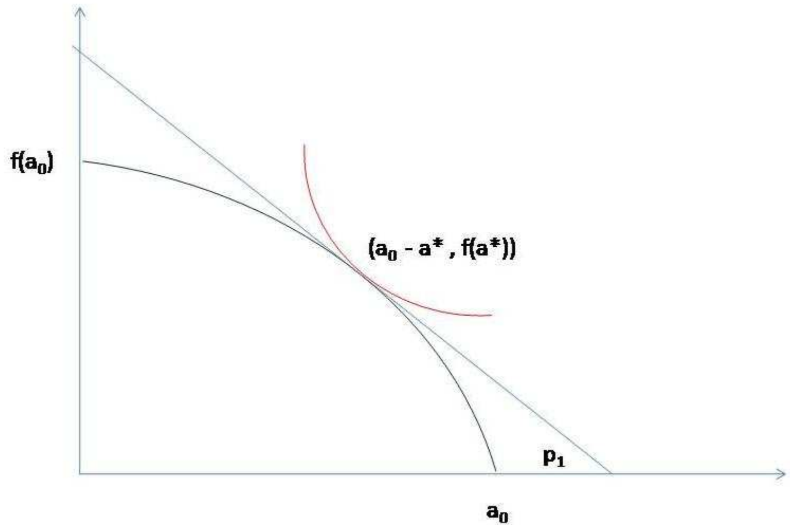
The output of the production center is a consumption bundle.

The consumption center takes $(a_0 - a, f(a))$ **as endowment**, **and finds the optimal consumption allocation that it can afford according to the prices declared by the pricing center**.

The prices declared by the pricing center are chosen to create harmony between the other two centers in the sense that the consumption center finds the outcome of the production center's activity, $(a_0 - a, f(a))$, **optimal given the announced prices**.

a. **Show that under Rule 1**, **the economic agent consumes the bundle** $(a_0 - a^*, f(a^*))$ **which maximizes his preferences**.

The solution corresponds to the point on the production possibility set where preferences are maximized.



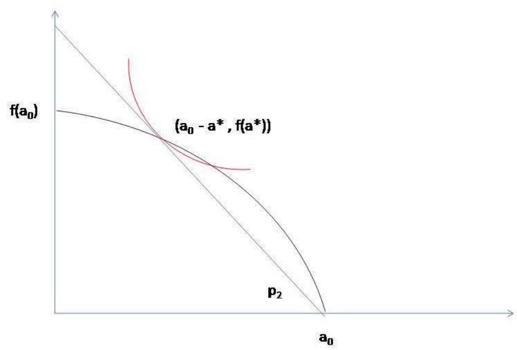
Since the production possibility set is strictly convex, and preferences are convex we know that there is a unique maximum.

Now choose a price vector such that the price line is tangent to this set and the indifference line exactly at the maximum.

By construction, profit is maximized given prices, and preferences are maximized at the intersection point for given prices and endowment point $(a_0 - a^*, f(a^*))$.

b. **What is the economic agent's consumption with Rule 2**?

The economic agent chooses $(a_0 - a^*, f(a^*))$ with maximal a^* subject to the constraint that preferences are maximized at this point when we take the line connecting this point to $(a_0, 0)$ as the price line. By construction, production is maximized here subject to the constraint that there are no losses with the given prices. Also the point is chosen to guarantee that the consumer preferences are maximized at the budget set with the same prices.

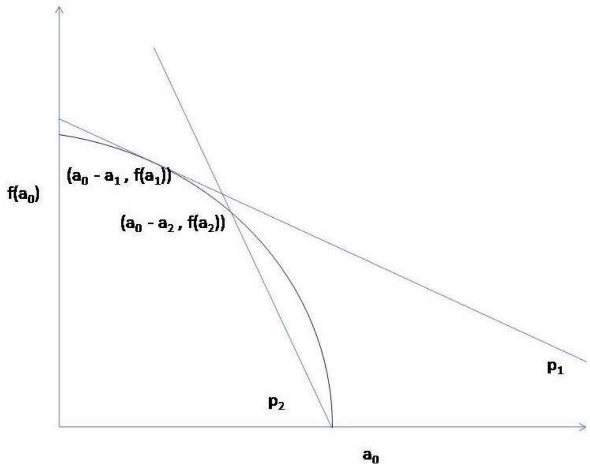


c. **State and prove a general conclusion about the comparison between the**

behavior of two individuals, **one whose production center operates with Rule 1 and one whose production center activates Rule 2**.

Claim: Individual using Rule 2 will always produce more, i.e. for a_1, p^1 and a_2, p^2 denoting the solutions under Rule 1 and Rule 2, $f(a_1) \leq f(a_2)$.

Assume for contradiction that $a_1 > a_2$. This means that the solution with Rule 2 is strictly to the right of the solution with Rule 1. Since f is strictly concave and monotonic, if solution with Rule 2 is to the right of solution with Rule 1, we must have $\frac{p_1^1}{p_2^1} < \frac{p_1^2}{p_2^2}$.



Note that $(a_0 - a_1, f(a_1))$ affordable (and strictly interior) in the budget set defined by p^2. Also $(a_0 - a_2, f(a_2))$ is affordable (and strictly interior) in the budget set defined by p^1.

$\Rightarrow (a_0 - a_2, f(a_2)) \succ (a_0 - a_1, f(a_1))$

$\Rightarrow (a_0 - a_1, f(a_1)) \succ (a_0 - a_2, f(a_2))$

which is a contradiction.

Problem Set 7 - Expected Utility

Problem 1.

Consider the following preference relations that were described in the text: “**the size of the support**” **and** “**comparing the most likely prize**".

a. **Check carefully whether they satisfy axioms I and C**.

Both preference relations violate both axioms:

the size of the support

Let p_t be the lottery: $p_t(z_1) = t$ and $p_t(z_2) = 1 - t$.

not I: $[z_1] \succ p_{1/2}$, but for $1 > \alpha > 0$, $\alpha[z_1] \oplus (1-\alpha)p_{1/2} \sim p_{1/2}$.

not C: For any $1/n > 0$, $p_{1/n} \sim p_{1/2}$, but in the limit $[z_1] \succ p_{1/2}$.

comparing the most likely prize

Assume that z_1 is better than z_2 and that "ties are broken in favor" of z_1:

not I: $[z_1] \succ [z_2]$, but $p_{1/4} \sim [z_2]$.

not C: $p_{1/2-1/n} \sim [z_2]$ for all n but in the limit $p_{1/2} \succ [z_2]$.

b. **These preference relations are not immune to a certain** "**framing problem**". **Explain**.

Both preference relations strictly prefer the lottery \$50 with probability 0.4 and \$100 with probability 0.6 to the lottery \$50 with probability 0.4, one blue note of \$100 with probability 0.3 and one green note of \$100 with probability 0.3, even though the lotteries seem to be the "same".

Problem 2.

One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery L **on the basis of two numbers**: $Ex(L)$, **the expectation of** L, **and** $var(L)$, L**'s variance**. **Such a construction may or may not be consistent with vNM assumptions**.

a. **Show that** $u(L) = Ex(L) - (1/4)var(L)$ **induces a preference relation that is not consistent with the vNM assumptions**. (**For example**, **consider the mixtures of each of the lotteries** $[1]$ **and** $0.5[0] \oplus 0.5[4]$ **with the lottery** $0.5[0] \oplus 0.5[2]$.)

$[1] \sim 0.5[0] \oplus 0.5[4]$ since $u([1]) = u(0.5[0] \oplus 0.5[4]) = 1$.

However, for $\alpha = 1/2$:

$\alpha[1] + (1-\alpha)[0.5[0] \oplus 0.5[2]] \succ \alpha[0.5[0] \oplus 0.5[4]] + (1-\alpha)[0.5[0] \oplus 0.5[2]]$

since the utility of the left lottery, $7/8$, is greater than the utility of the right lottery, $13/16$.

b. **Show that** $u(L) = Ex(L) - (Ex(L))^2 - var(L)$ **is consistent with vNM assumptions**.

Using the formula $var(X) = Ex(X^2) - (ExX)^2$ we get $u(L) = Ex(L) - (Ex(L))^2 - var(L) = Ex(L) - (Ex(L))^2 - (\sum_{z \in Z} L(z)z^2 - (Ex(L))^2) = \sum_{z \in Z} L(z)(z - z^2)$

is an expected utility function with vNM value $v(z) = z - z^2$.

Problem 3.

A decision maker has a preference relation $\succsim$ **over the space of lotteries** $L(Z)$ **having a set of prizes** Z. **On Sunday he learns that on Monday he will be told whether he has to choose between** L_1 **and** L_2 (**probability** $1 > \alpha > 0$) **or between** L_3 **and** L_4 (**probability** $1-\alpha$). **He will make his choice at that time**. **Let us compare between two possible approaches the decision maker can take**.

Approach 1: **He delays his decision to Monday** ("**why bother with the decision now when I can make up my mind tomorrow**.").

Approach 2: **He makes a contingent decision on Sunday regarding what he will do on Monday**, **that is**, **he decides what to do if he faces the choice between** L_1 **and** L_2 **and what to do if he faces the choice between** L_3 **and** L_4 ("**On Monday morning I will be so busy**. . .").

a. **Formulate Approach 2 as a choice between lotteries**.

The DM chooses one of the four "compound" lotteries in the set

$$\{\alpha L_i \oplus (1-\alpha)L_j \mid i \in \{1,2\}, j \in \{3,4\}\}.$$

b. **Show that if the preferences of the decision maker satisfy the independence axiom**, **then his choice under Approach 2 will always be the same as under Approach 1**.

Let L_i (L_j) be the preferred lottery in $\{L_1,L_2\}$ ($\{L_3,L_4\}$), and L_{-i} (L_{-j}) be the other lottery. Under approach 1, the DM selects L_i (L_j) if the choice set on Monday is $\{L_1,L_2\}$ ($\{L_3,L_4\}$). Let $\succsim$ be the DM's preferences over the compound lotteries in (a). By I,

$\alpha L_i \oplus (1-\alpha)L_j \succ \alpha L_i \oplus (1-\alpha)L_{-j} \succ \alpha L_{-i} \oplus (1-\alpha)L_{-j}$ and

$\alpha L_i \oplus (1-\alpha)L_j \succ \alpha L_{-i} \oplus (1-\alpha)L_j$. Thus $\alpha L_i \oplus (1-\alpha)L_j$ is the best of the "compound" lotteries.

Problem 4.

A decision maker is to choose an action from a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α and any $z \in Z \setminus \{z^*\}$ is realized with probability $r(a,z) = (1-\alpha)q(a,z)$.

a. **Assume that after making his choice he is told that z^* will not occur and is given a chance to change his decision**. **Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms**, **he will not change his decision**.

By the vNM Theorem, preferences exhibit expected utility representation. Before learning the information, the DM solves

$$\max_{a \in A}\left[\sum_{z \in Z \setminus \{z^*\}} r(a,z)v(z) + \alpha v(z^*)\right].$$

After learning that z^* will not occur, the DM updates his beliefs so that $r'(a,z) = r(a,z)/(1-\alpha) = q(a,z)$ for $z \in Z \setminus \{z^*\}$ and the DM solves $\max_{a \in A} \sum_{z \in Z \setminus \{z^*\}} r'(a,z)v(z)$, which yields the same solution.

b. **Give an example where a decision maker who follows nonexpected utility preference relation or obyes a non-Bayesean updating rule is not time consistent**.

Example 1. Assume the DM has a “worst case” preference relation, where z_1 is the best prize, z_2 is the second best and z^* is the worst. Let action a_1 yield z_1 for sure and action a_2 yield z_1 and z_2 with equal probability, conditional on z^* not occurring. Then the DM will initially be indifferent between a_1 and a_2, but will strictly prefer a_1 after the information is revealed.

Example 2. Assume that $Z = \{1,2,3,z^* = 0\}$ and that $v(z) = z$. Assume that initially his beliefs are: $q(a_1,2) = 1$, $q(a_2,3) = 0.4$ and $q(a_2,1) = 0.6$. Contingentally the DM chooses a_1. If he updates his beliefs and after he was lucky to avoid z^* he believes that he will be fortunate again, that is $q'(a_2,3) = 1$, then he will change his mind and choose a_2.

Problem 5. Assume there is a finite number of income levels. An income distribution specifies the proportion of individuals at each level. Thus, an income distribution has the same methematical structure as a lottery. Consider the binary relation "one distribution is more egalitarian than another".

a. Why is the von Neumann-Morgenstern independence axiom inappropriate for characterizing this type of relation?

Assigning all members of the society the income 1 is as egalaterian as assigning all of them the income 2 and under the independence axiom, $0.5[1] \oplus 0.5[2]$ should be as egalaterian as $[1]$, but our intuition is that $0.5[1] \oplus 0.5[2]$ is less egalitarian than assigning equal income to all members of the society.

b. Suggest and formulate a property that is appropriate, in your opinion, as an axiom for this relation. Give two examples of preference relations that satisfy this property.

If p and q are identical distributions, except that the highest (lowest) income level in p is less (more) than in q, then p is more egaletrian than q.

Example 1: $p \succsim q$ if $Var(p) \leq Var(q)$.

Example 2: $p \succsim q$ if $\max_{z \in supp\ p(z)} z - \min_{z \in supp\ p(z)} z \leq \max_{z \in supp\ q(z)} z - \min_{z \in supp\ q(z)} z$.

Problem 6.

A decision maker faces a trade-off between longevity and quality of life. His preference relation ranks lotteries on the set of all certain outcomes of the form (q,t) defined as "a life of quality q and length t" (where q and t are nonnegative numbers). Assume that the preference relation satisfies von Neumann-Morgenstern assumptions and that it also satisfies the following:

(**i**) **There is indifference between any two certain lotteries $[(q,0)]$ and $[(q',0)]$.**

(**ii**) **Risk neutrality with respect to life duration: an uncertain lifetime of expected duration T is equally preferred to a certain lifetime duration T when q is held fixed.**

(**iii**) **Whatever quality of life, the longer the life the better.**

a. **Show that the preference relation derived from maximizing the expectation of the function $v(q)t$, where $v(q) > 0$ for all q, satisfies the assumptions.**

(i) If $t = 0$, then $v(q)t = 0$ for all q.

(ii) Let p be a lottery over t with expectation T. Then $\sum_t p(t)v(q)t = v(q)\sum_t tp(t) = v(q)T$.

(iii) $v(q)t' > v(q)t$ for all $t' > t$.

b. **Show that all preference relations satisfying the above assumptions can be represented by an expected utility function of the form $v(q)t$, where v is a positive function.**

Since $\succsim$ satisfies the v-NM axioms, then $\succsim$ is represented by an expected utility function with values $w(q,t)$.

By the second property, $w(q,t)$ is a affine tranformation of t, that is $w(q,t) = v(q)t + b(q)$.

By propery (i) it must be that $b(q) = b$ as otherwise for some q and q' we would have $w(q,0) \neq w(q',0)$.

By (iii) $v(q) > 0$ for all q.

Problem 7. **Consider a decision maker who systematically calculates that** $2 + 3 = 6$. **Construct a** ”**money pump**“ **argument against him**. **Discuss the argument**.

Tell the DM: “If you pay me \$5.99, I will give you two checks, one for \$2 and another for \$3.” The DM will take the offer since he thinks he profits \$0.01. Then buy from him the checks for \$2.01 and \$3.01 and so on...

Problem Set 8 – Risk Aversion

Problem 1.

a. **Show that a sequence of numbers** $(a_1,..,a_K)$ **satisfies that** $\sum a_k x_k \geq 0$ **for all vectors** $(x_1,..,x_K)$ **such that** $x_k > 0$ **for all** k **iff** $a_k \geq 0$ **for all** k.

$\Rightarrow$If $a_{k^*} < 0$ then take $x_{k^*} = 1$ and $x_k = \varepsilon > 0$. For ε small enough $\sum a_k x_k < 0$.

$\Leftarrow$If $a_k \geq 0$ for all k then $\sum a_k x_k \geq 0$ for all vectors $(x_1,..,x_K)$ such that $x_k \geq 0$.

b. **Show that a sequence of numbers** $(a_1,..,a_K)$ **satisfies that** $\sum a_k x_k \geq 0$ **for all vectors** $(x_1,..,x_K)$ **such that** $x_1 > x_2 > \ldots x_K > x_{K+1} = 0$ **iff** $\sum_{k=1}^{l} a_k \geq 0$ **for all** l.

It follows, like in part (a), from the equality:

$\sum a_k x_k = \sum_{k=1}^{K} a_k \sum_{l=k}^{K} (x_l - x_{l+1}) = \sum_{l=1}^{K} (x_l - x_{l+1}) \sum_{k=1}^{l} a_k$.

Problem 2.

We say that p second-order stochastically dominates q and denote this by pD_2q if $p \succsim q$ for all preferences $\succsim$ satisfying the vNM assumptions, monotonicity and risk aversion.

a. **Explain why pD_1q implies pD_2q.**

If pD_1q, then $p \succsim q$ for all preferences satisfying the vNM assumptions and monotonicity. Thus $p \succsim q$ for all preferences satisfying the vNM assumptions, monotonicity *and* risk aversion.

b. **Let p and ϵ be lotteries. Define $p+\epsilon$ to be the lottery that yields the prize t with the probability $\sum_{\alpha+\beta=t} p(\alpha)\epsilon(\beta)$. Interpret $p+\epsilon$. Show that if ϵ is a lottery with expectation 0, then for all p, $pD_2(p+\epsilon)$.**

$p+\epsilon$ is the combination of two independent lotteries p and ϵ. Let the agent satisfy vNM assumptions, monotonicity and risk aversion. Then

$$\begin{aligned} U(p+\epsilon) &= \sum_{\alpha\in Z} p(\alpha) \sum_{\beta\in Z} \epsilon(\beta)u(\alpha+\beta) \\ &\leq \sum_{\alpha\in Z} p(\alpha)u\Big(\sum_{\beta\in Z} \epsilon(\beta)(\alpha+\beta)\Big) && \text{by } u \text{ concave} \\ &= \sum_{\alpha\in Z} p(\alpha)u(\alpha) && \text{by } Ex(\epsilon) = 0 \\ &= U(p). \end{aligned}$$

c. **Show that pD_2q iff for all $t < K$, $\sum_{k=0}^{t}[G(p,x_{k+1}) - G(q,x_{k+1})][x_{k+1} - x_k] \geq 0$, where $x_0 < \ldots < x_K$ are all the prizes in the support of either p or q and $G(p,x) = \sum_{z\geq x} p(z)$.**

Let $\succsim$ satisfy the vNM axioms, monotonicity and risk aversion. Then $\succsim$ is represented by $U(p) = \sum_{k=0}^{K} u(x_k)p(x_k)$, with u increasing and concave. Define

$$\alpha_k = \begin{cases} \frac{u(x_{k+1})-u(x_k)}{x_{k+1}-x_k} & \text{if } k < K \\ 0 & \text{if } k = K \end{cases}$$

By u increasing and concave, $\alpha_k \geq \alpha_{k+1}$ for all k. Then

$$
\begin{aligned}
U(p) - U(q) &= \sum_{k=0}^{K} (p(x_k) - q(x_k))u(x_k) \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][u(x_{k+1}) - u(x_k)] && \text{by algebra} \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k]\alpha_k && \text{by def. of } \alpha_k \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] \sum_{t=k}^{K-1} (\alpha_t - \alpha_{t+1}) && \text{telescopic sum and } \alpha_K = 0 \\
&= \sum_{t=0}^{K-1} (\alpha_t - \alpha_{t+1}) \sum_{k=0}^{t} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] && \text{by algebra} \\
&\geq 0 \qquad \text{by } \alpha_t \geq \alpha_{t+1} \text{ and } \sum_{k=0}^{t} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] \geq 0.
\end{aligned}
$$

By contradiction, assume $\sum_{k=0}^{T} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] < 0$ for some $T < K$. Let $\alpha \in (0, 1)$ and define

$$
u(x) = \begin{cases} x & \text{if } x \leq x_{T+1} \\ x_{T+1} + \alpha(x - x_{T+1}) & \text{if } x > x_{T+1}. \end{cases}
$$

By the above $U(p) < U(q)$ for α small enough.

Problem 3.

Consider a phenomenon called preference reversal. **Let** $L_1 = 8/9[4] \oplus 1/9[0]$ **and** $L_2 = 1/9[40] \oplus 8/9[0]$.

Discuss the phenomenon that many people prefer L_1 **to** L_2 **but when asked to evaluate the certainty equivalence of these lotteries they attach a lower value to** L_1 **than to** L_2.

People often prefer L_1, but they are not willing to pay $4 to play. Nevertheless, some people are willing to pay $4 to play L_2. It seems that people tend to over estimate small probabilities when they evaluate a lottery. See http://www.encyclopedia.com/doc/1O87-preferencereversal.html.

Problem 4.

Consider a consumer's preference relation over K-tuples describing quantities of K uncertain assets. Denote the random return on the kth asset by Z_k. Assume that the random variables $(Z_1,\ldots,Z_K)$ are independent and take positive values with probability 1. If the consumer buys the combination of assets $(x_1,\ldots,x_K)$ and if the vector of realized returns is $(z_1,\ldots,z_K)$, then the consumer's total wealth is $\sum_k x_k z_k$. Assume that the consumer satisfies vNM assumptions, that is, there is a function v (over the sum of his returns) so that he maximizes the expected value of v. Assume that v is increasing and concave. The consumer preferences over the space of the lotteries induce preferences on the space of investments. Show that the induced preferences are monotonic and convex.

Monotonic: Let $x \geq x'$. Whenever the random variable $\sum_k x_k Z_k$ gets a certain value the random variable $\sum_k x'_k Z_k$ an higher value and thus $Ev(\sum_k x_k Z_k) \geq Ev(\sum_k x'_k Z_k)$.

Convex: Let x, x' be two investment combinations, $\lambda \in [0,1]$ and $x'' = \lambda x + (1-\lambda)x'$. By the concavity of v, $v(x'' \cdot z) \geq \lambda v(x \cdot z) + (1-\lambda)v(x' \cdot z)$ for all z, and thus $Ev(\sum_k x''_k Z_k) \geq \lambda Ev(\sum_k x_k Z_k) + (1-\lambda)Ev(\sum_k x'_k Z_k))$. The expectation of v is thus quasi-concave, and preferences are convex.

Problem 5.

Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete ($t = 1, 2, \ldots$) and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that:

a) Adam likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.

b) Adam is impatient. He would be delighted to increase his consumption on day t from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later.

c) In any day in which he does not have an apple, he prefers to get one apple immediately in exchange for two apples tomorrow.

d) Adam expects to live for 120 years.

Show that if (poor) Adam is offered a stream of 2 apples starting in day 4 for the rest of his expected life, he would be willing to exchange that offer for one apple right away.

The following is a sequence of streams, in an increasing ordering:

(0,0,0,2,2,......,2)

(0,0,1,0,2,......,2). and continuing in this way until:

(0,0,1,1,1,......1,0)

(0,0,2,0,2,0...,2,0,0)

(0,1,0,1,0,....1,0,1,0,0,0) and "folding from the end":

(0,1,0,1,0,.1,0..2,0,0,0,0,0)

(0,1,0,1,0,.1,1,0,0,0,0,0,0)...until we reach:

(0,2,0,...0)

(1,0,....)

Problem 6.

In this problem you will encounter Quiggin and Yaari's functional, one of the main alternatives to expected utility theory.

Recall that expected utility can be written as $U(p) = \sum_{k=1}^{K} p(z_k)u(z_k)$ **where** $z_0 < z_1 < \ldots < z_K$ **are the prizes in the support of** p. **Let** $W(p) = \sum_{k=1}^{K} f(G_p(z_k))[z_k - z_{k-1}]$, **where** $f : [0,1] \to [0,1]$ **is a continuous increasing function and** $G_p(z_k) = \sum_{j \geq k} p(z_j)$. ($p(z)$ **is the probability that the lottery** p **yields** z **and** G_p **is the "anti-distribution" of** p.)

a. **The literature often refers to** W **as the dual expected utility operator**. **In what sense is** W **dual to** U?

Recall that $Ex(p) = \sum_{k=1}^{K} p(z_k)z_k = \sum_{k=1}^{K} G_p(z_k)[z_k - z_{k-1}]$

While the expected utility functional transforms the prize numbers whereas Quiggin-Yaari functional transofrms the anti-distribution numbers.

b. **Show that** W **induces a preference relation on** $L(Z)$ **that may not satisfy the independence axiom**.

Let $K = 2$, $f(x) = x^2$, $z_0 = 0$, $z_1 = 1$ and $z_2 = 4$. Define lotteries $p = .75[z_0] \oplus .25[z_2]$ and $p' = .5[z_0] \oplus .5[z_1]$. Then

$$U(p) = 4f(\frac{1}{4}) = \frac{1}{4} = f(\frac{1}{2}) = U(p')$$

but

$U(\frac{1}{2}p \oplus \frac{1}{2}[z_1]) = f(\frac{5}{8}) + 3f(\frac{1}{8}) = \frac{28}{64} < \frac{9}{16} = f(\frac{3}{4}) = U(\frac{1}{2}p' \oplus \frac{1}{2}[z_1])$.

c. **What are the difficulties with a functional form of the type** $\sum_z f(p(z))u(z)$? **(See Handa (1977))**

(1) If the DM is indifferent between prizes z_1 and z_2, then $[z_1]$ and $0.5[z_1] \oplus 0.5[z_2]$ need not be indifferent. If $f(1/2) \neq 1/2$, then $U([z_1]) \neq U(0.5[z_1] \oplus 0.5[z_2])$.

(2) The DM might be "worse" off if probability weight is shifted to a more preferred alternative:

if $f(1/2) > 1/2$, $0.5[1-\varepsilon] \oplus 0.5[1] \succ [1]$ for $\varepsilon > 0$ small enough while and–

if $f(1/2) < 1/2$, $0.5[1+\varepsilon] \oplus 0.5[1] \prec [1]$ for $\varepsilon > 0$ small enough.

Problem 7. The two envelopes paradox.

Assume that a number 2^n is chosen with probability $2^n/3^{n+1}$ and the amounts of money 2^n, 2^{n+1} are put into two envelopes. One envelope is chosen randomly and given to you and the other is given to your friend. Whatever the amount of money in your envelope, the expected amount in your friend's envelope is larger (verify it). Thus, it is worthwhile for you to switch envelopes with him even without opening the envelope! What do you think about this paradoxical conclusion?

Note that this is indeed a probability distribution: $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = 1.$

Assume that in your envelope the sum of money is 2^n. For any $n \geq 1$ this can be either the smaller amount or the larger one. If it is the smaller, then the other envelope contains 2^{n+1} and changing envelopes means a gain of 2^n. The probability for this event is $2^n/3^{n+1}$. If your amount is the larger one, then the other envelope contains 2^{n-1} and changing means a loss of 2^{n-1}. The probability for this event is $2^{n-1}/3^n$. Hence, the expected gain when changing envelopes is $\frac{2^n \cdot 2^n/3^{n+1} - 2^{n-1} \cdot 2^{n-1}/3^n}{2^n/3^{n+1} + 2^{n-1}/3^n} = \frac{2^n}{10}$ which is positive for any n. For $n = 0$, your envelope is surely the smaller one, hence changing envelopes is profitable for any $n \geq 0$. Thus, we can conclude that you should change envelopes without even opening yours.

Note that the random expected amount of money in any envelope, your own and the other one, is $\frac{1}{3} + \sum_{n=1}^{\infty} 2^n \cdot 5 \cdot \frac{2^{n-1}}{3^{n+1}} = \infty$. It is possible to show that if the problem is constructed such that the expected value of each envelope is finite, this paradox does not arise.

Random variables with infinite expectation create many paradoxes. For example, after every draw of such a random variable the decision maker who is risk neutral would prefer to replace the outcome in hand with another draw...

Problem Set 9 – Social Choice

Problem 1.

Assume that the set of social alternatives, X, includes only two alternatives. Define a social welfare function to be a function that attaches a preference to any profile of preferences (allow indifference for the SWF and the individuals' preference relations). Consider the following axioms:

Anonymity: If σ is a permutation of N and if $p = \{\succsim_i\}_{i\in N}$ and $p' = \{\succsim'_i\}_{i\in N}$ are two profiles of preferences on X so that $\succsim'_{\sigma(i)} = \succsim_i$, then $\succsim(p) = \succsim(p')$.

Neutrality: For any preference $\succsim_i$, define $(-\succsim_i)$ as the preference satisfying $x(-\succsim_i)y$ iff $y \succsim_i x$. Then, $\succsim(\{-\succsim_i\}_{i\in N}) = -\succsim(\{\succsim_i\}_{i\in N})$.

Positive Responsiveness: If the profile $\{\succsim'_i\}_{i\in N}$ is identical to $\{\succsim_i\}_{i\in N}$ with the exception that for one individual j either ($x \sim_j y$ and $x \succ'_j y$) or ($y \succ_j x$ and $x \sim'_j y$) and if $x \succsim y$, then $x \succ' y$.

a. Interpret the axioms.

A: The social aggregation treats any two individuals symmetrically.

N: The social aggregation treats the two alternatives symmetrically.

PR: The fact that one individual changed his mind in favor of an alternative cannot harm (and in some cases is required to improve) the social status of that alternative.

b. Show that majority rule satisfies all of them.

A: trivially satisfied.

N: Let $N(x,p)$ be the number of individuals that strictly prefer x to y in profile p. If $x \succsim(p)y$, then $N(x,p) \geq N(y,p)$, and thus $y \succsim(-p)x$, proving N.

PR: p' is identical to p, except that one individual "increases" his preference for x, and $N(x,p) \geq N(y,p)$ then $N(x,p') > N(y,p')$, and thus $x \succ(p')y$.

c. Prove May's theorem by which the majority rule is the only SWF satisfying the above axioms.

Assume $N(x,p) = N(y,p)$. Let σ be a permutation such that $\sigma(i)$'s preference is the reverse of i's preference. Let $p' = \{\succsim'_i\}_{i\in N}$ be a profile so that $\succsim'_{\sigma(i)} = \succsim_i$. By A $x \succsim(p)y$ iff $x \succsim(p')y$. Since $p' = -p$ by N $x \succsim(p')y$ iff $y \succsim(p)x$ and thus $x \sim(p)y$.

Assume p a profile such that $N(x,p) > N(y,p)$. Assume $y \succsim(p)x$. We can move from p to a profile p' by changing the preferences of $N(x,p) - N(y,p)$ individuals who prefer x to y to indifference. By PR we would get $y \succ(p')x$ although $N(x,p') = N(y,p')$, a contradiction.

d. **Are the above three axioms independent**?

Yes.

A and N, but not PR: $x \sim (p)y$ for all p.

A and PR, but not N: Let $X = \{a,b\}$. $a \succ (p)b$ if $N(a,p) > N(b,p) + 1$, otherwise $b \succ (p)a$.

N and PR, but not A: For any profile of preferences we attach the "lexicographic" preferences with some fixed priority of the individuals (For example,

$x \succsim (p)y$ if [$x \succ_1 y$] or [$x \sim_1 y$ and $x \succ_2 y$] and so on...)

Problem 2.

Assume that the set of alternatives, X, **is the interval** $[0,1]$ **and that each individual's preference is single-peaked**, **i.e.**, **for each** i **there is an alternative** a_i^* **such that if** $a_i^* \geq b > c$ **or** $c > b \geq a_i^*$, **then** $b \succ_i c$. **Show that for any odd** n, **if we restrict the domain of preferences to single-peaked preferences**, **then the majority rule induces a** "**well-behaved**" **SWF**.

Let $x \succsim (p)y$ and $y \succsim (p)z$. By contradiction, assume that $z \succ (p)x$.

(i) $x < y < z$. If a majority find x as good as y then all those strictly prefer y to z and there is a majority who strictly prefer x over z.

(i) $x < z < y$. If a majority find y as good as z then all those strictly prefer z to x and there is a majority who strictly prefer y over x.

(i) $z < x < y$. If a majority find z as good as x then all those strictly prefer x to y and there is a majority who strictly prefer z over y.

Problem 3.

Each of N **individuals chooses a single object from among a set** X, **interpreted as his recommendation for the social action**. **We are interested in functions that aggregate the individuals' recommendations** (**not preferences**, **just recommendations**!) **into a social decision** (**i**.**e**., $F : X^N \to X$). **Discuss the following axioms**:

Par: **If all individuals recommend** x^*, **then the society chooses** x^*.

I: **If the same individuals support an alternative** $x \in X$ **in two profiles of recommendations**, **then** x **is chosen in one profile if and only if it chosen in the other**.

a. **Show that if** X **includes at least three elements**, **then the only aggregation method that satisfies** P **and** I **is a dictatorship**.

L1: For any recommendation profile $(x_1,\ldots,x_N)$, $F(x_1,\ldots,x_N) \in \{x_1,..,x_N\}$.

Otherwise, by I, $F(c,,,.c) = F(x_1,\ldots,x_N)$ for any $c \neq F(x_1,\ldots,x_N)$, contradicting P.

L2: If $F(x_1,\ldots,x_N) = a$ and $\{i|\ x_i = a\} \subset \{i|\ y_i = a\}$ then $F(y_1,\ldots,y_N) = a$.

If not than by L1 $F(y_1,\ldots,y_N) = b \neq a$ where b is one of the alternatives in y.

Let c be a third alternative. Let $(z_1,\ldots,z_N)$ be the same as y with any $i \notin \{i|\ x_i = a\}$ with $y_i = a$ is changed to $z_i = c$. Then, $\{i|\ y_i = b\} = \{i|\ z_i = b\}$and $\{i|\ x_i = a\} = \{i|\ z_i = a\}$ thus by I $F(z)$ should be both b and a. A contradiction.

Let us say that G is *decisive* with respect to $x^* \in X$, if [for all $i \in G$, $x_i = x^*$] then [$F(x_1,\ldots,x_N) = x^*$].

By L2 if $F(x_1,..,x_N) = a$ then $\{i|\ x_i = a\}$ is decisive with respect to a.

L3: If G is decisive with respect to a with $|G| \geq 2$, then for any b there exists $\emptyset \subset G' \subset G$ such that G' is decisive with respect to a or b.

Let c be a third alternative. Since G is decisive with respect to a then $F(x_1,\ldots,x_N) = a$ where $x_i = a$ for $i \in G$ and $x_i = c$ otherwise

Let G_1, G_2 be a partition of G and

$$y_i = \begin{cases} a & \text{if } i \in G_1 \\ b & \text{if } i \in G_2 \\ c & \text{if } i \in N \setminus G \end{cases}$$

By I $F(y_1,\ldots,y_N)$ is not c and by L1 it is either a or b . Thus, by L2 either G_1 or G_2 are decisive with respect to a or b.

L4: There is a singleton i^* who is decisive with respect to some alterative a.

L5: i^* is decisive regarding any alterative b.

Let $x_{i^*} = a$ and $x_i = c$ for all other i. By I, $F(x) = a$.

Let $y_{i^*} = b$ and $y_i = c$ for all other i, By I $F(y) \neq c$ and by L1 $F(y) = b$ and by L2 it is decisive regarding b.

b. **Show the necessity of the three conditions P, I, and $|X| \geq 3$ for this conclusion**.

Choosing the most popular recommendation (with pre-specified tie breaking rule) satisfies P and I when $|X| = 2$.

When $|X| \geq 3$, the most popular recommendation (with pre-specified tie breaking rule)satisfies P but fails I.

Always choosing action $x^* \in X$, regardless of recommendation, satisfies I but fails P.

Problem 4.

Some proofs of Arrow's theorem use the notion of decisive and almost decisive coalitions.

Given the SWF we say that:

i) **a coalition** G **is decisive with respect to** x,y **if** [**for all** $i \in G$, $x \succ_i y$] **implies** [$x \succ y$], **and**

ii) **a coalition** G **is almost decisive with respect to** x,y **if** [**for all** $i \in G$, $x \succ_i y$ **and for all** $j \notin G$, $y \succ_j x$] **implies** [$x \succ y$].

Note that if G **is decisive with respect to** x,y, **then it is also almost decisive with respect to** x,y, **since** "**almost decisiveness**" **refers only to the subset of profiles in which all members of** G **prefer** x **to** y **and all members of** $N - G$ **prefer** y **to** x.

We say that a coalition G **is decisive if it is decisive with respect to all** x,y.

Let F **be an SWF satisfying** Par **and** IIA.

a. **Prove the** "**Field Expansion Lemma**": **If** G **is almost decisive with respect to** x,y, **then** G **is decisive with respect to** x,z **and with respect to** y,z.

Consider a profile $(\succ_1, \ldots, \succ_n)$ such that $x \succ_i z$ for all $i \in G$.

Consider another profile $(\succ_1^*, \ldots, \succ_n^*)$ which ranks x,y and z as follows:

if $i \in G$	*if* $i \in N \setminus G$ and $z \succ_i x$	*if* $i \in N \setminus G$ and $x \succ_i z$
x	y	y
y	z	x
z	x	z

Since G is almost decisive with respect to x,y, then $x \succ^* y$. By *Par*, $y \succ^* z$. By transitivity $x \succ^* z$. By IIA also $x \succ z$.

Now, consider a profile $(\succ_1, \ldots, \succ_n)$ such that $z \succ_i y$ for all $i \in G$.

Consider another profile $(\succ_1^*, \ldots, \succ_n^*)$ which ranks x,y and z as follows:

if $i \in G$	*if* $i \in N \setminus G$ and $z \succ_i y$	*if* $i \in N \setminus G$ and $y \succ_i z$
z	z	y
x	y	z
y	x	x

Since G is almost decisive with respect to x,y, then $x \succ^* y$. By *Par*, $z \succ^* x$. By transitivity $z \succ^* y$. By IIA also $z \succ y$.

b. **Conclude that if G is almost decisive with respect to x,y, then G is decisive**.

Let w, z be any two alternatives.

If G is almost decisive with respect to x,y, then by part a it is decisive with respect to x, z. Thus, G is also almost decisive with respect to x, z. By part a, G is decisive with respect to w, z.

c. **Prove the** "**Group Contraction Lemma**": **If G is decisive.and $|G|\geq 2$, then there exists $G' \subset G$ such that G' is decisive**.

Let $G = G_1 \cup G_2$, where G_1 and G_2 are nonempty and $G_1 \cap G_2 = \emptyset$. By the Field Expansion Lemma it is enough to show that G_1 or G_2 is almost decisive with respect to some alternatives.

Take three alternatives x,y, and z and a profile of preference relations $(\succ_i)_{i\in N}$ satisfying:

if $i \in G_1$	*if* $i \in G_2$	*if* $i \in N \setminus \{G_1 \cup G_2\}$
z	x	y
x	y	z
y	z	x

The coalition G is decisive, thus $x \succ y$.

If G_1 is not almost decisive with respect to z,y, then there is a profile $(\succ_i')_{i\in N}$ such that $z \succ_i' y$ for all $i \in G_1$ and $y \succ_i' z$ for all $i \notin G_1$, such that $F(\succ_1', \ldots, \succ_n')$ determines $y \succ' z$. Therefore, by IIA, $y \succ z$.

Similarly, if G_2 is not almost decisive with respect to x, z, then $z \succ x$. Thus, by transitivity $y \succ x$, but since G is decisive, $x \succ y$, a contradiction. Thus, G_1 or G_2 is almost decisive with respect to some alternatives.

d. **Show that there is an individual i^* such that $\{i^*\}$ is decisive**.

By *Par*, the set N is decisive. By the Group Contraction Lemma, every decisive set that includes more than one member has a proper subset that is decisive. Thus, there is a set $\{i^*\}$ that is decisive, which means that $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

Problem 5

Who is an economist? Departments of economics are often sharply divided over this question. Investigate the approach according to which the determination of who is an economist is treated as an aggregation of the views held by department members on this question.

Let $N = \{1, \ldots, n\}$ **be a group of individuals** ($n \geq 3$). **Each** $i \in N$ **"submits" a set** E_i**, a proper non empty subset of** N**, which is interpreted as the set of "real economists" in his view. An aggregation method** F **is a function that assigns a proper non empty subset of** N **to each profile** $(E_i)_{i=1,\ldots,n}$ **of proper subsets of** N. $F(E_1, \ldots, E_n)$ **is interpreted as the set of all members of** N **who are considered by the group to be economists. (Note that we require that all opinions be proper subsets of** N.**)**

Consider the following axioms on F:

Consensus: If $j \in E_i$ **for all** $i \in N$**, then** $j \in F(E_1, \ldots, E_n)$ **and if** $j \notin E_i$ **for all** $i \in N$**, then** $j \notin F(E_1, \ldots, E_n)$.

Independence: If $(E_1, \ldots, E_n)$ **and** $(G_1, \ldots, G_n)$ **are two profiles of views so that for all** $i \in N$, $[j \in E_i$ **iff** $j \in G_i]$, **then** $[j \in F(E_1, \ldots, E_n)$ **iff** $j \in F(G_1, \ldots, G_n)]$.

a. Interpret the two axioms.

C: If all individuals include (omit) an economist from their list, then that option is included (omitted) from the aggregation.

I: If all individuals in profiles $\{E_i\}$ and $\{G_i\}$ have the same opinion regarding j, then j is either included or excluded in both aggregations.

b. Find one aggregation method that satisfies C **but not** I **and one that satisfies** I **but not** C.

Select the set with the smallest number of elements suggested by one of the members (for tie breaking rule, from among the sets with minimal number of individuals choose the one suggested by the member with the smallest index) This methods satisfies C but not I.

Always selecting $F(E_1, \ldots, E_n) = \{1\}$ satisfies I but not C.

c. Provide a proof similar to that of Arrow's Impossibility Theorem of the claim that the only aggregation methods that satisfy the above two axioms are those for which there is a member i^* **such that** $F(E_1, \ldots, E_n) = E_{i^*}$.

L1: Assume that G is almost decisive regarding j (that is j is an economist whenever the group of people who consider j an economist is precisely G) then it is almost decisive regarding any h (and thus will be almost decisive).

Consider the profile where the supporters of j are the members of G, the supporters of i (a third member) are $N - G$ and everybody supports all other members. By C those must be in E and since E is a proper subset and $j \in E$, it must be that i is not.

Now consider the profile where the set of supporters of h is G, the supporters of i are $N - G$ and nobody considers any other member to be an economist. By C all those members besides i and h are not in E. By I i is not in E. Since E is a proper subset it must be that h is in E. Thus, G is almost decisive regarding h.

L2: If G is almost decisive and contains more than one element then there is a proper subset of G which is almost decisive.

Partition G to G_1 and G_2. Consider the profile where 1 is considered an economist by exactly the members of G_1, 2 is considered an economist by exactly the members of G_2 and all other individuals are considered economists by the members of $N - G$ only. As we have seen in L1, since G is almost decisive then $N - G$ is not and thus all individuals besides 1 and 2 are not determined by the aggregator to be economists. It follows that either 1 or 2 must be an economist and thus (using I) either G_1 or G_2 is almost decisive in regard 1 and 2 and by L1 at least one of them is almost decisive.

L3: There is an i^* who is almost decisive.

L4: If i is supported by G which contains i^* then i is an economist.

Consider the profile where i is supported by G, j by $N - \{i^*\}$ and all the rest by no one. By C all members $N - \{i^*, j\}$ are not economists. From the argument above neither is j so i must be an economist with respect to this profile and by I with respect to any profile.

Review Problems

A. Choice:

Problem A1 (**Princeton 2000**. **Based on Fishburn and Rubinstein** (**1982**).)

Let $X = \Re^{+} \times \{0,1,2,\ldots\}$, **where** (x,t) **is interpreted as receiving** $\$x$ **at time** t. **A preference relation on** X **has the following properties**:

(**i**) **There is indifference between receiving** $\$0$ **at time** 0 **and receiving** $\$0$ **at any other time**.

(**ii**) **It is better to receive any positive amount of money as soon as possible**.

(**iii**) **Money is desirable**.

(**iv**) **The preference between** (x,t) **and** $(y,t+1)$ **is independent of** t.

(**v**) **Continuity**.

1. **Define formally the continuity assumption for this context**.

For any (x,t) and (y,s) with $(x,t) \succ (y,s)$, there is $\varepsilon > 0$ such that for any x' such that $|x - x'| < \varepsilon$ and any y' such that $|y - y'| < \varepsilon$, $(x',t) \succ (y',s)$.

2. **Show that the preference relation has a utility representation**.

Claim 1: For any pair (x,t), there is a unique number $v(x,t) \in \Re_{+}$ such that $(x,t) \sim (v(x,t),0)$.

Proof: By (iii) it is enough to show that for every (x,t) there is y with $(x,t) \sim (y,0)$.

For $x = 0$, by (i) $(0,t) \sim (0,0)$.

For any pair (x,t) with $x \neq 0$, define the set $B(x,t) = \{z \in \Re_{+} | (x,t) \succsim (z,0)\}$. By (i) and (iii), $0 \in B(x,t)$. By (ii) and (iii) $B(x,t)$ is bounded above by x. Let $x^* = \sup B(x,t)$.Then by continuity $(x,t) \sim (x^*,0)$.

Claim 2: The preference relation is represented by $u(x,t) = v(x,t)$.

Proof:

$$\begin{aligned} u(x,t) \geq u(y,s) &\Leftrightarrow v(x,t) \geq v(y,s) \\ &\Leftrightarrow (v(x,t),0) \succsim (v(y,s),0) \text{ (by (ii))} \\ &\Leftrightarrow (x,t) \succsim (y,s) \text{ (by def of } v. \text{ and transitivity)} \end{aligned}$$

3. **Verify that the preference relation represented by the utility function** $v(x,t) = u(x)\delta^{t}$ (**with** $\delta < 1$ **and** u **continuous**, **increasing and** $u(0) = 0$) **satisfies the above properties**.

(i): For all t, t', $v(0,t) = u(0)\delta^{t} = 0 = u(0)\delta^{s} = v(0,s)$. Thus, $(0,t) \sim (0,s)$.

(ii): For any x and $t < s$, $v(x,t) = \delta^t u(x) > \delta^s u(x) = v(x,s)$. Thus, $(x,t) \succ (x,s)$.
(iii): This holds since $u(\bullet)$ is increasing.
(iv): For all x, y, t, s,

$$\begin{aligned}(x,t) \succsim (y,t+1) &\Leftrightarrow \delta^t u(x) > \delta^{t+1} u(y) \\ &\Leftrightarrow \delta^s u(x) > \delta^{s+1} u(y) \\ &\Leftrightarrow (x,s) \succsim (y,s+1)\end{aligned}$$

(v): Continuity follows from u being continuous.

4. **Formulize a concept** "**one preference relation is more impatient than another**".
$\succsim^1$ is more impatient than $\succsim^2$ if for any (x,t), and any (y,s) with $s > t$, $y > x$

$$(y,s) \succsim^1 (x,t) \Rightarrow (y,s) \succsim^2 (x,t).$$

5. **Discuss the claim that preferences represented by $u_1(x)\delta_1^t$ are more impatient than preferences represented by $u_2(x)\delta_2^t$ if and only if $\delta_1 < \delta_2$.**
This would be true when $u_1 = u_2 = u$: then for any $s > t$, $y > x$, $\delta_1^s u(y) \geq \delta_1^t u(x) \Leftrightarrow \delta_1^{(s-t)} \geq u(x)/u(y)$ implying $\delta_2^{(s-t)} \geq u(x)/u(y) \Leftrightarrow \delta_2^s u(y) \geq \delta_2^t u(x)$
However, the claim is not necessarily true if $u_1 \neq u_2$. For instance, suppose $u_1(x) = \ln x$, $u_2(x) = x$, $\delta_1 = 1/2$ and $\delta_2 = 1/\sqrt{3} > 1/2$. Then person 2 is indifferent between getting \$1 at time 0 and \$3 at time 2, since

$$\delta_2^2 u_2(3) = 1 = \delta_2^0 u_2(1)$$

However, person 1, the supposedly more impatient (lower δ) person, prefers to wait for the \$3:

$$\delta_1^2 u_1(3) = 1/4 \ln 3 > \delta_1^0 u_1(1) = \ln 1.$$

Problem A2 (**Tel Aviv 2003**. **Based on Gilboa and Schmeidler** (**1995**).)

An agent must decide whether to do something, Y, **or not to do it**, N.
A history is a sequence of results for past events in which the agent chose Y; **each result is either a success** S **or a failure** F. **For example**, (S,S,F,F,S) **is a history with five events in which the action was carried out**. **Two of them** (**events** 3 **and** 4) **ended in failure while the rest were successful**.
The decision rule D **is a function that assigns the decision** Y **or** N **to every possible history**.
Consider the following properties of decision rules:
$A1$: **After every history that contains only successes**, **the decision rule will dictate** Y **and after every history that contains only failures**, **the decision rule will dictate** N.
$A2$: **If the decision rule dictates a certain action following some history**, **it will dictate the same action following any history that is derived from the first history by reordering its members**. **For example**, $D(S,F,S,F,S) = D(S,S,F,F,S)$.
$A3$: **If** $D(h) = D(h')$, **then this will also be the decision following the concatenation of** h **and** h'. (**Reminder**: **The concatenation of** $h = (F,S)$ **and** $h' = (S,S,F)$ **is** (F,S,S,S,F)).
1. **For every** $i = 1,2,3$, **give an example of a decision rule that does not fulfill property** Ai **but does fulfill the other two properties**.
A2 and A3 but not A1: Choose always Y. A1 is violated since $D(F) = Y$.
A1 and A3 but not A2: Choose Y if the first experience was S and choose N otherwise. A2 is violated since $D(S,F) = Y$ while $D(F,S) = N$.
A1 and A2 but not A3: Choose Y after (S) or if the experience does not contain two failures. A3 is violated since $D(F,S) = Y$ but $D(F,S,F,S) = N$.

2. **Give an example of a decision rule that fulfills all three properties**.
For a given $\alpha > 0$, choose Y iff at least α of the cases in the experience were S.

3. (**Difficult**) **Characterize the decision rules that fulfill the three properties**.
Claim: A decision rule D satisfies the three axioms iff there is $1 \geq \alpha \geq 0$, such that it chooses Y if the proportion of S is above α and chooses N if the proportion of S is below α.
Proof: Denote by $q(h)$ the proportion of S in the history h, let $n(h)$ be its length and let kh

be the replication of h, k times. Assume by contradiction that there are two histories x and y such that $q(x) \geq q(y)$, $D(y) = Y$ and $D(x) = N$.
Using A3 $D(n(y)x) = N$, $D(n(x)y) = Y$ and the length of the two histories is the same $(n(x)n(y))$. Thus, we can assume that $n(x) = n(y)$.
By A2 we can also assume WLOG that all the failures appear in the sequences at the end of the sequences. Thus we get something like
$x = (S,S,S,S,S,F,F)$
$y = (S,S,S,F,F,F,F)$.
The number of S's in x is larger by $k > 0$ than in y. Thus by A3 $D(x,kF) = N$ and $D(kS,y) = Y$ though $(x,kF) = (kS,y)$ a contradiction.
Note that for any $\alpha \in \mathbb{Q}$, the decision rule has to specify also a choice for the case when the proportion of S is exactly α. For $1 > \alpha > 0$, the decision rule can be either Y or N, for $\alpha = 1$ it has to be Y and for $\alpha = 0$ it has to be N. Since the proportion of S is a rational number, it will never be equal to α if $\alpha \notin \mathbb{Q}$.

Problem A3 (**NYU 2005**)

Let X **be a finite set containing at least three elements**. **Let** C **be a choice correspondence**. **Consider the following axiom**:

If $A, B \subseteq X$, $B \subseteq A$ **and** $C(A) \cap B \neq \emptyset$, **then** $C(B) = C(A) \cap B$.

1. **Show that the axiom is equivalent to the existence of a preference relation** $\succsim$ **such that** $C(A) = \{x \in A \mid x \succsim a \; for$ **all** $a \in A\}$.

It's enough to show that the Choice Axiom in the question is equivalent to the Weak Axiom.

Choice Axoim $\Rightarrow$ WA. Let $x \in C(A)$, $y \in C(B)$ and $x, y \in A \cap B$. Define $B' = A \cap B$. Clearly, $C(A) \cap B' \neq \emptyset$ and $C(B) \cap B' \neq \emptyset$, so the Choice Axiom implies that $C(A) \cap B = C(A) \cap B' = C(B') = C(B) \cap B' = C(B) \cap A$. But x clearly is in $C(A) \cap B$, which implies that $x \in C(B) \cap A$, which itself implies that $x \in C(B)$.

WA $\Rightarrow$ Choice Axiom. Let $B \subseteq A$ and $C(A) \cap B \neq \emptyset$. Let $x \in C(B)$ and $y \in C(A) \cap B$. Clearly, $x, y \in A \cap B$. But then the WA implies that $x \in C(A)$, and therefore, $x \in C(A) \cap B$ and $y \in C(B)$. But this means that $C(B) \subseteq C(A) \cap B$ and $C(A) \cap B \subseteq C(B)$, which is equivalent to $C(B) = C(A) \cap B$.

2. **Consider a weaker axiom**:

If $A, B \subseteq X$, $B \subseteq A$ **and** $C(A) \cap B \neq \emptyset$, **then** $C(B) \subseteq C(A) \cap B$.

Is this sufficient for the above equivalence?

No, it's not sufficient. Consider the following example. Let $X = \{a, b, c\}$. Suppose $C(X) = X$, $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$, $C(\{a, c\}) = \{c\}$. If there was a preference $\succsim$ such that $C(A) = \{x \in A \mid x \succ a$ for all $a \in A\}$ then $a \succ b$, $b \succ c$ and $c \succ a$, a contradiction.

Problem A4 (**NYU 2007**. **Based on Plott** (**1973**).)

Let X be a set and C be a choice correspondence defined on all non-empty subsets of X.

We say that C satisfies Path Independence (PI) if for every two disjoint sets A and B, we have $C(A \cup B) = C(C(A) \cup C(B))$.

We say that C satisfies Extension (E) if $x \in A$ and $x \in C(\{x,y\})$ for every $y \in A$ implies that $x \in C(A)$ for all sets A.

1. Interpret PI and E.

PI: The agent's choice set is identical, regardless if he (1) chooses directly from the set A or (2) first partitions A into two subsets, chooses from each of the subsets and then makes a subsequent choice from these two choice sets.

E: If $x \in A$ is chosen when compared (pairwise) with every other alternative in A, then x is in the choice set of A.

2. Show that if C satisfies both PI and E, then there exists a binary relation $\succsim$ that is complete, reflexive and satisfies $x \succ y$ and $y \succ z$ implies $x \succ z$, such that $C(A) = \{x \in A \mid$ for no $y \in A$ is $y \succ x\}$.

For all $x,y \in X$, define $x \succsim y$ if $x \in C(\{x,y\})$. Clearly, $\succsim$ is complete and reflexive. By contradiction, assume $x \succ y$ and $y \succ z$, but $z \succsim x$, that is, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, and $z \in C(\{x,z\})$. Define $A = \{x,y,z\}$, and note that PI implies that

$$C(A) = C(C(\{x,y\}) \cup C(\{z\})) = C(\{x,z\}) \Rightarrow z \in C(A) \text{ and}$$

$$C(A) = C(C(\{x\}) \cup C(\{y,z\})) = C(\{x,y\}) \Rightarrow z \notin C(A),$$

a contradiction. Therefore, $x \succ z$.

Finally, if $x \succsim y$ for all $y \in A$, then $x \in C(A)$ by E, and thus $C(A) \supseteq \{x \in A \mid$ for no $y \in A, y$

Conversely, if there exists a $y \in A$ such that $y \succ x$, then by PI

$$C(A) = C(C(\{x,y\}) \cup C(A \setminus \{x,y\})) = C(\{y\} \cup C(A \setminus \{x,y\})),$$

and thus $x \notin C(A)$. Thus, $C(A) = \{x \in A \mid$ for no $y \in A, y \succ x\}$.

3. Give one example of a choice correspondence satisfying PI but not E, and one satisfying E but not PI.

Let $\succsim$ be an ordering over X.

PI and not E: Define $C(A)$ to equal the best and second best elements in A according to $\succsim$.

Satisfies PI: The best and second best elements of $A \cup B$ are contained in $C(A) \cup C(B)$, and thus $C(C(A) \cup C(B))$ equals these first and second best elements in $A \cup B$.
Fails E: Every $x \in A$ is in $C(\{x,y\})$ for all $y \in A$. Nevertheless, only the two best elements in A are included in $C(A)$.

PI and not E: Define $C(A)$ to be the best element in A if $A \neq X$, and $C(X) = X$.
Satisfies E: If $x \in C(\{x,y\})$ for all $y \in A$, then there is no $y \succ x$, and thus $x \in C(A)$.
Fails PI: Let A and B be a partition of X. Then $C(X) \neq C(C(A) \cup C(B))$.

Problem A5 (NYU 2008. Based on Eliaz, Richter, and Rubinstein (2011).)

Let X be a (finite) set of alternatives. Given any choice problem A (where $|A| \geq 2$), the decision maker chooses a set $D(A) \subseteq A$ of two alternatives which he wants to examine more carefully before making the final decision.
The following are two properties of D:

A1: If $a \in D(A)$ and $a \in B \subset A$ then $a \in D(B)$.
A2: If $D(A) = \{x,y\}$ and $a \in D(A - \{x\})$ for some a different than x and y, then $a \in D(A - \{y\})$.

Answer the following four questions. A full proof is required only for the last question:
1. Find an example of a D function which satisfies both A1 and A2.
Let $\succ$ be a strict preference relation on X. Let $D(A)$ be the set of the two $\succ$–best elements in A.

2. Find a function D which satisfies A1 and not A2.
Let $\succ$ be a strict preference relation on X. Let $D(A)$ be the set containing the $\succ$–best element and the $\succ$–worst element in A.

3. Find a function D which satisfies A2 and not A1.
Let $\succ$ be a strict preference relation on X. Let $D(A)$ be the set containing the second and third $\succ$–best elements in A (if $|A| \geq 3$) and $D(A) = A$ otherwise.

4. Show that for any function D satisfying A1 and A2 there exists an ordering $\succ$ of the elements of X s.t. $D(A)$ is the set of the two $\succ$–best elements in A.
Let D be a choice correspondence satisfying the three axioms. Let us construct inductively a count of all elements in X.
Let $D(X) = \{x_1, x_2\}$.
Assume we constructed the sequence $x_1, x_2, .., x_K, x_{K+1}, x_{K+2}$ such that $D(X - \{x_1, x_2 ..., x_k\}) = \{x_{k+1}, x_{k+2}\}$ for all $0 \leq k \leq K$.
By $A1$ $x_{K+2} \in D(X - \{x_1, x_2, .., x_K, x_{K+1}\})$. Define $x_{K+3}$by $D(X - \{x_1, x_2, .., x_{K+1}\}) = \{x_{K+2}, x_{K+3}\}$.
Define $x_i \succ x_j$ if $i < j$.

For any set A let $x_{i(A)}$ and $x_{j(A)}$ be the two $\succ$-top elements in A. The set $D(A)$ must contain the $\succ$-top element in A since by definition $D(X-\{x_1,..,x_{i(A)-1}\}) = \{x_{i(A)},x_{i(A)+1}\}$ and as $A \subseteq X-\{x_1,..,x_{i(A)-1}\}$ then by $A1$ $x_{i(A)} \in D(A)$.

Assume by contradiction that there exists a set A such that $D(A)$ contains $x_{i(A)}$, but not $x_{j(A)}$. Consider such a set A with minimum $j(A)-i(A)$. We will show that $j(A)-i(A) = 1$ which leads to contradiction (since by construction $D(\{x_k|k \geq i(A)\}) = \{x_{i(A)},x_{i(A)+1}\}$ and by $A1$ $D(A) = \{x_{i(A)},x_{i(A)+1}\}$).

If $j(A) > i(A)+1$ consider the set $B = A \cup \{x_{i(A)+1}\}$. Let $D(A) = \{x_{i(A)},x_{h(A)}\}$ (and $j(A) < h(A)$). By construction $D(\{x_k|k \geq i(A)\}) = \{x_{i(A)},x_{i(A)+1}\}$ and by $A1$ $D(B) = \{x_{i(A)},x_{i(A)+1}\}$. By $A1$ $x_{i(A)+1} \in D(B-\{x_{i(A)}\})$. Since $x_{h(A)} \in D(B-\{x_{i(A)+1}\}) = D(A)$. Let $C = B-\{x_{i(A)}\}$. By $A2$ $D(C) = \{x_{i(A)+1},x_h\}$. Since $i(C) = i(A)+1$ and $j(C) = j(A)$ we have $j(C)-i(C) < j(A)-i(A)$.

Problem A6 (**Tel Aviv 2009**. **Inspired by Mandler**, **Manzini and Mariotty**, (**2010**)).

Consider a decision maker who is choosing an alternative from subsets of a finite set X **using the following procedure**:

Following a fixed list of properties (**a checklist**), **he examines one property at a time and deletes from the set all the alternatives that do not satisfy this property**. **When only one alternative remains**, **he chooses it**.

1. **Show that if this procedure induces a choice function**, **then it is consistent with the rational man model**.

Let $a \in B \subset A$ such that a is the chosen alternative from A. Any other alternative in A, and in B in particular, fails to satisfy at least one of the first m properties that a does satisfy. Therefore, a is the surviving alternative also in B and condition α holds.

2. **Show that any rational decision maker can be described as if he follows this procedure**.

Order all the alternatives in X in ascending order. For each alternative $x \in X$, define the property "not being x" and order the properties according to the same order. Clearly, the only alternative for which "not being x" does not hold is x itself.

Let $A \subset X$ be the set from which the individual chooses. In the first stage, he deletes from A the worst element in X, if it belongs to A, and does nothing otherwise. Similarly, at any subsequent stage, if the alternative is not in A he continues to the next stage and if it does belong to A he deletes it. Hence, at each stage he deletes the worst alternative from his choice set. This process continues until he is left with the best alternative in A.

Problem A7 (Tel Aviv 2010)

A decision maker has a preference relation over $\mathbb{R}^n_+$. A vector (x_1,x_2) is interpreted as an income combination where x_i is the dollar amount the decision maker receives at period i.

Let P be the set of all preference relations satisfying:

(**i**) **Strong Monotonicity** (**SM**) **in** x_1 **and** x_2 .

(**ii**) **Present preference** (**PP**): $(x_1+\varepsilon, x_2-\varepsilon) \succsim (x_1,x_2)$ **for all** $\varepsilon > 0$.

Define $(x_1,x_2)D(y_1,y_2)$ **if** $(x_1,x_2) \succsim (y_1,y_2)$ **for all** $\succsim \in P$.

1. **Interpret the relation** D. **Is it a preference relation**?

D is a domination relation: x dominates y if for every monotonic present biased preference relation, x is considered at least as good as y.

D is not a preference relation: although it is transitive, it is not complete.

2. **Is it true that** $(1,4)D(3,3)$? **What about** $(3,3)D(1,4)$?

$(1,4)D(3,3)$ is false: consider the preference relation represented by the utility function $u(x_1,x_2) = x_1 + x_2$. It satisfies the two properties, but $(1,4) \prec (3,3)$.

$(3,3)D(1,4)$ is true:.for any preference relation which satisfies the two properties, by PP $(3,3) \succsim (1.5,4.5)$ and by SM, $(1.5,4.5) \succ (1,4)$

3. **Find and prove a proposition of the following type**:

$(x_1,x_2)D(y_1,y_2)$ **if and only if** [**put here a condition on** (x_1,x_2) **and** (y_1,y_2)].

Proposition: $(x_1,x_2)D(y_1,y_2)$ Iff $x_1 \geq y_1$ and $(x_1+x_2) \geq (y_1+y_2)$.

Proof:

$\Leftarrow$ Assume $x_1 \geq y_1$ and $(x_1+x_2) \geq (y_1+y_2)$.

Let $\succsim \in P$, i.e. a preference relation satisfying SM and PP. If $x_2 \geq y_2$, by SM $x \succsim y$. If $y_2 > x_2$ let $\epsilon = y_2 - x_2$.

It follows that $x_1 \geq y_1 + \epsilon$. By SM $(x_1,x_2) \succsim (y_1+\epsilon, x_2)$ and by PP $(y_1+\epsilon, x_2) = (y_1+\epsilon, y_2-\epsilon) \succsim (y_1,y_2)$.

$\Rightarrow$ Assume $(x_1,x_2)D(y_1,y_2)$.

The condition $x_1 \geq y_1$ is necessary: let a and b be two vectors such that aDb and $a_1 < b_1$. Consider the preferences $\succsim_t$ represented by $u(x_1,x_2) = tx_1 + x_2$ where $t > 1$. Obviously they satisfy PP and SM. For t large enough $ta_1 + a_2 < tb_1 + b_2$ and thus $b \succ_t a$.

The condition $x_1 + x_2 \geq y_1 + y_2$ is necessary: let a and b be two vectors such that aDb and $a_1 + a_2 < b_1 + b_2$. Then $b \succ_1 a$.

Problem A8 (NYU 2011.)

Let X be a finite set of alternatives.

A decision maker of type 1 uses the following choice procedure. He has a subset of "satisfactory alternatives" in mind. Whenever he chooses from a set A, then (i) if there are satisfactory elements in A, he is happy to choose any satisfactory alternative which comes to his mind and (ii) If there are none, he is happy with any of the non-satisfactory alternatives.

A decision maker of type 2 has in mind a set of strict orderings. Whenever he chooses from a set A, he is happy with any alternative that is the maxima in A of at least one ordering.

1. Define formally the two types of decision makers as choice correspondences.

1: Let S be a set of "satisfactory alternatives"and let $N = X - S$ be the set of "non-satisfactory alternatives".

$$C^S(A) == \begin{cases} A \cap S & if \;\; A \cap S \neq \emptyset \\ A & otherwise \end{cases}$$

2: Let $\Lambda = \{\succ_i\}$ be a set of strict orderings.

$$C^\Lambda(A) = \left\{a \in A \;:\; for some \succ_i \;\in \Lambda \; we \;\; have \;\; a \succ_i y \quad \forall y \in A \backslash \{a\}\right\}$$

2. Show that any decision maker of type 1 can also be described as a decision maker of type 2.

Let C^S be a choice correspondence of type 1 characterized by a set S of satisfactory elements.

Let Λ be the set of all orderings which place all elements in S above all elements in N. Let C^Λ be the choice correspondence of type 2 DM who has Λ in mind. We need to prove that $C^S(A) = C^\Lambda(A)$ for all $A \subseteq X$.

1. $C^S(A) \subseteq C^\Lambda(A)$.

Let $a \in C^S(A)$. Either $a \in S$ or $a \in N$.

If $a \in S$, then a is also the maximum in A of an ordering in Λ which places a first.

If $a \in N$, then $A \cap S \neq \emptyset$ and therefore a is the maximum in A of an ordering in Λ which puts a above all elements in N.

Hence, $a \in C^\Lambda(A)$.

2. $C^S(A) \supseteq C^\Lambda(A)$.

Let $a \in C^\Lambda(A)$. Either $a \in S$ or $a \in N$.

If $a \in S$, then $a \in C^S(A)$.

If $a \in N$, then there is no element from S in A since if there was, a would not be a maximum of any of the relations in Λ. Thus, $a \in C^S(A)$.

3. **Show that there is a decision maker of type** 2 **who cannot be described as a decision maker of type** 1.

Consider the following example: $X = \{a, b, c\}$ and $\Lambda = \{\succ\}$ where $a \succ b \succ c$.

Suppose this DM can be represented as type A. Under this set of strict orderings, $C^{\Lambda}(\{a, b\}) = \{a\}$ which implies that $b \in N$. Also, $C^{\Lambda}(\{b, c\}) = \{b\}$ which implies that $b \in S$ a contradiction.

Problem A9 (Tel Aviv 2012. Based on de Clippel (2011).)

Consider a decision maker (DM) who has in mind two orderings on a finite set X. The first ordering, $\succ_L$, expresses his long-term goals, and the second, $\succ_S$, expresses his short-term goals.

When choosing from a set $A \subseteq X$ the DM chooses the best alternative according to his long-term preferences, unless there are "too many" alternatives that are better than this alternative according his short-term preferences. More precisely, given a choice problem $A \subseteq X$, he excludes all alternatives which are not among the k best alternatives in A according to his short-term preferences, and out of the remaining he chooses the best one according to $\succ_L$.

1. Show that the above description always defines a choice function.

For any $A \subseteq X$, let $CS(A)$ (the consideration set from A) be the set of the best k alternatives according to $\succ_S$ in A (if $|A| < k$ let $CS(A) = A$). Clearly, $CS(A)$ is not empty. The above procedure is equivalent to choosing the best alternative from $CS(A)$ according to $\succ_L$, which is always well defined.

2. Show that it may be that the same alternative is chosen from both A and B, but is not chosen from $A \cup B$ nor from $A \cap B$

Consider the following example: $X = \{a,b,c,d\}$, $A = \{a,b,c\}$, $B = \{a,b,d\}$, $k = 2$, $a \succ_L b \succ_L c \succ_L d$ and $d \succ_S c \succ_S b \succ_S a$.

$CS(A) = \{b,c\}$ and $CS(B) = \{b,d\}$. Thus, $C(A) = b$ and $C(B) = b$.

However, $CS(A \cap B) = CS(\{a,b\}) = \{a,b\}$ and thus, $C(A \cap B) = a$. Furthermore, $CS(A \cup B) = CS(X) = \{c,d\}$ and thus $C(A \cup B) = c$.

3. Conclude that this type of behavior conflicts with the rational man paradigm.

Obviously, if the rational agent maximizes a preference relation, then if a is the maximizer in both A and B it is also the maximizer in $A \cup B$ and $A \cap B$.

Let N be a set of individuals who behave according to the above procedure with $k = 2$. All individuals share the same long-term goals but may differ in their short-term goals.

Consider a situation in which the N individuals must choose together only one alternative from the set X and that for each alternative $x \in X$, there is one individual $r(x)$ who has the right to force x. An equilibrium is an alternative y such

that no individual wants to exercise his right to force one of the alternatives that he can force. That is, for any agent i**, the alternatives** y **is the one chosen by the agent from the set** $\{y\} \cup \{x | r(x) = i\}$.

4. Show that if there are more individuals than alternatives then it is possible to assign the "forcing rights" such that whatever are the individuals' short-term goals and whatever are the common long-term goals, the only equilibrium is the top $\succ_L$ **alternative. Explain why this is not necessarily correct if the number of alternatives is larger than the number of individuals.**

If there are more individuals than alternatives, we can assign the rights such that at most one exclusive alternative is assigned to each individual.

Now, in a particular world (configuration of common long and short term preferences) let y^* be the top $\succ_L$ alternative.

First note that y^* is an equilibrium: for any agent i there is at most one other alternative y such that $r(y) = i$. Agent i can choose only from $\{y, y^*\}$, which are both in his consideration set, and thus he chooses y^*.

Consider any other $y \neq y^*$. This alternative is not an equilibrium since agent $i = r(y^*)$ faces a choice from $\{y, y^*\}$, and, choosing according to $\succ_L$, he forces y^*.

As to the last part of the question: consider a world with $X = \{x, y, z\}$ and two individuals. It must be that two alternatives, let us say, x and y, are assigned to one of the individuals, let us say 1. For the configuration $y \succ_L z \succ_L x$ and $z \succ_{i,S} x \succ_{i,S} y$ for both i we get that z is an equilibrium becasue 1 can choose from X and he chooses z.

Problem A10 (**NYU 2013**)

Consider the following procedure which yields a choice function C **over subsets of a finite set** X:

The decision maker has in mind a set $\{\succ_i\}_{i=1,..n}$ **of orderings over** X **and a set of positive weights** $\{\alpha_i\}_{i=1,..n}$. **Facing a choice set** $A \subseteq X$, **the decision maker calculates a score for each alternative** $x \in A$ **by summing the weights of those orderings that rank** x **first from among the members of** A **and then chooses the alternative with the highest score**.

1. **Explain why a rational choice function is consistent with this procedure**.

Suppose that the decision maker is rational, i.e. he chooses the $\succeq$-maximal element for some preference relation $\succeq$. Then let $\alpha_1 = 1$ and $\succ_1 \; = \succeq$.

2. **Give an example to show that the procedure can produce a choice function which is not rationalizable**.

Consider the following situation: $X = \{a,b,c\}$ and

$$\begin{array}{ll} a \succ_1 b \succ_1 c & \alpha_1 = 5 \\ b \succ_2 c \succ_2 a & \alpha_2 = 4 \\ c \succ_3 b \succ_3 a & \alpha_3 = 3 \end{array}$$

Then, $C(X) = a,\ C(\{a,b\}) \; = b \in X$, and therefore C violates condition α.

3. **Show that for** $|X| = 3$ **all choice functions are consistent with the procedure**.

Let $X = \{a,b,c\}$. If the choice function is rationalizable, then by part 1.we are done. If C is not rationalizable, then condition α is violated and therefore WLOG we can assume that $C(X) = a,\ C(\{a,b\}) \; = b$. Define $R(\{x,y\})$ to be the element that is rejected from $\{x,y\}$ and define the orderings as:

$$\begin{array}{ll} a \succ_1 C(\{b,c\}) \succ_1 R(\{b,c\}) & \alpha_1 = 5 \\ b \succ_2 C(\{a,c\}) \succ_2 R(\{a,c\}) & \alpha_2 = 4 \\ c \succ_3 b \succ_3 a & \alpha_3 = 3 \end{array}$$

4. **Explain why it is not generally true that a choice function** C **which is derived from this procedure satisfies the condition that if** $x = C(A) = C(B)$, **then** $x = C(A \cup B)$.

Our example in part 2 violates this condition since $b = C(\{a,b\}) = C(\{b,c\})$, but $b \neq C(\{a,b,c\})$. The issue here is that while b is preferred to a and to c in two preference orderings each (whose sum of weights is greater than 5), it is only the maximal element of one ordering whose weight is not the highest. Condition α is not satisfied here since adding new elements can lead to other elements having a higher relative score.

5) (**More Difficult**) **Can you find a non-trivial property that is satisfied by choice functions which are derived from this procedure but not by all choice functions**? **Is there any choice function that cannot be explained by this procedure**?

There is no such non-trivial property since any choice function can be explained by this procedure.

The proof is by induction on $|X| = m$. The statement holds trivially for $m = 2$.

For the inductive step, suppose that it holds for all $m < M$. Let $|X| = M$. For each $x \in X$, let O_x be a set of orderings $O^x = \{\succ_1^x, \ldots, \succ_{n_a}^x\}$ and w^x be a vector of weights $w^x = (a_1^x, \ldots, a_{n_x}^x)$ such that:

- x is the top element in each ordering $\succ_j^x$.
- The orderings and weights explain the choice function on $X \backslash \{x\}$.

We can also normalize the weights such that $\sum_j \alpha_j^x = 1$.

Let $C(X) = a$. To explain the choice function on X we will use all the orderings in $\cup_{x \in X} O^x$. The weight attached to ordering $\succ_j^x$ will be α_j^x but the orderings $\succ_j^a$ will get an additional positive weight of ϵ/n_a where $\epsilon > 0$ is small enough (let $score^z(x)$ be the total score of x in O^z and choose ϵ smaller than all differences $|score^z(x) - score^z(y)|$).

To check that the procedure with these orderings and weights explains the choice function we need to examine the following cases:

Case (i): $A = X$. The element a scores $1 + \epsilon$ and every other element in X scores 1. Thus, a is chosen by the procedure.

Case (ii): $a \notin A \neq X$. Each element $y \in A$ scores 1 in O^y and 0 in any other O^z where $z \in A - \{y\}$. In any other group of orderings O^z, where $z \notin A$, the element $C(A)$ receives the highest score (because $z \notin A$, by definition, O^z together with w^z explain the choice from A). Thus, by summing all the scores, $C(A)$ is selected by the procedure.

Case (iii): $a \in A \neq X$. Each element $y \in A - \{a\}$ scores 1 in O^y and a scores $1 + \epsilon$ in O^a. In any other group of orderings O^z, where $z \notin A$, the element $C(A)$ receives a strictly higher score than any other element. ϵ was selected to be small enough such that the advantage of a does not change the order the total scores, and by summing all the scores, $C(A)$ is selected by the procedure.

Problem A11 (**NYU 2013**)

An agent makes a binary comparison of pairs of numbers. His real interest is to maximize the sum $x_1 + x_2$. **When he compares** (x_1, x_2) **and** (y_1, y_2) **he always makes the right decision if one of the pairs dominates the other. When this is not the case he might make a mistake. The technology of mistakes is characterized by a function** $\alpha(G, L)$ **with the interpretation that if the gain in one dimension is** $G \geq 0$ **and the loss in the other is** $L \geq 0$, **then the probability of a mistake is** $\alpha(G, L)$.

1. Suggest reasonable and workable assumptions for the function α (**such as** $\alpha(G, L) \leq 1/2$ **for all** G **and** L).

Some possible assumptions on the function:

a. $\alpha(G, L) \leq \frac{1}{2}$

b. $\alpha(G, 0) = \alpha(0, L) = 0$

c. $\alpha(G, L) = 0$ if $G = L$ (this expresses the fact that there can be no mistake in this case).

d. $\alpha(G, L)$ is decreasing in G and decreasing in L (i.e. as the gain gets larger relative to the loss, the agent is more likely to choose the correct pair).

e. $\alpha(G, L) \leq \alpha(G', L')$ if $|G - L| \geq |G' - L'|$ (the agent is less likely to make a mistake when the difference between the optimal and suboptimal pair is larger).

f. $\alpha(G, L) \leq \alpha(G', L')$ when $|G - L| = |G' - L'|$ and $G' \geq G, L' \geq L$ so that the agent is more likely to make a mistake when he is dealing with larger numbers.

2. Suggest a formal notion which expresses the phrase "agent 1 **is more accurate in his choices than agent** 2."

We can say that agent 1 makes less mistakes than agent 2 if for any G, L, we have $\alpha_1(G, L) \leq \alpha_2(G, L)$ and for some (G, L), the inequality is strict.

3. Show that according to the notion you defined in 2 the probability that three binary comparisons on the triple $(7, 2)$, $(3, 10)$, $(0, 6)$ **yields a cycle is smaller for the agent who is more accurate in his choices**.

First, note that $(3, 10)$ strictly dominates $(0, 6)$ so that each agent will always make the correct choice $(3, 10) \succ (0, 6)$.

The only possible cycle is:

$$(7, 2) \succ (3, 10) \succ (0, 6) \succ (7, 2)$$

The probability of this cycle is $\alpha_i(4, 8) \cdot \alpha_i(7, 4)$. If $\alpha_1(G, L) \leq \alpha_2(G, L)$, then the probability of a cycle for $1 = \alpha_1(4, 8) \cdot \alpha_1(7, 4) \leq \alpha_2(4, 8) \cdot \alpha_2(7, 4) =$ probability of a cycle for 2.

4. **Show that the probability of the binary comparisons yielding a cycle on a general triple of pairs is not necessarily smaller for the agent who is more accurate**.

Consider the triple $(7,4),(1,7),(0,6)$. Since $(1,7)$ dominates $(0,6)$, each agent will always choose $(1,7) \succ (0,6)$. Thus, the only possible cycle is

$$(7,4) \succ (1,7) \succ (0,6) \succ (7,4)$$

The probability of this cycle is $(1-\alpha_i(6,3))\alpha_i(7,2)$. We can then set the probabilities of the two agents to be

Agent	$\alpha_i(6,3)$	$\alpha_i(7,2)$
$i=1$	$\frac{1}{4}$	$\frac{1}{4}$
$i=2$	$\frac{1}{2}$	$\frac{1}{4}$

Then, the probability that agent 2's answers exhibit a cycle is $(1-\frac{1}{2})\frac{1}{4} = \frac{1}{8}$ and the probability that agent 1's answers exhibit a cycle is $(1-\frac{1}{4})\frac{1}{4} = \frac{3}{16}$.

Problem A12 (**Tel Aviv 2014**)

Consider a world in which the grand set X is the entire plane and choice sets can only be less than 180 degree closed arcs of the unit circle. Denote a choice set by $B(\alpha,\beta)$ where α and β, are the two angles that confine the arc which are numbers between 0 and 360. For example, $B(0,90)$ is one-quarter of a circle contained in the positive quadrant.

a. Give an example of a choice function that does not satisfy the weak axiom of revealed preference.

Assume that the agent always chooses the middle of the arc. Then, WA is violated. For example, the agent chooses the middle bundle x from the quarter of a circle $B(0,60)$, which lies on the 30° ray, while from the smaller arc $B(0,40)$, which still contains x, he chooses the bundle that lies on the 20-degree ray.

b. Give an example of a choice function that satisfies the weak axiom of revealed preference and yet is not rationalizable.

Assume that the agent always chooses the leftmost bundle on the arc (looking from the origin). The bundles chosen in $B(0,120)$, $B(120,240)$ and $B(240,0)$ create a cycle.

However, WA is never violated: Assume that the agent chooses x while y is another point on the arc. Since each arc is less than 180° and x is the leftmost point on the arc, we can conclude that y is less than 180° right of x. Therefore, x is more than 180° right of y. Thus, if y is chosen as the leftmost point on a given arc, then x cannot be on that arc as well. Thus, if x is revealed to be preferred to y, then y is never revealed to be preferred to x.

Assume now that the choice sets are only arcs in the positive quadrant (i.e. the two angles that define the choice sets are between 0° and 90°) and that the agent maximizes a monotonic, continuous and strictly convex preference relation.

c. Show that the agent's choice function is well defined.

The choice set is compact and therefore the continuity of the preferences implies that a solution to the agent's problem in $B(\alpha,\beta)$ does exist.

Assume, in contradiction, that the two alternatives x and y are solutions. Then, $x \sim y$ and by the strict convexity of the preferences, $\alpha x+(1-\alpha)y \succ y$. However, $\alpha x+(1-\alpha)y$ is strictly below the budget set and by the monotonicity of the preferences there exists a bundle on $B(\alpha,\beta)$ that is strictly preferred to $\alpha x+(1-\alpha)y$. By transitivity, this bundle is also preferred to y. A contradiction.

d. **Explain how one could identify the agent's choice function from the indirect preference relation** (**defined over the parameters of the choice sets**).

Consider $B(\alpha, \beta)$. Let α^* be the maximum angle x between α and β such that $B(\alpha, \beta)$ and $B(x, \beta)$ are indifferent according to the indirect preferences. Then, the point on $B(\alpha, \beta)$ at angle α^* is the chosen alternative from $B(\alpha, \beta)$.

B. **The Consumer and The Producer**

Problem B1 (**Tel Aviv 1998**)

A consumer with wealth $w = 10$ "must" obtain a book from one of three stores. **Denote the prices at each store as p_1, p_2, p_3. All prices are below w in the relevant range**. **The consumer has devised a strategy**: **he compares the prices at the first two stores and purchases the book from the first store if its price is not greater than the price at the second store**. **If $p_1 > p_2$, he compares the prices of the second and third stores and purchases the book from the second store if its price is not greater than the price at the third store**. **He uses the remainder of his wealth to purchase other goods**.

1. **What is this consumer's "demand function"**?

Denote a bundle by (m, x_1, x_2, x_3) where m is remaining wealth, and x_i is 1 if he purchases the book from the i'th store and 0 otherwise.

The demand function $x(w, p_1, p_2, p_3)$ is
$$\begin{array}{ll} (w - p_1, 1, 0, 0) & \text{if } p_1 \le p_2 \\ (w - p_2, 0, 1, 0) & \text{if } p_1 > p_2 \le p_3 \\ (w - p_3, 0, 0, 1) & \text{if } p_1 > p_2 > p_3 \end{array}$$

2. **Does this consumer satisfy "rational man" assumptions**?

No. Under the price vector $(3, 3, 1)$, the consumer buys the book in the first shop and his consumption bundle is $(w - 3, 1, 0, 0)$ whereas $(w - 1, 0, 0, 1)$ was feasible.

Under the price vector $(3, 2, 1)$, the consumer buys the book in the third shop, the bundle $(w - 1, 0, 0, 1)$ whereas $(w - 3, 1, 0, 0)$ was feasible.

3. **Consider the function $v(p_1, p_2, p_3) = w - p_{i^*}$, where i^* is the store from which the consumer purchases the book if the prices are (p_1, p_2, p_3). What does this function represent**?

Since the consumer must buy the book, his welfare is measured by the money left in his pocket. Thus, v could be thought of as an indirect utility function.

4. **Explain why $v(\cdot)$ is not monotonically decreasing in p_i. Compare with the indirect utility function of the classic consumer model**.

If $p_3 < p_1 < p_2$ then an increase in p_1 will improve the consumer's welfare. In contrast, an indirect utility function which is derived from a maximization of a utility function is always non-increasing in a price.

Problem B2 (**Princeton 2001**)

1 Define a formal concept for "$\succsim_1$ *and* $\succsim_0$ **are closer than** $\succsim_2$ *and* $\succsim_0$ ".

Let us say that $\succsim_1$ and $\succsim_0$ are closer than $\succsim_2$ and $\succsim_0$ if $x \succsim_2 y$ and $x \succsim_0 y$ imply $x \succsim_1 y$, and $x \succ_2 y$ and $x \succ_0 y$ imply $x \succ_1 y$.

2 Apply your definition to the class of preference relations represented by $U_1 = tU_2 + (1-t)U_0$, **where the function** U_i **represents** $\succsim_i$ $(i = 0,1,2)$.

Let $0 \le t \le 1$. If $U_2(x) \ge [>]U_2(y)$ and $U_0(x) \ge [>]U_0(y)$, then $U_1(x) = tU_2(x) + (1-t)U_0(x) \ge [>]tU_2(y) + (1-t)U_0(y) = U_1(y)$. So the preferences represented by U_1 and the preferences represented by U_0 are closer than the preferences represented by U_2 and the preferences represented by U_0.

3. **Consider the above definition in the consumer context**. **Denote by** $x_k^i(p,w)$ **the demand function of** $\succsim_i$ **for good** k. **Show that** $\succsim_1$ *and* $\succsim_0$ **may be closer than** $\succsim_2$ *and* $\succsim_0$, **and nevertheless** $|x_k^1(p,w) - x_k^0(p,w)| > |x_k^2(p,w) - x_k^0(p,w)|$ **for some commodity** k, **price vector** p **and wealth level** w.

Let $K = 3$, $U_0(x) = \min\{x_1, x_2\}$ and $U_2(x) = \min\{x_2, x_3\}$ and let U_1 be defined as in (b) with $t = 1/2$. Then for $p = (1,1,1)$ and $w = 1$, the demands associated with U_0, U_1 and U_2 are $x_0 = (1/2, 1/2, 0)$, $x_1 = (1/3, 1/3, 1/3)$ and $x_2 = (0, 1/2, 1/2)$ respectively. By (b), the preferences induced by U_1 and the preferences induced by U_0 are colser than the preferences induced by U_2 and the preferences induced by U_0, but $|x_2^1 - x_2^0| = 1/6 > 0 = |x_2^2 - x_2^0|$.

Problem B3 (**Princeton 2002**)

Consider a consumer with a preference relation in a world with two goods, X (**an aggregated consumption good**) **and** M ("**membership in a club**," **for example**), **which can be consumed or not**. **In other words**, **the consumption of** X **can be any nonnegative real number**, **while the consumption of** M **must be either** 0 **or** 1. **Assume that the consumer's preferences are strictly monotonic**, **continuous and satisfy the following property**:

Property E: **For every**, x **there is** y **such that** $(y,0) \succ (x,1)$ (**that is**, **there is always some amount of the aggregated consumption good that can compensate for the loss of membership**).

1. **Show that any consumer's preference relation can be represented by a utility function of the type**:

$$u(x,m) = \begin{cases} x & if \quad m = 0 \\ x + g(x) & if \quad m = 1 \end{cases}.$$

For any x let $h(x)$ satsfies $(x,1) \sim (h(x),0)$. Notice that such $h(x)$ always exists and it is unique. This is because $(0,0) \prec (x,1)$ by monotonicity and $(y,0) \succ (x,1)$ for some y by property E so continuity implies that $(x,1) \sim (z,0)$ for some z. Also, it must be unique because of monotonicity. The function h is increasing.

Let $g(x) = h(x) - x$.

Let's verify that u actually represents $\succsim$.

Case 1 $(x,0) \succsim (x',0)$, iff $x \geq x'$ iff $u(x,0) = x \geq x' = u(x',0)$

Case 2 $(x,1) \succsim (x',1)$ iff $x \geq x'$ iff $h(x) \geq h(x')$ iff $u(x,1) = h(x) \geq h(x') = u(x',1)$

Case 3 $(x,1) \succsim (\precsim)(x',0)$ iff $(h(x),0) \succsim (\precsim)(x',0)$ iff $h(x) \geq x'$ iff $u(x,1) = h(x) \geq (\leq)x' = u(x',0)$

Therefore, u represents $\succsim$.

2. (**Less easy**) **Show that the consumer's preference relation can also be represented by a utility function of the type**:

$$u(x,m) = \begin{cases} f(x) & if \quad m = 0 \\ f(x) + v & if \quad m = 1 \end{cases}.$$

Define $h^0(x) = x$ and $h^{n+1}(x) = h(h^n(x))$. Since $(x,1) \sim (h(x),0) \succ (x,0)$, $h^{n+1}(x) > h^n(x)$ for all n.

(1) For $x \in [0, h(0)]$ let $f(x)$ be an arbitrary increasing function with $f(0) = 0$. Let $v = f(h(0))$.

(2) Continue inductively: assume that $f(x)$ has been already defined for $x \in [0, h^{n-1}(0)]$ such that $f(x)$ is increasing in the region $[h(0), h^{n-1}(0)]$ and $f(h^k(0)) = kv$ for all $k \leq n-1$. Define $f(x)$ for $x \in (h^{n-1}(0), h^n(0)]$, such that $f(x) = v + f(h^{-1}(x))$. Since h is an increasing function, h^{-1} exists, increasing and $h^{-1}(x) \in (0, h(0)]$. The function f is increasing in $x \in [h^{n-1}(0), h^n(0)]$ and $f(h^n(0)) = nv$.

(3) Let $u(x,0) = f(x)$ and $u(x,1) = f(x) + v$.

We have to make sure that f is actually defined for all $s \geq 0$. Since $\{h^n(0)\}$ is an increasing sequence, $f(x)$ is not defined twice or more in the above steps. Therefore, it is sufficient to show that $\lim_{n\to\infty} h^n(0) = \infty$.

Suppose this is not, since $\{h^n(0)\}$ is an increasing sequence, let $sup_n h^n(0) = K$ and $h^n(0) < K$ for all n. This means that $(h^n(0), 1) \sim (h^{n+1}(0), 0) \prec (K, 0)$ for all n where the first indifference comes from the definition of h and the second preference comes from monotonicity. By continuity and since $h^n(0) \to K$, $(K,1) \precsim (K,0)$, contradicting monotonicity.

Finally, we need to confirm that this u actually represents $\succsim$. Notice that u represents $\succsim$ correctly between $(x,0)$ and $(x',0)$ (or $(x,1)$ and $(x',1)$) because f is an increasing function. Therefore, we need to show that $(x,1) \succsim (x',0)$ if and only if $f(x) + v \geq f(x')$.

Suppose that $(x,1) \succsim (x',0)$. Then, by the definition of h, $h(x) \geq x'$. Since f is increasing, we have $u(h(x),0) = f(h(x)) \geq f(x') = u(x',0)$. By construction of f, we have $f(h(x)) = v + f(h^{-1}(h(x))) = v + f(x) = u(x,1)$. Therefore, $u(x,1) \geq u(x',0)$. The same argument can be applied for the case when $(x,1) \precsim (x',0)$. Hence, we conclude that u defined above actually represents $\succsim$.

3. Explain why continuity and strong monotonicity (without prperty E) are not sufficient for (a)

Consider the lexicographic preference $\succsim$ which gives a priority to M. (so $(x,m) \succ (x',m')$ if and only if $m > m'$ or "$m = m'$ and $x > x'$").

This is clearly strictly monotonic both in x and m.

Continuity: To see this, take any $(x,m) \succ (x',m')$.

Suppose $m = 1$ and $m' = 0$. Take $\epsilon = 1/2$, then $(a,n) \in B_\epsilon((x,m))$ implies $n = 1$ and $(a',n') \in B_\epsilon((x',m'))$ implies $n' = 0$. Therefore, $(a,n) \succ (a',n')$. WHen $m = m'$ (so $x > x'$) let $\epsilon = (x - x')/3$. Then, $(a,n) \in B_\epsilon(x,m)$ and $(a',n') \in B_\epsilon((x',m'))$ imply that $n = n' = m$ but $a > a'$, so we have $(a,n) \succ (a',n')$.

However, $\succsim$ cannot be represented by a utility function with the form given in (A) since (0,1) would be indifferent to $(g(0),0)$.

4. **Calculate the consumer's demand function**.

For simplicity assume $p_x = 1$. By monotonicity, the consumer always spends all wealth on X or M. If $p_m \leq w$, his choice is between "buying M and spending all the remaining wealth on X" and "Spending all wealth on X" . If $p_m > w$, he has no choice except "spending all the wealth on X". Therefore, his demand function is characterized by

$$(x(p,w),m(p,w)) = \begin{cases} (w,0) & \text{if } (w,0) \succsim ((w-p_m)/p_x,1) \text{ or } p_m > w \\ (w-p_m,1) & \text{if } (w,0) \precsim ((w-p_m)/p_x,1) \text{ and } p_m \leq w \end{cases} .$$

5. **Taking the utility function to be of the form described in** (**1**), **derive the consumer's indirect utility function**. **For the case where the function** g **is differentiable**, **verify Roy's identity with respect to commodity** M.

If the consumer's utility function is given by a differentiable utility function as in part (A), then his/her indirect utility function is

$$v(p,w) = \begin{cases} w & \text{if } w \geq (w-p_m)+g((w-p_m)) \text{ or } p_m > w \\ (w-p_m)+g((w-p_m)) & \text{if } (w,0) \leq ((w-p_m),1) \text{ and } p_m \leq w \end{cases}$$

In the first case,

$$\frac{\partial v/\partial p_m}{\partial v/\partial w} = 0 = m(p,w)$$

and in the second case,

$$\frac{\partial v/\partial p_m}{\partial v/\partial w} = -\frac{(-1)-g'(w-p_m)}{1+g'(w-p_m)} = 1 = m(p,w)$$

so the Roy equality holds.

Problem B4 (**Tel Aviv 2003**)

Consider the following consumer problem: **There are two goods**, **1 and 2**. **The consumer has a certain endowment**. **His preferences satisfy monotonicity and continuity**. **Before the consumer are two** "**exchange functions**": **he can exchange** x **units of good 1 for** $f(x)$ **units of good 2**, **or he can exchange** y **units of good 2 for** $g(y)$ **units of good 1**. **Assume the consumer can only make one exchange**.

1. **Show that if the exchange functions are continuous**, **then a solution to the consumer problem exists**.

Denote by w the initial endowment. the budget set includes $\{(x_1, x_2)|\ x_1 \leq w_1$ and $(x_2 - w_2) = f(w_1 - x_1)$, or, $x_1 \geq w_1$ and $(x_1 - w_1) = g(w_2 - x_2)\}$. This is a compact set and thus the consumer's preferences has a maximum point.

2. **Explain why strong convexity of the preference relation is not sufficient to guarantee a unique solution if the functions** f **and** g **are increasing and convex**.

Take, for example, $w = (4,4)$, $f(x) = 2x$ and $g(y) = 2y$. Consider the utility function $u(x_1, x_2) = x_1 x_2$. Clearly both $(6,3)$and $(3,6)$are solutions.

3. **Interpret the statement** "**the function** f **is increasing and convex**".

The more units the consumer will exchange of commodity 1 the better will be is exchange rate.

4. **Suppose both functions** f **and** g **are differentiable and concave and that the product of their derivatives at point** 0 **is** 1. **Suppose also that the preference relation is strongly convex**. **Show that under these conditions**, **the agent will not find two different exchanges**, **one exchanging good 1 for good 2**, **and one exchanging good 2 for good 1**, **optimal**.

Since $f'(0) = 1/g'(0)$, the consumer's budget line is differentiable at (w_1, w_2), which is his original endowment (note that f is a function from good 1 to good 2 while g is a function from good 2 to good 1). The functions f and g are both concave, thus the budget line is convex. Assume, in contradiction, that the consumer finds two exchanges to be optimal, one that yields the consumption bundle (x_1, x_2) and one that yields (y_1, y_2). Since the budget line is convex, the bundle $(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2})$ is in the consumer's budget set. Since the consumer is indifferent between (x_1, x_2) and (y_1, y_2), by the strict convexity of his preference relation he would strictly prefer the bundle $(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2})$, in contradiction to

(x_1, x_2) and (y_1, y_2) being optimal.
If it would be the case it would mean that there are two exchanges $(-a, +f(a))$ and $(+g(b), -b)$ which are better than $(0, 0)$. By the concavity of f and g, $f(a) < a$ and $g(b) < b$. Let $\lambda/(1-\lambda) = b/a$. By convexity the change $(-\lambda a + (1-\lambda)g(b), +\lambda f(a) - (1-\lambda)b)$ is improving but $-\lambda a + (1-\lambda)g(b) < -\lambda a + (1-\lambda)b = 0$ and $\lambda f(a) - (1-\lambda)b < \lambda a - (1-\lambda)b = 0$ a contradiction to the monotonicity of the preferences.

5. **Now assume** $f(x) = ax$ **and** $g(y) = by$. **Explain this assumption**. **Find a condition that will ensure it is not profitable for the consumer to make more than one exchange**.
If the consumer sells one unit of the first commodity he gets a of the second and exchanging it back to commodity 2 he will get ab units of commodity 1. It is necessary that $ab < 1$. Of course if $ab > 1$ he would be able to reach any bundle.

Problem B5 (**NYU 2005**)

A consumer has preferences which satisfy monotonicity, **continuity and strict convexity**, **in a world of** K **goods**. **The goods are split into two categories**, 1 **and** 2, **of** K_1 **and** K_2 **goods respectively** $(K_1 + K_2 = K)$. **The consumer receives two types of money**: w_i **units of money of type** i, **which can only be exchanged for goods in the i'th category given a price vector** p_i.

Define the induced preference relation over the two-dimensional space (w_1, w_2). **Show that these preferences are monotonic**, **continuous and convex**.

The consumer's problem is to mximize the preferences $\succsim$ over the set $\{(x_1, x_2)| x_1 \in \mathbb{R}^{K_1}, x_2 \in \mathbb{R}^{K_2},\ p_1x_1 \leq w_1$ and $p_2x_2 \leq w_2\}$. Denote its solution by $x(w_1, w_2)$. Define the preference relation over the space (w_1, w_2) as

$$(w_1, w_2) \succsim^* (w_1', w_2') \Leftrightarrow x(w_1, w_2) \succsim x(w_1', w_2')$$

Now let's prove the properties of $\succsim^*$.

Monotonicity: Let $(w_1', w_2') \gg (w_1, w_2)$. This implies that

$$p_1x_1(w_1, w_2) < w_1' \text{ and } p_2x_2(w_1, w_2) < w_2'.$$

The budget conatraints contain a bundle y which "dominates" $x(w_1, w_2)$. Therefore, by Monotonicity of $\succsim$ we get $x(w_1', w_2') \succ x(w_1, w_2)$, that is $(w_1', w_2') \succ^* (w_1, w_2)$.

Convexity: Let $(w_1', w_2') \succsim^* (w_1, w_2)$. This implies that $x(w_1', w_2') \succsim x(w_1, w_2)$. Let $w^\lambda = \lambda(w_1', w_2') + (1-\lambda)(w_1, w_2)$ and $x^\lambda = \lambda x(w_1', w_2') + (1-\lambda)x(w_1, w_2)$ for some $\lambda \in [0,1]$. We know that

$$p_1x_1(w_1, w_2) \leq w_1 \text{ and } p_2x_2(w_1, w_2) \leq w_2$$
$$p_1x_1(w_1', w_2') \leq w_1' \text{ and } p_2x_2(w_1', w_2') \leq w_2'$$

which implies that

$$p_1x_1^\lambda \leq w_1^\lambda,\ p_2x_2^\lambda \leq w_2^\lambda.$$

Moreover, convexity of $\succsim$ implies that $x^\lambda \succsim x(w_1, w_2)$. Therefore, it must be that $x(w_1^\lambda, w_2^\lambda) \succsim x^\lambda \succsim x(w_1, w_2)$, which implies that $(w_1^\lambda, w_2^\lambda) \succsim^* (w_1, w_2)$.

Continuity: To show this we'll first show that the demand function x is continuous. So, let $(w_1^n, w_2^n) \to (w_1, w_2)$. We need to show that $x(w_1^n, w_2^n) \to x(w_1, w_2)$. Note first that the sequence $x(w_1^n, w_2^n)$ is inside of some compact set: define

$$\overline{w}_1 = \sup\{w_1^n\}, \overline{w}_2 = \sup\{w_2^n\}.$$

It's clear that for any n we must have

$$x(w_1^n, w_2^n) \in \{x \in \Re_+^{k_1+k_2} : p_1x_1 \leq \overline{w}_1 \text{and } p_2w_2 \leq \overline{w}_2\}.$$

So our sequence is indeed entirely contained in a compact set. Now suppose that

$x(w_1^n, w_2^n) \nrightarrow x(w_1, w_2)$. This implies that there exists a subsequence $x(w_1^k, w_2^k)$, such that $x(w_1^k, w_2^k) \to y \neq x(w_1, w_2)$. The fact that $x(w_1^k, w_2^k) \to y$ implies that

$$p_1 y_1 \leq w_1,\ p_2 y_2 \leq w_2.$$

By the strict convexity, $x(w_1, w_2) \succ (y_1, y_2)$. Continuity of $\succsim$ implies that there exists $z \ll x(w_1, w_2)$ such that $z \succ (y_1, y_2)$. But then, continuity of $\succsim$ together with the fact that $p_1 z_1 < w_1$ and $p_2 z_2 < w_2$ imply that for k large enough

$$p_1 z_1 < w_1^k, p_2 z_2 < w_2^k, z \succ x(w_1^k, w_2^k)$$

which contradicts the optimality of $x(w_1^k, w_2^k)$.

Now that we know that the demand function is continuous we can easily show that $\succsim^*$ is continuous. Let $(w_1^n, w_2^n) \to (w_1, w_2)$ and $(\hat{w}_1^n, \hat{w}_2^n) \to (\hat{w}_1, \hat{w}_2)$ be such that $(w_1^n, w_2^n) \succsim^* (\hat{w}_1^n, \hat{w}_2^n)$ for all n. This implies that $x(w_1^n, w_2^n) \succsim x(\hat{w}_1^n, \hat{w}_2^n)$ for all n. By continuity of $\succsim$ and continuity of the demand function x, $x(w_1, w_2) \succsim x(\hat{w}_1, \hat{w}_2)$, which is equivalent to $(w_1, w_2) \succsim^* (\hat{w}_1, \hat{w}_2)$.

Problem B6 (**NYU 2005**. **Inspired by Chen**, **Lakshminarayanan and Santos** (**2005**).)

In an experiment, **a monkey is given** $m = 12$ **coins which he can exchange for apples or bananas**. **The monkey faces** m **consecutive choices in which he gives a coin either to an experimenter holding** a **apples or another experimenter holding** b **bananas**.

1. **Assume that the experiment is repeated with different values of** a **and** b **and that each time the monkey trades the first** 4 **coins for apples and the next** 8 **coins for bananas**.

Show that the monkey's behavior is consistent with the classical assumptions of consumer behavior (**namely**, **that his behavior can be explained as the maximization of a monotonic**, **continuous and convex preference relation on the space of bundles**).

We can model the problem of the monkey as the following. It has wealth m and can buy two commodities, apples and bananas. The price of one apple is $p_a = 1/a$ and the price of one banana is $p_b = 1/b$. What the experiment shows is that the monkey always spends $1/3$ of its wealth on apples and $2/3$ on bananas. This behavior is consistent with Cobb-Douglas preferences with weights $1/3$ and $2/3$.

2. **Assume that it was later observed that when the monkey holds an arbitrary number of coins**, **m**. **Then**, **irrespective of the values of** a **and** b, **he exchanges the first** 4 **coins for apples and the remaining** $m - 4$ **coins for bananas**. **Is this behavior consistent with the rational consumer model**?

No. Assume, by contradiction, that $\succsim$ is monotonic and rationalizes the monkey behavior. Let $m = 4$, $a = 2$ and $b = 4$. In this case the monkey would buy the following bundle $(8,0)$. Since the monkey could also buy $(4,8)$, it must be the case that $(8,0) \succsim (4,8)$. Monotonicity of $\succsim$ implies that $(9,1) \succ (4,8) \succsim (4,6)$.

Now let's look at the monkey's choice when $m = 10$, $a = 1$ and $b = 1$. In this case the monkey would chose $(4,6)$ though $(9,1)$ is feasible. A contradiction.

Problem B7 (**NYU 2006**)

Consider a consumer in a world of 2 commodities who has to make choices from budget sets parametrized by (p,w), **with the additional constraint that the consumption of good 1 is limited by some external bound** $c \geq 0$. **That is**, **in his world**, **a choice problem is a set of the form** $B(p,w,c) = \{x|px \leq w \text{ and } x_1 \leq c\}$. **Denote by** $x(p,w,c)$ **the consumer's choice from** $B(p,w,c)$.

1. **Assume that** $px(p,w,c) = w$ **and** $x_1(p,w,c) = \min\{0.5w/p_1, c\}$. **Show that this behavior is consistent with the assumption that demand is derived from a maximization of some preference relation**.

Maximization the preferences $\succsim$ represented by the Cobb-Douglas utility function $u(x) = x_1^{0.5}x_2^{0.5}$ leads to the given demand function $x(p,w,c)$. There are two different cases to consider.

Case 1: $0.5w/p_1 \leq c$

In this case, $x(p,w,c)$ is equal to the point $(0.5w/p_1, 0.5w/p_2)$, which is the unique maximizer of u on the usual budget set $B(p,w) = \{x \mid px \leq w\}$. In particular, $x(p,w,c)$ is the unique maximizer of u on $B(p,w,c)$.

Case 2: $0.5w/p_1 > c$

Suppose by contradiction that $x(p,w,c) = (c,(w-p_1c)/p_2)$ is not the unique maximizer of u on $B(p,w,c)$. Then there is a $y \in B(p,w,c)$ with $y \neq x(p,w,c)$ such that $y \succsim x(p,w,c)$. By strict monotonicity of u we must have $y_1 < c$, otherwise we would have $x(p,w,c) \succ y$. Moreover, we can assume $py = w$. Define $\bar{x} = (0.5w/p_1, 0.5w/p_2)$. Now, since $y_1 < c < 0.5w/p_1$, $x(p,w,c)$ can be written as a strict convex combination of the points y and $\bar{x}$. Since $\bar{x}$ is the unique maximizer of u on $B(p,w)$, by strict convexity of $\succsim$ we must have $x(p,w,c) \succ y$, a contradiction.

2. **Assume that** $px(p,w,c) = w$ **and** $x_1(p,w,c) = \min\{0.5c, w/p_1\}$. **Show that this consumer's behavior is inconsistent with preference maximization**.

Suppose by contradiction that $x(p,w,c)$ is consistent with maximization of a preference relation $\succsim$. Fix a price vector p and wealth level w. Pick a $c < w/p_1$, so that $x(p,w,c) = (0.5c,(w-p_1 0.5c)/p_2)$. Now we can pick a c', sufficiently close to c, such that

$$0.5c' < c < c' < w/p_1.$$

Since $0.5c' < w/p_1$, $x(p,w,c') = (0.5c',(w-p_1 0.5c')/p_2) \neq x(p,w,c)$. Moreover, since $0.5c' < c$, we have $x(p,w,c') \in B(p,w,c)$, and hence $x(p,w,c') \prec x(p,w,c)$, a contradiction to the optimality of $x(p,w,c')$.

3. **Assume that the consumer chooses his demand for x by maximizing the utility function** $u(x)$. **Denote the indirect utility by** $V(p, w, c) = u(x(p, w, c))$. **Assume** V **is "well-behaved". Outline the idea of how one can derive the demand function from the function** V **in case that** $\partial V/\partial c(p, w, c) > 0$.

Fix a parameter vector $t^* = (p^*, w^*, c^*)$ and assume $\partial V(t^*)/\partial c > 0$. We claim that $x_1(t^*) = c^*$.

Suppose by contradiction that $x_1(t^*) < c^*$. However, since $\partial V(t^*)/\partial c > 0$, there exists an $\epsilon > 0$ such that $V(p^*, w^*, c^* - \epsilon) < V(t^*)$ and $c^* - \epsilon > x_1(t^*)$. Since $p^* x(t^*) = w^*$, it follows that $x(t^*) \in B(p^*, w^*, c^* - \epsilon)$. Hence, $V(t^*) = u(x(t^*)) \le V(p^*, w^*, c^* - \epsilon)$, a contradiction to $\partial V(t^*)/\partial c > 0$.

Problem B8 (**Tel Aviv 2006**)

Imagine a consumer who lives in a world with $K+1$ **commodities and behaves in the following manner**: **The consumer is characterized by a vector** D, **consisting of the commodities** $1,..,K$. **If he can purchase** D, **he will consume it and spend the rest of his income on commodity** $K+1$. **If he is unable to purchase** D, **he will not consume commodity** $K+1$ **and will purchase the bundle** tD ($t \leq 1$) **where** t **is as large as he can afford**.

1. **Show that there exists a monotonic and convex preference relation which explains this pattern of behavior**.

Consider the preferences represnted by

$$u(x) = \begin{cases} 1 + x_{K+1} & \text{if } x \geq D \\ \min\{x_1/d_1,\ldots,x_K/d_K\} & \text{otehrwise} \end{cases}$$

Since $\min\{x_1/d_1,\ldots,x_K/d_K\} \leq 1$ for all bundles in the second range, the consumer will prefer to be in the first range so whenever he can purchase D he will choose a bunlde in the first range where it is optimal for him to purchase as much as possible on the $K+1$'th good. If he cannot afford D then his behvaior will be as a amximizer of $\min\{x_1/d_1,\ldots,x_K/d_K\}$ and thus he will consume a bundle tD with the highest t he can afford.

2. **Show that there is no monotonic and continuous preference relation that explains this pattern of behavior**.

If there was such a preference relation it would have to satsify $(d_1,\ldots,d_{K,}2) \succ (d_1+1,\ldots,d_K+1,0)$. However, $(d_1,\ldots,d_{K,}0) \succ (d_1-\delta,d_2,\ldots,d_{K,}2)$ for all $\delta > 0$, thus by continuity $(d_1,\ldots,d_K,0) \succsim (d_1,\ldots,d_{K,}2)$ contradicting monotonicity.

Problem B9 (**NYU 2007**)

A consumer in a world of K **commodities maximizes the utility function** $u(x) = \sum_k x_k^2$.

1. **Calculate the consumer's demand function** (**whenever it is uniquely defined**).

Demand is uniquely defined when there is a unique minimal price p_k. The consumer will set $x_k(p,w) = \frac{w}{p_k}$ and $x_j(p,w) = 0$ for all $j \neq k$.

2. **Give another preference relation** (**not just a monotonic transformation of** u) **which induces the same demand function**.

Preferences represented by $v(x) = \sum_k x_k$.

3. **For the original utility function** u, **calculate the indirect preferences for** $K = 2$. **What is the relationship between the indirect preferences and the demand function**? (**It is sufficient to answer for the domain where** $p_1 < p_2$.)

The indirect preferences are represented by $v(p,w) = [\frac{w}{\min\{p_1,p_2\}}]^2$.

When $p_1 < p_2$, demand can be induced from v as follows:

$$x(p,w) = (-\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w}, -\frac{\partial v(p,w)/\partial p_2}{\partial v(p,w)/\partial w}) = (-\frac{-2w/p_1^2}{2/p_1}, 0) = (\frac{w}{p_1}, 0).$$

4. **Are the preferences in** (**1**) **differentiable** (**according to the definition given in class**)?

No. Let $K = 2$ and consider the bundle $(1,1)$. The vectors $d = (1,-1)$ and $d' = (-1,1)$ are both improvement directions, but for any vector of values v, if $v \cdot d > 0$, then $v \cdot d' < 0$.

Problem B10 (**NYU 2008**)

A decision maker has a preference relation over the pairs (x_{me}, x_{him}) **with the interpretation that** x_{me} **is an amount of money he will get and** x_{him} **is the amount of money another person will get**. **Assume that**

(**i**) **for all** (a,b) **such that** $a > b$ **the decision maker strictly prefers** (a,b) **over** (b,a).

(**ii**) **if** $a' > a$ **then** $(a',b) \succ (a,b)$.

The decision maker has to allocate M **between him and another person**.

1. **Show that these assumptions guarantee that he will never allocate to the other person more than he gives to himself**.

Let $B(M) = \{(a,b)|a+b \leq M\}$ be the set of feasible allocations, and $x(M) = (x_1,x_2)$ be the chosen allocation from the feasible set, that is $x(M) \succsim (a,b)$ for any $(a,b) \in B(M)$. For any $a < b$ by (i) $(b,a) \succ (a,b)$, and (b,a) is feasible so that (a,b) is not optimal. Therefore $x_1 \geq x_2$.

2. **Assume** (**i**), (**ii**) **and**

(**iii**) **The decision maker is indifferent between** (a,a) **and** $(a-\varepsilon, a+4\varepsilon)$ **for all** a **and** $\varepsilon > 0$.

Show that nevertheless he might allocate the money equally.

Suppose the preferences are represented by, for example

$$u(x,y) = \begin{cases} 4x+y & \text{if } y \geq x \\ 2x+3y & \text{if } x \geq y \end{cases}$$

These preferences satisfy monotonicity.
The preferences satisfy (iii) since $\forall a$, $u(a,a) = 5a = 4(a-\varepsilon)+(a+4\varepsilon) = u(a-\varepsilon, a+4\varepsilon)$.
They also satisfy (i) since if $x_2 > x_1$, then $u(x_1,x_2) = 4x_1+x_2 = 5x_1+(x_2-x_1) < 5x_1+2(x_2-x_1) = 2x_2+3x_1 = u(x_2,x_1)$.
In this case, the utility would be maximized by setting $x = y = \frac{M}{2}$.

3. **Assume** (**i**), (**ii**), (**iii**) **and**

(**iv**) **The decision maker's preferences are also differentiable** (**according to the definition given in class**).

Show that in this case, **he will allocate to himself** (**strictly**) **more than to the other**.

Assume by contradiction the DM chooses to allocate $x(M) = (\frac{M}{2}, \frac{M}{2})$. By differentiability of the preferences, $v(x) = (4,1)$ (i.e. the hyperplane separating the improving directions would be the hyperplane $4x+y = M$). But then $(1,-1)$ is a strictly improving direction, and the bundle $(\frac{M}{2}+\epsilon, \frac{M}{2}-\epsilon) \succ (\frac{M}{2}, \frac{M}{2}) = x(M)$ would be affordable for small $\epsilon > 0$, a contradiction.

Problem B11 (**Tel Aviv 2010**)

A basketball coach considers buying players from a set A. **Given a budget** w **and a price vector** $(p_a)_{a \in A}$ **the coach can purchase any set such that the total cost of the players in it is not greater than** w. **Discuss the rationality of each of the following choice procedures**, **defined for any budget level** w **and price vector** P:

(**P1**) **The consumer has in mind a fixed list of the players in** A.: $a_1, \ldots, a_n$. **Starting at the beginning of the list**, **when he arrives to the** $i's$ **player he adds him to the team if his budget allows him to after his past decisions**, **and then continues to the next player on the list with his remaining budget**. **This continues until he runs out of budget or has gone through the entire list**.

Let $\{1, \ldots n\}$ be the list of players. Identify a set B with a vector $x(B)$ of 0's and 1's such that $x(B)_i = 1$ if $i \in B$ and $x(B)_i = 0$ if $i \notin B$. The consumer's choice is rationalized by the preferences: $B \succsim C$ if $x(B) \succsim_L x(C)$, where $\succsim_L$ are the standard lexicographic preferences on $\mathbb{R}^n$.

(**P2**) **He purchases the combination of players that minimize the excess budget he is left with**.

The procedure is NOT rationalizable since it is even does not induce a choice from a choice set:

The choice set is $\{\{a_1\}, \{a_2\}\}$ for both sets of parameters:

$p_1 = 2$, $p_2 = 3$ and $w = 3$

and

$p_1 = 3$, $p_2 = 2$ and $w = 3$.

But in one case $\{a_1\}$ is chosen and in the other $\{a_2\}$.

Problem B12 (**NYU 2010**)

A consumer in a two commodity world operates in the following manner:

The consumer has a preference relation $\succsim_S$ **on** $\mathbb{R}^2_+$. **His father has a preference relation** $\succsim_F$ **on the space of his son's consumption bundles**. **Both relations satisfy strong monotonicity**, **continuity and strict convexity**. **The father does not allow his son to purchase a bundle which is not as good** (**from his perspective**) **as the bundle** $(M,0)$. **The son**, **when choosing from a budget set**, **maximizes his own preferences subject to the constraint imposed by his father**. **In the case that he cannot satisfy his father's wishes**, **he feels free to maximize his own preferences**.

1. **Prove that the behavior of the son is rationalizable**.

Define $\succ$ as follows: $a \succ b$ iff (i) $a \succsim_F (M,0)$ and $(M,0) \succ_F b$, or (ii) both $a,b \succsim_F (M,0)$ and $a \succ_S b$ or (iii) both $(M,0) \succ_F a,b$ and $a \succ_S b$.

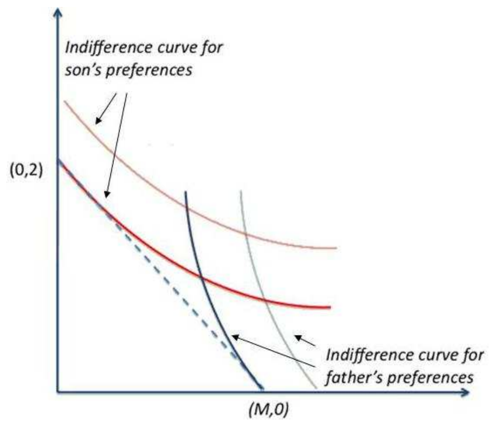
$\succ$ can easily be shown to be complete and transitive.

2. **Prove that the preferences which rationalize this kind of behavior are monotonic**.

Take any x,y st. $x_1 \geq y_1$ and $x_2 \geq y_2$. Since $\succsim_S$,$\succsim_F$ are monotonic, $x \succsim_S y$ and $x \succsim_F y$. Thus by construction of $\succ$, $x \succsim y$.

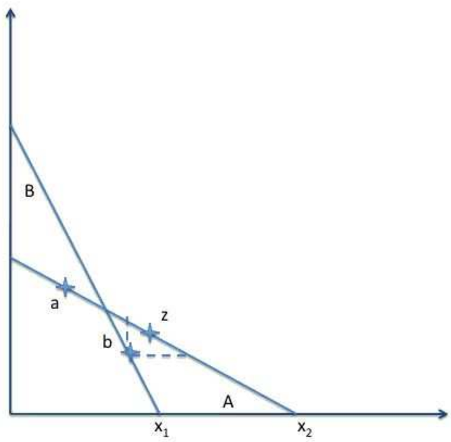
3. **Show that the preferences which rationalize this kind of behavior are not necessarily continuous nor convex** (**you can demonstrate this diagrammatically**).

To see possible violation of continuity and convexity consider the example below.



Note that $(M,0) \succ (0,2)$ since $(0,2)$ is below the father's indifference curve passing through $(M,0)$. However we can see from the son's preferences that for any $\alpha \in (0,1)$, $(0,2) \succ \alpha(0,2) + (1-\alpha)(M,0)$ violating convexity and continuity.

4. (**Bonus**) **Assume that the father's instructions are that given the budget set (p,w) the son is not to purchase any bundle which is $\succsim_F$ -worse than $(w/p_1,0)$. The son seeks to maximize his preferences subject to satisfying his father's wishes. Show that the son's behavior satisfies the Weak Axiom of Revealed Preferences.**



Assume there is a violation of the WARP. Then there must be two overlapping budget sets as shown above such that a is chosen from set A and b is chosen from set B.
It must be that $a \succsim_F (x_2,0)$ and $b \succsim_F (x_1,0)$. By monotonicity, $(x_2,0) \succsim_F (x_1,0)$ and thus $a \succsim_F (x_1,0)$ Since b is chosen over a in set B, $b \succ_S a$ By monotonicity, there exists $z \in B$ st $z \succsim_S b \succ_S a$ Also by convexity $z \succsim_F x_2$, contracting a being optimal in set A.

Problem B13 (NYU 2012)

A consumer operates in a world with K commodities. He has in mind a list of consumption priorities, a sequence (k_n, q_n) where $k_n \in \{1,\ldots,K\}$ is a commodity and q_n is a quantity (commodities may appear more than once in the sequence). When facing a budget set (p,w) he purchases the goods according to the order of priorities in the list, until his budget is exhausted. (In the case that his money is exhausted during the n'th stage he purchases whatever proportion of the quantity q_n that he can afford).

1. How does the demand for the k'th commodity responds to the p_k, p_j ($j \neq k$) and w?

Let y^n be the bundle which gets the value 0 for all commodities besides k_n and the value q_n for commodity k_n.

Given a budget set (p,w) let L be minimum l for which $\sum_{n=1}^{l} p_n q_n > w$. Then

$$x_k(p,w) = \begin{cases} \sum_{n=1}^{L-1} y_k^n & \text{if } k \neq k_L \\ \sum_{n=1}^{L-1} y_k^n + \frac{w - \sum_{n=1}^{L-1} p_{k_n} q_n}{p_k} & \text{if } k = k_L \end{cases}$$

In other words to demand is the highest feasible bundle on the line which connects the bundles

$0, y^1, y^1 + y^2, y^1 + y^2 + y^3 \ldots$

Increasing w weakly increases the demand to each of the commodities. Increasing each of the prices (weakly) decreases the demand to each of the goods.

2. Suggest an increasing utility function which rationalizes the consumer's behavior.

Define $u(x)$ to be the largest number t for which $\sum_{n=1}^{[t]} y^n + (t - [t]) y^{[t]+1} \leq x$ (where the inequality is inequality of vectors and $[t]$ is the greatest integer of t.) In other words, $u(x)$ is the number of tasks (could be 6.3) that can be fulfilled with the assets in x.

Clearly the consumer's behavior is derived from the maximization of this utility function.

3. Using the utility function you suggested in (ii) prove the Roy equality for this consumer at (p,w) where the consumer exhausts his entire budget while satisfying his n'th goal.

The indirect utility function $v(p,w)$ which is induced from this utility function is the number of stages which could be obtained given the income w and the price vector p.

Assume that at (p,w) the consumer is in the midst of the n'th stage. Changing the price of good k by a small ε changes the expense of the $k'th$ commodity by $\varepsilon x_k(p,w)$ and thus changes the stage of he can obtain by $-(\varepsilon x_k(p,w)/_{q_n p_n})$

Thus, $\frac{\partial v}{\partial p_k}(p,w) = -x_k(p,w)/_{q_n p_n}$.

A change in ε in the wealth allows a change of $(\varepsilon/_{q_n p_n})$ in his indirect utility and thus

Thus, $\frac{\partial v}{\partial w}(p,w) = 1/_{q_n p_n}$. It follows that:

$$-\frac{\frac{\partial v}{\partial p_k}(p,w)}{\frac{\partial v}{\partial w}(p,w)} = x_k(p,w)$$

Problem B14 (**Tel Aviv 2013**)

Consider a consumer in a world with two commodities. He has two continuous strictly-increasing evaluation functions v_1 and v_2 with a range $[0,\infty)$. Facing a budget set $B(p_1,p_2,w)$, the consumer compares between $v_1(w/p_1)$ and $v_2(w/p_2)$ and spends all of his resources on the good that yields a higher evaluation (**in the case of a tie he arbitrarily chooses one of the goods**).

1. Show that this behavior is consistent with maximizing continuous, monotonic and convex preferences over R^2_+.

This behavior is consistent with preferences that have linear (though not necessarily parallel) indifference curves such that any two bundles $(x_1,0)$ and $(0,x_2)$ where $v_1(x_1) = v_2(x_2)$ are on the same indifference curve. By the strict monotonicity of v_1 and v_2, these lines do not intersect.

2. Show that this behavior is inconsistent with maximizing continuous, monotonic and strictly convex preferences over R^2_+.

Assume, to the contrary, that this behavior is consistent with maximizing continuous, monotonic, and strictly convex preferences. Let x_1^* and x_2^* be some quantities of the two commodities such that $v_1(x_1^*) = v_2(x_2^*)$.

First, note that $(x_1^*,0) \sim (0,x_2^*)$: If $(x_1^*,0) \succ (0,x_2^*)$, then by continuity there exists $x_1 < x_1^*$ such that $(x_1,0) \succ (0,x_2^*)$ as well. By the monotonicity of v_1, it holds that $v_1(x_1) < v_2(x_2^*)$ which implies that for a budget set where $p_1x_1 = p_2x_2^* = w$ an agent that follows this procedure will choose $(0,x_2^*)$. This is inconsistent with $(x_1,0) \succ (0,x_2^*)$. Similarly, it cannot be that $(x_1^*,0) \prec (0,x_2^*)$ and therefore $(x_1^*,0) \sim (0,x_2^*)$.

Let $B(p_1,p_2,w)$ be a budget set such that $p_1x_1^* = p_2x_2^* = w$. The agent is indifferent between $(x_1^*,0)$ and $(0,x_2^*)$, the two corners of the budget set, and by strict convexity prefers any point between the two corners, $(\alpha x_1^*,(1-\alpha)x_2^*)$, to the corners themselves. This is inconsistent with the procedure, which requires choosing one of the corners.

3. Does the demand function satisfy the "law of demand" (**according to which a decrease in the price of a commodity weakly increases the demand for it**)?

Yes. If the price of good i decreases, i.e. $p_i' < p_i$, then the consumer can buy more of it, i.e. $w/p_i' > w/p_i$, while the amount of commodity j he can buy remains unchanged. Thus, his evaluation of commodity j ($v_j(w/p_j)$) remains the same while his evaluation of commodity i increases from $v_i(w/p_i)$ to $v_i(w/p_i')$.

If under p_i the consumer did not consume commodity i, then his demand cannot decrease and the law of demand holds.

Otherwise, the consumer buys w/p_i units of commodity i and we can conclude that $v_j(w/p_j) \leq v_i(w/p_i)$. Clearly, we now have that $v_j(w/p_j) < v_i(w/p_i')$ and the consumer continues to consume commodity i. His consumption of commodity i increases from w/p_i to w/p_i' and the law of demand again holds.

Problem B15 (**NYU 2013**)

Imagine a consumer who operates in two stages when facing a budget set $B(p,w)$ **in a world with the commodities** $1,\ldots,K$ **split into two exclusive non-empty groups** A **and** B:

Stage 1: **He allocates** w **between the two groups by maximizing a function** v **on the set of pairs** (w_A,w_B).

Stage 2: **He chooses an** A-**bundle maximizing a function** u_A **defined over the** A-**bundles given** w_A, **and independently chooses a** B-**bundle that maximizes a function** u_B **defined over the** B-**bundles given** w_B.

1. **Show that if the consumer is interested in choosing a bundle** (**over the** K **commodities**) **that in the end maximizes the** (**ridiculous**) **utility function** $\Pi_{k=1,..,K}x_k^{\alpha_k}$ (**where** $\alpha_k > 0\ \forall k$ **and** $\sum_{k=1}^K \alpha_k = 1$), **then he can attain this goal by following the procedure above with some functions** (v,u_A,u_B).

Let $A = \{1,\ldots,L\},\ B = \{L+1,..,K\}$. This Cobb-Douglas utility function yields the demand function $x_k(p,w) = \frac{w\alpha_k}{p_k}$. Consider the following procedure:

- Stage 1: Any function whose maximization yields $w_A = w\sum_{i=1}^L \alpha_i$ and $w_B = w\sum_{j=L+1}^K \alpha_j$ will work. For example, consider

$$v(w_A,w_B) = min\{w_A\sum_{j=L+1}^{K}\alpha_j, w_B\sum_{i=1}^{L}\alpha_i\}\ \ s.t. w_A+w_B \leq w$$

- Stage 2: Choose

$$u_A(p,w_A) = \prod_{i=1}^{L} x_i^{\alpha_i} \Rightarrow x_{A,i}(p,w_A) = \frac{w_A\alpha_i}{p_i\sum_{i=1}^L \alpha_j}$$

$$u_B(p,w_B) = \prod_{j=L+1}^{K} x_j^{\alpha_j} \Rightarrow x_{B,j}(p,w_A) = \frac{w_B\alpha_j}{p_j\sum_{j=L+1}^K \alpha_j}$$

Since the first maximization yields $w_A = w\sum_{i=1}^L \alpha_i$ and $w_B = w\sum_{j=L+1}^K \alpha_j$, we get

$$x_{A,i}(p,w_A) = \frac{w\alpha_i}{p_i\alpha}$$

$$x_{B,j}(p,w_A) = \frac{w\alpha_j}{p_j\alpha}$$

which yields the same demand as our original problem.

2. **Show that the claim in** (**1**) **is not true in general**. **For example**, **you might** (**but don't have to**) **look at the case** $K=4,\ A=\{1,2\},\ B=\{3,4\}$ **and the utility function** $max\{x_1x_3,x_2x_4\}$. (**Note that this is the max**, **not the min function**.)

Consider a consumer with the utility function $U(x) = max\{x_1x_3,x_2x_4\}$. This consumer purchases only good $1,3$ if $p_2p_4 > p_1p_3$ and purchases only good $2,4$ if $p_1p_3 > p_2p_4$. Assume that there are utility functions v, u_A, u_B such that the outcome of the procedure coincides with the maximization of U. Let x_A, x_B be demand functions derived from u_A, u_B respectively.

If $p = (1,1,2,1)$, then since $p_1p_3 > p_2p_4$ it must be that $x_B((2,1), w_B)$ contains only good 4. Now let $p' = (1,3,2,1)$. Then, since $p_1p_3 < p_2p_4$, it must be that $x_B((2,1), w_B)$ contains only good 3. But this cannot be since the optimization problem over B-bundles has not changed, so x_B will not change.

3. (**More Difficult**) **Show that if the consumer follows the above procedure**, **then it might be that his overall choice cannot be rationalized**. (**For the first stage**, **you can choose a simple function like** $v = min\{w_A, w_B\}$.)

Let $v(w_A, w_B) = min\{w_A, w_B\}$ (so that maximization of v leads to $w_A = w_B = \frac{w}{2}$). Consider the following two utility functions for $A = \{1,2\}, B = \{3,4\}$:

$$u_A(p,w) = \begin{cases} x_1 & if\, x_1 < 1 \\ 1 + x_2 & if\, x_1 \geq 1 \end{cases}$$

$$u_B(p,w) = \begin{cases} x_3 & if\, x_3 < 1 \\ 1 + x_3 & if\, x_3 \geq 1 \end{cases}$$

Let $(p,w) = ((1.1, 0, 1.1, 0), 2)$ and $(p', w') = ((1,1,1,1), 3)$. Then we obtain the following demand functions:

$$\begin{aligned} x(p,w) &= (x_A(p, \frac{w}{2}), x_B(p, \frac{w}{2})) &= (\frac{10}{11}, 0, \frac{10}{11}, 0) \\ x(p'w') &= (x_A(p', \frac{w'}{2}), x_B(p', \frac{w'}{2})) &= (1, \frac{1}{2}, 1, \frac{1}{2}) \end{aligned}$$

This violates the weak axiom since $x(p', w')$ is feasible in $B(p,w)$ and $x(p,w)$ is feasible in $B(p', w')$ since:

$$\begin{aligned} p \cdot x(p', w') &= \frac{10}{11} + 0 + \frac{10}{11} + 0 = \frac{20}{11} < 2 = w \\ p' \cdot x(p,w) &= \frac{10}{11} + 0 + \frac{10}{11} + 0 = \frac{20}{11} < 3 = w'. \end{aligned}$$

Problem B16 (NYU 2014)

A DM needs to decide how to allocate a budget between two activities: 1 and 2. A combination of activities is a pair (a_1, a_2) where a_i is the level of activity i. The DM's problem is to choose a combination of activities given a budget w and a vector of prices for the activities (p_1, p_2).

Two consultants, A and B, are involved in the DM's process. Each consultant submits to the DM a recommendation which is the outcome of maximizing a "classical" and differentiable preference relation defined over the set of all activity combinations. Assume that whatever the "budget set" is, consultant A always recommends a (weakly) higher level of activity 1 than B does. Formally, assume that at each combination of activities (a_1, a_2) the "marginal rate of substitution" (the ratio of local values) of A is strictly larger than that of B.

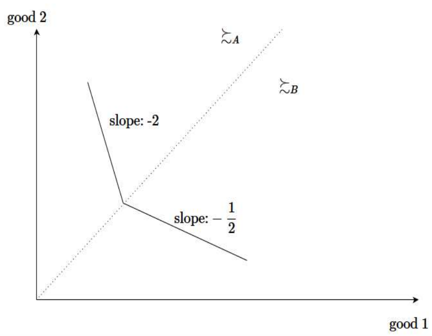
The DM collects the two recommendations and then:

If both recommend that the level of a certain activity i should be higher than that of the other activity, then the DM follows the more "moderate recommendation", namely the one which is closer to the main diagonal.

If consultant A recommends a higher level of activity 1 and B recommends a higher level of activity 2, then the DM spends his entire budget such that he consumes equal levels of the two activities (i.e., a combination on the main diagonal).

1. Assume that A aims to maximize $2a_1 + a_2$ (and in the case of indifference recommends only activity 1) and B seeks to maximize $a_1 + 2a_2$ (and in the case of indifference recommends only activity 2). Is the DM's behavior rationalizable in the sense that there exists a convex and monotonic preference relation that rationalizes the DM's behavior?

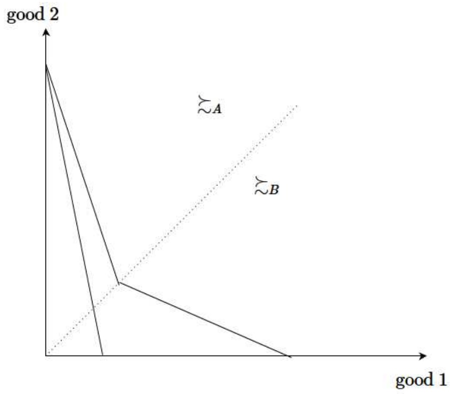
You can rationalise the demand function with the following preferences, the indifference curves of which are drawn below.



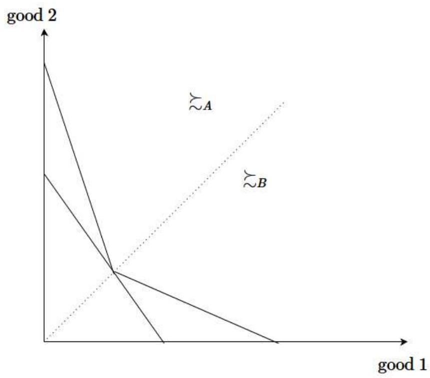
The part of the indifference curve above the diagonal is given by consultant A's indifference curve and the part below the diagonal is given by consultant B's indifference curve. Formally, for any bundle x, let $(d_A(x), d_A(x))$ be the unique point on the diagonal which satisfies $(d_A(x), d_A(x)) \ \sim_A x$ and similarly for consultant B. Then preferences can be represented by a utility function

$$u(x) = \begin{cases} d_A(x) & if \quad x_1 \leq x_2 \\ d_B(x) & if \quad x_1 > x_2 \end{cases}$$

To argue these preferences are consistent with the demand function, consider first $p_1 > 2p_2$. Then both consultants recommend spending the entire budget on good 2. This is also the u-maximal bundle in the budget set as depicted below.



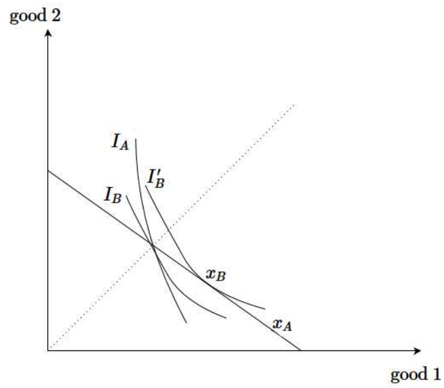
The case $p_2 > 2p_2$ follows analogously. We are left with the case where no good is more than twice expensive than the other. Then consultant A recommends spending everything on good 1 and consultant B recommends spending everything on good 2, so the demand function picks the intersection point of the diagonal and the budget line. This is also the u-maximal bundle in the budget set.



2. **Extend your answer to any two consultants that satisfy the question's assumptions**.

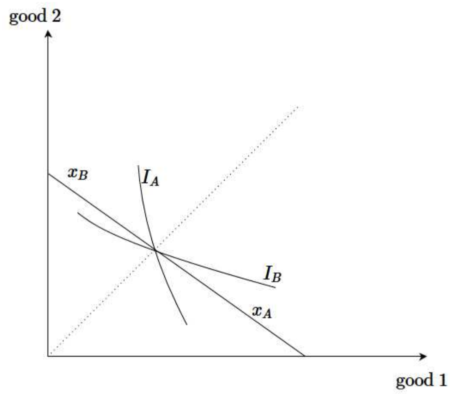
We will use the preferences from part (a) to rationalise the demand function in the general case as well. Let (d, d) denote the intersection point of the diagonal and the budget line.

Firstly, suppose $\frac{p_1}{p_2} \leq \frac{v_B^1(d,d)}{v_B^2(d,d)}$. In this case both consultants' indifference curves I_A and I_B are steeper than the budget line at (d, d). Therefore, the consultants recommend points x_A and x_B below the diagonal. The demand function picks x_B as it is closer to the diagonal. Now the highest attainable indifference curve for the consumer is the one which coincides with I_B' below the diagonal. Hence, the preference-maximising bundle is x_B, as required.



We proceed similarly if $\frac{p_1}{p_2} \geq \frac{v_A^1(d,d)}{v_A^2(d,d)}$.

Finally, suppose $\frac{v_A^1(d,d)}{v_A^2(d,d)} \leq \frac{p_1}{p_2} \leq \frac{v_B^1(d,d)}{v_B^2(d,d)}$. Then A recommends x_A below the diagonal and B recommends x_B above the diagonal. Thus, the demand functions picks (d, d). This is also the preference-maximal bundle because the highest attainable indifference curve consists of I_A above the diagonal and I_B below the diagonal.



C. Uncertainty:

Problem C1 (Princeton 1997)

A decision maker forms preferences over the set X of all possible distributions of a population over two categories (such as living in two locations). An element in X is a vector (x_1, x_2) where $x_i \geq 0$ and $x_1 + x_2 = 1$. The decision maker has two considerations in mind:

He thinks that if $x \succsim y$, then for any z, the mixture of $\alpha \in [0,1]$ of x with $(1-\alpha)$ of z should be at least as good as the mixture of α of y with $(1-\alpha)$ of z.

He is indifferent between a distribution that is fully concentrated in location 1 and one that is fully concentrated in location 2.

1. Show that the only preference relation that is consistent with the two principles is the degenerate indifference relation ($x \sim y$ for any $x, y \in X$).

Let $x = (1,0)$, $y = (0,1)$ and $(\alpha, 1-\alpha)$ be an arbitrary distribution, $0 \leq \alpha \leq 1$. By the second consideration, $x \sim y$, and in particular $x \succeq y$. By the first consideration, the mixture α of x and $(1-\alpha)$ of y is at least as good as the mixture α of y and $(1-\alpha)$ of y (choosing z to be y). Thus, $(\alpha, 1-\alpha) \succeq y$. Similarly, the mixture α of x and $(1-\alpha)$ of x is at least as good as the mixture α of x and $(1-\alpha)$ of y (choosing z to be x). Thus, $x \succeq (\alpha, 1-\alpha)$. Hence, any distribution $(\alpha, 1-\alpha)$ is indiffernet to x and y, since $x \succeq (\alpha, 1-\alpha) \succeq y$.

2. The decision maker claims that you are wrong because his preference relation is represented by a utility function $|x_1 - 1/2|$. Why is he wrong?

The preferences represented by the utility function $u(x_1, x_2) = |x_1 - \frac{1}{2}|$ satisfy the first consideration, since $u(1,0) = |1 - \frac{1}{2}| = |0 - \frac{1}{2}| = u(0,1)$. However, these preferences fail to satisfy the second consideration: although $(1,0) \succeq (0,1)$, mixing both sides with $\alpha = \frac{1}{2}$ of $(0,1)$ yields $(\frac{1}{2}, \frac{1}{2}) \prec (0,1)$, since $u(\frac{1}{2}, \frac{1}{2}) = |\frac{1}{2} - \frac{1}{2}| < |0 - \frac{1}{2}| = u(0,1)$.

Problem C2 (**Tel Aviv 1999**)

Tversky and Kahneman (**1986**) **report the following experiment**: **each participant receives a questionnaire asking him to make two choices**, **the first from** $\{a, b\}$ **and the second from** $\{c, d\}$:

a. **A sure profit of** $\$240$.

b. **A lottery between a profit of** $\$1000$ **with probability** 25% **and** 0 **with probability 75**%.

c. **A sure loss of** $\$750$.

d. **A lottery between a loss of** $\$1000$ **with probability** 75% **and** 0 **with probability 25**%.

The participant will receive the sum of the outcomes of the two lotteries he chooses. **73**% **of the participants chose the combination a and d**. **Is their behavior sensible**?

The combination of a+d is the lottery $0.75[-760\$] + 0.25[+240]$ whereas the combination b+c is the lottery $0.75[-750\$] + 0.25[+250]$ which first order stochastically dominates $a + d$.

Problem C3 (**Princeton 2001**)

A consumer has to make a choice of a bundle before he is informed whether a certain event, which is expected with probability α and affects his welfare, has happened or not. He assigns a vNM utility $v(x)$ to the consumption of the bundle x when the event occurs, and a vNM utility $v'(x)$ to the consumption of x should the event not occur. Having to choose a bundle the consumer maximizes his expected utility $\alpha v(x) + (1-\alpha)v'(x)$. Both v and v' induce preferences on the set of bundles satisfying the standard assumptions about the consumer. Assume also that v and v' are concave.

1. Show that the consumer's preference relation is convex.

The function $\alpha v(x) + (1-\alpha)v'(x)$ is a convex combination of concave functions, thus it is quasi-concave and induces a preference relation which is convex.

2. Find a connection between the consumer's indirect utility function and the indirect utility functions derived from v and v'.

Let $x(p,w)$ denote the demand and f, f_v and $f_{v'}$ the indirect utility functions. Then, $f(p,w) = \alpha v(x(p,w)) + (1-\alpha)v'(x(p,w)) \le \alpha f_v(p,w) + (1-\alpha)f_{v'}(p,w)$. The interpretation of this property is that the decision maker is better off if he makes his choice after rather than before the uncertainty is resoleved.

3. A new commodity appears on the market: "A discrete piece of information that tells the consumer whether or not the event occurred". The commodity can be purchased prior to the consumption decision. Use the indirect utility functions to characterize the demand function for the new commodity.

Suppose that v and v' are continuous and monotonic so that f, f_v and $f_{v'}$ are strictly increasing and continuous in w. Then for given (p,w), by (b), there is a unique $\beta^*(p,w)$ that satisfies:

$$f(p,w) = \alpha f_v(p, w - \beta^*(p,w)) + (1-\alpha)f_{v'}(p, w-\beta^*(p,w)).$$

The demand for this good is 1 if its price is below $\beta^*(p,w)$ and 0 otherwise.

Problem C4 (**NYU 2006**)

Consider a world with balls of K **different colors**. **An object is called a bag and is specified by a vector** $x = (x_1, .., x_K)$ (**where** x_k **is a non-negative integer indicating the number of balls of color** k). **For convenience denote by** $n(x) = \sum x_k$ **the number of balls in bag** x.

An individual has a preference relation over bags of balls.

1. **Suggest a context where it will make sense to assume that**:

i. **For any integer** λ, $x \sim \lambda x$.

ii. **If** $n(x) = n(y)$, **then** $x \succsim y$ **iff** $x + z \succsim y + z$.

Suppose that the decision maker chooses a ball from bag x, and he gets a prize depending on the color of the ball he chose. Then we can identify a bag x with a lottery $p(x)$ in which the prize associated with the color k is received with probability $x_k/n(x)$. Since $p(x) = p(\lambda x)$, we should expect $x \sim \lambda x$. Moreover, ii would follow in this case from the independence axiom.

2. **Show that any preference relation which is represented by** $U(x) = \sum x_k v_k/n(x)$ **for some vector of numbers** $(v_1, .., v_k)$ **satisfies the two axioms**.

A1: For any natural number λ,

$$U(\lambda x) = \sum_k \frac{\lambda x_k}{n(\lambda x)} v_k = \sum_k \frac{\lambda x_k}{\lambda n(x)} v_k = U(x).$$

A2: If $n(x) = n(y)$, then

$$\begin{aligned} U(x) \geq U(y) &\Leftrightarrow \sum_k x_k v_k \geq \sum_k y_k v_k \\ &\Leftrightarrow \sum_k (x_k + z_k) v_k \geq \sum_k (y_k + z_k) v_k \\ &\Leftrightarrow U(x+z) \geq U(y+z) \end{aligned}$$

where in the last equivalence we use that fact that $n(x + z) = n(y + z)$.

3. **Find a preference relation which satisfies the two properties that cannot be represented in the form suggested in** (**2**).

Let $K = 3$ and let $\succsim_L$ be the usual lexicographic relation on $\Re^3$. Define the preference relation $\succsim$ on X as

$$x \succsim y \Leftrightarrow (x_1/n(x), x_2/n(x), x_3/n(x)) \succsim_L (y_1/n(y), y_2/n(y), y_3/n(y)).$$

A1: Note that for any x and any natural number λ, $x_i/n(x) = \lambda x_i/\lambda n(x)$ for all i. Hence,

$x \sim \lambda x$.

A2: $x \succsim y$ and $n(x) = n(y)$ iff $(x_1, x_2, x_3) \succsim_L (y_1, y_2, y_3)$ iff $(x_1 + z_1, x_2 + z_2, x_3 + z_3) \succsim_L (y_1 + z_1, y_2 + z_2, y_3 + z_3)$ iff $x + z \succsim y + z$ (since $n(x + z) = n(y + z)$).

To see that $\succsim$ does not admit a representation of the form given in part (2), suppose to the contrary that $U(x)$ represents $\succsim$. Note that $(1, 0, n) \succ (0, 1, 0)$ for all n. Hence, $v_1/(n + 1) + v_3 n/(n + 1) > v_2$ for all n. Passing to limit as $n \to \infty$ gives $v_3 \geq v_2$. On the other hand, $(0, 1, 0) \succ (0, 0, 1)$ implies $v_2 > v_3$, a contradiction.

Problem C5 (**NYU 2007**)

Identify a professor's lifetime with the interval $[0,1]$. **There are** $K+1$ **academic ranks**, $0,..,K$. **All professors start at rank** 0 **and eventually reach rank** K. **Define a career as a sequence** $t = (t_1,....,t_K)$ **where** $t_0 = 0 \le t_1 \le t_2 \le ... \le t_K \le 1$ **with the interpretation that** t_k **is the time it takes to get the** k**'th promotion**. (**Note that a professor can receive multiple promotions at the same time**.) **Denote by** $\succsim$ **the professor's preferences on the set of all possible careers**.

For any $\epsilon > 0$ **and for any career** t **such that** $t_K \le 1-\epsilon$, **define** $t+\epsilon$ **to be the career** $(t+\epsilon)_k = t_k+\epsilon$ **for all k** (**i**.**e**. **all promotions are delayed by** ϵ).

Following are two properties of the professor's preferences:

Monotonicity: **For any two careers** t **and** s, **if** $t_k \le s_k$ **for all** k **then** $t \succsim s$ **and if** $t_k < s_k$ **for all** k, **then** $t \succ s$.

Invariance: **For every** $\epsilon > 0$ **and every two careers** t **and** s **for which** $t+\epsilon$ **and** $s+\epsilon$ **are well defined**, $t \succsim s$ **iff** $t+\epsilon \succsim s+\epsilon$.

1. **Formulate the set** L **of careers in which a professor receives all** K **promotions at the same time**. **Show that if** $\succsim$ **satisfies continuity and monotonicity**, **then for every career** t **there is a career** $s \in L$ **such that** $s \sim t$.

$L = \{s \mid s = (\alpha,....,\alpha) \text{ for some } \alpha \in [0,1]\}$.

By monotonicity, $(0,...,0) \succsim t \succsim (1,....,1)$. By continuity, there exists a bundle s on the interval connecting $(0,...,0)$ and $(1,...,1)$ such that $s \sim t$. Clearly, $s \in L$ since it is on the main diagonal.

2. **Show that any preference which is represented by the function** $U(t) = -\sum \Delta_k t_k$ (**for some** $\Delta_k > 0$) **satisfies Monotonicity**, **Invariance and Continuity**.

MON: If $t_k \le s_k$ for all k, then $-\Delta_k t_k \ge -\Delta_k s_k$ for all k. Consequently, $U(t) = \sum -\Delta_k t_k \ge \sum -\Delta_k s_k = U(s)$, and thus $t \succsim s$, and analogously in the strict case.

INV: $t \succsim s \Leftrightarrow -\sum \Delta_k t_k \ge -\sum \Delta_k s_k \Leftrightarrow -\sum \Delta_k t_k - \epsilon \sum \Delta_k \ge -\sum \Delta_k s_k - \epsilon \sum \Delta_k \Leftrightarrow$

$\Leftrightarrow -\sum \Delta_k (t_k+\epsilon) \ge -\sum \Delta_k (s_k+\epsilon) \Leftrightarrow t+\epsilon \succsim s+\epsilon$.

CON: U is continuous, so $\succsim$ is continuous.

3. **One professor evaluates a career by the maximum length of time one has to wait for a promotion and the smaller this number the better**. **Show that these preferences cannot be represented by the utility function described in** (**2**).

$(.2,.8) \sim (.1,.7)$, and thus preferences fail monotonicity.

Problem C6 (NYU 2008)

An economic agent has to choose between projects. The outcome of each project is uncertain. It might yield a failure or one of K "types of success". Thus, each project z can be described by a vector of K non-negative numbers, $(z_1,\ldots,z_K)$ where z_k stands for the probability that the project success will be of type k.
Let $Z \subset \Re_+^K$ be the set of feasible projects. Assume Z is compact, convex and satsifies "free disposal".
The decision maker is an Expected Utility maximizer.
Denote by u_k the vNM utility from the k–th type of success, and attach 0 to failure. Thus the decision maker chooses a project (vector) $z \in Z$ in order to maximize $\sum z_k u_k$.
1. First, formalize the decision maker's problem. Then, formalize (and prove) the claim: If the decision maker suddenly values type k success higher than before, he would choose a project assigning a higher probability to k.
The DM solves:

$$max_{z\in Z} z \bullet u = max_{z\in Z} \sum_{k=1}^{K} z_k u_k$$

Claim: Let $u_i' = u_i$ for every $i \neq k$, $u_k' > u_k$. Then $z_k^* \leq z_k^*$ (where z^* and $z^{*\prime}$ are the solutions for u and u' respectively.
Proof: Equivalent to the proof of "The Law of Demand (or Supply)". $(z^* - z^{*\prime}) \bullet [u - u'] = [z^* - z^{*\prime}] \bullet u + [z^{*\prime} - z^*] \bullet u' \geq 0$, since $z^* \bullet u \geq z^{*\prime} \bullet u$, and $z^{*\prime} \bullet u' \geq z^* \bullet u'$. Since $u - u' = (0,\ldots,0,u_k - u_k',0,\ldots,0) < 0$, then $z_k^* - z_k^{*\prime} \leq 0$ QED.

2. Apparently, the decision maker realizes that there is an additional uncertainty. The world may go "one way or another". With probability α the vNM utility of the k'th type of success will be u_k and with probability $1-\alpha$ it will v_k. Failure remains 0 in both contingencies.
First, formalize the decision maker's new problem. Then, formalize (and prove) the claim: Even if the decision maker would obtain the same expected utility, would he have known in advance the direction of the world, the existence of uncertainty makes him (at least weakly) less happy.
The DM solves:

$$max_{z\in Z} z \bullet [\alpha u + (1-\alpha)v] = max_{z\in Z} \sum_{k=1}^{K} z_k[\alpha u_k + (1-\alpha)v_k]$$

Claim: The maximal expected utility in the uncertain world is weakly less than the maximal expected utility when the direction of the world is known.
Proof: Denote the DM's chosen projects under the two seperate directions (with vNM utility u or v) as z^u and z^v.
Then
$max_{z \in Z} z \cdot [\alpha u + (1-\alpha)v] \leq \alpha[max_{z' \in Z} z' \cdot u] + (1-\alpha)[max_{z' \in Z} z' \cdot v] = \alpha z^u \cdot u + (1-\alpha) z^v \cdot v$.
Even if $z^u \cdot u = z^v \cdot v = E$, we get $max_{z \in Z} z \cdot [\alpha u + (1-\alpha)v] \leq E$.

Problem C7 (NYU 2009)

For any non negative integer n and a number $p \in [0,1]$ let (n,p) be the lottery which gets the prize $\$n$ with probability p and $\$0$ with probability $1-p$. Let us call those lotteries "simple lotteries". Consider preference relations on the space of simple lotteries.

We say that such a preference relation satisfies Independence if $p \succeq q$ iff $\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r$ for any $\alpha > 0$, and any simple lotteries p,q,r for which the compound lotteries are also simple lotteries.

Consider a preference relation satisfying the Independence axiom, strictly monotonic in money and continuous in p. Show that:

1. (n,p) is monotonic in p for $n > 0$, i.e. for all $p > p'$ $(n,p) \succ (n,p')$

Observation 1: By monotonicity, $(n,1) \succ (m,1)$ for all $m < n$.

Observation 2: For all n, $(n,0) \sim (0,1)$ since both lotteries give 0 w.p. 1.

Proof of Claim:

By observation 1 and 2, $(n,1) \succ (n,0)$. By independence $p'/p(n,1) \oplus (1-p'/p)(n,1) \succ p'/p(n,0) \oplus (1-p'/p)(n,1) \Rightarrow (n,1) \succ (n,p'/p)$. Using independence again, $p(n,1) \oplus (1-p)(n,0) \succ p(n,p'/p) \oplus (1-p)(n,0) \Rightarrow (n,p) \succ (n,p')$.

2. For all n there is a unique $v(n)$ such that $(1,1) \sim (n,1/v(n))$

By observations above for $n > 1$, $(n,1) \succ (1,1) \succ (n,0)$. Since (n,p) is continuous, and monotonic in p, there exists a unique p_n such that $(1,1) \sim (n,p_n)$. Denote $v(n)$ such that $v(n) = 1/p_n$, and $v(0) = 0$, and naturally $v(1) = 1$

3. It can be represented with the expected utility formula: that is there is an increasing function v such that $pv(n)$ is a utility function which represents the preference relation.

Claim : For $n > m$, $v(n) > v(m)$

By monotonicity in money, $(n,1) \succ (m,1)$. By independence $(n,1/v(n)) \sim (1,1) \succ (m,1/v(n)) \Rightarrow (m,1/v(m)) \succ (m,1/v(n))$. By monotonicity in p, $1/v(m) > 1/v(n)$

Now lets check that $u(n,p) = v(n)p$ represents preferences over these lotteries. Note that, $(n,1/v(n)) \sim (m,1/v(m))$. By independence $(n,v(m)/v(n)q) \sim (m,q)$. Then (n,p) relates to (m,q) like p relates to $v(m)/v(n)q$. Thus $(n,p) \succ (m,q)$ iff $v(n)p > v(m)q$.

Problem C8 (**Tel Aviv 2012**)

A decision maker has in mind a function CE, **with the interpretation that for every lottery** p, $CE(p)$ **is the certainly equivalence of** p. **Following are two procedures for deriving the function**.

Procedure 1: **The decision maker has in mind an increasing vNM utility function** u **and his answer satisfies** $Eu(p) = u(CE(p))$.

Procedure 2: **The decision maker has in mind two increasing**, **continuous and concave functions** g (**for gains**) **and** l (**for losses**) **which satisfy** $g(0) = l(0) = 0$. $CE(p)$ **is a number** x **which equalizes the expected** "**loss**" **with the expected** "**gain**", **that is satisfies** $\sum_{y<x} p(y)l(x-y) = \sum_{y>x} p(y)g(y-x)$.

1. **Explain why** pD_1q **implies under the two procedures that** $CE(p) \geq CE(q)$.

Procedure 1: If pD_1q then for any utility function u it holds that $Eu(p) \geq Eu(q)$. Therefore, $u(CE(p)) \ \geq u(CE(q))$, and by the monotonicity of u, $CE(p) \geq CE(q)$.

Procedure 2: Let $x^* = CE(p)$, that is, $\sum_{y>x} p(y)g(y-x^*) - \sum_{y<x} p(y)l(x^*-y) = 0$.

Given this x^*, define an increasing vNM utility function $u(y) = \begin{cases} g(y-x^*) \text{ if } y \geq x^* \\ -l(x^*-y) \text{ if } y < x^* \end{cases}$.

If pD_1q we can conclude that $Eu(p) \geq Eu(q)$. Moreover, because $Eu(p) = 0$ then $Eu(q) \leq 0$, which implies $\sum_{y>x} q(y)g(y-x^*) - \sum_{y<x} q(y)l(x^*-y) \leq 0$. Given q, the expression $\sum_{y>x} q(y)g(y-x) - \sum_{y<x} q(y)l(x-y)$ is decreasing in x, therefore $CE(q) \leq x^*$.

2. **Explain why the first procedure allows behavior which is not possible under procedure 2**.

Let p be a lottery and let $p+k$ be a lottery in which all prizes are increased by k (Formally, $p(x) = (p+k)(x+k)$ for any x in the support of p). Clearly, in procedure 2, if $x = CE(p)$ then $x+k = CE(p+k)$. This is not necessarily the case for the first procedure.

For example, let $u(x) = \sqrt{x}$ and let p be lottery yielding 0 and 1 with equal probabilities. $Eu(p) = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{1} = \frac{1}{2}$. Therefore, $\sqrt{CE(p)} = \frac{1}{2}$, or $CE(p) = \frac{1}{4}$.

Now, consider the lottery yielding 1 and 2 with equal probabilities. $Eu(p+1) = \frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{2} = 1.207$. Therefore, $\sqrt{CE(p)} = 1.207$, or $CE(p) = 1.46 \neq 1.25$

3. (**More Difficult**) **Can any individual who operates by Procedure 2 be described as working through procedure 1**?

Not necessarily.

Assume $l(x) = 2x$ and $g(x) = x$. That is, the losses are twice more significant than the gains.

Let p be a lottery yielding $-\frac{1}{2}$ and 1 with equal probabilities. By procedure 2, $CE(p) = 0$.

Let q be a lottery yielding 0 and 3 with equal probabilities. By procedure 2, $CE(q) = 1$.

Assume there is a utility function $u(x)$ such that yields the same CE for these two lotteries when using procedure 1. That is:

By lottery p, $\frac{1}{2}u(-\frac{1}{2}) + \frac{1}{2}u(1) = u(0)$, and by lottery q, $\frac{1}{2}u(0) + \frac{1}{2}u(3) = u(1)$.

Substituting $u(1)$ we get $\frac{1}{2}u(-\frac{1}{2}) + \frac{1}{4}u(0) + \frac{1}{4}u(3) = u(0)$, which implies that 0 is the certainly equivalent of the lottery $r = \frac{1}{2}[-\frac{1}{2}] \oplus \frac{1}{4}[0] \oplus \frac{1}{4}[3]$.

However, by procedure 2 we can see that 0 is not the $CE(r)$ since $\sum_{y<0} r(y)l(0-y) = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \neq \frac{1}{4} \cdot 1 \cdot 3 = \sum_{y>0} r(y)g(y-0)$.

Problem C9 (**NYU 2012**)

Consider a decision maker in the world of lotteries, **with** $Z = R$ **being monetary prizes**. **The decision maker assigns a number** $v(z)$ **to each amount of money** z. **The function** v **is continuous and increasing**. **The decision maker evaluates each lottery** p **according to**:

$U(p) = \alpha[max\{v(z)|z \in supp(p)\}] + (1-\alpha)[\min\{v(z)|z \in supp(p)\}]$.

1. **Characterize the decision makers of this type who are** "**risk averse**".

Let us see that for $0 < \alpha \leq 1$ the decision maker does not exhibit risk aversion. Fix a, c such that $a < c$. For any α we can find a number $b \in (a,c)$ such that $\alpha v(c) + (1-\alpha)v(a) > v(b)$. Let λ be a number such that $\lambda c + (1-\lambda)a < b$. Consider the lottery p which receives the value c with probability λ and the value a with probability $(1-\lambda)$. Then, $U_\alpha([Ep]) = v(Ep) < v(b) < U_\alpha(p) = \alpha v(c) + (1-\alpha)v(a)$. Thus, $\succsim$ does not exhibit risk aversion.

If $\alpha = 0$, then whatever $v(z)$ is the relation $\succsim$ is risk averse since always $U_0(p) = min_{z \in supp(p)} v(z) \leq v(Ep)$.

2. **Show that if two decision makers of this type**, **with** $\alpha = 1/2$, **hold the functions** v_1 **and** v_2 **and** $v_1 \circ v_2^{-1}$ **is concave**, **then decision maker 1 is more risk averse than decision maker 2**.

Assume that $p \succsim_1 c$. Let $v_1(M) = \max\{v_1(z) \mid z \in supp(p)\}$ and $v_1(m) = \min\{v_1(z) \mid z \in supp(p)\}$. Then, $[v_1(M) + v_1(m)]/2 \geq v_1(c)$. Since $\varphi = v_2 \circ v_1^{-1}$ is convex, then

$$[v_2(M) + v_2(m)]/2 = [\varphi(v_1(M)) + \varphi(v_1(m))]/2) \geq \varphi([v_1(M) + v_1(m)]/2) \geq \varphi(v_1(c)) = v_2(c)$$

That is, $p \succsim_2 c$.

3. **Assume that the two decision makers use** $\alpha = 1/2$. **Is the concavity of** $v_1 \circ v_2^{-1}$ **a necessary condition for decision maker 1 to be more risk averse than decision maker 2**.

We will first show that $\phi \equiv v_1 \circ v_2^{-1}$ is half concave. Following the notation in solution for part (b), fix M, m such that $M > m$, and pick a $c \in (m, M)$ such that $v_2(c) = [v_2(M) + v_2(m)]/2$. We will show $v_1(c) \geq [v_1(M) + v_1(m)]/2$. If not, i.e. $v_1(c) < [v_1(M) + v_1(m)]/2$, then for some $\epsilon > 0$ we have $v_2(c) > [v_2(M-\epsilon) + v_2(m)]/2$ and $v_1(c) < [v_1(M-\epsilon) + v_1(m)]/2$. A contradiction to the fact that DM1 is more risk averse than DM2. Thus $v_1(c) \geq [v_1(M) + v_1(m)]/2$, or in other words,

$\phi(v_2(c)) = v_1(c) \geq [v_1(M) + v_1(m)]/2 = [\phi(v_2(M)) + \phi(v_2(m))]/2$. ϕ is half concave.

From the fact that if a continuous and increasing function is half concave, then it's concave, we know that ϕ is also concave. Hence, its concavity is also a necessary condition.

Problem C10 (**NYU 2014**)

Consider the following family of preference relations defined over $L(Z)$ (**the set of all lotteries with prizes in some finite set** Z):

The DM has in mind a function which assigns to each prize $z \in Z$ **a value** $v(z)$. **He partitions** Z **into two sets** G **and** B **such that if** $g \in G$ **and** $b \in B$ **then** $v(g) > v(b)$. **He evaluates any lottery** p **by**

$p(Supp(p) \cap G) \max_{z \in Supp(p) \cap G} v(z) + p(Supp(p) \cap B) \min_{z \in Supp(p) \cap B} v(z)$.

These evaluations form his preferences over $L(Z)$ (**where** $p(A) = \sum_{z \in A} p(z)$).

1. **Explain the procedure in words**.

The DM classifies the lottery outcomes in two categories: good (G) and bad (B). He is optimistic about the good outcomes and pessimistic about the bad outcomes. He evaluates a lottery p by the expected value where he treats all the G (B) outcomes in the support of p as if they are the best (worst) outcomes in the support of p.

2. **Show that such a preference relation satisfies neither the Independence axiom nor the Continuity axiom**.

Independence: Consider $Z = \{1,2,3\}$, $v(x) = x, B = \{1\}$ and $G = \{2,3\}$. Denote by $u(p)$ the DM's evaluation of the lottery p. The DM is indifferent between the lottery $p = [2]$ and the lottery $q = 0.5[1] + 0.5[3]$.

However $u(0.5p + 0.5[3]) = 3 > u(0.5q + 0.5[3]) = u(0.25[1] + 0.75[3]) = 2.5$, which violates independence.

Continuity: $[3]$ is better than $[2]$ but for any $\varepsilon > 0$, $\varepsilon[3] + (1-\varepsilon)2$ is indifferent to and not better than $[3]$.

3. **Show that a weaker independence property holds**: **If** $Supp(p) = Supp(q)$ **then for every** $1 > \alpha > 0$ **and every** r, $p \succsim q$ **iff** $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.

If the support of two lotteries is the same, then the best G-outcome and the worst B-outcome are the same in both lotteries p_1 and p_2. Hence, the comparison between the lotteries is the same as the comparison of the aggregate probabilities they assign to good outcomes. Now also supp$(\alpha p_1 + (1-\alpha)p_3) =$ supp$(\alpha p_2 + (1-\alpha)p_3) = S$. Thus,

$$\alpha p_1 + (1-\alpha)p_3 \succsim \alpha p_2 + (1-\alpha)p_3 \text{ iff}$$
$$[\alpha p_1 + (1-\alpha)p_3](S \cap G) \geq [\alpha p_2 + (1-\alpha)p_3](S \cap G) \text{ iff}$$
$$p_1(\text{supp}(p_1) \cap G) \geq p_2(\text{supp}(p_1) \cap G) \text{ iff}$$
$$p_1 \succsim p_2$$

4. **Describe in words and then formally define a** "**monotonicity property**" **that holds**.

Monotonicity: If $[a] \succ [b]$ then if p and q are two lotteries such that $p(a) > q(a)$, $p(b) < q(b)$ and $p(x) = q(x)$ for any other x then $p \succsim q$.

The proof is quite trivial, as:

(i) the best outcome in $supp(p) \cap G$ is at least as good as the best outcome in $supp(q) \cap G$.

(ii) The worst outcome in $supp(p) \cap B$ is at least as good as the worst outcome in $supp(q) \cap B$.

(iii) $p(supp(p) \cap G) \geq q(supp(q) \cap G)$.

D. **Social Choice**:

Problem D1 (**Princeton 2000**)

Consider the following social choice problem: **a group has** n **members who must choose from a set containing** 3 **elements** $\{A,B,L\}$, **where** A **and** B **are prizes and** L **is the lottery which yields each of the prizes** A **and** B **with equal probability**. **Each member has a strict preference over the three alternatives that satisfies vNM assumptions**.

1. **Show that there is a non-dictatorial social welfare function which satisfies the independence of irrelevant alternatives axiom** (**even the strict version** I^*) **and the Pareto axiom** (**Par**).

Since the preference relations is strict and sastifies vNM assumptions, each of them must be of one of two types

$A \succ_A L \succ_A B$ or $B \succ_B L \succ_B A$. Let $\sim$ be the preference $A \sim L \sim B$ (which also satsifies the vNM assumptions).

The majority rule assigns the relation $\begin{array}{ll} \succ_A & if\ \{i|\ \succ_i\ =\ \succ_A\} > \{i|\ \succ_i\ =\ \succ_B\} \\ \sim & if\ \{i|\ \succ_i\ =\ \succ_A\} = \{i|\ \succ_i\ =\ \succ_B\} \\ \succ_B & if\ \{i|\ \succ_i\ =\ \succ_B\} > \{i|\ \succ_i\ =\ \succ_A\} \end{array}$.

Clearly, this satisfies the Pareto axiom - if every individual prefers x to y, then so does society, according to the majority rule.

It also satisfies IIA: consider two profiles $\{\succsim_i\}$ and $\{\succsim_i'\}$, and pick two pairs of alternatives (a,b) and (x,y) s.t. $a \succsim_i b$ iff $x \succsim_i' y$. Note that every preference relation in the domain of the SWF corresponds to a distinct preference over any two alternatives. Thus if $a \succ_i b$ iff $x \succ_i' y$ then the relation $\succ_{ab}$ where $a \succ_{ab} b$ is most popular iff the relation and $\succ_{xy}$ where $x \succ_{xy} y$ is most popular.

2. **Reconcile this fact with Arrow's Impossibility Theorem**.

Arrow's impossibility theorem says that if there are at least three alternatives, then the only SWF with unrestricted domain satisfying the Pareto and IIA axioms is the dictatorship. But here the domain is restricted to only two possible preference relations.

Problem D2 (**NYU 2009**)

We will say that a choice function C is consistent with the majority vetoes a dictator procedure if there are three preference relations $\succ_1$, $\succ_2$ and $\succ_3$ such that $c(A)$ is the $\succ_1$ maximum unless both $\succ_2$ and $\succ_3$ agree on another alternative being the maximum in A.

1. **Show that such a choice function might not be rationalizable**.

We will show that the choice function violates property α. Consider the following preferences on a, b, c:

$$a \succ_1 b \succ_1 c$$
$$b \succ_2 a \succ_2 c$$
$$c \succ_3 b \succ_3 a$$

According to these preferences

$$C(\{a,b,c\}) = a$$
$$C(\{a,b\}) = b$$

2. **Show that such a choice function satisfies the following property**: **If** $c(A) = a$, $c(A - \{b\}) = c$ **for** b **and** c **different than** a **then** $c(B) = c$ **for all** B **which contains** c **and is a subset of** $A - \{b\}$.

Claim: a is $\succ_1$ maximal in A. Assume not for contradiction. Then, a must be $\succ_2$ and $\succ_3$ maximal in A. But since $A - \{b\}$ is a subset of A, a must be $\succ_2$ and $\succ_3$ maximal there, too. But according to the majority veto dictator rule it must also be chosen in $A - \{b\}$ which contradicts $c(A - \{b\}) = c$.

By claim, and $A - \{b\} \subset A$, we know that a is $\succ_1$ maximal in $A - \{b\}$. Since $c(A - \{b\}) = c$, we know that c must be $\succ_2$ and $\succ_3$ maximal in $A - \{b\}$. (Otherwise a would be chosen in $A - \{b\}$.) But c must be maximal $\succ_2$ and $\succ_3$ in any B which contains c and is a subset of $A - \{b\}$. Then c must be chosen in any of these subsets.

3. **Show that not all choice functions could be explained by the majority vetoes a dictator procedure**.

Any choice function satisfying $C(\{a,b,c,d\}) = a$, $C(\{a,c,d\}) = c$, and $C(\{a,c\}) = a$ violates the property in part (2) and thus cannot be explained by the majority vetoes a dictator procedure.

Problem D3 (**Tel Aviv 2009**. **Inspired by Miller** (**2007**)).

Lately we have been using the term a "reasonable reaction" quite frequently. **In this problem we assume that this term is defined according to the opinions of the individuals in the society with regard to the question**:"**What is a reasonable reaction**?".

Assume that in a certain situation, **the possible set of reactions is** X **and the set of individuals in the society is** N.

A "**reasonability perception**" **is a non**-**empty set of possible reactions that are perceived as reasonable**.

The social reasonability perception is determined by a function f **which attaches a reasonability perception** (**a non**-**empty subset of** X) **to any profile of the individuals**' **reasonability perception** (**a vector of non**-**empty subsets of** X).

1. **Formalize the following proposition**:

Assume that the number of reactions in X **is larger than the number of individuals in the society and that** f **satisfies the following four properties**:

A. **If in a certain profile all the individuals do not perceive a certain reaction as reasonable**, **then neither does the society**.

B. **All the individuals have the same status**.

C All the reactions have the same status.

D. **Consider two profiles that are different only in one individual**'**s reasonability perception**. **Any reaction that** f **determines to be reasonable in the first profile**, **and regarding which the individual did not change his opinion from reasonable to unreasonable in the second profile**, **remains reasonable**.

Then f **determines that a reaction is socially reasonable if and only if at least one of the individuals perceives it as reasonable**.

Denote by S_i the reasonability perception of individual i.

Proposition:

Assume $|X| > |N|$. Let f be a function that satisfies:

A. $\forall i \in N. x \notin S_i \Rightarrow x \notin f(\{S_i\}_{i\in N})$.

B. Let σ be a permutation of N. If $\{S_i\}_{i\in N}$ and $\{S_i^{'}\}_{i\in N}$ are two reasonability perception profiles such that for every i: $S_i^{'} = S_{\sigma(i)}$ then $f(\{S_i\}_{i\in N}) = f(\{S_i^{'}\}_{i\in N})$.

C. Let $\sigma^{'}$ be a permutation of X. If $\{S_i\}_{i\in N}$ and $\{S_i^{'}\}_{i\in N}$ are two reasonability perception profiles such that for every x and for every i it holds that $x \in S_i \Leftrightarrow \sigma^{'}(x) \in S_i^{'}$, then $x \in f(\{S_i\}_{i\in N}) \Leftrightarrow \sigma^{'}(x) \in f(\{S_i^{'}\}_{i\in N})$.

D. Let $\{S_i\}_{i\in N}$ and $\{S_i^{'}\}_{i\in N}$ be two reasonability perception profiles such $S_i^{'} = S_i$ that for

any $i \neq j$. Let $x \in f(\{S_i\}_{i \in N})$. If $x \neq S_j$ or $x \in S_j^{'}$, then $x \in f(\{S_i^{'}\}_{i \in N})$.
Then, $x \in f(\{S_i\}_{i \in N}) \Leftrightarrow \exists i \text{ such that } x \in S_i$.

2. Show that all four properties are necessary for the proposition.

1. The fixed function $f(\bullet) = X$ satisfies B, C and D but not A.
2. The function $f(\{S_i\}_{i \in N}) = S_i$ for some fixed i (a dictatorship) satisfies A, C and D but not B.
3. A function which determines that a reaction is reasonable if and only if at least one of the individuals perceives it as such, except for one specific reaction for which it is necessary that two individuals perceive it as reasonable, satisfies A, B and D but not C.
4. A function which determines as reasonable the reaction(s) that the largest number of individuals perceive as reasonable (the most popular reaction(s)), satisfies A, B and C but not D.

3. Prove the proposition.

$\Rightarrow$

Let f be a function satisfying A, B, C and D. Let $\{S_i\}_{i \in N}$ be a reasonability perception profile. Let y be a reaction that is perceived as reasonable by at least one individual, denoted by j.

Define the profile $\{T_i\}_{i \in N}$ by arbitrarily assigning one alternative to each individual, with no repetitions (that is $T_i = \{x_i\}$ and $x_i \neq x_k$ for any $i \neq k$), such that $x_j = y$. This is possible since there are more alternatives than individuals.

Claim: In the profile $\{T_i\}_{i \in N}$, all reactions are determined to be socially reasonable.

Proof: $f(\{T_i\}_{i \in N})$ is non-empty and therefore for some x, $x \in f(\{T_i\}_{i \in N})$. By property A, there is an individual k such that $x = x_k \in T_k$. Let $i \neq k$ and let σ be a permutation of N that switches between i and k. Now $\{x_k\} = T_i$ and $\{x_i\} = T_k$ and by property B, $x_k \in f(\{T_{\sigma(i)}\}_{i \in N})$. Let $\sigma^{'}$ be a permutation of X that switches between x_i and x_k. By property C, x_i is now socially reasonable, but in fact we are back to the original profile $\{T_i\}_{i \in N}$. Therefore, for every i, it holds that $x_i \in f(\{T_i\}_{i \in N})$.

The above claim implies that $y \in f(\{T_i\}_{i \in N})$.

Define $\{R_i\}_{i \in N}$ such that for every i, $R_i = S_i \cup T_i$. One can transform $\{T_i\}_{i \in N}$ into $\{R_i\}_{i \in N}$ by adding one reaction to one individual at a time. By property D, in each of these stages, the reaction y remains socially reasonable and thus $y \in f(\{R_i\}_{i \in N})$.

If for some individual i the reaction that we chose arbitrarily in T_i is not in the original reasonability perception set S_i, then subtract it from R_i. After a finite number of subtractions, we will obtain the original profile $\{S_i\}_{i \in N}$. Since at no step did we change the status of y in any reasonability perception set, $y \in f(\{S_i\}_{i \in N})$ by D.

$\Leftarrow$

Trivial.

Problem D4 (Tel Aviv 2010)

Let $\succsim$ be a preference relation on R^n satisfying the following two properties:

Weak Pareto (WP): If $x_i \geq y_i$ for all i, then $x = (x_1,..,x_n) \succsim y = (y_1,..,y_n)$ and if $x_i > y_i$ for all i, then $(x_1,..,x_n) \succ (y_1,..,y_n)$.

Independence (IIA): Let $a,b,c,d \in \mathbb{R}^n$ be vectors such that in any coordinate $a_i > b_i$, $a_i = b_i$ or $a_i < b_i$ if and only if $c_i > d_i$, $c_i = d_i$ or $c_i < d$, accordingly. Then, $a \succsim b$ iff $c \succsim d$.

1. Find a preference relation different from those represented by $u_i(x_1,\ldots,x_n) = x_i$ which satisfies the two properties.

Lexicographic preferences such as: $x \succ y$ iff $x_{i^*} > y_{i^*}$ where $i^* = \min\{i | x_i \neq y_i\}$.

2. Show, for $n = 2$, that there is an i such that $a_i > b_i$ implies $a \succ b$.

Assume that $(4,2) \succ (2,4)$. By Pareto $(4,2) \succ (2,0)$.

Also $(4,2) \succ (2,2)$ since by by Pareto $(2,4) \succsim (2,2)$.

Now, consider two vectors $a = (a_1,a_2)$ and $b = (b_1,b_2)$ such that $a_1 > b_1$. By IIA the preference between a and b when $a_2 < b_2$, $a_2 = b_2$ or $a_2 > b_2$ is the same as between $(4,2)$ and $(2,4)$, $(2,2)$ or $(2,0)$ respectively, namely $a \succ b$.

3. Provide a "social choice" interpretation for the result in (2). Explain how it differs from Arrow's Impossibility Theorem.

We can interpret a point in R^n as an allocation of a desirable good between n individuals. The preferences of all individuals are fixed (each wants as much as possible). The independence property expresses a requirement that the social preference between any two alteratives a and b is a function of only the n comparisons between a_i and b_i.

4. Expand (2) for any n.

Let $A \subseteq \{1,\ldots,n\}$. We say that A is *decisive* if whenever for all $i \in A$, $x_i > y_i$ then $x \succ y$.

Let $A \subseteq \{1,\ldots,n\}$. We say that A is *almost decisive* if whenever for all $i \in A$, $x_i > y_i$ and for all $i \notin A$, $y_i > x_i$ then $x \succ y$

First, if A is almost decisive then it is decisive: By the independence it is enough to look at two vectors a and b such that $a_i = 3$ and $b_i = 1$ if $i \in A$, and all other a_j and b_j are either 1 or 3.

Let $c_i = 3$ if $i \in A$ and $c_i = 1$ otherwise and let $d_i = 1$ if $i \in A$ and $d_i = 3$ otherwise.

By the almost decisiveness of A, $c \succ d$. By Pareto $a \succsim c$ and $d \succsim b$, thus $a \succ b$.

Now let A be a decisive set. and let A_1 and A_2 be a partition of A. We will see that either A_1 or A_2 is almost decisive.
Assume not. Consider the three vectors:

	A_1	A_2	$N-A$
a	1	3	5
b	5	1	3
c	3	5	1

By A's decisiveness, $c \succ a$. If A_1 is not almost decisive then $a \succsim b$ and if A_2 is not almost decisive then $b \succsim c$. A contradiction.
Thus, there is i such that $\{i\}$ is decisive.

Problem D5 (NYU 2012. Based on Rubinstein (1980))

An individual is comparing pairs of alternatives within a finite set X ($|\mathbf{X}| \geq 3$). His comparison yields unambiguous results, such that either x is evaluated to be better than y (denoted $x \to y$) or y is evaluated to be better than x ($y \to x$). A ranking method assigns to each such relation $\to$ (namely, complete, irreflexive and antisymmetric relation, but not necessarily transitive) a preference relation $\succsim(\to)$ over X. Consider the following axioms with respect to ranking methods:

Neutrality: "the names of the alternatives are immaterial". (Formally, let σ be a permutation of X and let $\sigma(\to)$ be the relation defined by $\sigma(x)\sigma(\to)\sigma(y)$ iff $x \to y$. Then, $x \succsim (\to)y$ iff $\sigma(x) \succsim (\sigma(\to))\sigma(y)$.)

Monotonicity: if $x \succsim (\to)y$, then $x \succ (\to')y$ where $\to'$, differs from $\to$ only in the existence of one alternative z such that $z \to x$ and $x \to' z$.

Independence: The ranking between any two alternatives depends only on the results of comparisons that involve at least one of the two alternatives.

1. Define $N_\to(x) = |\{z|x \to z\}|$ (the number of alternatives beaten by x). Explain why the scoring method defined by $x \succsim (\to)y$ if $N_\to(x) \geq N_\to(y)$ satisfies the three axioms.

Neutrality: For any permutation σ of X, $x \succsim y \Leftrightarrow \sigma(x) \succsim (\sigma)\sigma(y)$ since the numbers of victories for x and y in $\to$ are the same as for $\sigma(x)$ and $\sigma(y)$ accordingly in $\to'$.

Monotonicity: If x has at least as many points as y in $\to$, then it will have strictly more victories in $\to'$ where it wins in one additional comparison (which it lost in $\to$).

Independence: The comparison between x and y depends only on the comparisons involving x and y.

2. For each of the properties, give an example of a ranking method which satsifies the other two properties but not that one.

The three axioms are independent.

(a) Consider a method that assigns to any $\to$ the same arbitrary fixed preference relation. This ranking method satisfies Monotonicity and Independence but violates Neutrality.

(b) Consider the ranking method according to $x \succsim (\to)y$ if $N_\to(x) \leq N_\to(y)$. It satisfies Neutrality and Independence but violates Monotonicity.

(c) Consider the ranking method defined by

$$x \succsim (\to)y \text{ if } \sum_{z\,|\,x\to z}(N_\to(z)+1) \geq \sum_{z\,|\,y\to z}(N_\to(z)+1)$$

It satisfies Neutrality and Monotonicity but violates Independence.

3. **Prove that the above scoring method is the only one that satisfies the three properties**.

We will first prove first the following :

Lemma: If a ranking method $\succsim$ satisfies Neutrality and Independence and if $N_{\to}(x) = N_{\to}(y)$, then $x \sim (\to)y$.

Proof: Define $A \equiv \{z \mid x \to z$ and $z \to y\}$ and $B \equiv \{z \mid z \to x$ and $y \to z\}$.

Since $x \to y$ and $N_{\to}(x) = N_{\to}(y)$ we have $|B| = |A| + 1$.

We shall prove by induction on $|A|$: Assume $|A| = 0$.

Let b be the unique element in B. The relation $\to$ cycles on $\{x, y, b\}$. By Independence we can assume that every element which beats both x and y beats b and every element beaten by both x and y is beaten by b. Then, by Neutrality $x \sim y \sim b$.

Assume that the induction hypothesis holds for $|A| = m - 1 \geq 0$ and let $|A| = m$.

Choose $a \in A$ and $b \in B$. By Independence, the ranking of x and y will not change if we assume that (i) $a \to b$ and (ii) $a \to z$ iff $x \to z$ for all $z \notin \{a, b, x, y\}$.

Then $x \to a$, $a \to b$ and $b \to x$, while for all $z \notin \{x, a, b\}$, $x \to z$ iff $a \to z$. It follows from the case in which $|A| = 0$, that $a \sim (\to)x$.

We also have $a \to y$ and $N_{\to}(a) = N_{\to}(y)$. The set $\{z | a \to z$ and $z \to y\}$ includes $m - 2$ elements. Therefore, by the induction hypothesis $a \sim (\to)y$. Thus $x \sim (\to)y$.

Finally, consider a ranking method that satisfies the three axioms on the set X. Suppose $x, y \in X$.

If $N_{\to}(x) > N_{\to}(y) \geq 1$, let E be a subset not including y whose $|\{z \mid x \to z\}| - |\{z \mid y \to z\}|$ elements are all beaten by x. Let $\to'$ denote the relation obtained from $\to$ by reversing the results of the comparisons of x to the elements in E. Then $x \sim (\to')y$ by the Lemma. Applying the Monotonicity axiom $|E|$ times, we have $x \succ y$.

If $N_{\to}(x) > N_{\to}(y) = 0$, let h be a third element. Let $\to'$ be the relation derived from $\to$ by reversing the results of the comaprison between y and h. Then, $N_{\to'}(x) \geq N_{\to'}(y) \geq 1$, and therefore $x \succsim (\to')y$ from the previous conclusion. If $y \succsim (\to)x$, then by Monotonicity we would have $y \succ (\to')x$ and hence $x \succ (\to)y$.

Problem D6 (**Tel Aviv 2013**)

Society often looks for a representative agent. Assume for simplicity that the number of agents in a society is a power of 2 (**1**,**2**,**4**,**8**....). **Each agent is one of a finite number of types** (**a member in a set T**). **A representative agent method** (**RAM**) **is a function** F **which attaches to any vector of types** $(t_1,..,t_n)$ (**where** $n = 2^m$ **and each** $t_i \in T$) **an element in** $\{t_1,..,t_n\}$.

Make the following assumptions about F:

(**i**) **Anonymity**: **For any** n **and for any permutation** σ of $\{1,..,n\}$, **we have** $F(t_1,..,t_n) = F(t_{\sigma(1)},..,t_{\sigma(n)})$.

(**ii**) **The** "**representative**" **is the** "**representative of the representatives**": $F(t_1,..,t_n) = F(F(t_1,..,t_{n/2}),F(t_{n/2+1},..,t_n))$

1. **Characterize the RAMs which satisfy the two axioms**.

Claim: an RAM satisfies the two axioms iff there is an ordering of the types in T, denoted by $\succ$, such that $F(t_1,..,t_n)$ is the $\succ$-maximal type in $\{t_1,..,t_n\}$.

Proof:

$\rightarrow$

Let F be an RAM satisfying the two axioms.

Define $t_i \succ t_j$ if $F(t_i,t_j) = t_i$. The relation $\succ$ is an ordering on T and has the following characteristics:

Asymmetry: by axiom (i), $F(t_i,t_j) = F(t_j,t_i)$ and therefore if $t_i \succ t_j$, then $F(t_j,t_i) \neq t_j$, which implies that $t_j \nsucc t_i$.

Completeness: By the assumption that $F(t_i,t_j) \in \{t_i,t_j\}$, either $F(t_i,t_j) = t_i$ or $F(t_j,t_i) = t_j$. Hence, either $t_i \succ t_j$ or $t_j \succ t_i$.

Transitivity: Assume that $t_i \succ t_j$ and $t_j \succ t_h$. If not $t_i \nsucc t_h$, then $F(t_h,t_i) = t_h$. By axiom (ii):

$F(t_i,t_j,t_h,t_h) = F(F(t_i,t_j),F(t_h,t_h)) \ = F(t_i,t_h) = t_h$ and

$F(t_j,t_h,t_i,t_h) = F(F(t_j,t_h),F(t_i,t_h)) \ = F(t_j,t_h) = t_j$.

However, by axiom (i) $F(t_i,t_j,t_h,t_h) = F(t_j,t_h,t_i,t_h)$, a contradiction.

Lastly, we can show that $F(t_1,..,t_n) = \ \succ$-maximal in $\{t_1,..,t_n\}$, by induction on m, where $n = 2^m$:

By definition this holds for $m = 1$. Assume that it is correct for $m = l-1$: $F(t_1,..,t_{2^{l-1}}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$.

Let $m = l$.

By axiom (ii), $F(t_1,..,t_{2^l}) = F(F(t_1,..,t_{2^{l-1}}),F(t_{2^{l-1}+1},..,t_{2^l}))$. By assumption, $F(t_1,..,t_{2^{l-1}}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$ and $F(t_{2^{l-1}+1},..,t_{2^l}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$. Denote these two maximal types by t' and t''.

By definition, $F(t',t'')$ is the $\succ$-maximal in $\{t',t''\}$ and clearly it is also the maximal in

$\{t_1, .., t_{2^l}\}$.

$\leftarrow$

(i). Trivial

(ii). The $\succ$-maximal type in $\{t_1, .., t_n\}$ is either in $\{t_1, .., t_{n/2}\}$ or in $\{t_{n/2+1}, .., t_n\}$. In either case, it is the $\succ$-maximal in its set and therefore it is chosen by F. Thus, this type is also in $\{F(t_1, .., t_{n/2}), F(t_{n/2+1}, .., t_n)\}$ and it will be chosen from $(t_1, .., t_n)$ by F.

2 Suggest an RAM that satisfies (i) but not (ii) and an RAM that satisfies (ii) but not (i).

(i) but not (ii): choosing the second-best type according to some ordering $\succ$.on T.

(ii) but not (i): choosing the type of the first agent: $F(t_1, .., t_n) = t_1$.

Problem D7 (**Tel Aviv 2014**)

We say that a binary relation P **over the space** $X = R^n$ **satisfies Property** I **if the statement** xPy (**the relation between** x **and** y) **depends only on the equalities between the components of the two vectors**. **Formally**, P **satisfies Property** I **if** $aPb \Leftrightarrow cPd$ **for any four vectors** a, b, c **and** d **that satisfy** (**i**) $a_i = a_j \Leftrightarrow c_i = c_j$, (**ii**) $b_i = b_j \Leftrightarrow d_i = d_j$ **and** (**iii**) $a_i = b_j \Leftrightarrow c_i = d_j$.

Denote $Y = \{x | \forall i \neq j,\ x_i \neq x_j\}$ **as the set of all vectors vectors that are composed of** n **different numbers**.

a. **Give an example** (**for** $n = 2$) **of non-degenerated preference relation on** X **that satisfies property** I.

Let P be the preference relation represented by: $U(x_1, x_2) = \begin{cases} 1 & if \quad x_1 \neq x_2 \\ 0 & if \quad x_1 = x_2 \end{cases}$. Clearly, it is a well defined preference relation and the relation between $U(x)$ and $U(y)$ depends only on the equalities between components of x and the equalities between components of y and thus the preference relation satisfies property I.

Show that any preference relation satisfying property I:

b. **is indifferent between the vector** $(1,2,3)$ **and any of the vectors** $(4,2,5)$, $(2,3,1)$ **and** $(4,5,6)$.

Note first that for any two vectors in Y (!), Property I implies that if $a_i = b_j \Leftrightarrow c_i = d_j$ for any i,j then $aPb \Leftrightarrow cPd$.

1. Let $x = (1,2,3)$ and $y = (4,2,5)$. Whatever can be said about x and y can be said about y and x ($x_i = y_j$ iff $y_i = x_j$) and thus xIy.

2. Let $x = (1,2,3), y = (2,3,1)$ and $z = (3,1,2)$. Note that $x_i = y_j \Leftrightarrow y_i = z_j \Leftrightarrow z_i = x_j$. Thus, xPy implies yPz and zPx; yPx implies zPy and xPz. Thus, in both cases, xIy.

3. Let $x = (1,2,3)$ and $y = (4,5,6)$. Again, $x_i = y_j$ iff $y_i = x_j$ for any i, j and thus xIy.

c. **is indifferent between any** $x,y \in Y$ **satisfying** $x_i \neq y_j$ **for any** i,j.

Choose x and y such that $x_i \neq y_j$ for any i, j. Assume xPy. As in 3 above, $x_i \neq y_j$ iff $y_i \neq x_j$ for any i, j. Thus, xIy.

d. **is indifferent between any** $x,y \in Y$ **where** x **is a permutation of** y.

Let x and y be two vectors such that y is a permutation of x. Thus, there exists a permutation on $\{1,\ldots,n\}$ such that $y_{\sigma(i)} = x_i$ for any i. Define a sequence of vectors $\{x^i\}$ such that $x^0 = x$ and $x^k_{\sigma(i)} = x^{k-1}_i$. Thus, any x^k is a permutation of x, $x^0 = x$ and $x^1 = y$. There exists a minimal integer K such that $x^K = x^0$.

Assume xPy, i.e., x^0Px^1. By definition, $x^0_i = x^1_j$ iff $j = \sigma(i)$. Since $x^1_i = x^2_j \Leftrightarrow j = \sigma(i)$ as well, x^1Px^2. Similarly x^2Px^3,...,$x^{K-1}Px^K = x^0$. By transitivity, x^1Px^0. Thus, $xPy \Rightarrow yPx$. An analgous argument applies in the case of yPx. Thus, xIy.

e. **Is indifferent between any** $x,y \in Y$.

Choose $z \in Y$ such that all components in z different from all componenets in x and y. By (c) xIz and yIz and thus xIy.

f. (**much more difficult**) **Characterize the set of preference relations satisfying Property** I.

See: Rubinstein, A. (2002). "Definable Preferences: Another Example (Searching for a Boyfriend in a Foreign Town)" in *In The Scope of Logic, Methodology and Philosophy of Science, Proceedings of the 11th International Congress of Logic*, ed. Peter Gardenfors et al. Kluwer, vol I, 235-243. (http://arielrubinstein.tau.ac.il/papers/65.pdf).