Expected Utility

Finally, games...

- One-person games :(

Lotteries

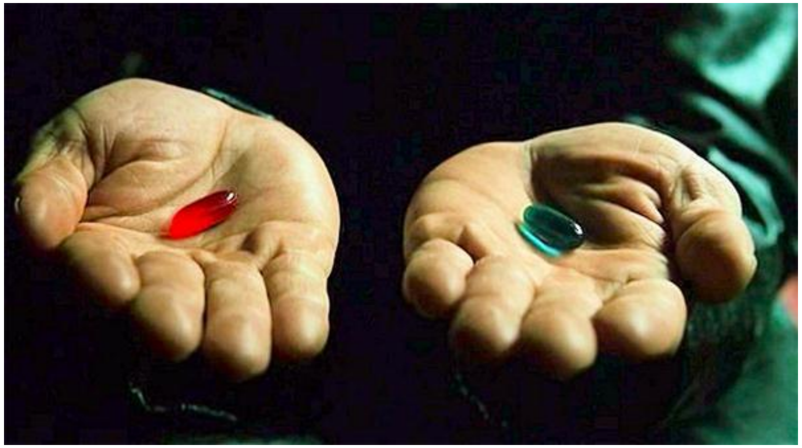
- To every action there is a consequence...

Lotteries

- So far, in our modeling situations each action <u>deterministically</u> leads to a particular consequence

Lotteries

- From now on, the correspondence between actions and consequences is **stochastic**
- The choice of an action is viewed as choosing a lottery where the prizes are the consequences



Lotteries

- We will be interested in preferences and choices over the set of lotteries

	Color	White	Red	Green	Yellow
(a)	Chance %	90	6	1	3
	Prize $	0	45	30	-15

	Color	White	Red	Green	Yellow
(b)	Chance %	90	7	1	2
	Prize $	0	45	-10	-15

Lotteries

- Let $\boldsymbol{Z}$ be a set of consequences (prizes)
  - for now, $\boldsymbol{Z}$ is finite
- A lottery is a probability measure on $\boldsymbol{Z}$
  - a lottery $\boldsymbol{p}$ is a function that assigns a nonnegative number $\boldsymbol{p(z)}$ to each prize $\boldsymbol{z}$, where $\boldsymbol{\Sigma_{z \in Z} p(z) = 1}$
  - The number $\boldsymbol{p(z)}$ is taken to be the objective probability of obtaining the prize $\boldsymbol{z}$ given the lottery $\boldsymbol{p}$

Lotteries

$p_a(0) = 0.9$

	Color	White	Red	Green	Yellow	
(a)	Chance %	90	6	1	3	p_a
	Prize $	0	45	30	-15	z_a

	Color	White	Red	Green	Yellow	
(b)	Chance %	90	7	1	2	p_b
	Prize $	0	45	-10	-15	z_b

Lotteries

- Which lottery do you prefer?

(a)

OR

(b)

$Z_a = \{\$35\}$

$p_a(\$35)=1$

$Z_b = \{\$20,\$80\}$

$p_b(\$20)=0.75$
$p_b(\$80)=0.25$

Lotteries

- Which lottery do you prefer?

(a)

(b)

OR

$Z_a = \{$  $\}$

$p_a($  $)=1$

$Z_b = \{$  ,  $\}$

$p_b($  $)=0.75$

$p_b($  $)=0.25$

Lotteries

- Denote by ***[z]*** the degenerate lottery for which $p(z) = 1$
- We will use the notation

$$\alpha x \oplus (1 - \alpha)y$$

to denote the lottery in which the prize x is realized with probability α and the prize y with probability $1 - \alpha$

Lotteries

- Denote by ***L(Z)*** the (infinite) space containing all lotteries with prizes in ***Z***
- A simplex in Euclidean space:

$$L(Z) = \{x \in \mathbb{R}_+^Z \mid \Sigma x_z = 1\}$$

where $\mathbb{R}_+^Z$ is the set of functions from ***Z*** into $\mathbb{R}_+$

Preferences over Lotteries

- Let us think about examples of “sound” preferences over a space ***L(Z)***
  - What makes a lottery better than the other?

Preferences over Lotteries

- Preference for uniformity
  - The decision maker prefers the lottery that is less disperse where dispersion is measured by

    $$\Sigma_z(p(z) - 1/|Z|)^2$$

  - Example:
    - a lottery over my music collection (all songs have the same chance of being played)

Preferences over Lotteries

- Preference for most likelihood
  - The decision maker prefers ***p*** to ***q*** if $\boldsymbol{max_z p(z)}$ is greater than $\boldsymbol{max_z q(z)}$
  - Example
    - Weather conditions before going out

Preferences over Lotteries

- The size of the support
  - The decision maker evaluates each lottery by the number of prizes that can be realized with positive probability

    $$supp(p) = \{z|p(z) > 0\}$$

  - He prefers a lottery ***p*** over a lottery ***q*** if

    $$|supp(p)| \leq |supp(q)|$$

  - Example
    - A multiple choice question in an exam

Preferences over Lotteries

- These three examples are <u>degenerate</u>
- **Ignored** the consequences and were dependent on the probability vectors alone

Preferences over Lotteries

- Increasing the probability of a “good” outcome
  - The set ***Z*** is partitioned into two disjoint sets ***G*** and ***B*** (good and bad), and between two lotteries the decision maker prefers the lottery that yields “good” prizes with higher probability
  - Example
    - Choosing a route from city A to city B

Preferences over Lotteries

- The worst case
  - The decision maker evaluates lotteries by the worst possible case
  - He attaches a number $v(z)$ to each prize z and $p \succsim q$ if $\min\{v(z) \mid p(z) > 0\} \geq \min\{v(z) \mid q(z) > 0\}$
- Example:
  - "This criterion is often used in computer science, where one algorithm is preferred to another if...
  - it functions better in the worst case independently of the likelihood of the worst case occurring"

Preferences over Lotteries

- Comparing the most likely prize
  - The decision maker considers the prize in each lottery that is most likely (breaking ties in some arbitrary way) and compares two lotteries according to a basic preference relation over $\boldsymbol{Z}$
- Example
  - Selecting a career

Preferences over Lotteries

- Lexicographic preferences:
  - The prizes are ordered $\boldsymbol{z_1, \ldots, z_K}$, and the lottery $\boldsymbol{p}$ is preferred to $\boldsymbol{q}$ if
  - $\boldsymbol{(p(z_1), \ldots, p(z_K)) \geq_L (q(z_1), \ldots, q(z_K))}$
- Example
  - choosing a movie to watch
    - explosions $\succ_1$ funny $\succ_2$ romance $\succ_3$ etc

Preferences over Lotteries

- Expected utility:
  - A number ***v(z)*** is attached to each prize, and a lottery ***p*** is evaluated according to its expected ***v***, that is, according to $\Sigma_z p(z)v(z)$
  - Thus,

$$p \succsim q \text{ if } U(p) = \Sigma_{z \in Z} p(z)v(z) \geq U(q) = \Sigma_{z \in Z} q(z)v(z)$$

- Example
  - Games in a casino

Preferences over Lotteries

- Examples could be even richer...

Preferences over Lotteries

- The richness of examples calls for the classification of preference relations over lotteries
- Study of properties that these relations satisfy

Preferences over Lotteries

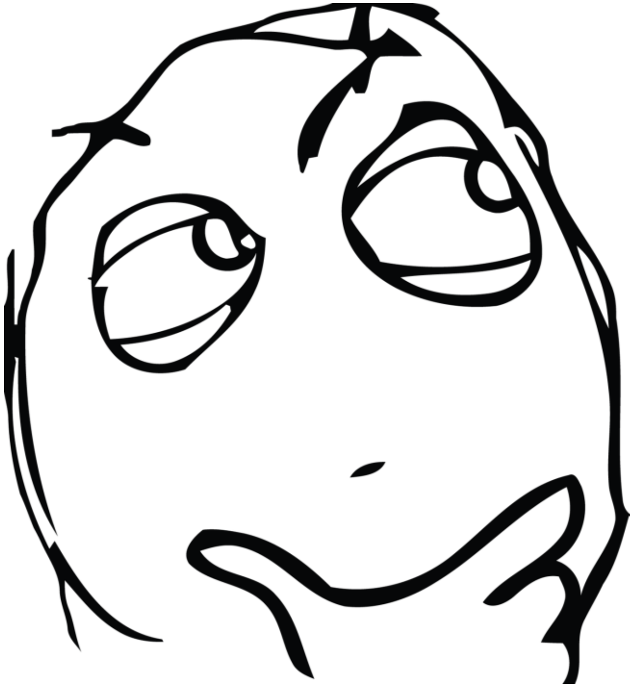
- Formally state general principles (axioms) that may apply to preferences over the space of lotteries
  - Consistency requirement
  - Procedural aspect of decision making

Preferences over Lotteries

- A set of axioms characterizing a family of preferences is a justification for focusing on that specific family

Preferences over Lotteries

- What are the desired properties of choice procedures over lotteries?



von Neumann and Morgenstern Axiomatization

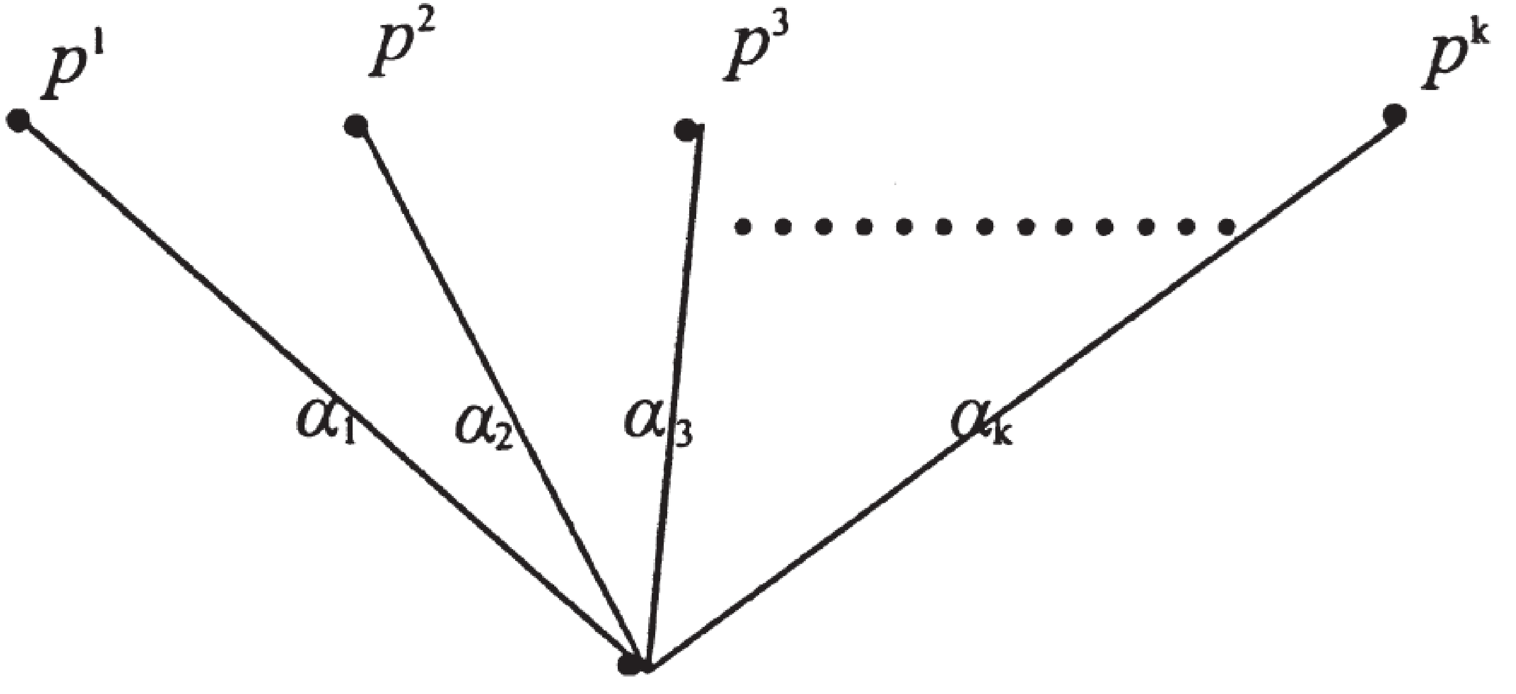
- Six axioms are usually presented
  - ordering of alternatives
  - reduction of compound lotteries
  - continuity
  - substitutability
  - transitivity
  - monotonicity

An axiomatic treatment of utility

- **Assumption 1**: ordering of alternatives
  - The **preference relation** $\succsim$ between two prizes $\boldsymbol{z_i}$ and $\boldsymbol{z_j}$ is transitive
  - Either $\boldsymbol{z_i} \succsim \boldsymbol{z_j}$ or $\boldsymbol{z_j} \succsim \boldsymbol{z_i}$
  - If $\boldsymbol{z_i} \succsim \boldsymbol{z_j}$ and $\boldsymbol{z_j} \succsim \boldsymbol{z_k}$, then $\boldsymbol{z_i} \succsim \boldsymbol{z_k}$
- Transitiveness is not always seen in data!
  - specially when people are presented with paired comparisons

Compound Lotteries

- $\bigoplus^{K}_{k=1}\alpha_k p^k$



An axiomatic treatment of utility

- **Assumption 2**: reduction of compound lotteries
  - A compound lottery
    $CL = \bigoplus^{K}_{k=1} \alpha_k p^k = (\alpha_1 p^1, \alpha_2 p^2, \ldots, \alpha_K p^K)$
  - Any compound lottery is indifferent to a simple lottery with the same prize list

    $(\alpha_1 p^1, \alpha_2 p^2, \ldots, \alpha_K p^K) \sim (p_1(z_1), p_2(z_2), \ldots, p_r(z_r))$

    where

    $p_i(z_i) = \alpha_1 p^1(z_i) + \alpha_2 p^2(z_i) + \ldots \alpha_K p^K(z_i)$

An axiomatic treatment of utility

- **Assumption 3**: continuity
- Each prize z_i is indifferent to some lottery ticket involving just z_1 (the best prize) and z_r (the worst)
- There exists a number p such that z_i is indifferent to $[\, pz_1, (1-p)z_r \,]$
- And for $Z = \{\$1, \$0.01, death\}$?

An axiomatic treatment of utility

- **Assumption 4**: <u>substitutibility</u>
  - If $\boldsymbol{z_i} \sim \boldsymbol{z_j}$, then one may substitute the other in a lottery

An axiomatic treatment of utility

- **Assumption 5**: <u>transitivity</u>
  - Preference and indifference among lotteries (or lottery tickets) are transitive relations

An axiomatic treatment of utility

- From these five assumptions
  - it is possible to find for any lottery ticket one to which it is indifferent and which only involves $\boldsymbol{z_1}$ and $\boldsymbol{z_r}$

An axiomatic treatment of utility

- **Assumption 6**: monotonicity
  - A lottery $[p(z_1), (1-p)z_r]$ is preferred or indifferent to $[p'(z_1), (1-p')z_r]$ if and only if...
  - *...* $p >= p'$
  - Two lotteries involving only two prizes, I should prefer the one which renders the most preferred prize more probable
    - But is it always?

von Neumann and Morgenstern Axiomatization

- Do these axioms make sense?
  - If we accept these six axioms, it turns out that we have no choice but to

    accept the existence of single-dimensional utility functions

    whose expected values agents want to maximize
  - If we do not want to reach this conclusion, we must therefore give up at least one of the axioms

von Neumann and Morgenstern Axiomatization

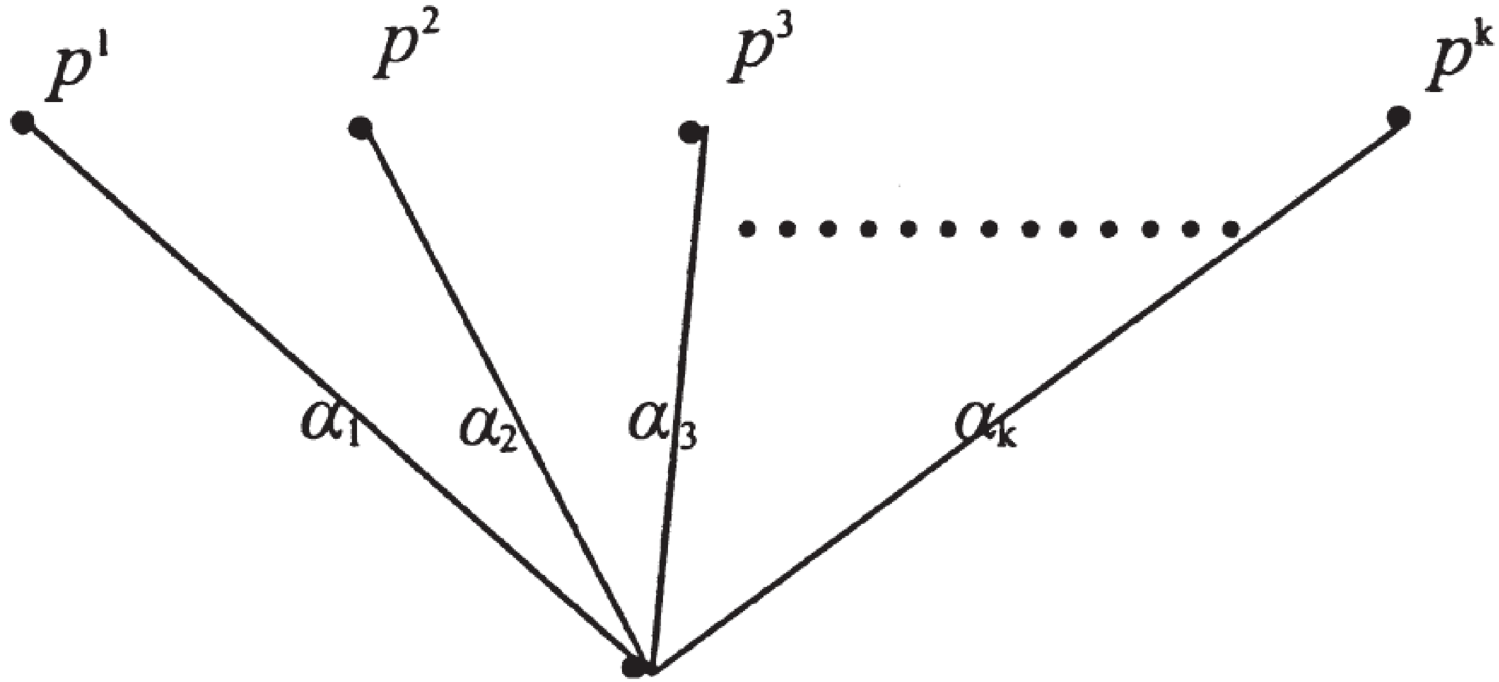
- Theorem (von Neumann and Morgenstern, 1944)
  - If $\succsim$ satisfies assumptions ***1*** through ***6***, there are numbers $\boldsymbol{u_i}$ associated with $\boldsymbol{z_i}$ such that, for two lotteries $\boldsymbol{p}$ and $\boldsymbol{q}$,
  - $p_1(z_1)\, u_1 + \ldots + p_r(z_r)\, u_r$ and $q_1(z_1)\, u_1 + \ldots + q_r(z_r)\, u_r$
  - reflect the <u>preferences</u> between the lotteries

von Neumann and Morgenstern Axiomatization

- Two axioms may do the job
  - **I:** independence
  - **C:** continuity

Compound Lotteries

- $\bigoplus^{K}{}_{k=1}\alpha_k p^k$



An axiomatic treatment of utility

- **I**: independence
  - For any $\boldsymbol{p, q, r \in L(Z)}$
  - and any $\boldsymbol{\alpha \in (0, 1)}$,
  - $\boldsymbol{p \gtrsim q}$ iff
  - $\boldsymbol{\alpha p \oplus (1 - \alpha)r \gtrsim \alpha q \oplus (1 - \alpha)r}$
  - No correlation between lotteries

An axiomatic treatment of utility

- **C**: continuity
  - If $\boldsymbol{p \succ q}$, then there are neighborhoods $\boldsymbol{B(p)}$ of $\boldsymbol{p}$ and $\boldsymbol{B(q)}$ of $\boldsymbol{q}$ such that for all $\boldsymbol{p' \in B(p)}$ and $\boldsymbol{q' \in B(q)}$, $\boldsymbol{p' \succ q'}$
  - Alternativelly, if $\boldsymbol{p \succ q \succ r}$, then there exists $\boldsymbol{\alpha \in (0, 1)}$ such that:
  - $\boldsymbol{q \sim [\alpha p \oplus (1 - \alpha) r]}$
  - Reality may be different...
    - if $\boldsymbol{r}$ is a lottery involving an extremely bad prize such as <u>‘death’</u>

Preferences over Lotteries

- Preference for most likelihood
  - The decision maker prefers ***p*** to ***q*** if $\max_z p(z)$ is greater than $\max_z q(z)$
  - Example
    - Choosing which clothes I should wear

An axiomatic treatment of utility

- Preferences for most likelihood
  - Satisfies ***C*** since the function $\max\{p_1, . . . , p_K\}$ that represents it is continuous in probabilities
  - It does not satisfy ***I*** since, for example,
  - $[z_1] \sim [z_2]$,
  - $[z_1] = 1/2[z_1] \oplus 1/2[z_1] \succ 1/2[z_2] \oplus 1/2[z_1]$

An axiomatic treatment of utility

- Expected utility:

$$
\begin{aligned}
U(\oplus_{k=1}^{K}\alpha_k p^k) &= \sum_{z\in Z}[\oplus_{k=1}^{K}\alpha_k p^k](z)v(z) = \sum_{z\in Z}\left[\sum_{k=1}^{K}\alpha_k p^k(z)\right]v(z) \\
&= \sum_{k=1}^{K}\alpha_k\left[\sum_{z\in Z}p^k(z)v(z)\right] = \sum_{k=1}^{K}\alpha_k U(p^k).
\end{aligned}
$$

  - It is linear, so it satisfies ***I***
  - It is continuous in the probability vector, so it satisfies ***C***

Utility Representation

- Theorem (vNM):
  - Let $\succsim$ be a preference relation over ***L(Z)*** satisfying ***I*** and ***C***
  - There are numbers $(v(z))_{z \in Z}$ such that

$$p \succsim q \text{ iff } U(p) = \Sigma_{z \in Z} p(z) v(z) \geq U(q) = \Sigma_{z \in Z} q(z) v(z)$$

U(p) is the utility of lottery ***p***

v(z) is the vNM utility, representing $\succsim$ over ***Z***

Utility Representation

- Let $\succsim$ be a preference relation over a set of lotteries
- If to each lottery ***p*** there is assigned a number ***U(p)*** such that $\boldsymbol{U(p) \geq U(q)}$ iff $\boldsymbol{p \succsim q}$,
- then there is a utility function ***U*** over ***L(Z)***
- When faced about uncertainty about which outcomes he will receive, the agent prefers outcomes that maximize ***U***

Utility Representation

- If ***I*** and ***C*** (or the six axioms presented previously) are met, then there is a <u>linear</u> vNM utility function v over the prizes
- $v(z_1) = 1$,
- $v(z_i) = v_i$, for $1 < i < r$
- $v(z_r) = 0$,
- $U(p(z_1), \ldots, p(z_r)) = p(z_1)v(z_1) + \ldots + p(z_r)v(z_r)$

Utility Representation

- Let $U(p)$ satisfy $\succsim$ over $L(Z)$ from a linear vNM utility function $v(z)$
- For some $\alpha>0$ and β, we can make
- $w(z) = \alpha v(z) + \beta$
- $W(p) = \Sigma_{z \in Z} p(z) w(z)$
- $W(p)$ will also satisfy $\succsim$ over $L(Z)$

Utility Representation

- What does this mean?
  - The absolute magnitudes of the utility function evaluated at different outcomes are unimportant
  - Instead, every positive affine transformation of a utility function yields another utility function for the same agent
    - In other words, if $\boldsymbol{u(o)}$ is a utility function for a given agent then $\boldsymbol{u'(o) = \alpha u(o) + \beta}$ is also a utility function for the same agent, as long as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constants and $\boldsymbol{\alpha}$ is positive

Interpersonal comparison of utility

- Very hard to define a universal value of utility
  - It changes from person to person
  - a gamble of $1 between a rich and a poor person
  - possible way to deal with it: normalize the maximum value to 1 and the lowest to 0