

# Figures Calculus III

Felix Claeys, Brecht Verbeken, Simon Verbruggen

October 9, 2025

### 1.2.3 Example evolute cycloid

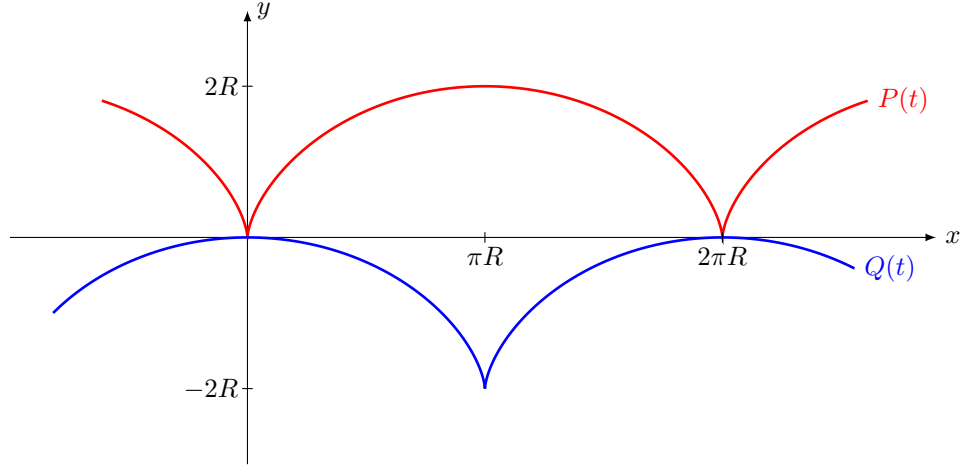


Figure 1: The cycloid  $P(t) = [R(t - \sin t), R(1 - \cos t)]$  and its evolute  $Q(t) = [R(t + \sin t), R(\cos t - 1)]$ , which is a translation of  $P(t)$ .

### 1.2.4 Example evolute catenary

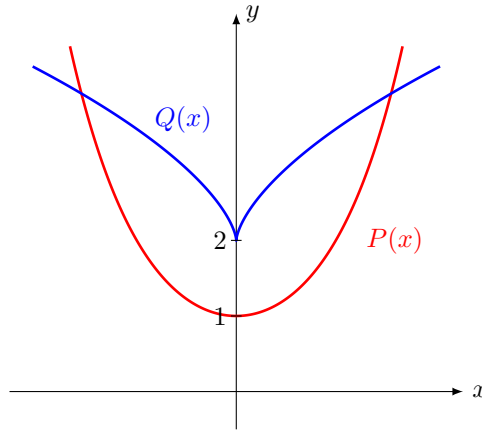


Figure 2: The catenary  $P(x) = [x, a \cosh(x/a)]$  and its evolute  $Q(x) = [x - \frac{a}{2} \sinh(2x/a), 2a \cosh(x/a)]$ .

### 1.2.5 Example involute catenary (tractrix)

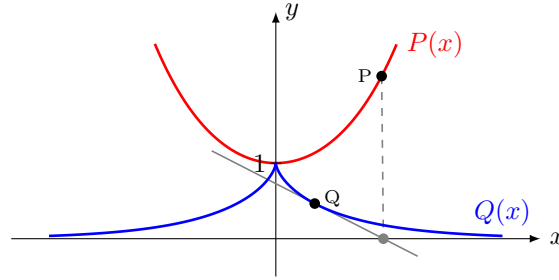


Figure 3: The catenary  $P(x) = [x, a \cosh(x/a)]$  and its involute: the tractrix  $Q(x) = [x - a \tanh(x/a), \frac{a}{\cosh(x/a)}]$ . For a point  $Q$  on the tractrix, the intersection of the tangent to  $Q$  with the  $X$ -axis coincides with the orthogonal projection of the corresponding point on the catenary  $P$ .

### 1.2.8 Example envelope family of straight lines

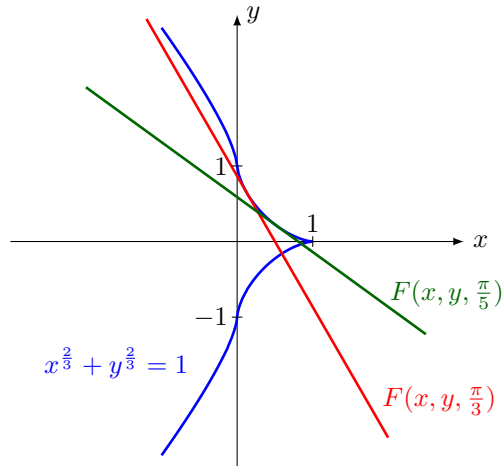


Figure 4: Some examples from the family of lines  $F(x, y, a) = \frac{x}{\cos(a)} + \frac{y}{\sin(a)} = 1$ , and the corresponding astroid:  $x^{2/3} + y^{2/3} = 1$ .

## 2.3 Gradient of scalar field

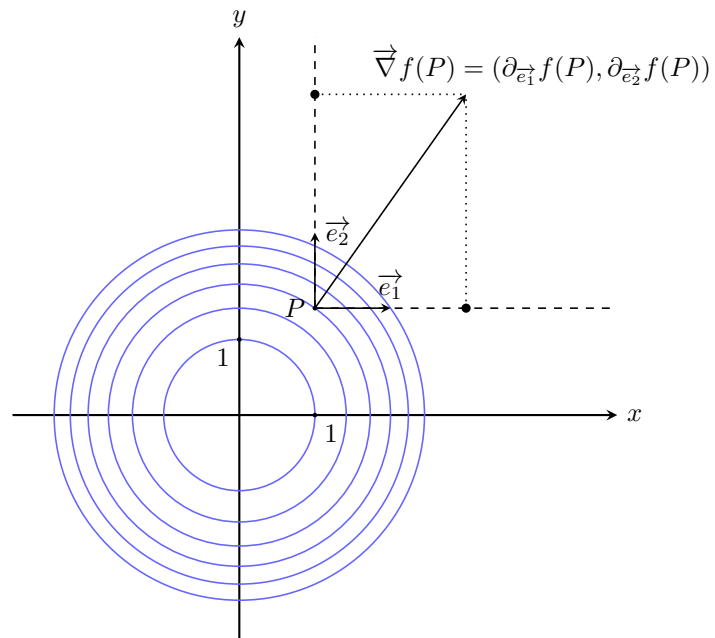


Figure 5:

### 3.1 Line integral of a scalar field

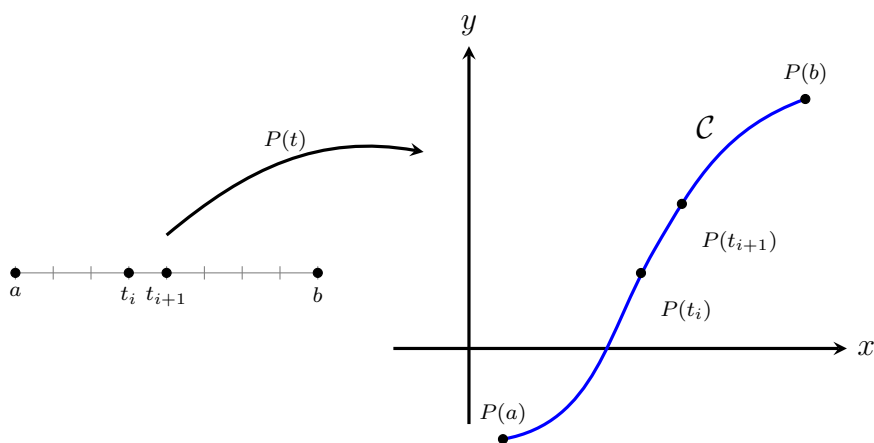


Figure 6: The line integral of a scalar field  $\phi$  over a smooth curve  $P(t)$  is given by:  $\int_{\mathcal{C}} \phi \, ds := \int_a^b \phi(P(t)) \left\| \frac{d\vec{P}}{dt} \right\| dt$ . In the figure, the displacement between different points along the curve,  $P(t_{i+1}) - P(t_i)$ , is shown. This relates to the infinitesimal displacement  $\left\| \frac{d\vec{P}}{dt} \right\| dt$ .

### 3.2 Line integral of a vector field

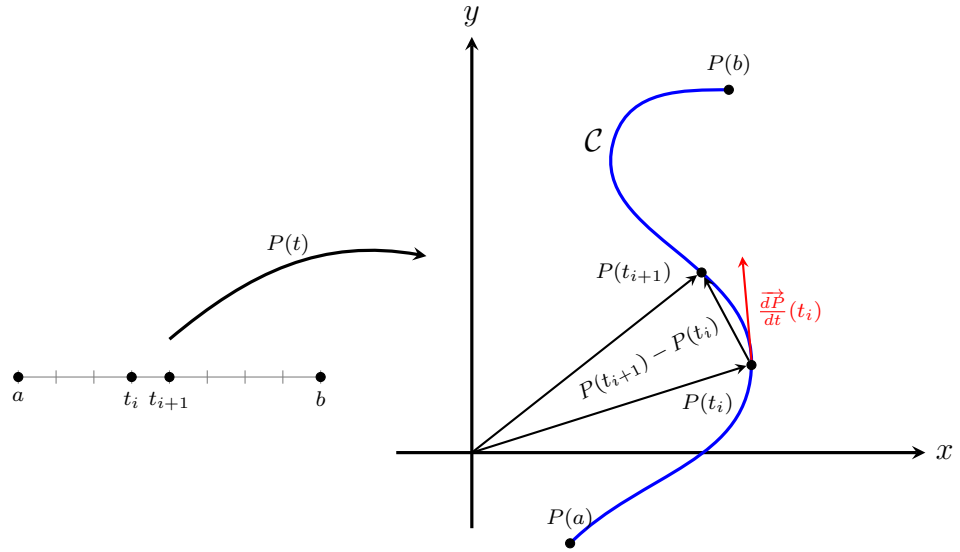


Figure 7: The line integral of a vector field  $\vec{F}$  over a smooth curve  $P(t)$  is given by:  $\int_C \vec{F} \cdot d\vec{P} := \int_a^b \vec{F}(P(t)) \cdot \frac{d\vec{P}}{dt} dt$ . In the figure, the displacement between different points along the curve,  $P(t_{i+1}) - P(t_i)$ , is shown. This relates to the infinitesimal displacement vector  $\frac{d\vec{P}}{dt} dt$ .

### 3.4.2 Conservative field along a curve

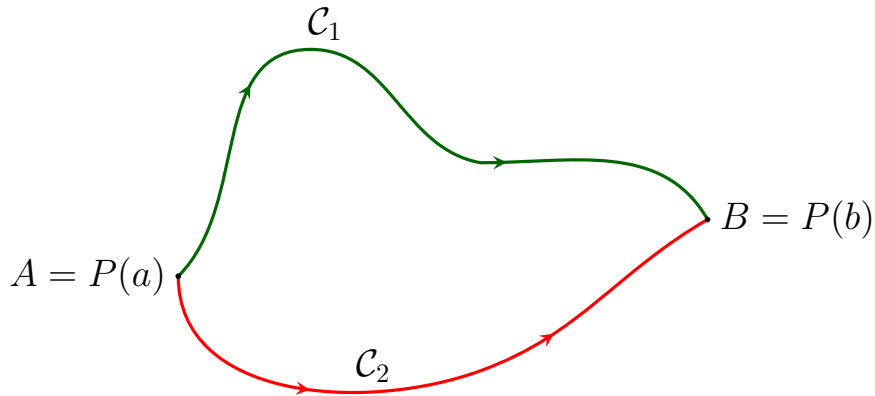


Figure 8: The points  $A$  and  $B$  lie on a closed curve  $\mathcal{C}$ . We can choose two arbitrary paths,  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , from  $A$  to  $B$ . If we evaluate the line integral of a conservative field  $\phi$  along both paths, the result will be the same, since the line integral of a conservative field is independent of the travelled path.

### 3.4.3 Proof conservative field

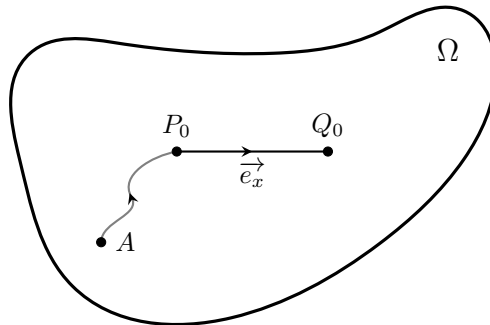


Figure 9: Proof that path-independence implies conservativeness in fields. Consider a path-independent vector field  $\vec{F}$  in an open, connected region  $\Omega$ . A potential function  $\phi(P_0)$  is constructed by integrating  $\vec{F}$  along an arbitrary curve  $\widehat{AP_0}$  from a fixed point  $A$  to  $P_0$ . The path independence ensures that  $\phi$  is well-defined, and its gradient reproduces  $\vec{F}$ .

### 3.5.1 Proof Greens theorem

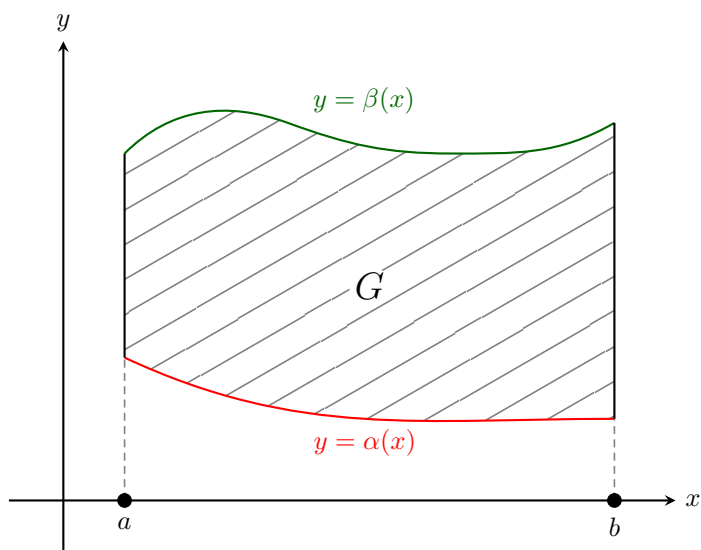


Figure 10: The normal region  $G$  can be projected onto the  $x$ -axis, with its upper and lower boundaries parametrized by  $y = \alpha(x)$  and  $y = \beta(x)$ .

### 3.5.2 Union of normal spaces

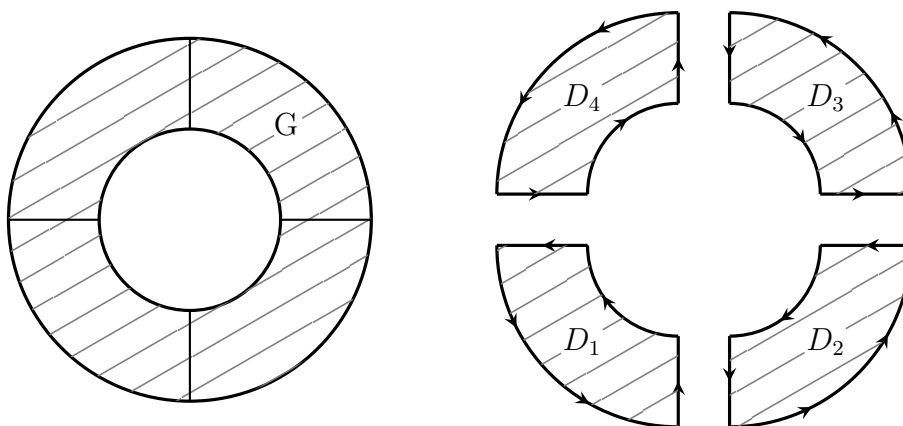


Figure 11: (Left) The annulus  $G$  is not a normal region, as it cannot be projected onto the  $X$ - or  $Y$ -axis. (Right) However,  $G$  is a union of normal regions: it can be split into four regions ( $D_i$ ) each of which can be projected onto the  $X$ - and  $Y$ -axis.



### 3.5.4 Alternative formulation Greens theorem

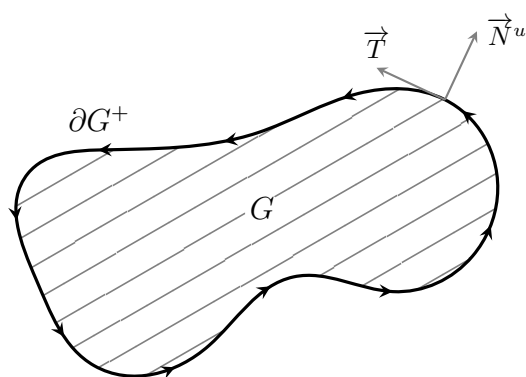


Figure 12: The alternative representation of Green's theorem:  $\int_{\partial G^+} \vec{F} \cdot \vec{N}^u ds = \int_G \vec{\nabla} \cdot \vec{F} dA$ , where  $\vec{T}$  is the tangent unit vector and  $\vec{N}^u$  the outward normal vector.

## 4.1 Surface integral of a scalar field

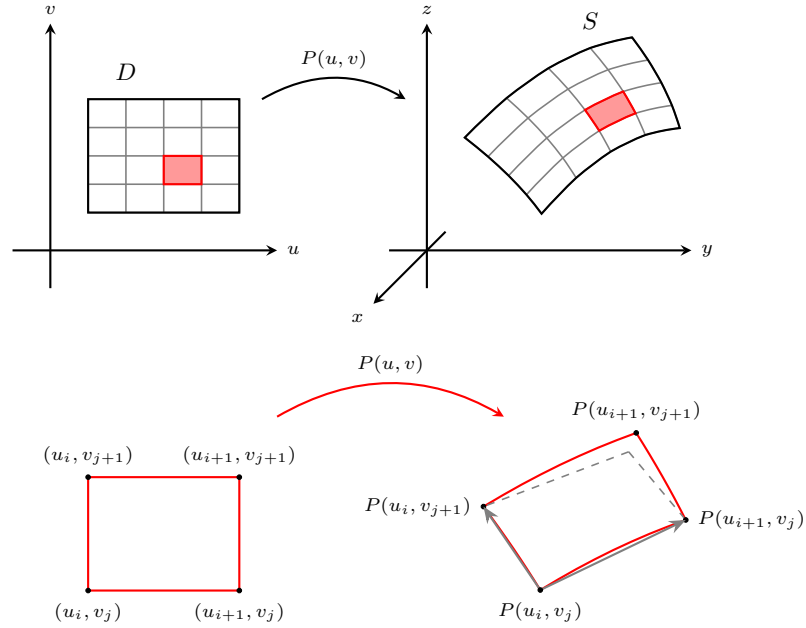


Figure 13: Construction of the surface integral of a scalar field  $\phi$  over a smooth surface  $S = P(u, v)$ :  $\int_S \phi \, d\sigma := \int_D \phi(P(u, v)) \|\overrightarrow{\partial_u P} \times \overrightarrow{\partial_v P}\| \, du \, dv$  (Below) An infinitesimal rectangle on  $D$  with area  $du_i \, dv_j$  is projected onto a curvilinear quadrilateral on  $S$ , which we can approximate by a parallelogram, with area:  $|(P(u_{i+1}, v_j) - P(u_i, v_j))(P(u_i, v_{j+1}) - P(u_i, v_j))| = \|\overrightarrow{\partial_u P}(u_i, v_j) \times \overrightarrow{\partial_v P}(u_i, v_j)\| \, du_i \, dv_j$ . This is represented in the integral by the absolute value of the Jacobian determinant associated with the parameterization:  $\|\overrightarrow{\partial_u P} \times \overrightarrow{\partial_v P}\|$ .

#### 4.4.1 The divergence theorem

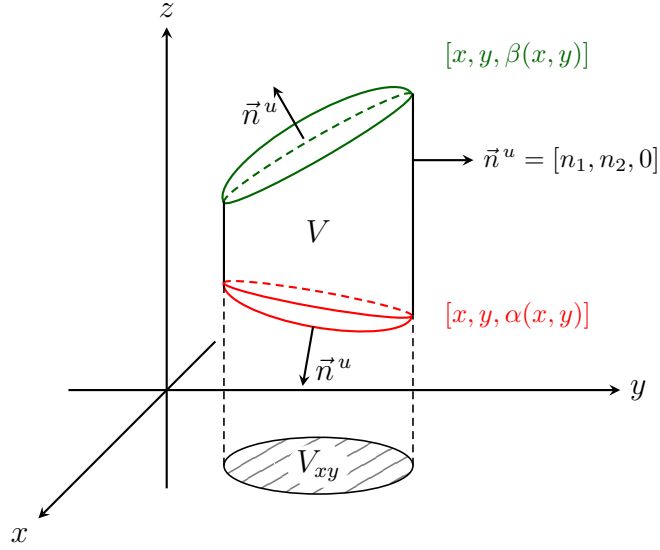


Figure 14: Proof of the divergence theorem. The normal region  $V$  is projected onto the  $XY$ -plane, with projection  $V_{xy}$ . The upper and lower boundaries of  $V$  are represented by a parametrization.

#### 4.6.0 The corkscrew rule

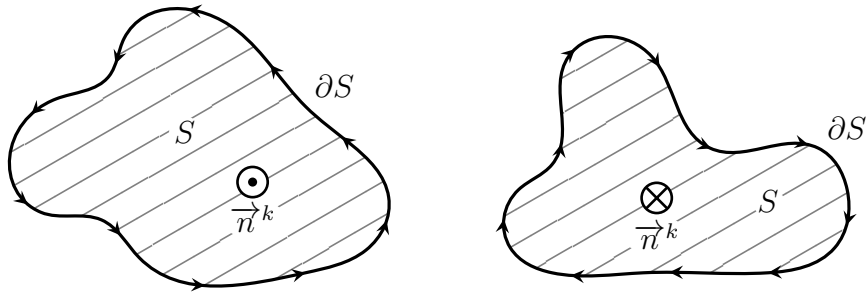


Figure 15: The unit vector along the surface normal,  $\vec{n}^k$ , is oriented such that its direction agrees with the orientation of  $\partial S$  according to the corkscrew rule (or right-hand rule).

### 4.6.1 Stokes theorem

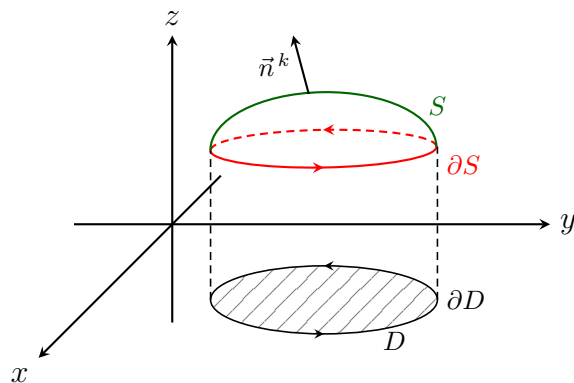


Figure 16: Proof Stokes' theorem.  $S$  is a smooth surface that can be projected onto the  $XY$ -plane, with projection  $D$ .

### 5.1 Inverse function

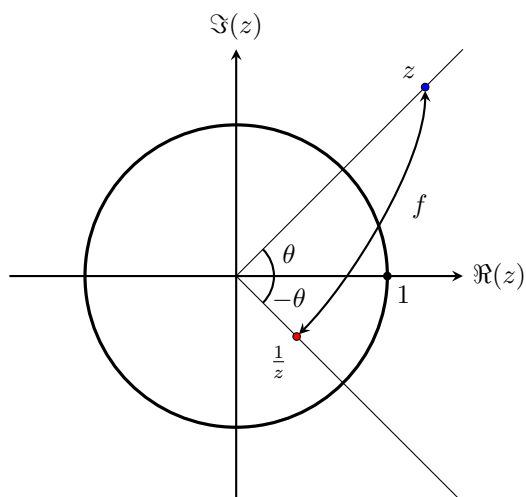


Figure 17:

## 5.1 Complex function

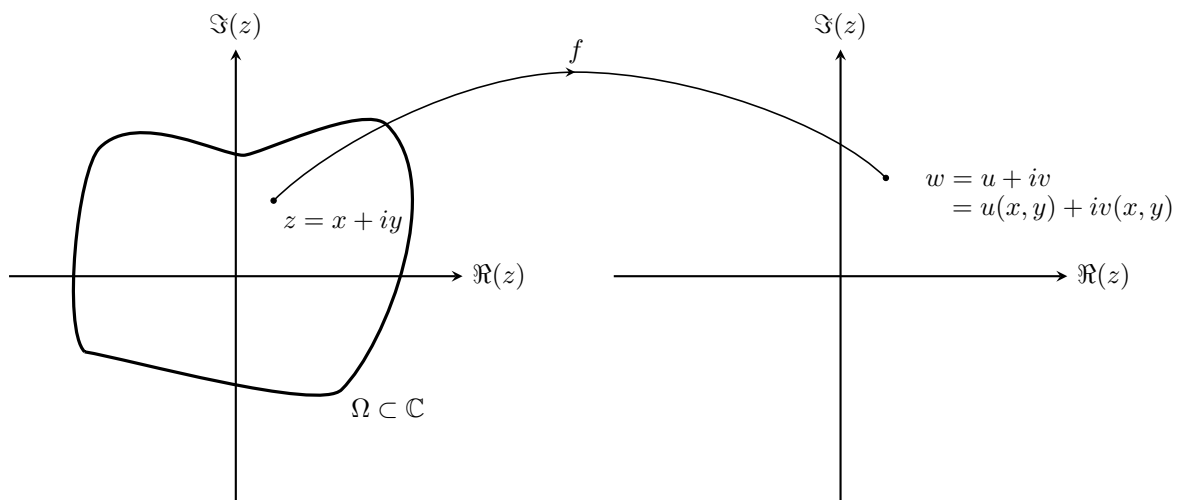


Figure 18:

## 5.2 Complex line integral

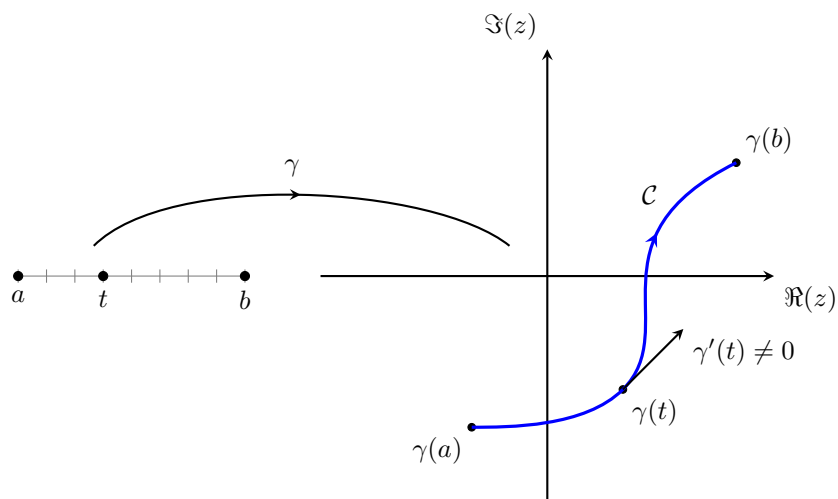


Figure 19:

### 6.2.1 Complex derivative

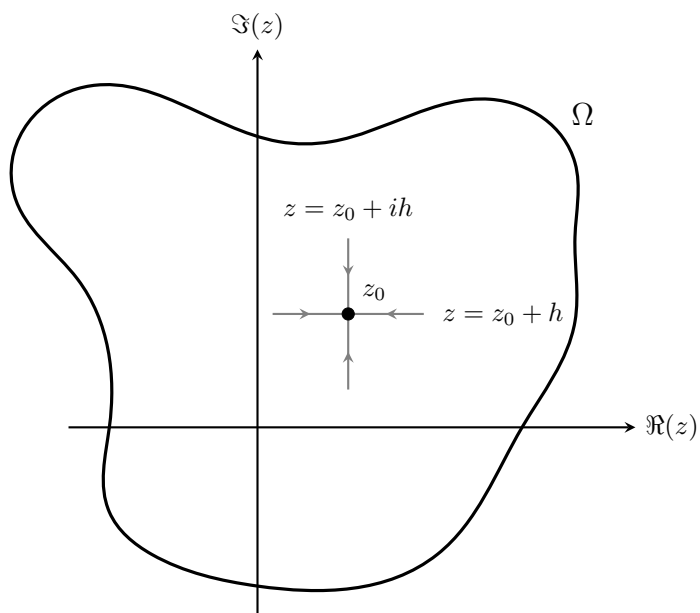


Figure 20: The complex derivative in a point  $z_0 = x_0 + iy_0$  for a complex function is:  $f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h}$ . For  $f'(z_0)$  to exist, this limit must be the same regardless of the direction in which  $h \rightarrow 0$ . In the figure, the limit is approached along the real and imaginary axes; equating the results of both approaches yields the Cauchy–Riemann conditions.

### 6.3 Cauchy Goursat theorem for multiply connected domains

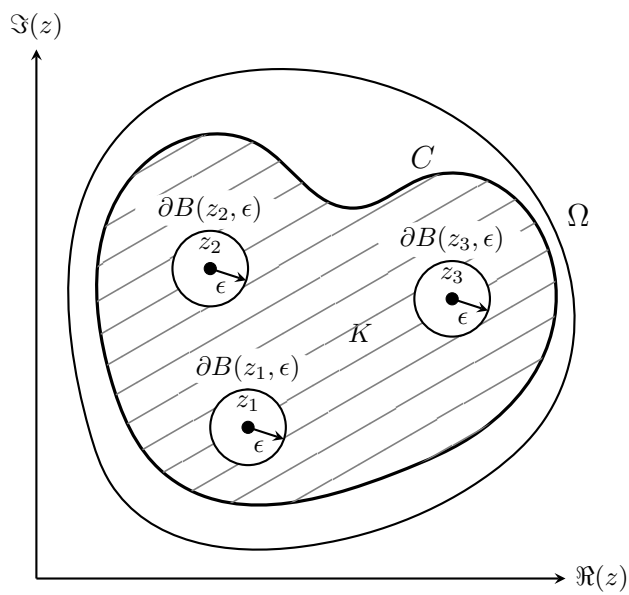


Figure 21:

### 6.3 Contour non simply connected

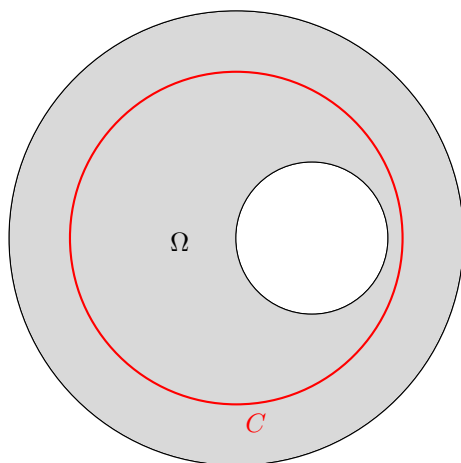


Figure 22:

### 6.3 Contour simply connected

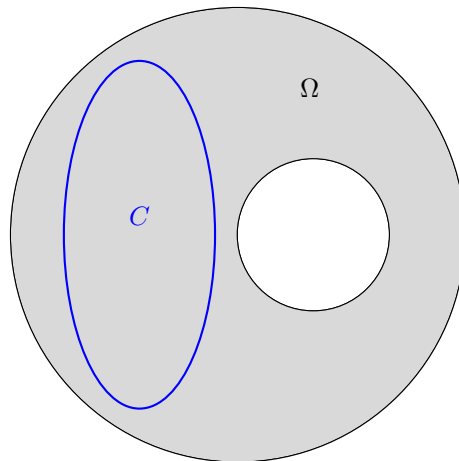


Figure 23:

#### 6.3.3 Proof integral formula Cauchy

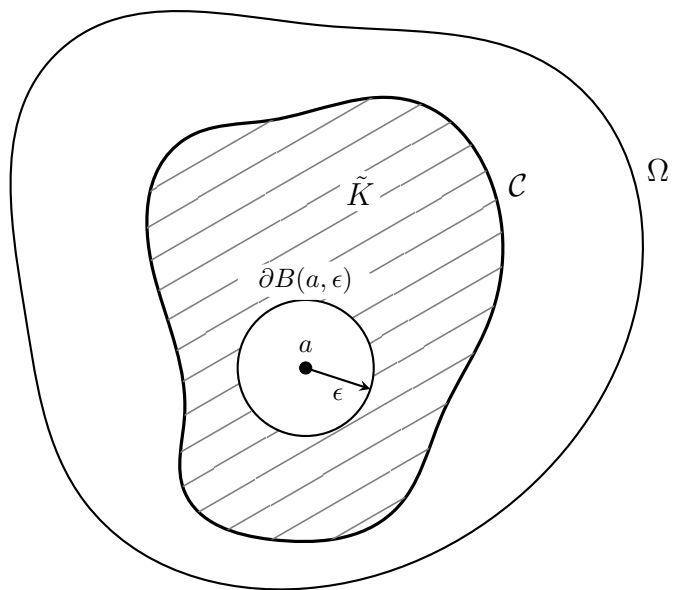


Figure 24:  $C$  is a bounded contour in  $\Omega$  which encloses a compact set  $K$  lying completely within  $\Omega$ . For a point  $a \in C \setminus K$ , we parametrize the circle  $\partial B(a, \epsilon)$ , which lies entirely in the interior of  $C$ .



### 7.2.4 Theorem convergence regions positive and negative power series

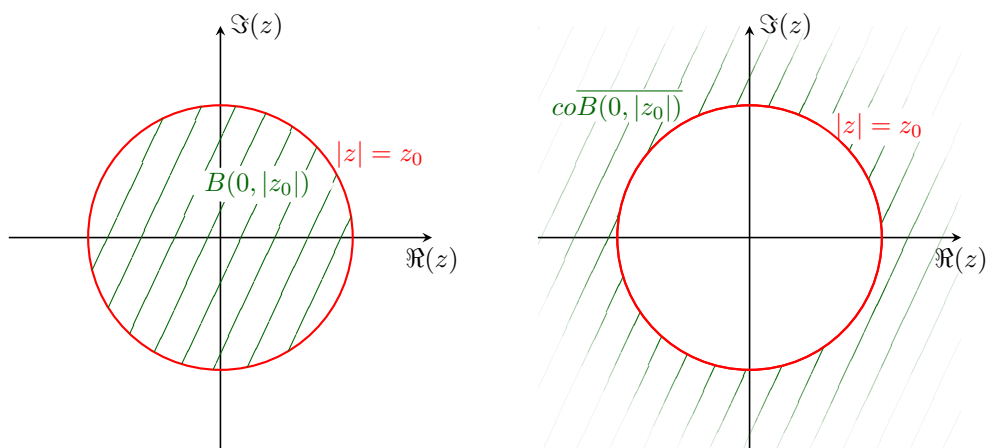


Figure 25: (Left) Region of convergence of a positive power series. (Right) Region of convergence of a negative power series.

### 8.2.1 Proof theorem Laurent series

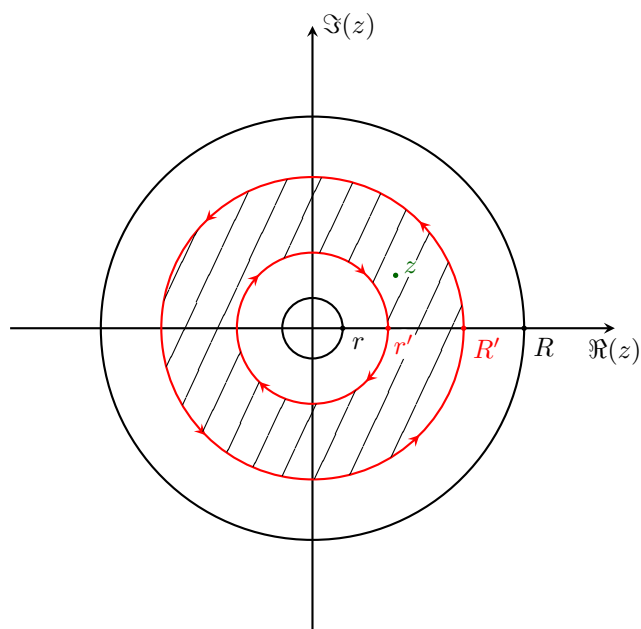


Figure 26: The region of convergence of the Laurent series is the annular region  $B(0, R) \setminus \overline{B(0, r)}$ . For every point  $z$ , it is possible to find  $R'$  and  $r'$  such that  $0 < r < r' < |z| < R' < R$ .

### 8.5.6 Residue theorem for region with multiple singularities

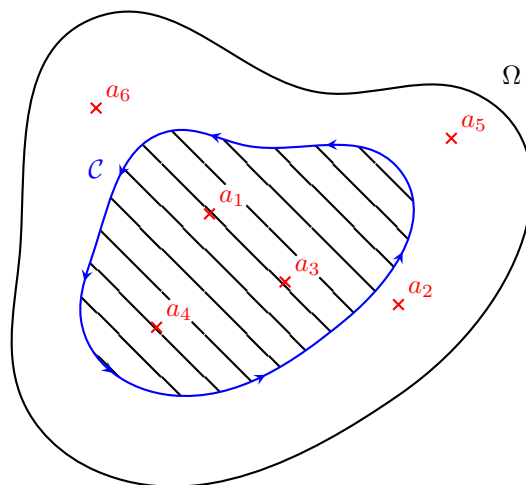


Figure 27: The contour  $\mathcal{C}$  passes through none of the singularities  $a_i$  of the complex function  $f$  and encloses a compact set that lies entirely within  $\Omega$ .  $\mathcal{C}$  contains a finite number of singularities of  $f$ .

## 9.3 Estimation lemmas

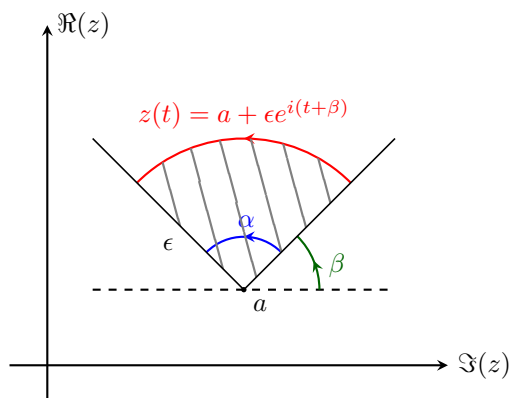


Figure 28: Small limit theorem: The circular arc  $C_\epsilon^t$  with center  $a$ , radius  $\epsilon > 0$ , and central angle  $\alpha$  can be described by the parametric equation  $z(t) = a + \epsilon e^{i(t+\beta)}$ ,  $t : 0 \rightarrow \alpha$ .

## 9.5 Summation of series

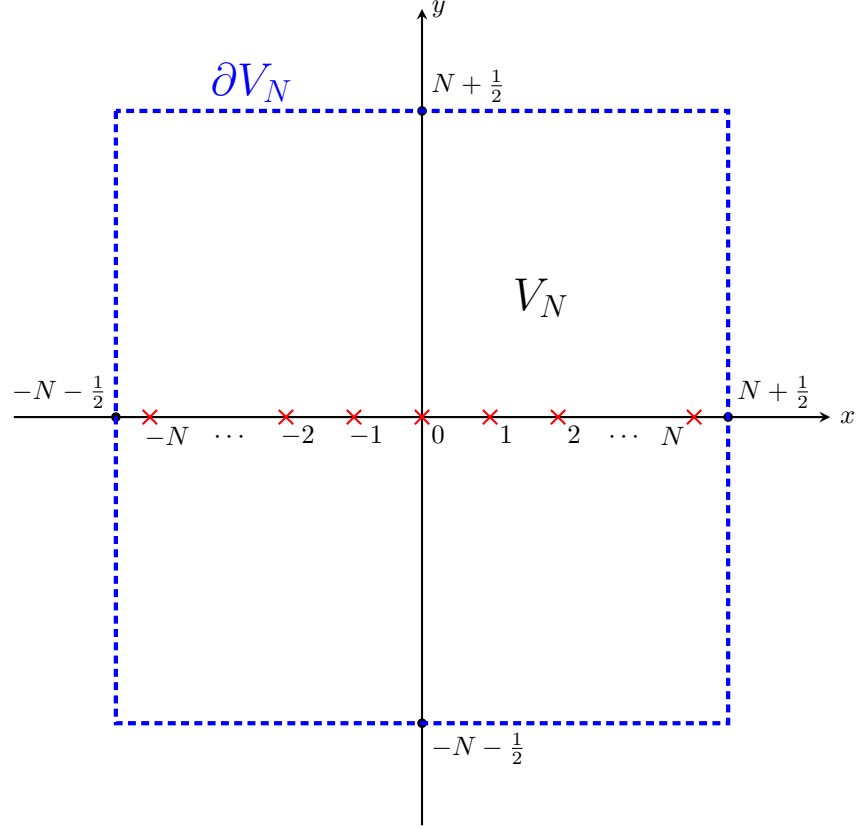


Figure 29: The function  $g(z) = \cot(\pi z)$  has simple poles at  $z = 0, \pm 1, \pm 2, \dots$ . If we consider the square  $V_N$  with center at the origin and side length  $2N + 1$ , where  $N \in \mathbb{N}$ , then we can compute the contour integral over  $\partial V_N^+$  using the residue theorem.