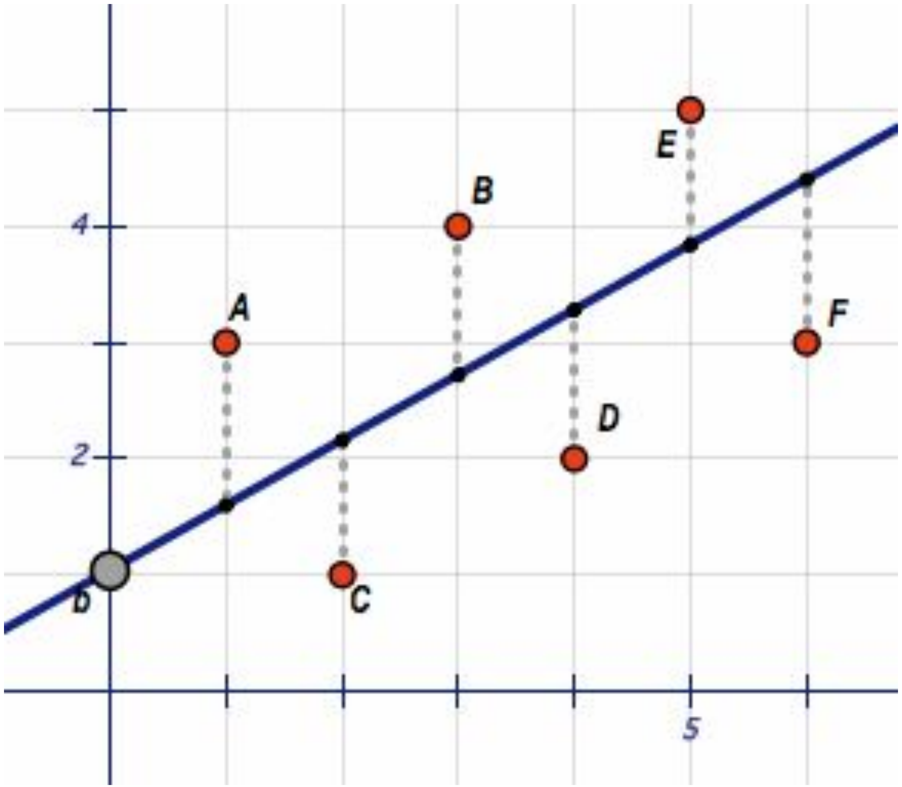
Algorithms Club

6.3.24

Least Squares

Input - Set of points in 2D space

Output - A line that minimizes the sum of squares of residuals



Cool Gif

https://www.r-bloggers.com/2020/12/least-squares-as-springs-the-shiny-app/

Good YT Video

https://www.youtube.com/watch?v=EDPCsD6BzWE

Pseudocode Overview

$$\mathbf{Ax = b}$$

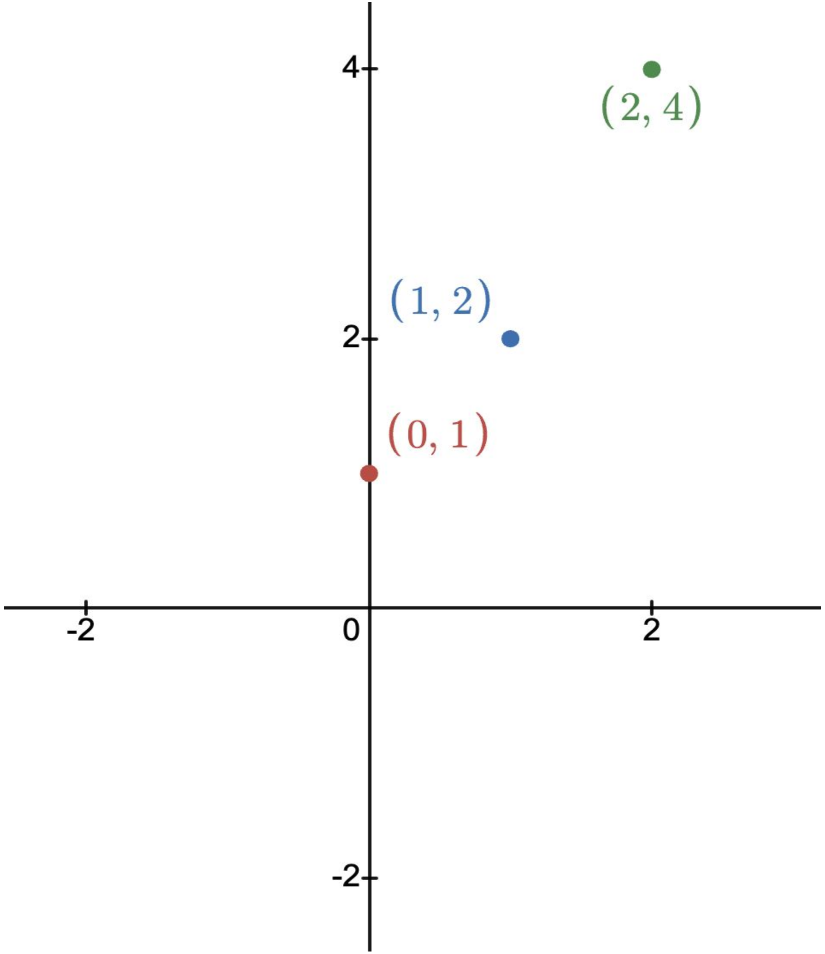
$$\big\downarrow$$

$$\mathbf{A^T A \hat{x} = A^T b}$$

1. Compute A^TA and A^Tb
2. Form augmented matrix
3. Row Reduce

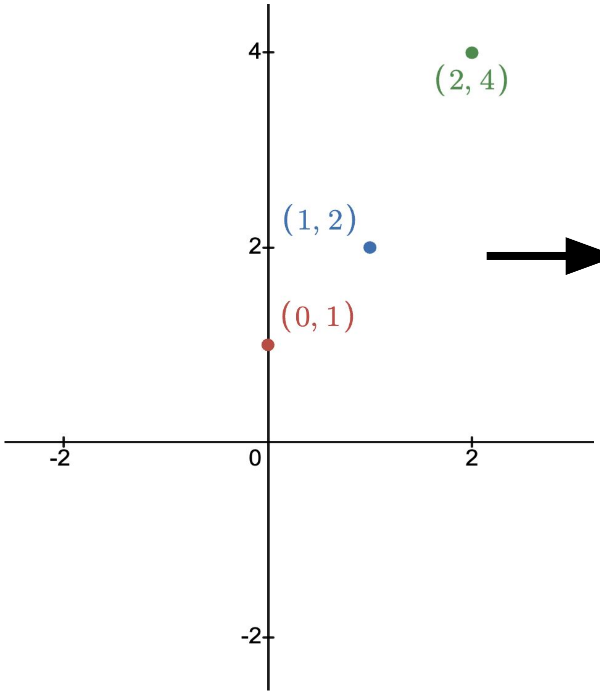
In Depth Pseudocode

1. **Input-** Set of points in 2D space



In Depth Pseudocode

1. **Input-** Set of points in 2D space



$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

2. Make A matrix and b vector

In Depth Pseudocode

3. Compute A^TA and A^Tb

$$A^TA$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^Tb$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

In Depth Pseudocode

3. Compute A^TA and A^Tb

4. Make augmented matrix

$$\begin{bmatrix} 3 & 3 & 7 \\ 3 & 5 & 10 \end{bmatrix}$$

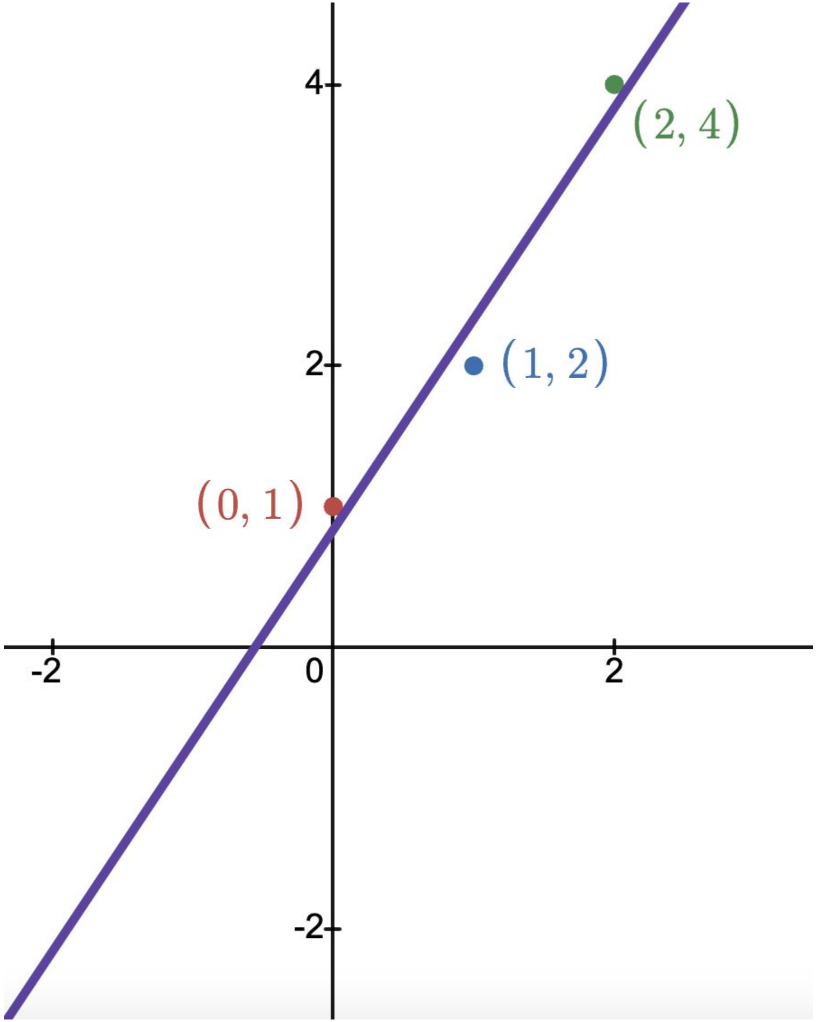
In Depth Pseudocode

3. Compute A^TA and A^Tb
4. Make augmented matrix
5. Rref of matrix

$$\begin{bmatrix} 3 & 3 & 7 \\ 3 & 5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5/6 \\ 0 & 1 & 3/2 \end{bmatrix}$$

In Depth Pseudocode

3. Compute A^TA and A^Tb
4. Make augmented matrix
5. Rref of matrix
6. Plot



$Ax = b$ – overdetermined

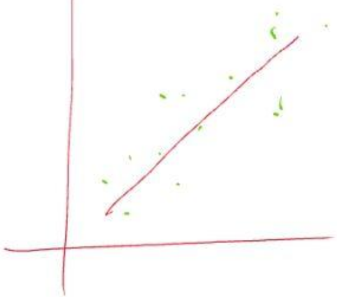
$$b = b_{\substack{\text{projection}\\ \text{Col}(A)}} + \underbrace{b - b_{\substack{\text{projection}\\ \text{col}(A)}}}_{\substack{\text{orthogonal to Col}(A)\\ = \text{orthogonal to Row}(A^T)\\ = \text{Null}(A^T)}}$$

Therefore:

$$A^TAx = A^Tb \quad \leftarrow \text{Normal equation}$$

$$Ax = \underbrace{A(A^TA)^{-1}A^T}_{\text{projection matrix}} b = b_{\substack{\text{projection}\\ \text{col}(A)}}$$

$$y_1 = \beta_0 + \beta_1 x_1$$
$$y_2 = \beta_0 + \beta_2 x_2$$
$$\vdots$$
$$y_n = \beta_0 + \beta_n x_n$$



$$\underbrace{\underset{\text{"b"}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{\text{"A"}}{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}} \underset{\text{"x"}}{\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}}}_{\substack{\text{This is overdetermined}\\ \text{so cannot solve!!}}}$$

Instead solve for

$$Ax = b_{\substack{\text{projection}\\ \text{Col}(A)}}$$

row echelon form (ref) of a matrix

Step 1. Begin with an $m \times n$ matrix A. If $A = 0$, go to Step 7.

Step 2. Determine the leftmost non-zero column.

Step 3. Use elementary row operations to put a 1 in the topmost position (we call this position pivot position) of this column.

Step 4. Use elementary row operations to put zeros (strictly) below the pivot position.

Step 5. If there are no more non-zero rows (strictly) below the pivot position, then go to Step 7.

Step 6. Apply Step 2-5 to the submatrix consisting of the rows that lie (strictly below) the pivot position.

Step 7. The resulting matrix is in row-echelon form

https://www.math.purdue.edu/~shao92/documents/Algorithm%20REF.pdf

1) $\begin{bmatrix} A \end{bmatrix}_{m\times n}$

2) $\begin{bmatrix} \boxed{\ } & A \end{bmatrix}$: find leftmost column

3) $\begin{bmatrix} 1 & \\ & A \end{bmatrix}$: Use elementary row operation

4) $\begin{bmatrix} 1 & \\ 0 & A \\ \vdots & \\ 0 & \end{bmatrix}$: Make them zeros with row operation
for example:
$$R_2 \longrightarrow R_2 - \begin{pmatrix}\text{Multi} \\ \text{factor}\end{pmatrix} * R_1$$

More along the column and perform same operation
- if any pivot element is zero, switch row. immidiately below, if not keep looking at the rows below when found switch rows

5) If no-more non-zeros (strictly) below the pivot position, You found a row echelon form

Reduced row echelon form (rref) of a matrix

we can reduce a Row Echelon Form to the Reduced Row Echelon Form:

Step 8. Determine all the leading ones in the row-echelon form obtained in Step 7.

Step 9. Determine the right most column containing a leading one (we call this column pivot column).

Step 10. Use elementary row operations to erase all the non-zero entries above the leading one in the pivot column.

Step 11. If there are no more columns containing leading ones to the left of the pivot column, then go to Step 13.

Step 12. Apply Step 9-11 to the submatrix consisting of the columns that lie to the left of the pivot column.

Step 13. The resulting matrix is in reduced row-echelon form.

https://www.math.purdue.edu/~shao92/documents/Algorithm%20REF.pdf

Solving Normal equation ??

$$A^T A x = A^T b$$