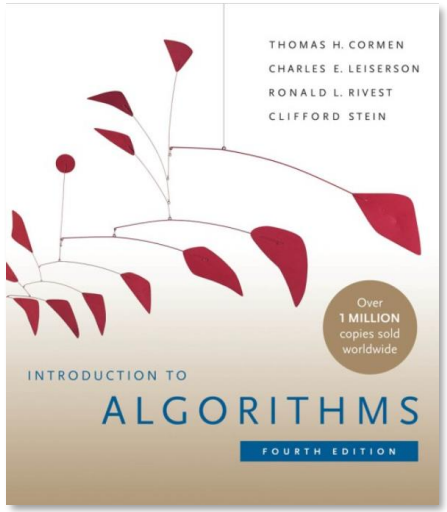


INTRODUCTION TO ALGORITHMS

FOURTH EDITION

LECTURE NOTES FOR CHAPTER 3

Characterizing Running Times

CHAPTER 3 OVERVIEW

Goals

- A way to describe behavior of functions in the limit. We're studying asymptotic efficiency.
- Describe growth of functions.
- Focus on what's important by abstracting away low-order terms and constant factors.
- How we indicate running times of algorithms.
- A way to compare "sizes" of functions:

$$
\begin{array}{ccc}
O & \approx & \leq \\
\Omega & \approx & \geq \\
\Theta & \approx & = \\
o & \approx & < \\
\omega & \approx & >
\end{array}
$$

O-notation

O-notation characterizes an ***upper bound*** on the asympototic behavior of a function: it says that a function grows ***no faster*** than a certain rate. This rate is based on the highest order term.

For example:

$f(n) = 7n^3 + 100n^2 - 20n + 6$ is $O(n^3)$, since the highest order term is $7n^3$, and therefore the function grows no faster than n^3.

The function $f(n)$ is also $O(n^5)$, $O(n^6)$, and $O(n^c)$ for any constant $c \geq 3$.

Ω-notation

Ω-notation characterizes a ***lower bound*** on the asymptotic behavior of a function.

For example:

$f(n) = 7n^3 + 100n^2 - 20n + 6$ is $\Omega(n^3)$, since the highest-order term, n^3, grows at least as fast as n^3.

The function $f(n)$ is also $\Omega(n^2)$, $\Omega(n)$ and $\Omega(nc)$ for any constant $c \leq 3$.

Θ-notation

Θ-notation characterizes a ***tight bound*** on the asympototic behavior of a function: it says that a function grows ***precisely*** at a certain rate, again based on the highest-order term.

If a function is is both $O(f(n))$ and $\Omega(f(n))$, then a function is $\Theta(f(n))$.

EXAMPLE: INSERTION-SORT

```
INSERTION-SORT(A, n)
1   for i = 2 to n
2       key = A[i]
3       // Insert A[i] into the sorted subarray A[1 : i − 1].
4       j = i − 1
5       while j > 0 and A[j] > key
6           A[j + 1] = A[j]
7           j = j − 1
8       A[j + 1] = key
```

EXAMPLE: INSERTION-SORT (continued)

First, show that INSERTION-SORT is runs in $O(n^2)$ time, regardless of the input:

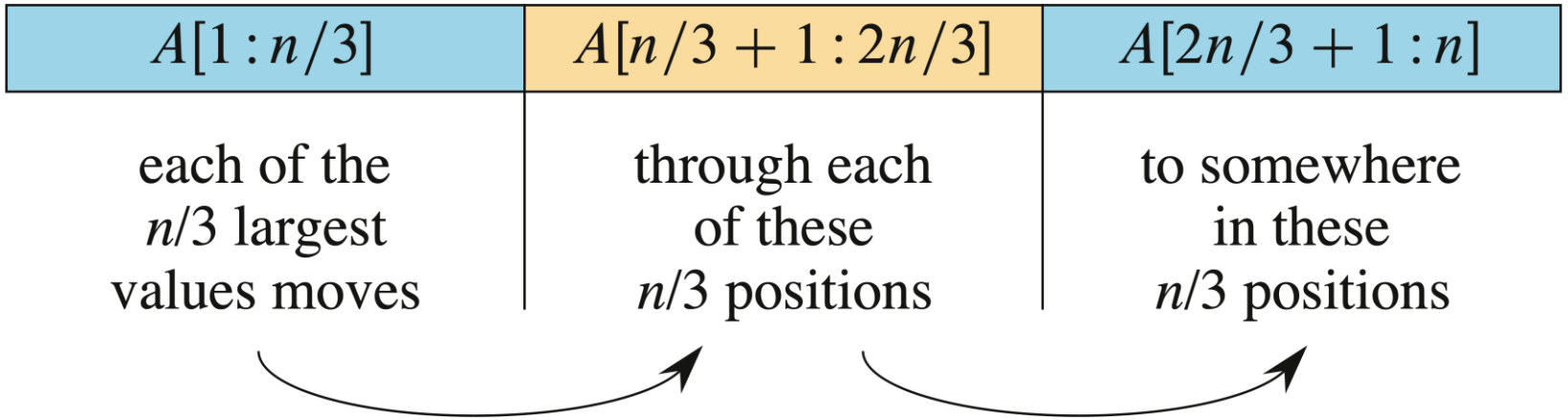
- The outer **for** loop runs $n - 1$ times regardless of the values being sorted.
- The inner **while** loop iterates at most $i - 1$ times.
- The exact number of iterations the **while** loop makes depends on the values it iterates over, but it will definitely iterate between 0 and $i - 1$ times.
- Since i is at most n, the total number of iterations of the inner loop is at most $(n - 1)(n - 1)$, which is less than n^2.

Each inner loop iteration takes constant time, for a total of at most cn^2 for some constant c, or $O(n^2)$.

EXAMPLE: INSERTION-SORT (continued)

Now show that INSERTION-SORT has a worst-case running time of $\Omega(n^2)$:

- Observe that for a value to end up k positions to the right of where it started, the line $A[j+1] = A[j]$ must have been executed k times.
- Assume that n is a multiple of 3 so that we can divide the array A into groups of $n/3$ positions.



EXAMPLE: INSERTION-SORT (continued)

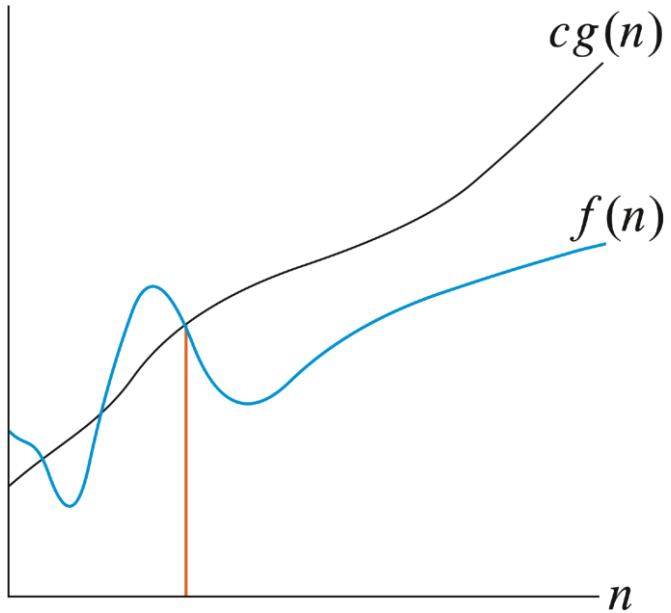
Because at least $n/3$ values must pass through at least $n/3$ positions, the line $A[j+1] = A[j]$ executes at least $(n/3)(n/3) = n^2/9$ times, which is $\Omega(n^2)$. For this input, INSERTION-SORT takes time $\Omega(n^2)$.

Since we have shown that INSERTION-SORT runs in $O(n^2)$ time in all cases and that there is an input that makes it take $\Omega(n^2)$ time, we can conclude that the worst-case running time of INSERTION-SORT is $\Theta(n^2)$.

The constant factors for the upper and lower bounds may differ. That doesn't matter.

O-notation

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.



$f(n) = O(g(n))$

$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

Example

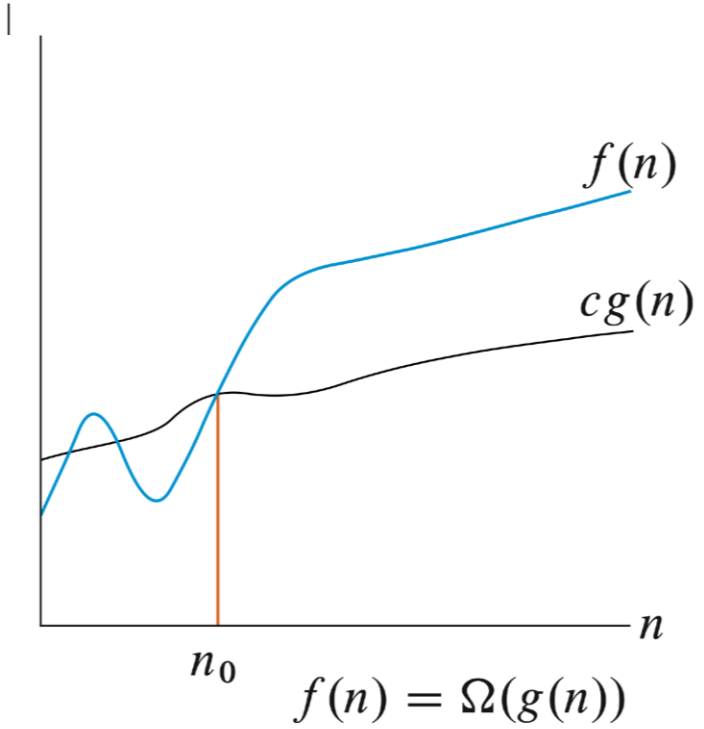
$2n^2 = O(n^3)$, with $c = 1$ and $n_0 = 2$.

Examples of functions in $O(n^2)$:

n^2
$n^2 + n$
$n^2 + 1000n$
$1000n^2 + 1000n$
Also,
n
$n/1000$
$n^{1.99999}$
$n^2/\lg\lg\lg n$

Ω-notation

$\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.



$f(n) = \Omega(g(n))$

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Example

$\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$.

Examples of functions in $\Omega(n^2)$:

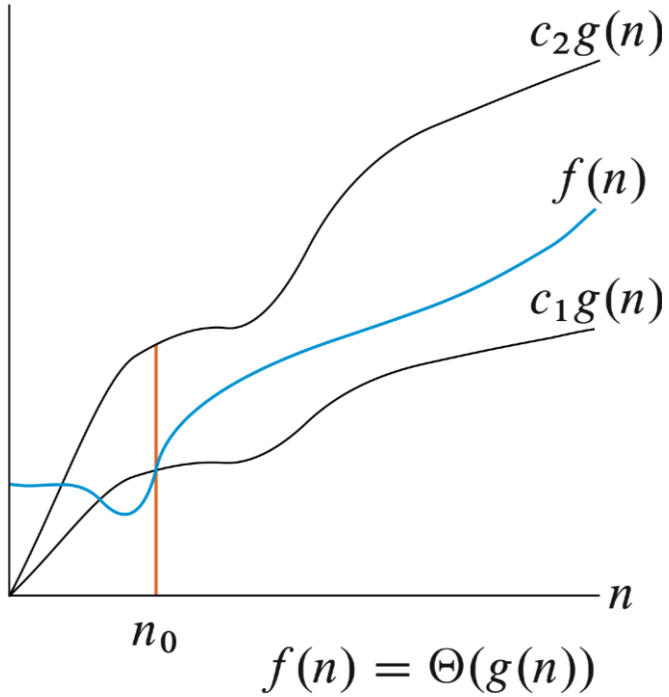
n^2
$n^2 + n$
$n^2 - n$
$1000n^2 + 1000n$
$1000n^2 - 1000n$

Also,

n^3
$n^{2.00001}$
$n^2 \lg\lg\lg n$
2^{2^n}

Θ-notation

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 such that
$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



Example

$n^2/2 - 2n = \Theta(n^2)$, with $c_1 = 1/4$, $c_2 = 1/2$, and $n_0 = 8$.

Theorem

$f(n) = \Theta(g(n))$ if and only if $f = O(g(n))$ and $f = \Omega(g(n))$.

Leading constants and low-order terms don't matter.

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

ASYMPTOTIC NOTATION AND RUNNING TIMES

Need to be careful to use asymptotic notation correctly when characterizing a running time. Asymptotic notation describes functions, which in turn describe running times. Must be careful to specify ***which*** running time.

For example:

The worst-case running time for insertion sort is $O(n^2)$, $\Omega(n^2)$, and $\Theta(n^2)$; all are correct. Prefer to use $\Theta(n^2)$ here, since it's the most precise.

The best-case running time for insertion sort is $O(n)$, $\Omega(n)$, and $\Theta(n)$; prefer $\Theta(n)$.

ASYMPTOTIC NOTATION AND RUNNING TIMES (continued)

But ***cannot*** say that the running time for insertion sort is $\Theta(n^2)$, with "worst-case" omitted. Omitting the case means making a blanket statement that covers ***all*** cases, and insertion sort does ***not*** run in $\Theta(n^2)$ time in all cases.

Can make the blanket statement that the running time for insertion sort is $O(n^2)$, or that it's $\Omega(n)$, because these asymptotic running times are true for all cases.

For merge sort, its running time is $\Theta(n \lg n)$ in all cases, so it's OK to omit which case.

ASYMPTOTIC NOTATION AND RUNNING TIMES (continued)

Common error: conflating O-notation with Θ-notation by using O-notation to indicate an asymptotically tight bound. O-notation gives only an asymptotic upper bound. Saying "an $O(n \lg n)$-time algorithm runs faster than an $O(n^2)$-time algorithm" is not necessarily true. An algorithm that runs in $\Theta(n)$ time also runs in $O(n^2)$ time. If you really mean an asymptotically tight bound, then use Θ-notation.

Use the simplest and most precise asymptotic notation that applies. Suppose that an algorithm's running time is $3n^2 + 20n$. Best to say that it's $\Theta(n^2)$. Could say that it's $O(n^3)$, but that's less precise. Could say that it's $\Theta(3n^2 + 20n)$ but that obscures the order of growth.

ASMPTOTIC NOTATION IN EQUATIONS

When on right-hand side:

$O(n^2)$ stands for some anonymous function in the set $O(n^2)$.

$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means $2n^2 + 3n + 1 = 2n^2 + f(n)$ *for some* $f(n) \in \Theta(n)$. In particular, $f(n) = 3n + 1$.

When on left-hand side:

No matter how the anonymous functions are chosen on the left-hand side, there is a way to choose the anonymous functions on the right-hand side to make the equation valid.

Interpret $2n^2 + \Theta(n) = \Theta(n^2)$ as meaning *for all* functions $f(n) \in \Theta(n)$, there exists a function $g(n) \in \Theta(n^2)$ such that $2n^2 + f(n) = g(n)$.

ASMPTOTIC NOTATION IN EQUATIONS
(continued)

Can chain together:

$$\begin{aligned} 2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\ &= \Theta(n^2) \ . \end{aligned}$$

Interpretation:

- First equation: There exists $f(n) \in \Theta(n)$ such that $2n^2 + 3n + 1 = 2n^2 + f(n)$.
- Second equation: For all $g(n) \in \Theta(n)$ (such as the $f(n)$ used to make the first equation hold), there exists $h(n) \in \Theta(n^2)$ such that $2n^2 + g(n) = h(n)$.

SUBTLE POINT: ASYMPTOTIC NOTATION IN RECURRENCES

Often abuse asymptotic notation when writing recurrences: $T(n) = O(1)$ for $n < 3$. Strictly speaking, this statement is meaningless. Definition of O-notation says that $T(n)$ is bounded above by a constant $c > 0$ for $n \geq n_0$, for some $n_0 > 0$. The value of $T(n)$ for $n < n_0$ might not be bounded. So when we say $T(n) = O(1)$ for $n < 3$, cannot determine any constraint on $T(n)$ when $n < 3$ because could have $n_0 > 3$.

What we really mean is that there exists a constant $c > 0$ such that $T(n) \leq c$ for $n < 3$. This convention allows us to avoid naming the bounding constant so that we can focus on the more important part of the recurrence.

O-notation

$o(g(n)) = \{f(n)$: for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.

Another view, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \neq o(n^2)$ (just like $2 \not< 2$)
$n^2/1000 \neq o(n^2)$

ω-notation

$$\omega(g(n)) = \{f(n) : \text{ for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\} .$$

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \neq \omega(n^2)$

COMPARISONS OF FUNCTIONS

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.

Same for O, Ω, o, and ω.

Reflexivity:

$f(n) = \Theta(f(n))$.

Same for O and Ω.

Symmetry:

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Transpose symmetry:

$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

$f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

COMPARISONS OF FUNCTIONS (continued)

Comparisons:

- $f(n)$ is ***asymptotically smaller*** than $g(n)$ if $f(n) = o(g(n))$.
- $f(n)$ is ***asymptotically larger*** than $g(n)$ if $f(n) = \omega(g(n))$.

No trichotomy. Although intuitively, we can liken O to $\leq$, Ω to $\geq$, etc., unlike real numbers, where $a < b$, $a = b$, or $a > b$, we might not be able to compare functions.

Example: $n^{1+\sin n}$ and n, since $1 + \sin n$ oscillates between 0 and 2.

LOGARITHMS

Notations:

$$\begin{aligned}
\lg n &= \log_2 n && \text{(binary logarithm)} ,\\
\ln n &= \log_e n && \text{(natural logarithm)} ,\\
\lg^k n &= (\lg n)^k && \text{(exponentiation)} ,\\
\lg \lg n &= \lg(\lg n) && \text{(composition)} .
\end{aligned}$$

Logarithm functions apply only to the next term in the formula, so that $\lg n + k$ means $(\lg n) + k$, and *not* $\lg(n + k)$.

In the expression $\log_b a$:

- Hold b constant $\Rightarrow$ the expression is strictly increasing as a increases.
- Hold a constant $\Rightarrow$ the expression is strictly decreasing as b increases.

LOGARITHMS (continued)

$$
\begin{aligned}
a &= b^{\log_b a} , \\
\log_c(ab) &= \log_c a + \log_c b , \\
\log_b a^n &= n \log_b a , \\
\log_b a &= \frac{\log_c a}{\log_c b} , \\
\log_b(1/a) &= -\log_b a , \\
\log_b a &= \frac{1}{\log_a b} , \\
a^{\log_b c} &= c^{\log_b a} .
\end{aligned}
$$