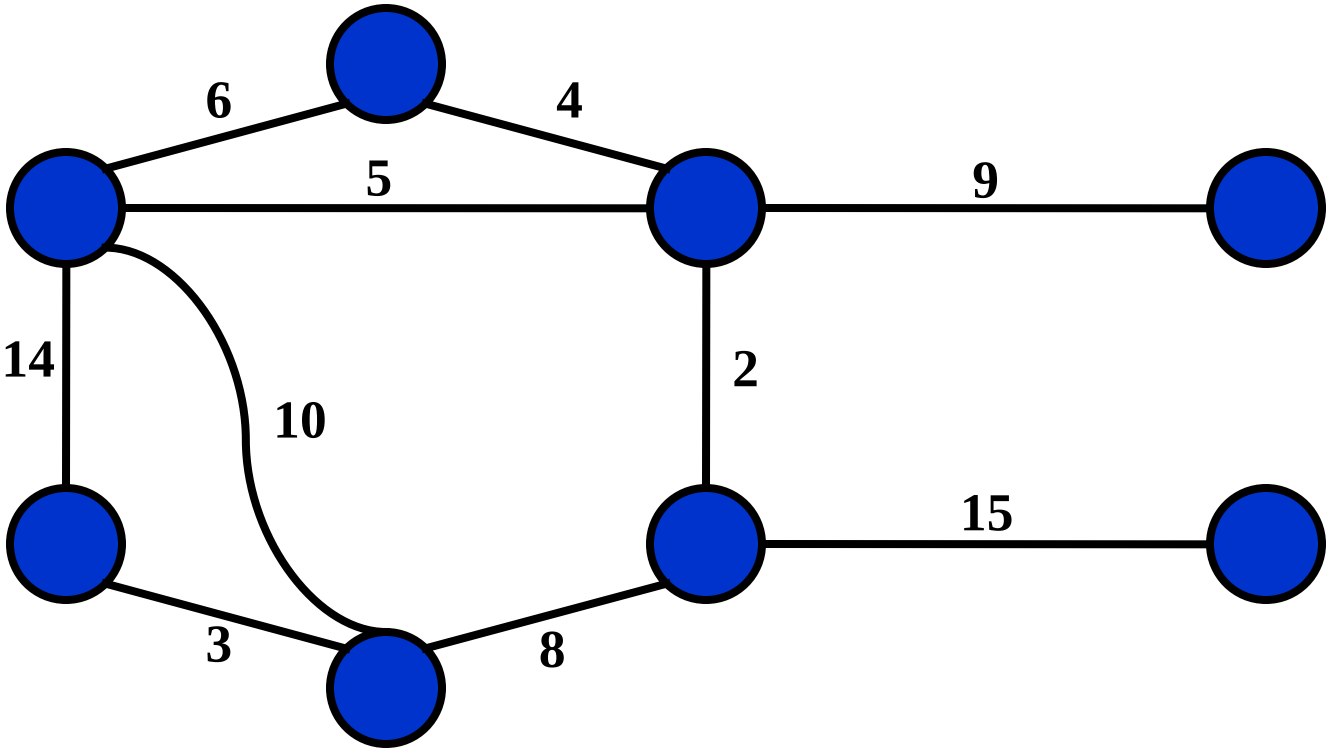
Minimum Spanning Trees

Chapter 23

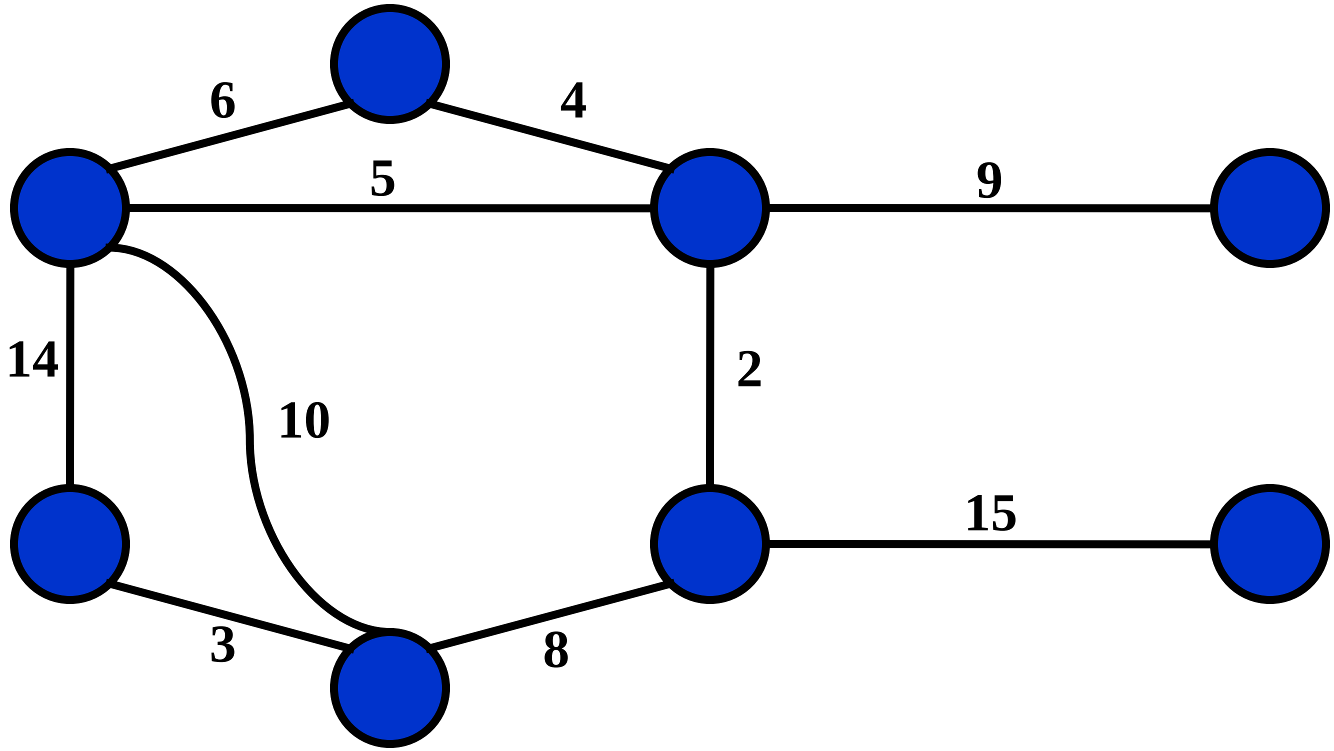
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph G(V,E) and a weight function w: E-> R



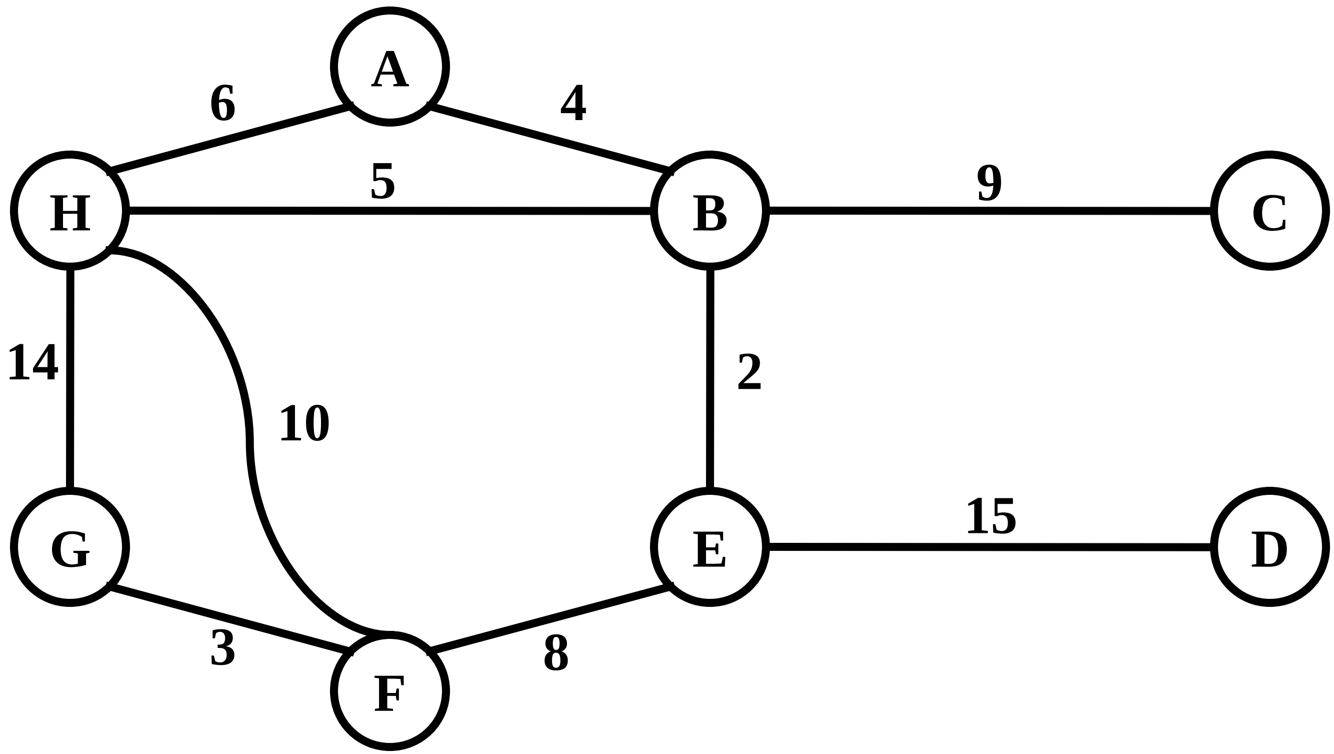
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a spanning tree T that connects all V of minimal weight



Minimum Spanning Tree

- Which edges form the minimum spanning tree (MST) of the below graph?



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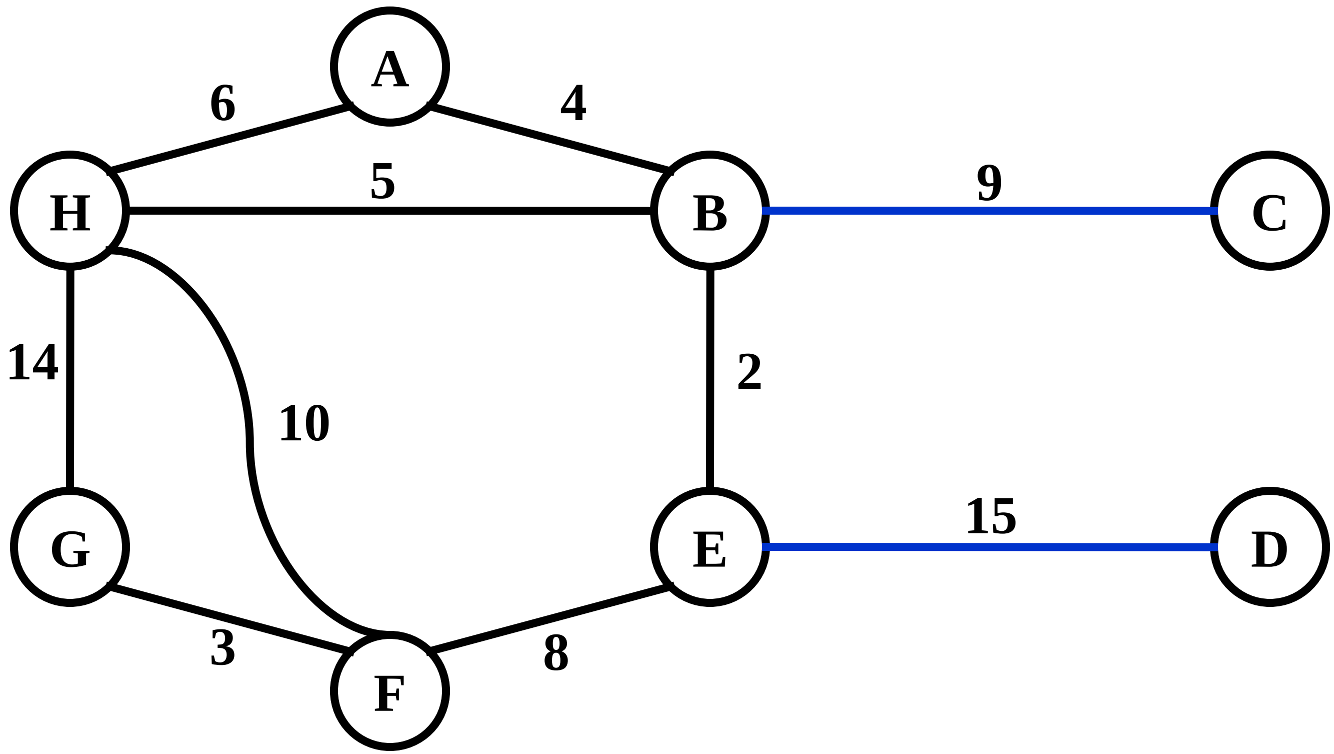

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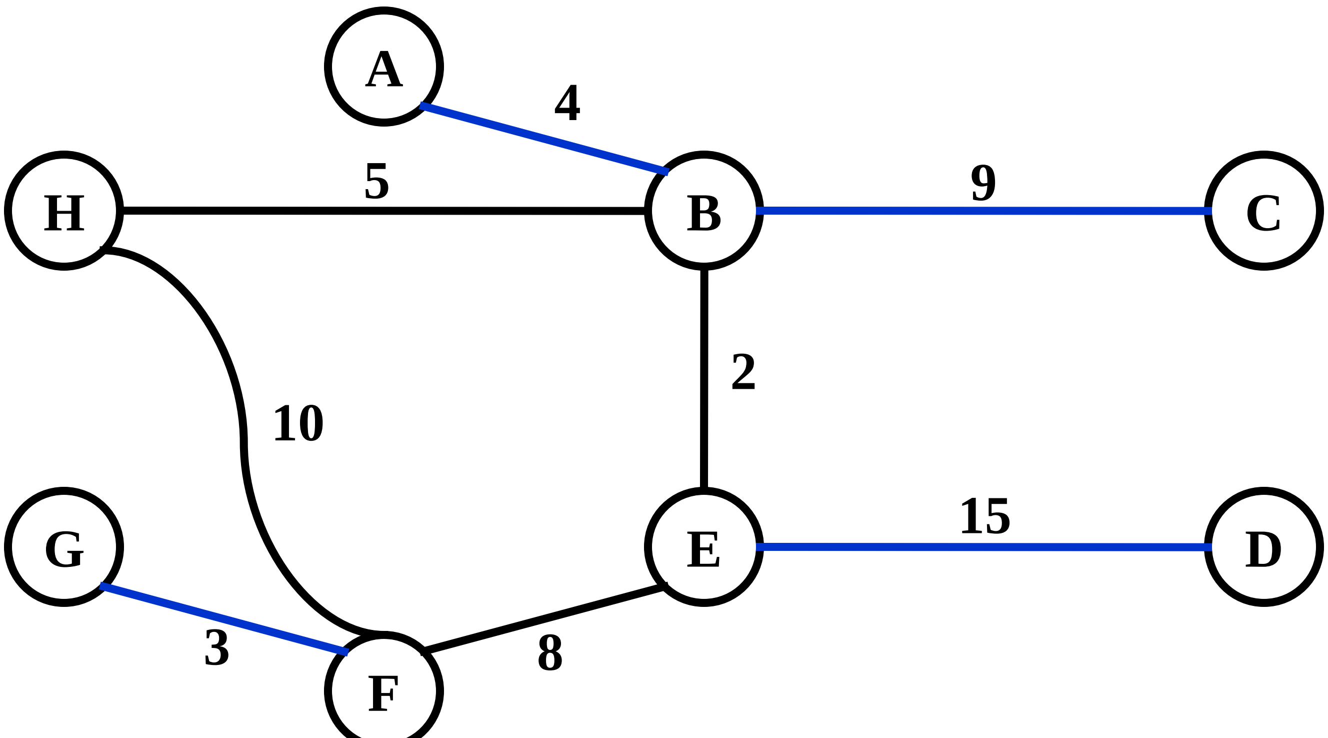

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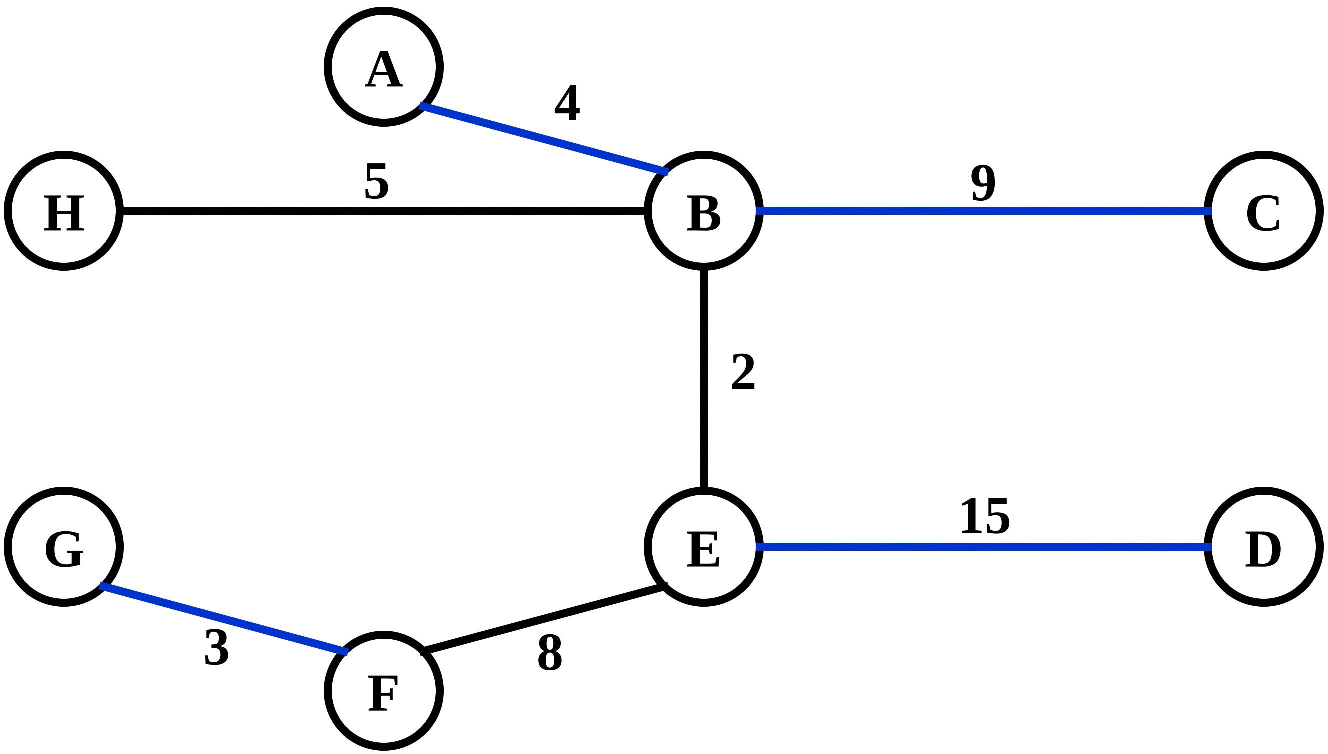
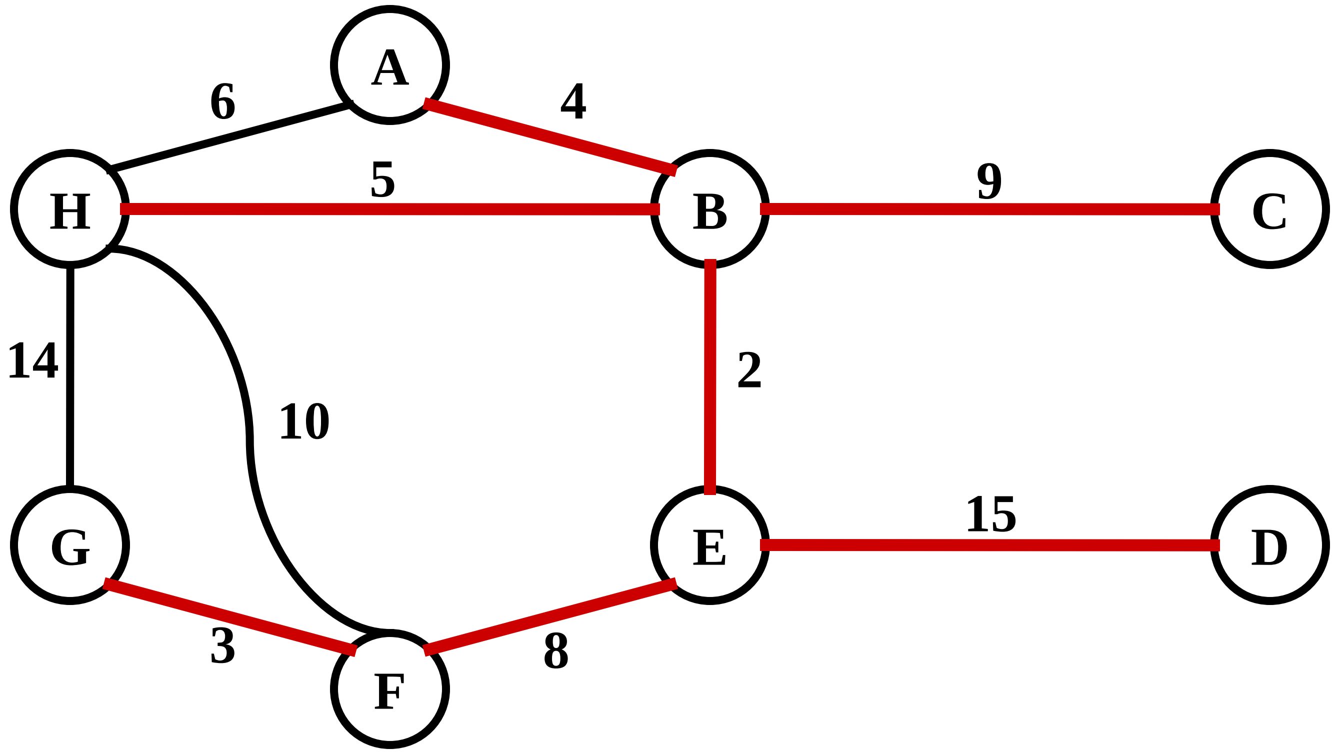


Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

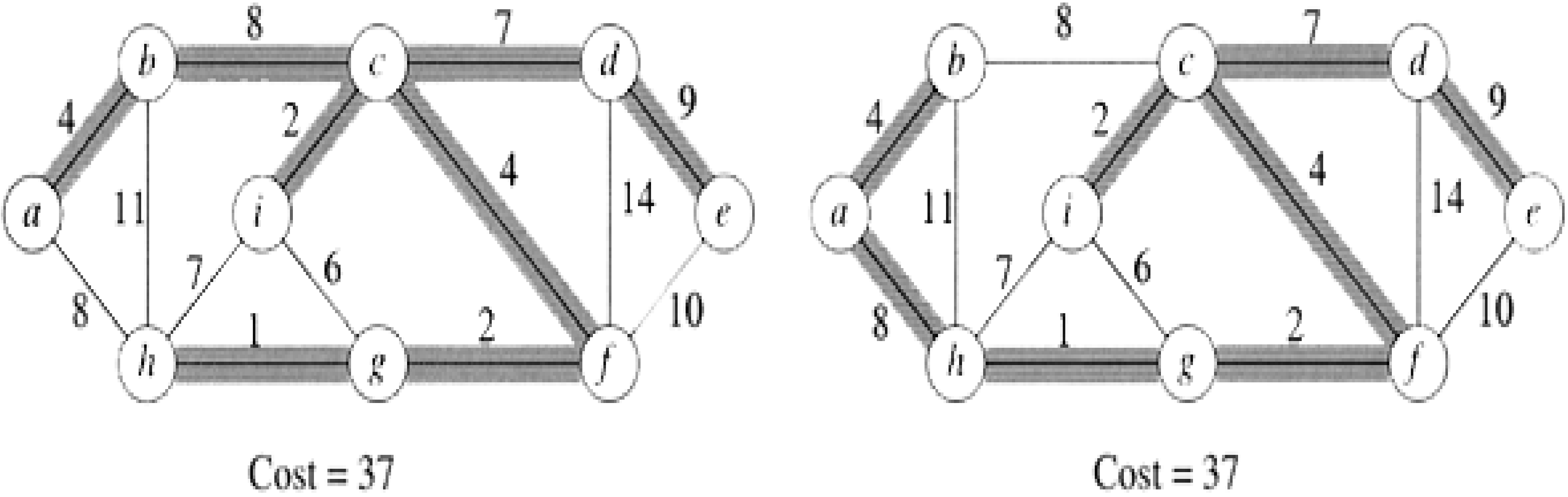


Figure 1: Minimum spanning tree.

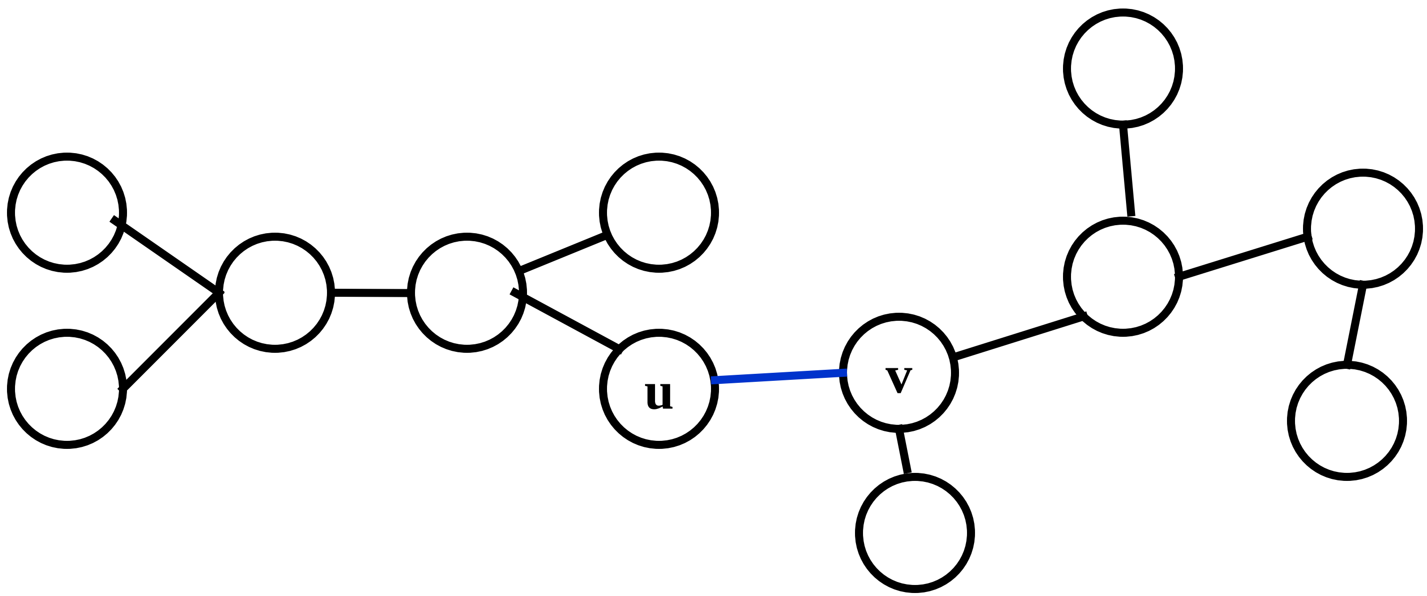
- Here we have two different MSTs for the same graph with equal costs and edge weights

Minimum Spanning Tree

- MSTs satisfy two powerful properties:
  - Optimal substructure
    - An optimal tree is composed of optimal subtrees
  - Greedy choice
    - A locally optimal choice is globally optimal

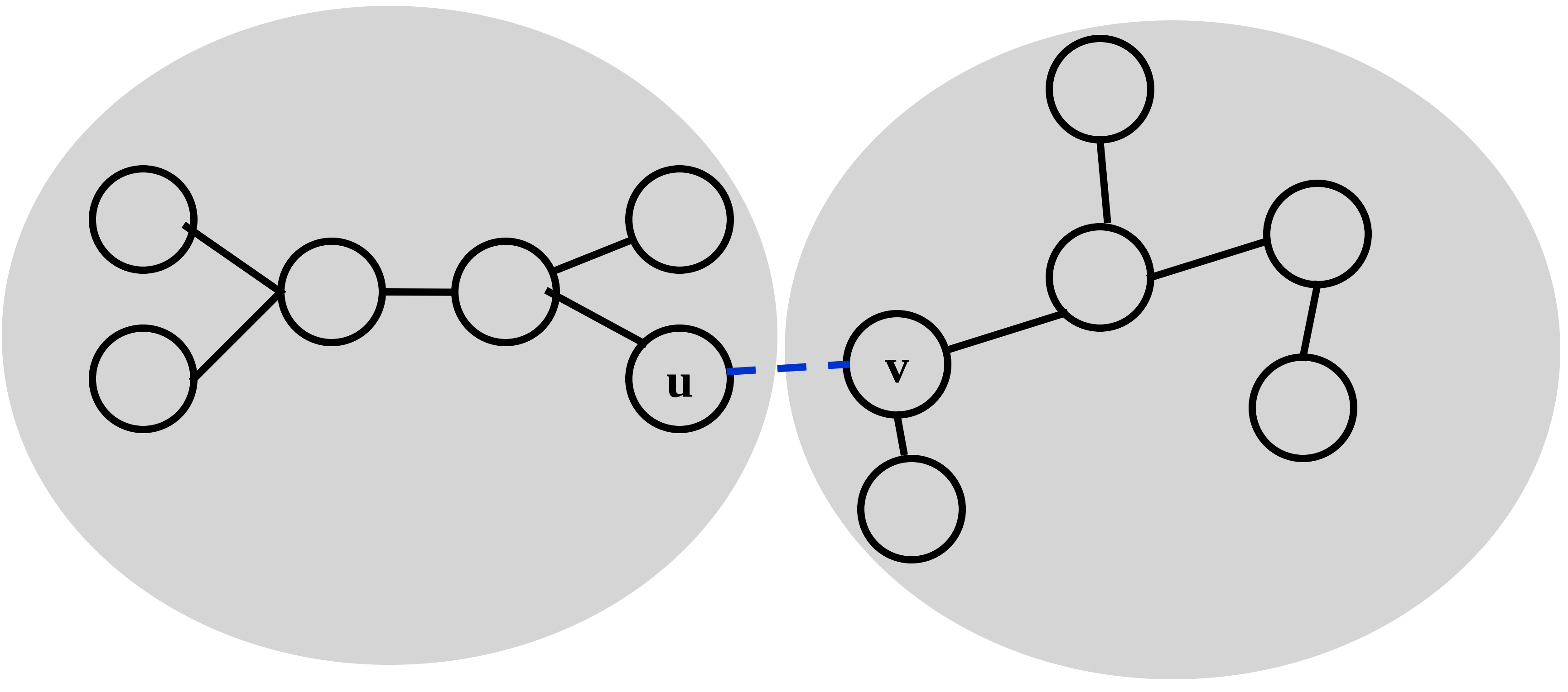
Optimal subtree property

- Optimal substructure property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2



Optimal subtree property

- Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$



Optimal subtree property

- Proof: cut and paste
- $w(T) = w(u,v) + w(T_1) + w(T_2)$
  (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Greedy Choice

- A locally optimal choice is globally optimal
- Thm:
  - Let T be MST of G, and let A $\subseteq$ E(T)
  - Let S be a subset of V such that E(S,V-S) has no intersection with A
  - Let (u,v) be min-weight edge connecting S to V-S

Then there is an MST of G, T' such that A $\subseteq$ E(T') and (u,v) $\in$ T'

Algorithms for MST

- Kruskal´s algorithm
  - Based on the idea of connected components
  - Starts with a forrest, and always adds to the forrest the min edge that connects two different components
- Prim´s algorithm
  - Starts with a tree, and always adds to the tree the min edge not yet in the tree

Algorithms for MST

- Kruskal´s algorithm
  - Based on the idea of connected components
    - Disjoint sets
  - Starts with a forest, and always adds to the forest the min edge that connects two different components
- Prim´s algorithm
  - Starts with a tree, and always adds to the tree the min edge not yet in the tree
    - Priority queues

Disjoint-Set Data Structures

- Want a data structure to support disjoint sets
  - Collection of disjoint sets $S = \{S_i\}$, $S_i \cap S_j = \varnothing$
- Need to support following operations:
  - MakeSet(x): $S = S \cup \{\{x\}\}$
  - Union(S_i, S_j): $S = S - \{S_i, S_j\} \cup \{S_i \cup S_j\}$
  - FindSet(X): return $S_i \in S$ such that $x \in S_i$
- Before discussing implementation details, we look at Kruskal´s algorithm

Kruskal's Algorithm

```
Kruskal()
{
   T = ∅;
   for each v ∈ V
      MakeSet(v);
   sort E by increasing edge weight w
   for each (u,v) ∈ E (in sorted order)
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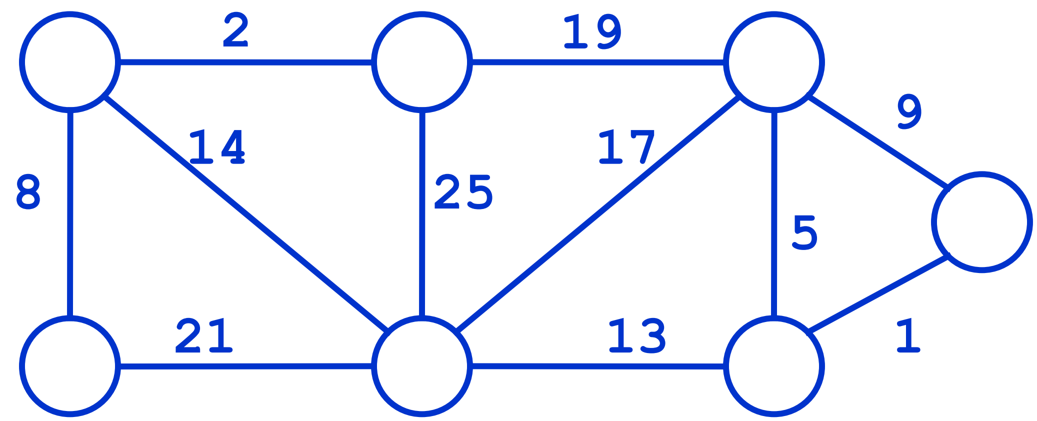

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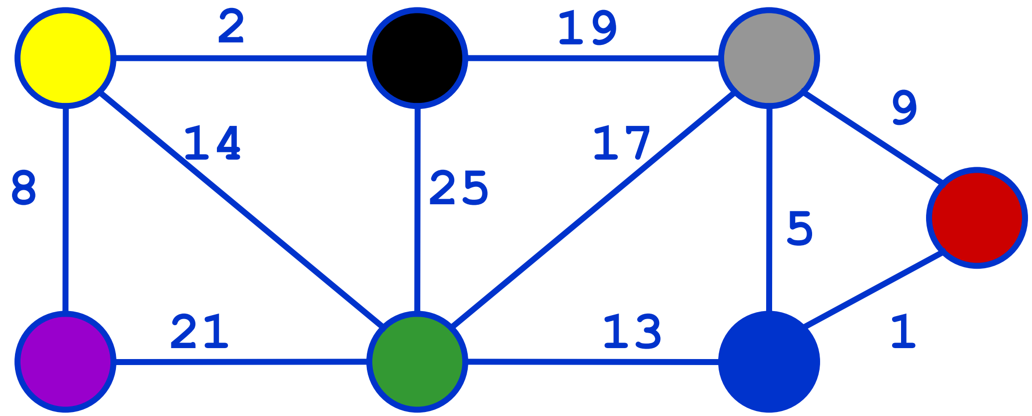

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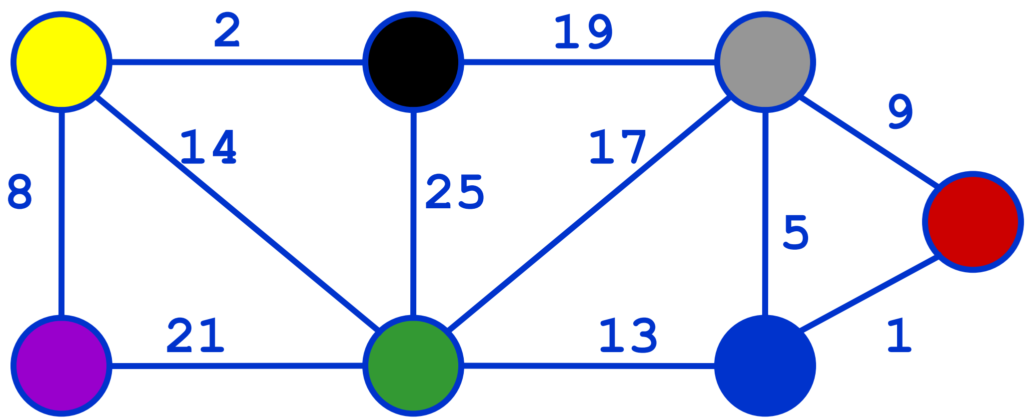

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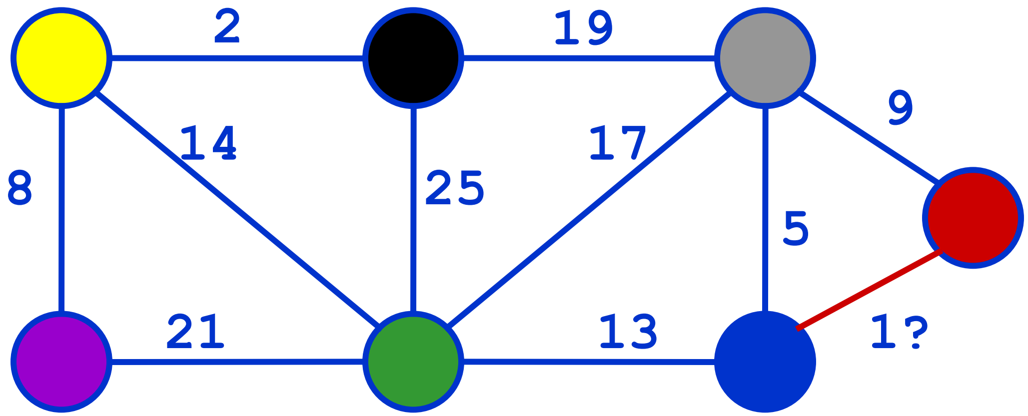

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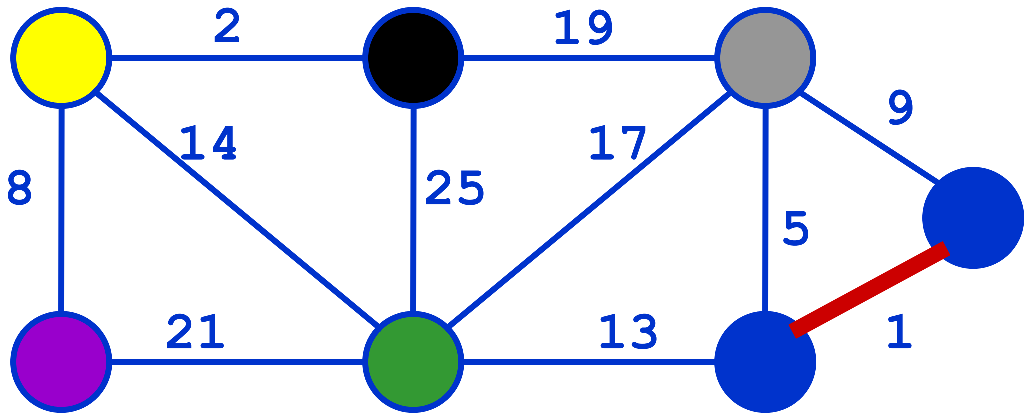

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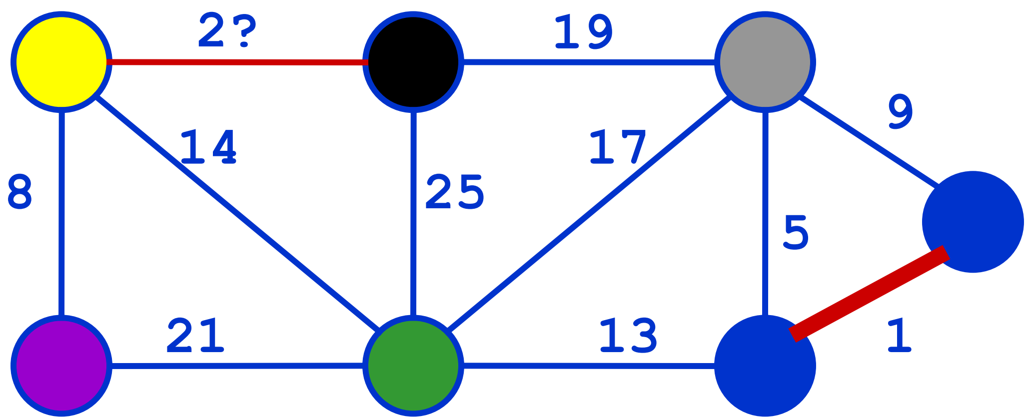

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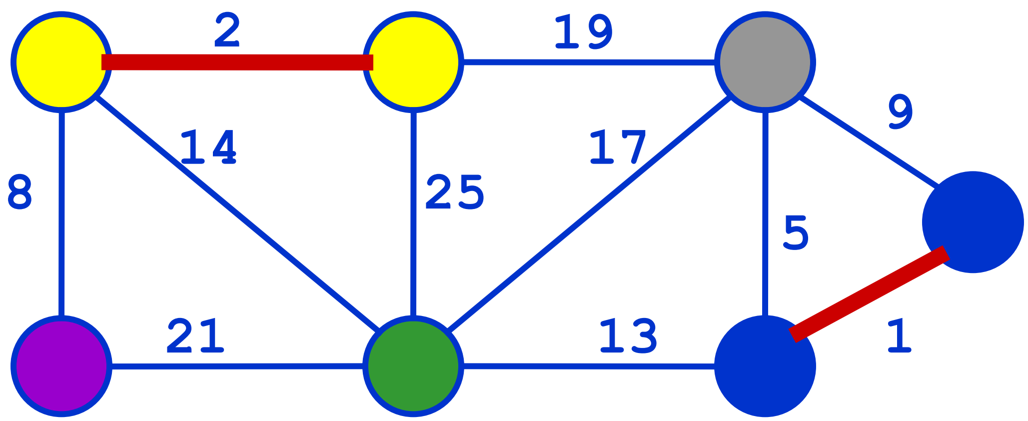

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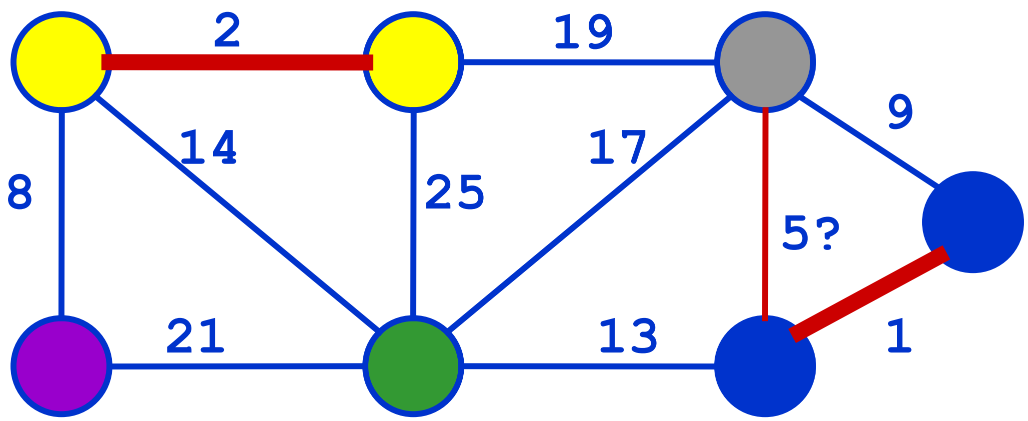


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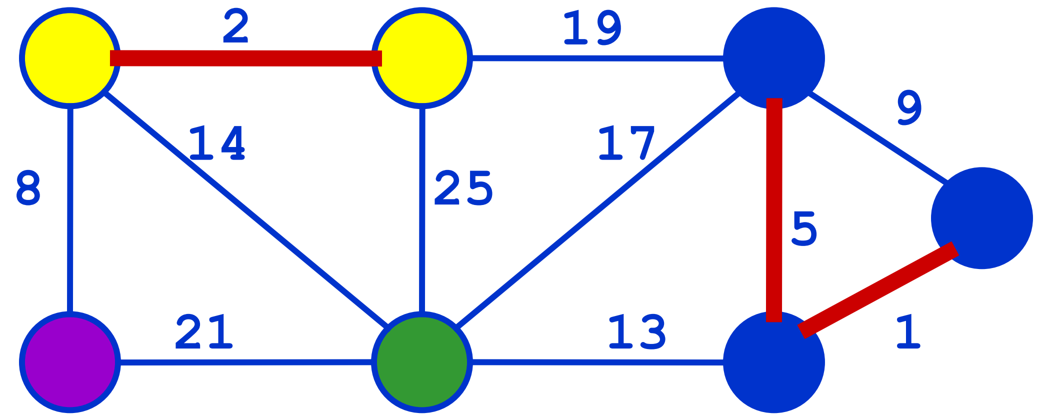


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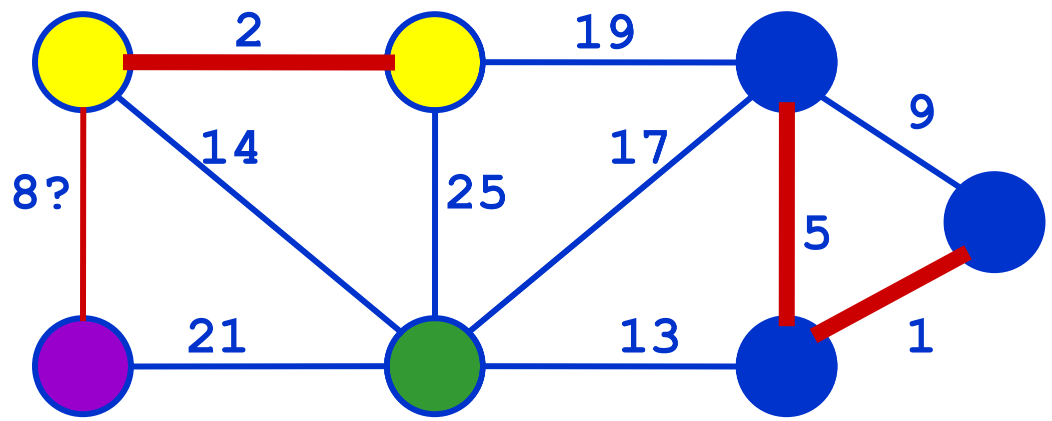

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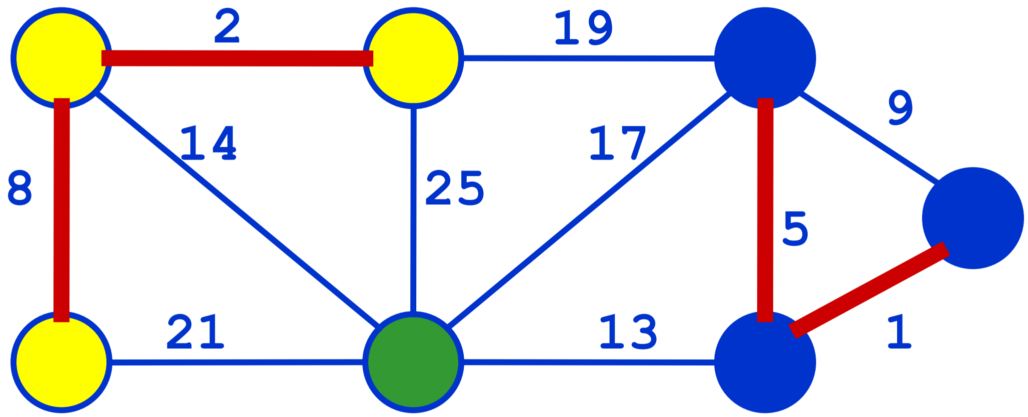


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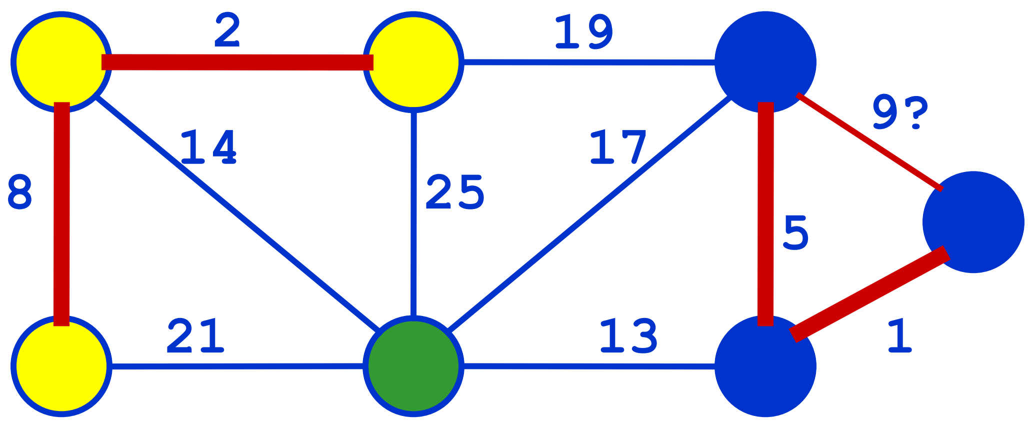

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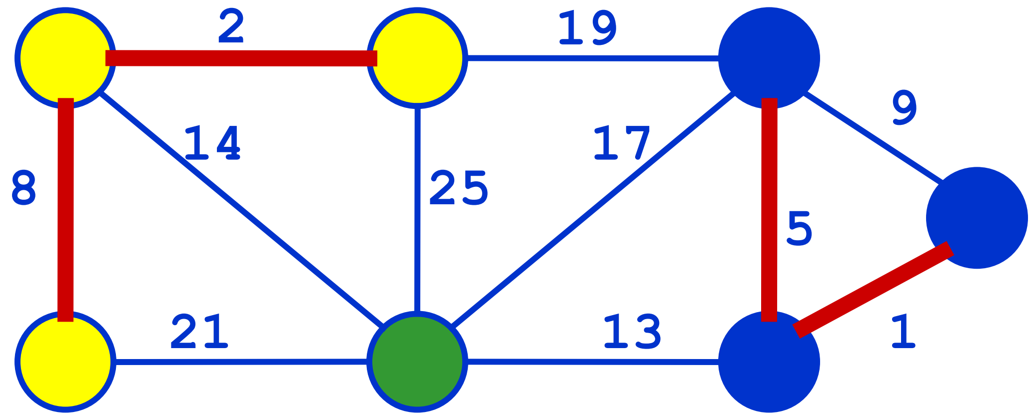


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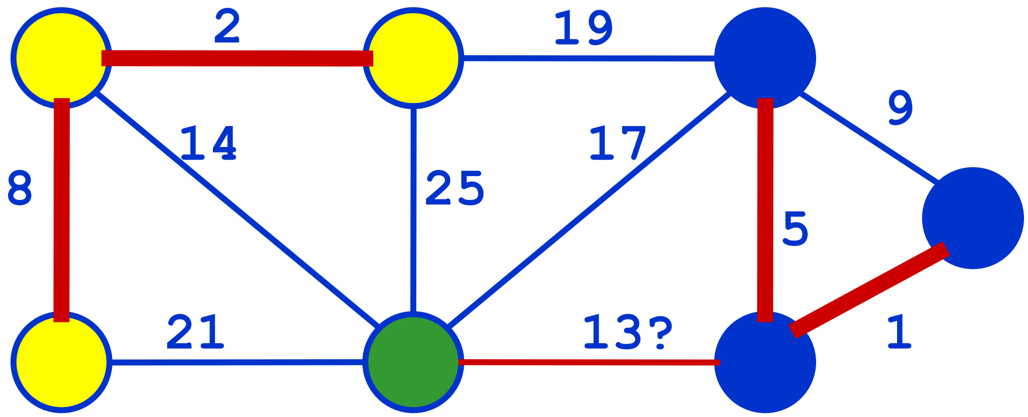

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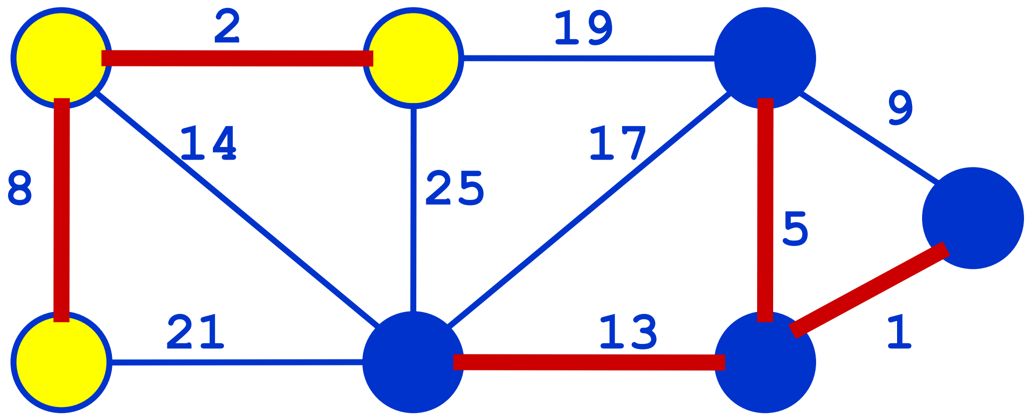

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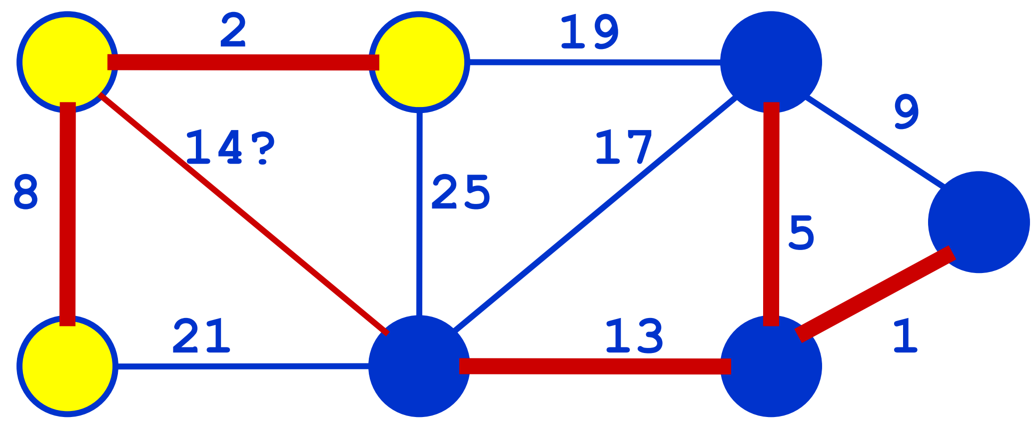


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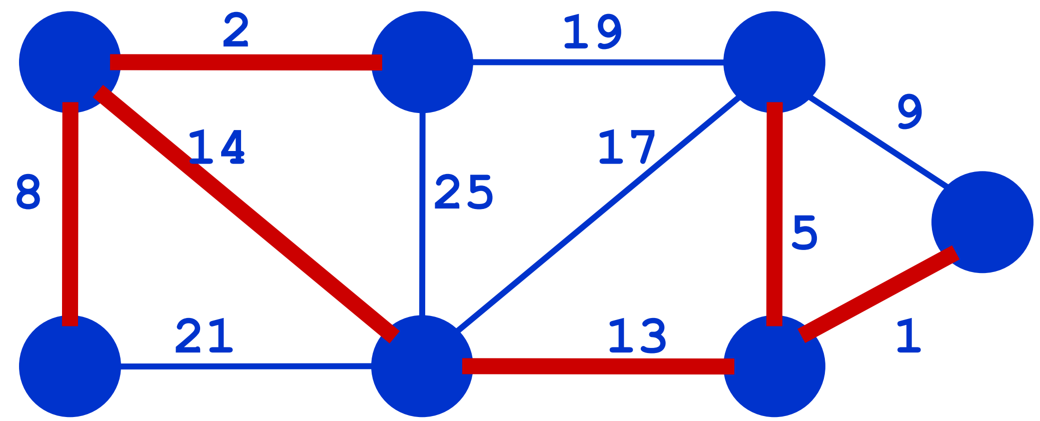


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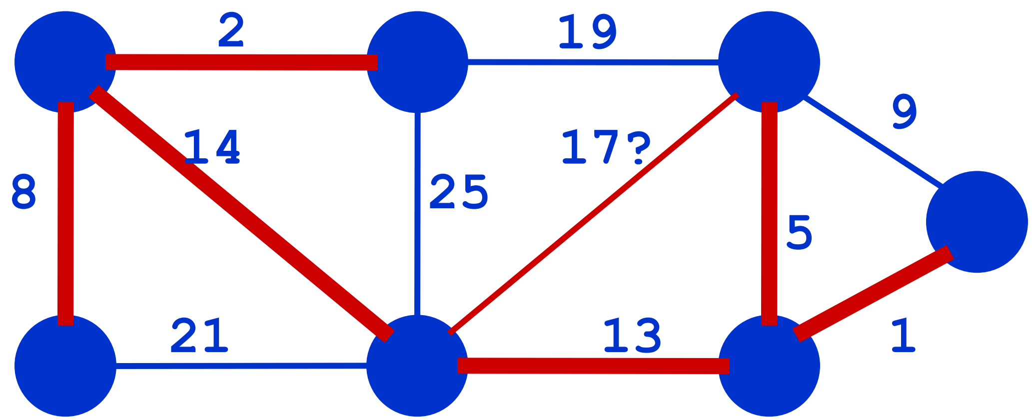


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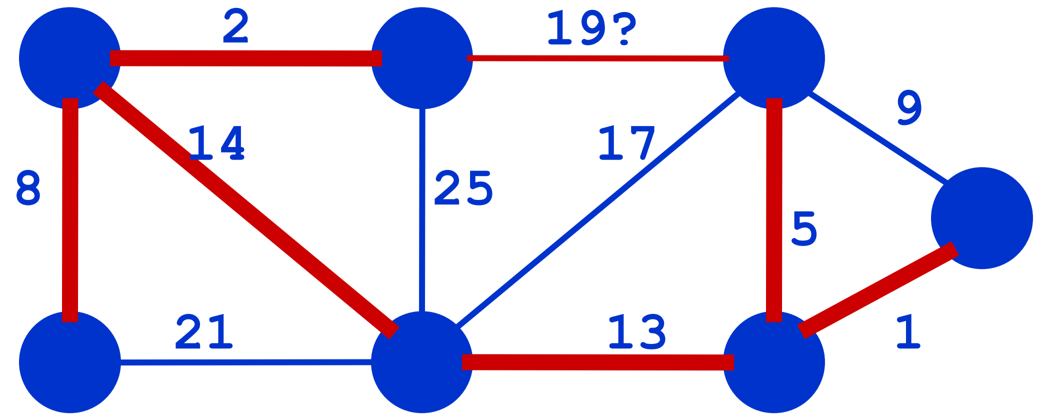

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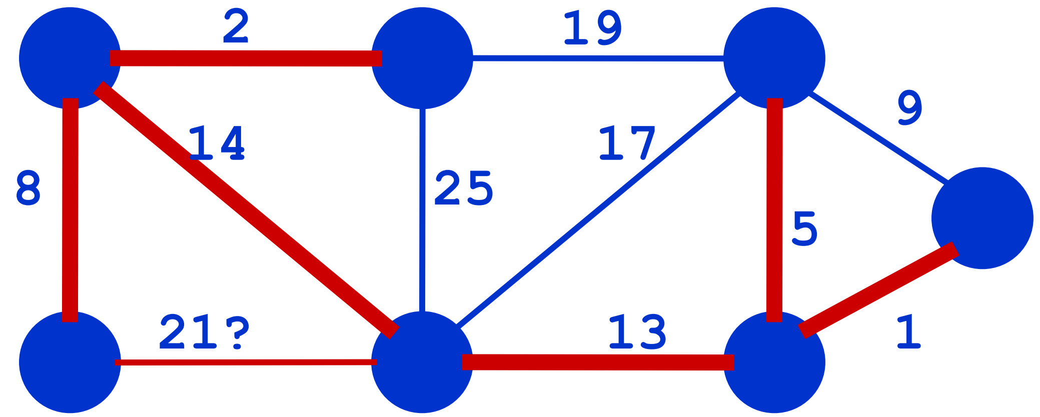

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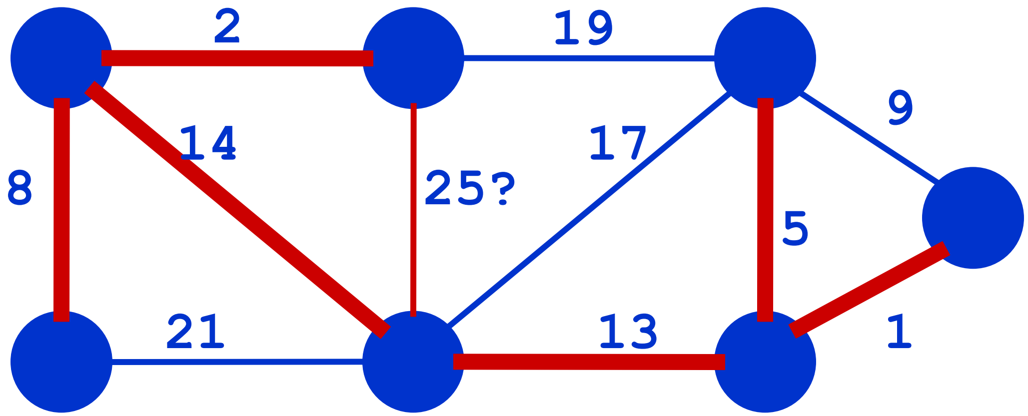


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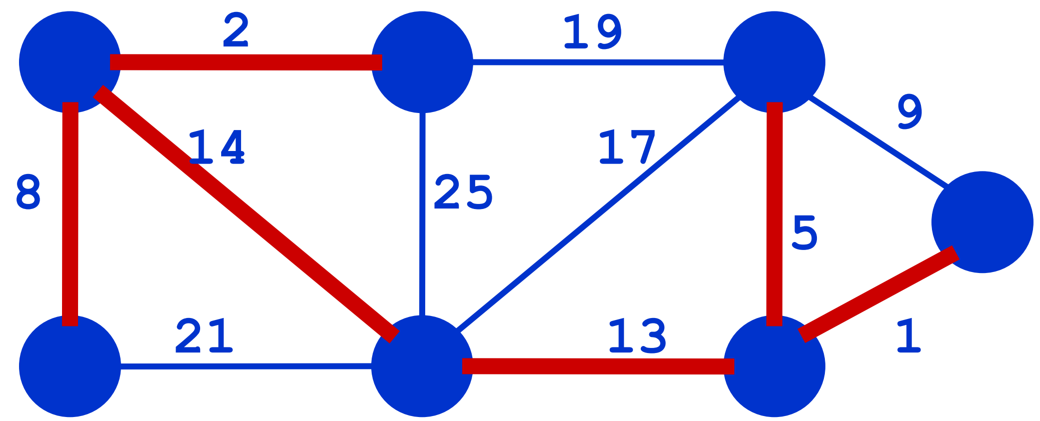

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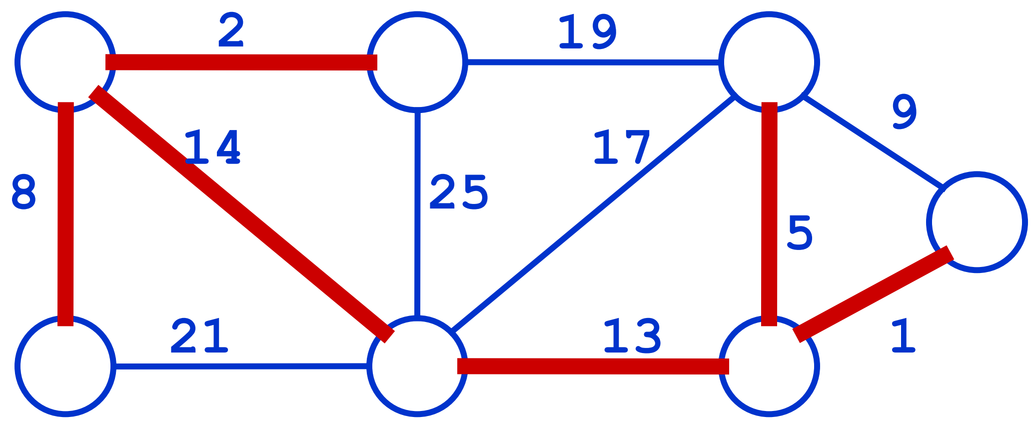

Correctness Of Kruskal’s Algorithm

- Sketch of a proof that this algorithm produces an MST for T:
  - Assume algorithm is wrong: result is not an MST
  - Then algorithm adds a wrong edge at some point
  - If it adds a wrong edge, there must be a lower weight edge to connect the subtrees
  - But algorithm chooses lowest weight joining two different components
  - Take S equal to one of the components and apply the theorem.

Kruskal's Algorithm

What will affect the running time?

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What will affect the running time?

1 Sort

O(V) MakeSet() calls

O(E) FindSet() calls

O(V) Union() calls

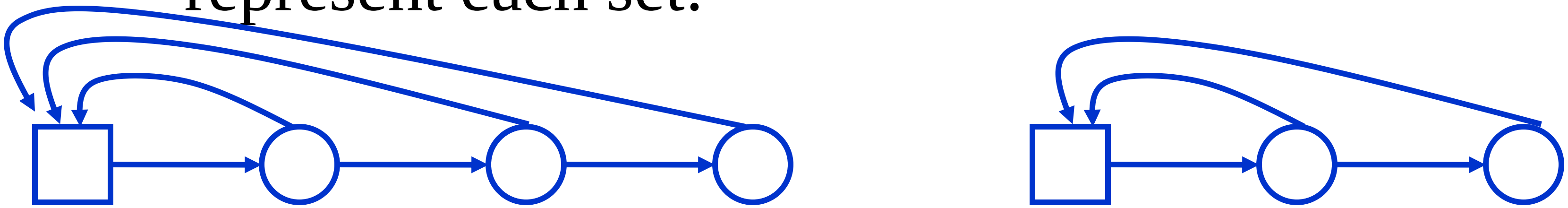
(Exactly how many Union()s?)

Kruskal's Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)
  - O(V) MakeSet()'s
  - O(E) FindSet()'s
  - O(V) Union()'s
- Upshot:
  - Best disjoint-set union algorithm makes above 3 operations take $O(E \cdot \alpha(E,V))$, α almost constant
  - Overall thus O(E lg V)

Disjoint Sets (Chapter 21)

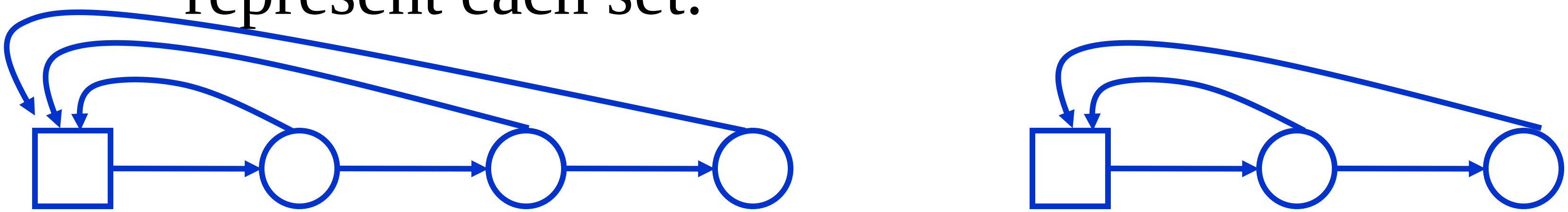
- So how do we implement disjoint-set union?
  - Naïve implementation: use a linked list to represent each set:



    - MakeSet(): ??? time
    - FindSet(): ??? time
    - Union(A,B): "copy" elements of A into B: ??? time

Disjoint Set Union

- So how do we implement disjoint-set union?
  - Naïve implementation: use a linked list to represent each set:



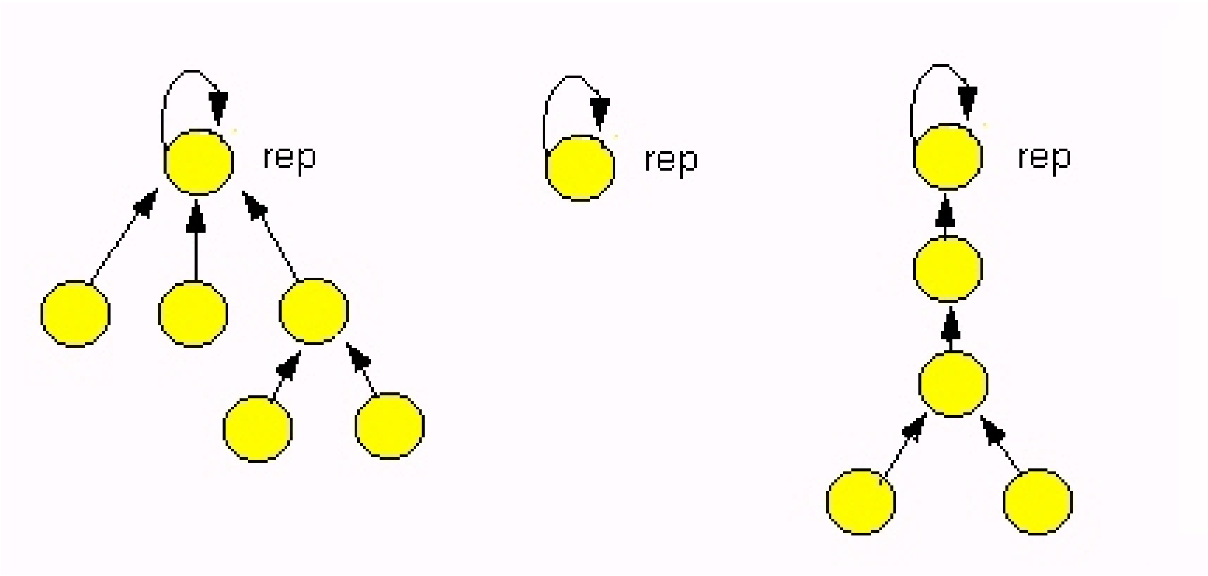
    - MakeSet(): O(1) time
    - FindSet(): O(1) time
    - Union(A,B): “copy” elements of A into B: O(A) time
  - How long will n Union()’s take?

Disjoint Set Union: Analysis

- Worst-case analysis: $O(n^2)$ time for n Union's
- Improvement: always copy smaller into larger
  - Maintains the length of the list
  - Weighted union heuristic
    - Union can still take $\Omega(n)$
- However, a sequence of m Make_Set, Union and FindSet operations, n of which are Make_Set, takes $O(m + n\lg n)$ (Theorm 21.1)

Disjoint-set union

- Another way to implement disjoint-set unions, and which is more efficient: Disjoint-set forests



Disjoint-set forests

- 2 heuristics make the operations efficient:
  - Union by rank
  - Path compression (FindSet)

- In this case, a sequence of m Make_Set, Union and FindSet operations, n of which are Make_Set, takes $O(m\,\alpha(n))$ (proof- Section 21.4)

Algorithms for MST

- Kruskal´s algorithm
  - Based on the idea of connected components
    - Disjoint sets
  - Starts with a forest, and always adds to the forest the min edge that connects two different components
- Prim´s algorithm
  - Starts with a tree, and always adds to the tree the min edge not yet in the tree
    - Priority queues

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G]; //elements not in the tree
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0; //least weight-edge connecting it to tree
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

Prim’s Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
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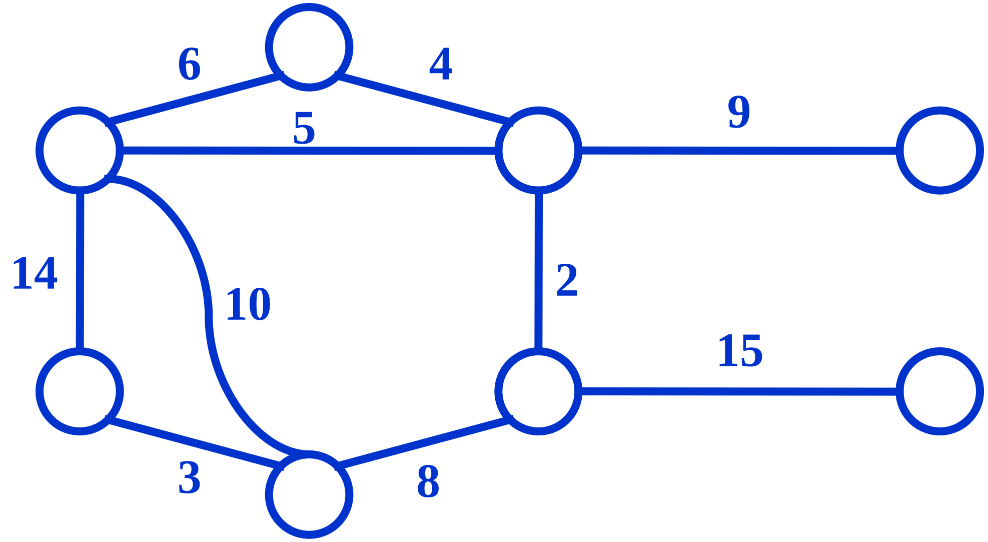

Run on example graph

Prim’s Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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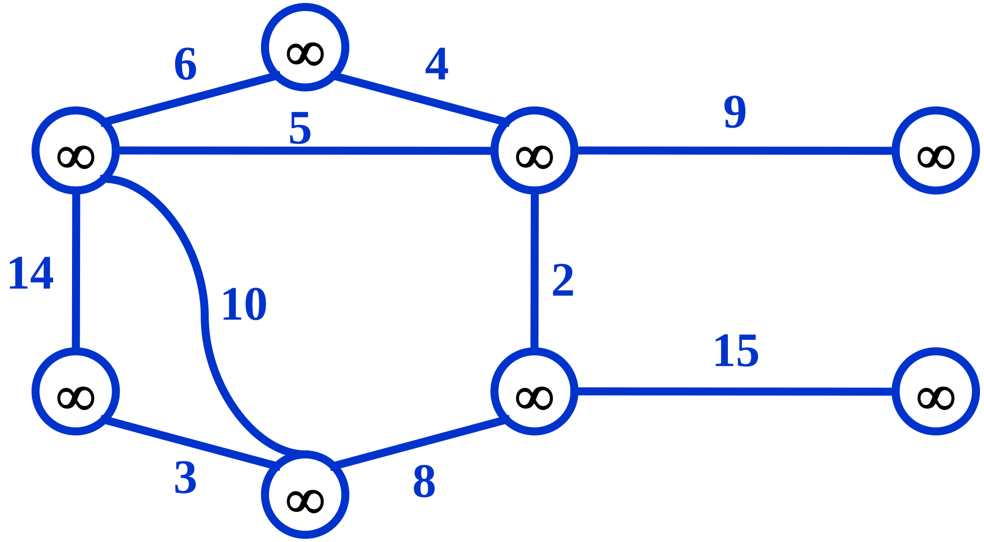

Run on example graph

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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    while (Q not empty)
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        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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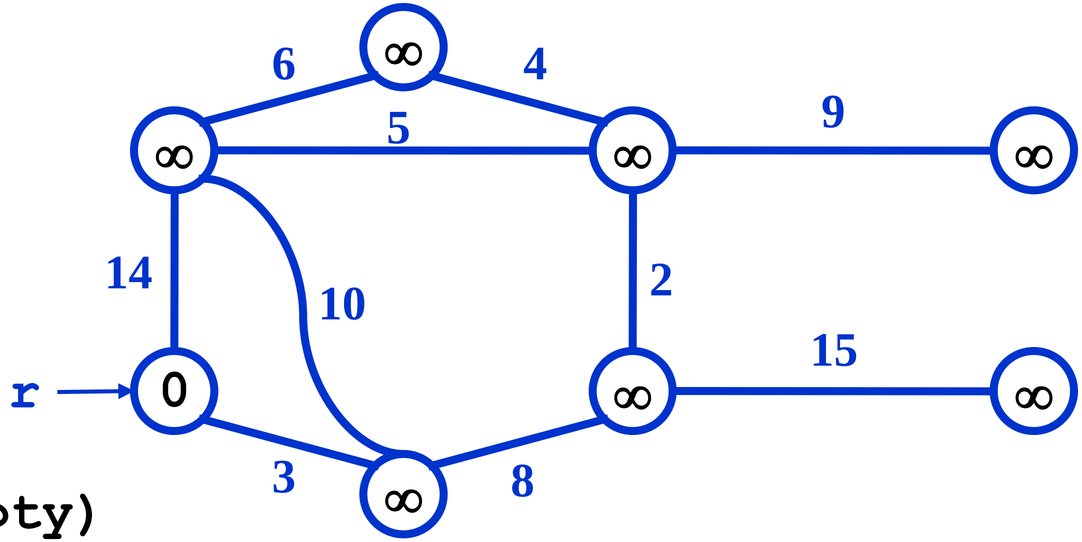
```



Pick a start vertex r

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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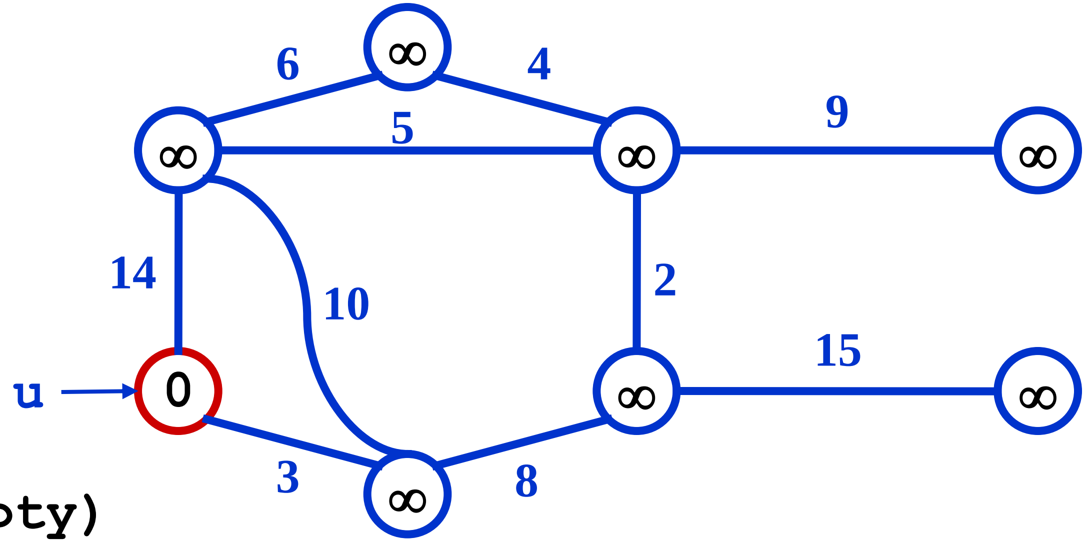
```



Red vertices have been removed from Q

Prim’s Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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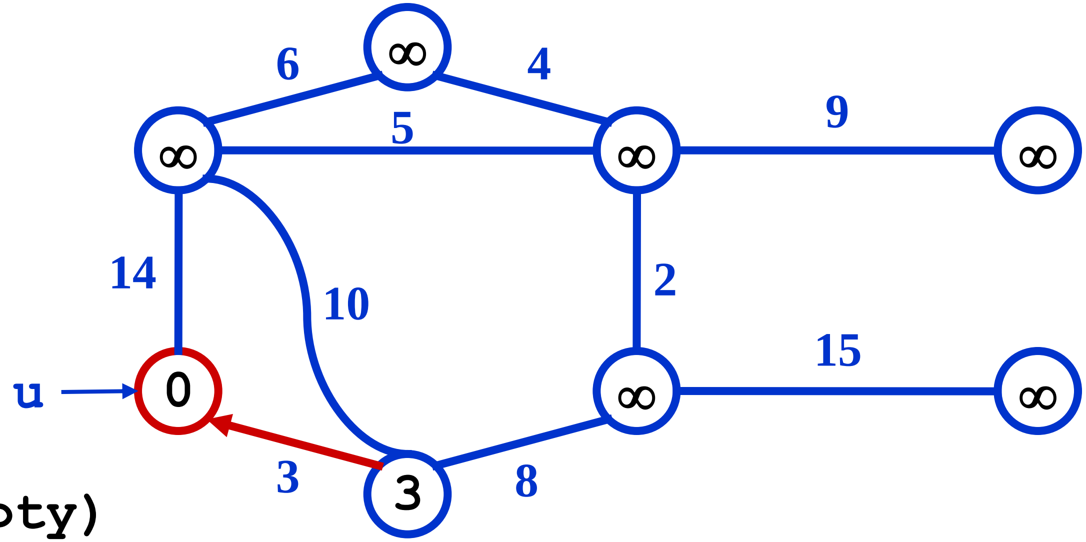
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Red arrows indicate parent pointers

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
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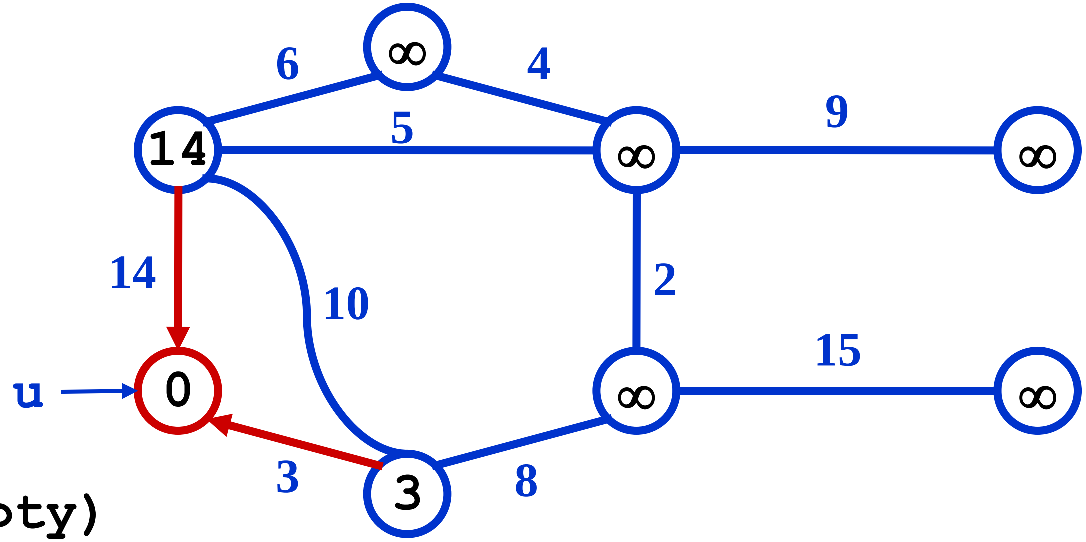

Prim's Algorithm

```
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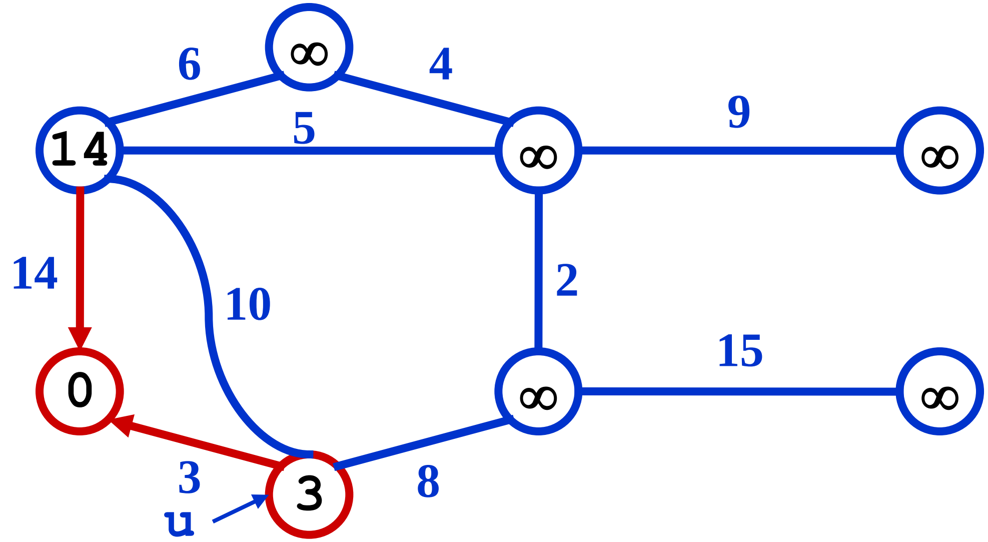

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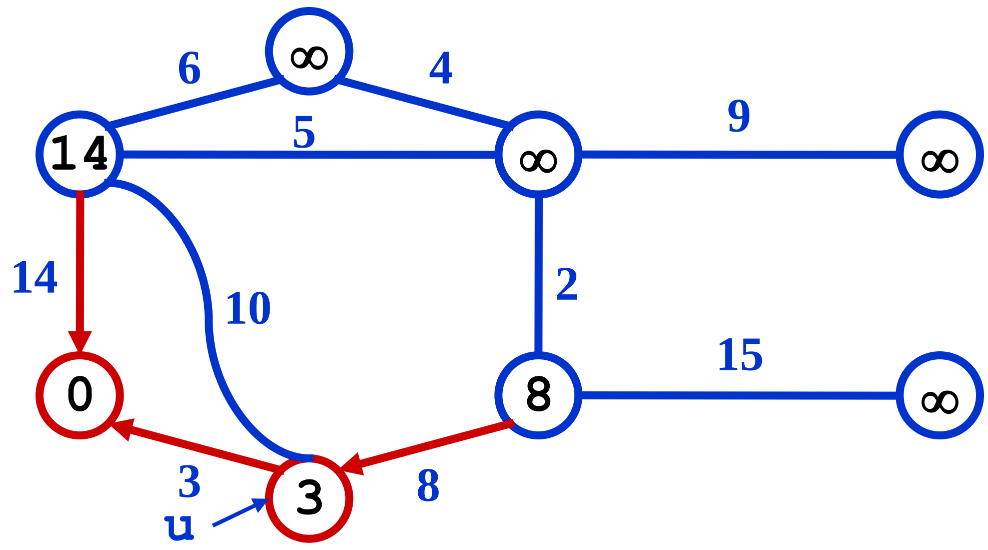

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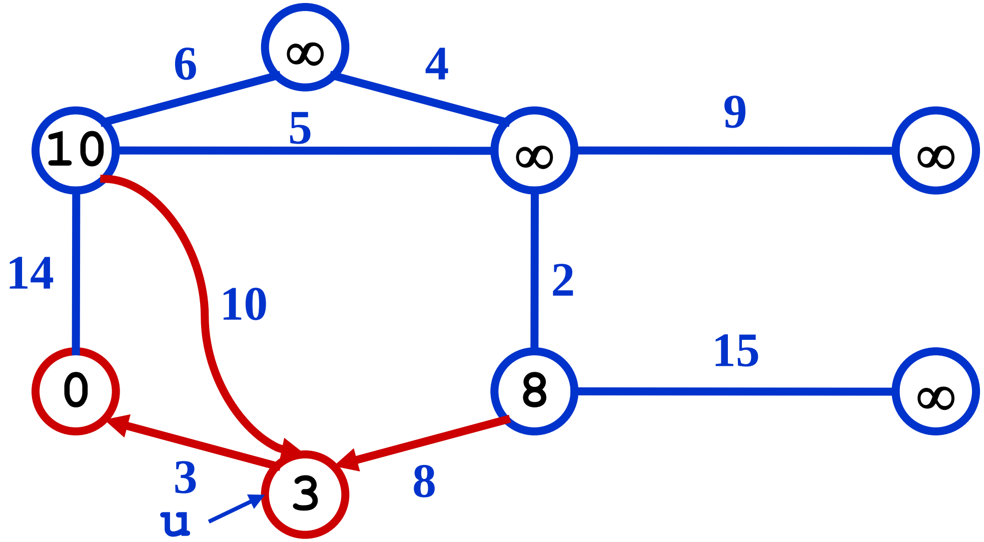

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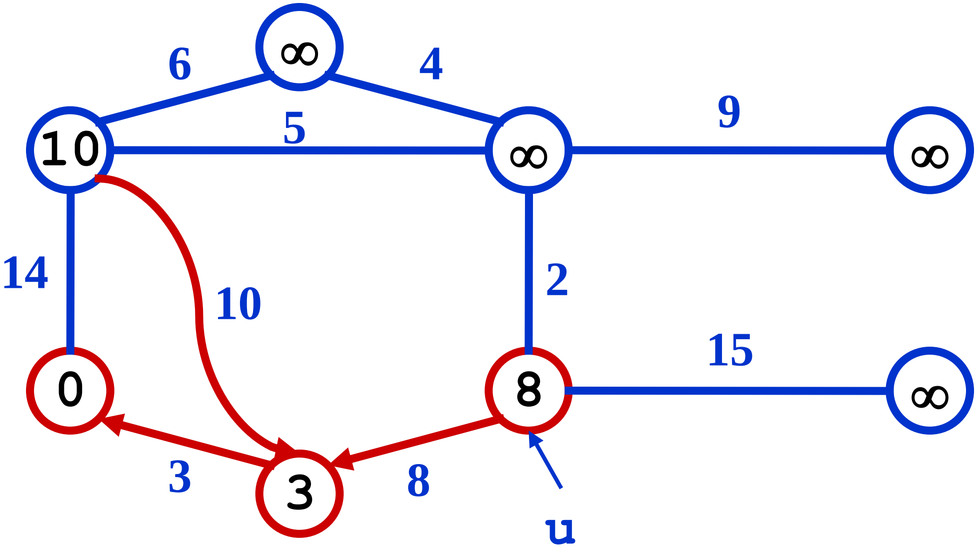

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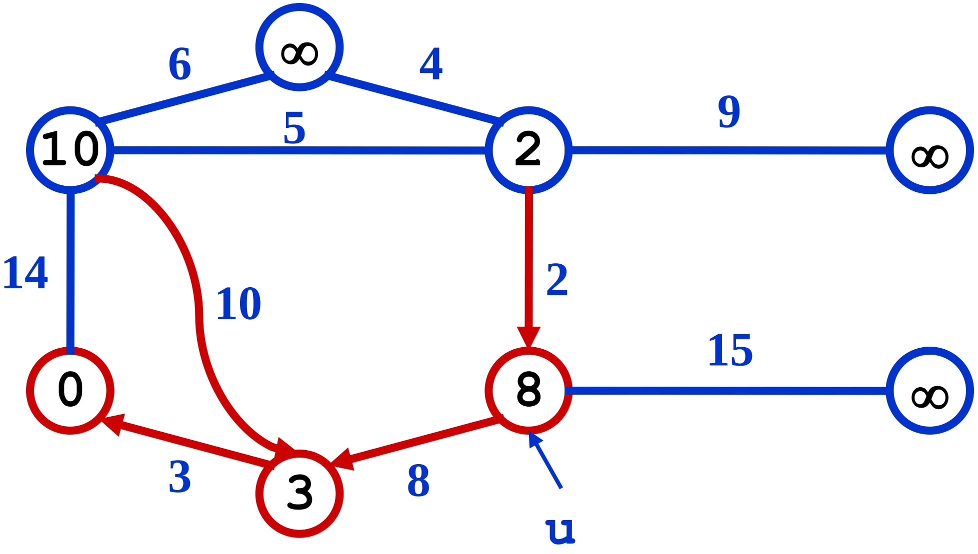

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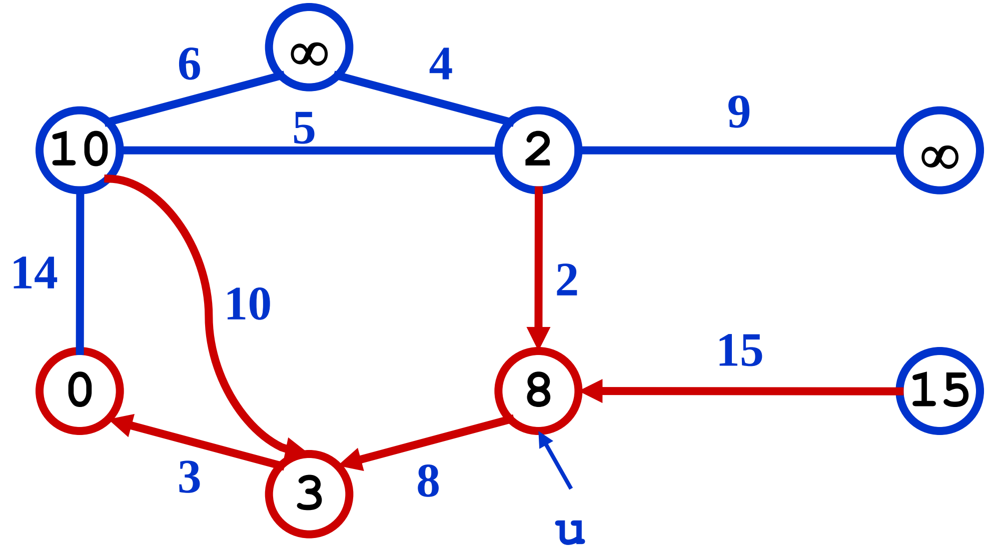

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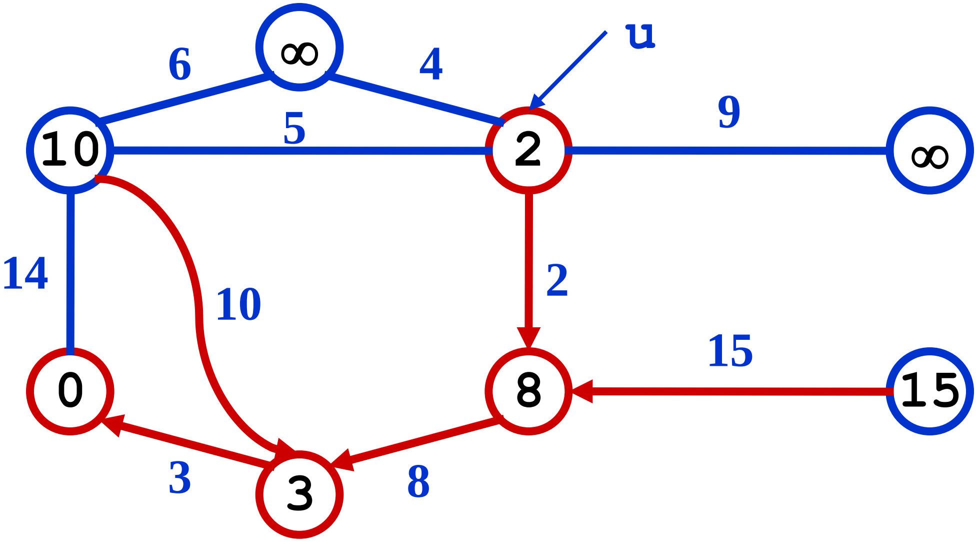

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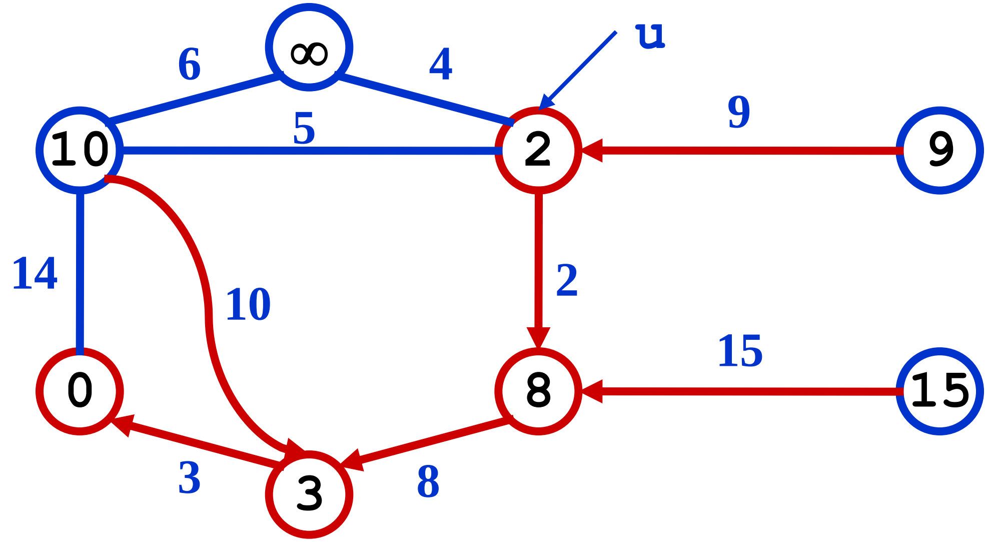

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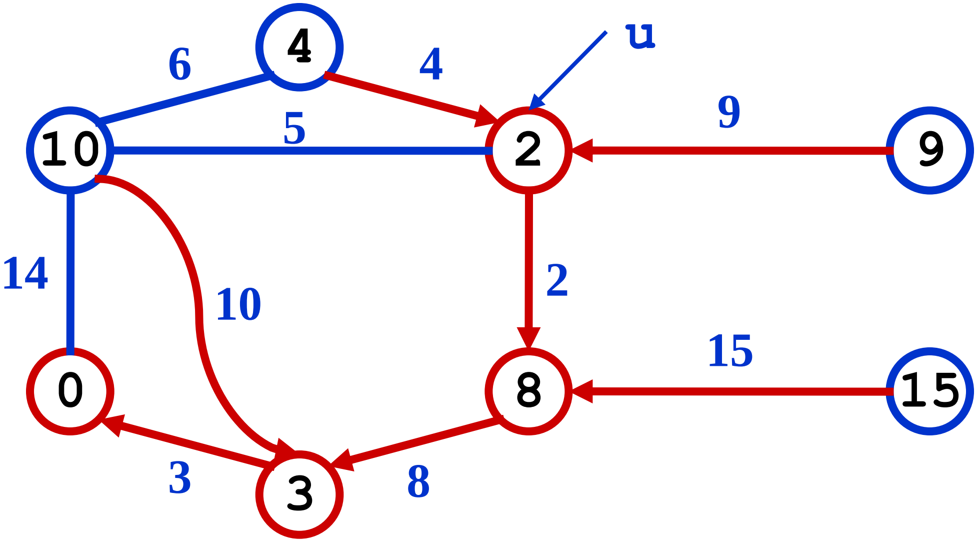

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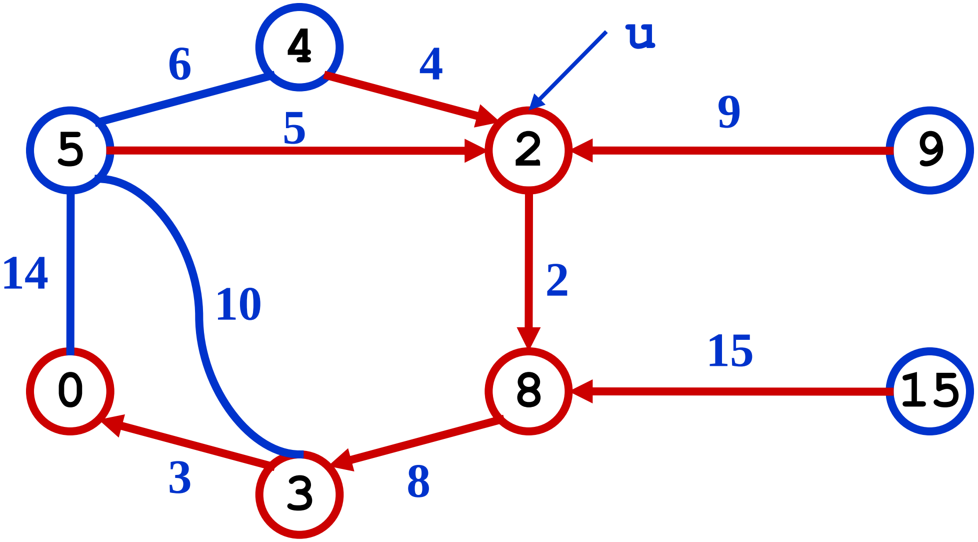

Prim’s Algorithm

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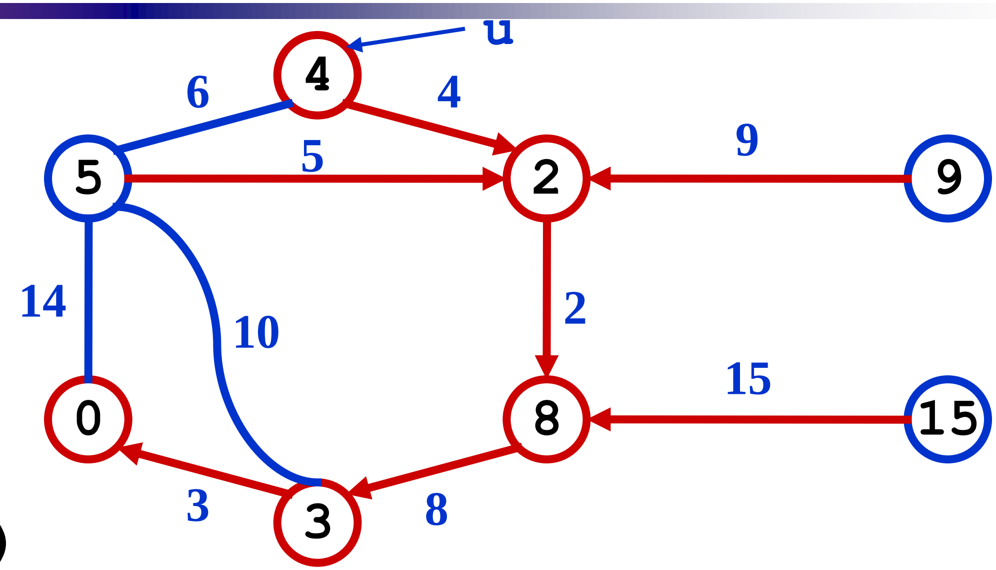

Prim's Algorithm

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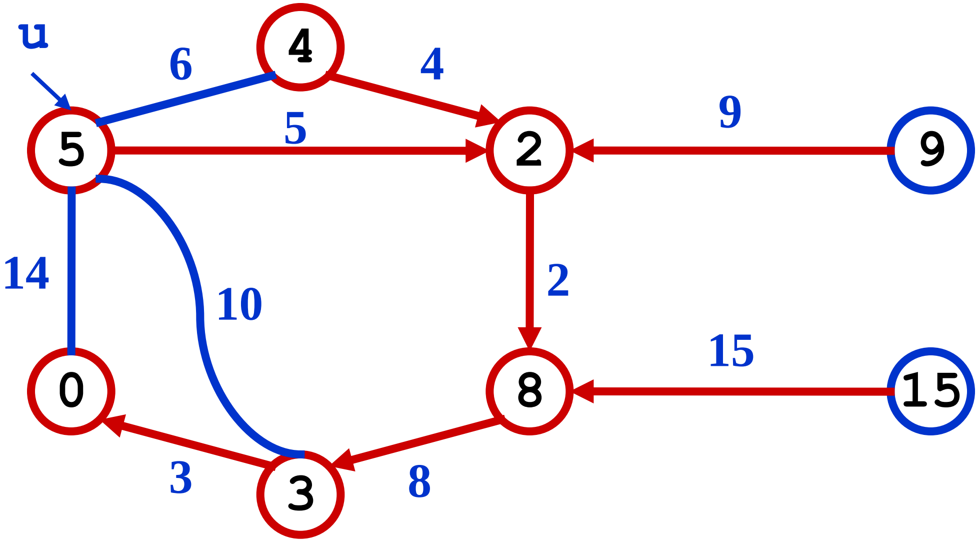

Prim's Algorithm

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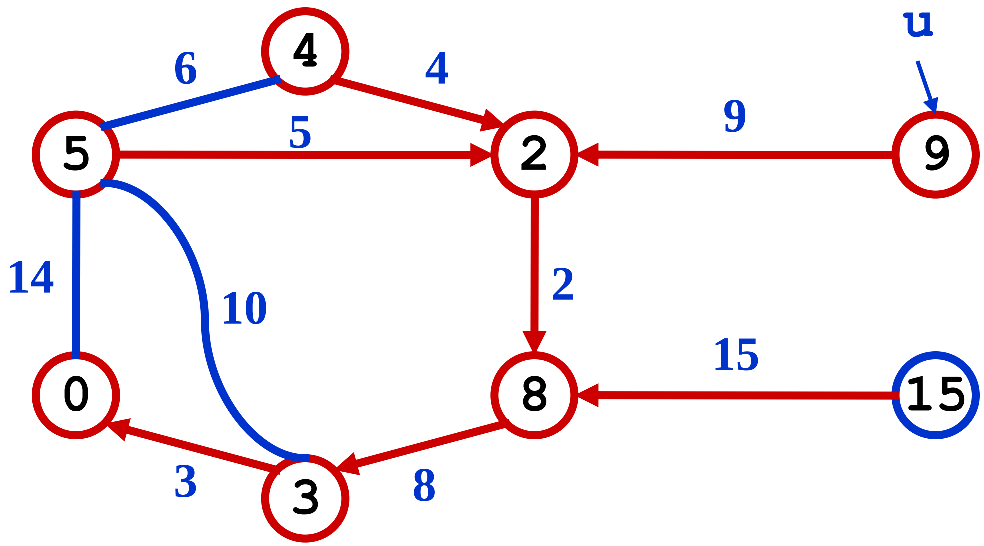

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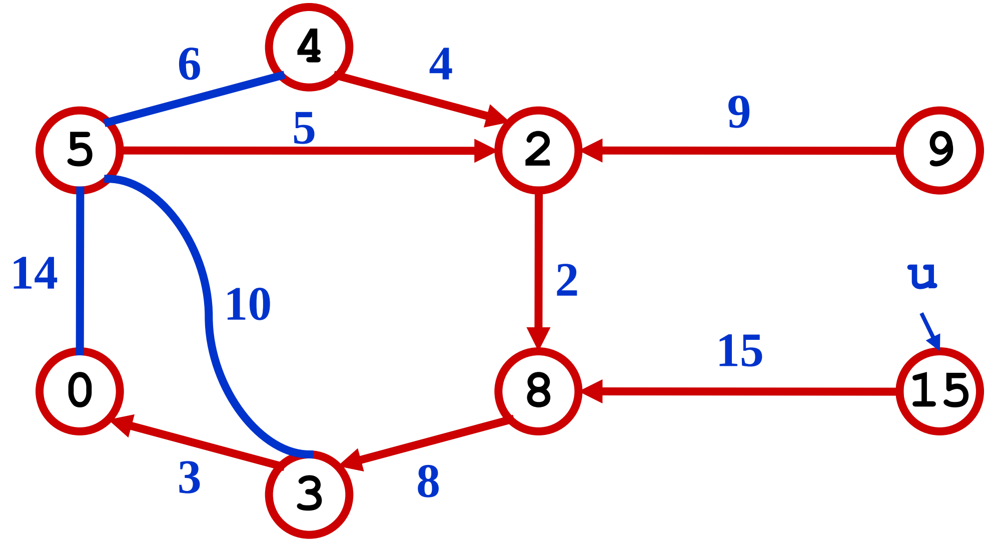

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        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

What is the hidden cost in this code?

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                DecreaseKey(v, w(u,v));
```

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q                    O(V)
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)               |V|
        u = ExtractMin(Q);
        for each v ∈ Adj[u]           Degree(u)
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                DecreaseKey(v, w(u,v));
```

How often is ExtractMin() called?

How often is DecreaseKey() called?

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

What will be the running time?

A: Depends on queue

Analysis

- Time = θ(V x T(ExtractMin)+E x T(DecreaseKey))
- Analysis according to Queue implementation:

Q	Textract	Tdecrease	Total
Array	O(V)	O(1)	$O(V^2)$
Binary Heap	O(lg V)	O(lg V)	O((E+V) lg V)
Fib Heap	O(logV)amort	O(1)amort	O(E+VlogV)