Figures Calculus III

Felix Claeys, Brecht Verbeken, Simon Verbruggen

October 9, 2025

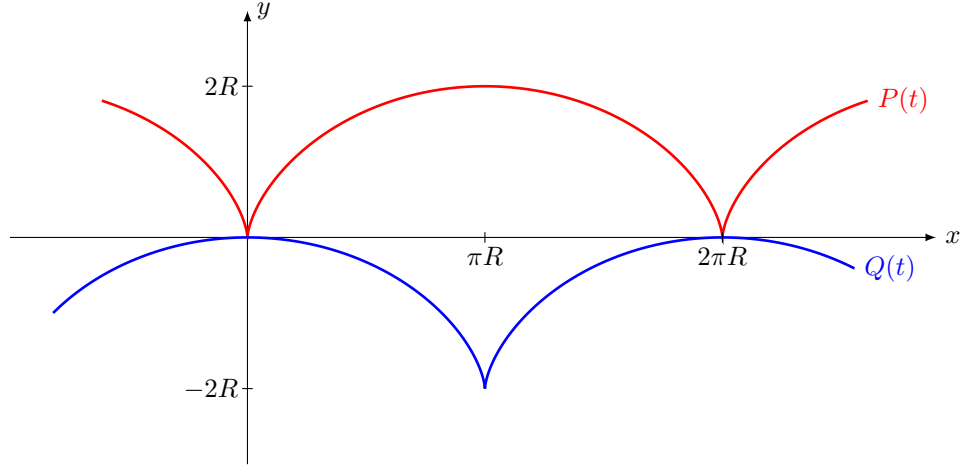### 1.2.3 Example evolute cycloid

Figure 1: The cycloid $P(t) = [R(t - \sin t), R(1 - \cos t)]$ and its evolute $Q(t) = [R(t + \sin t), R(\cos t - 1)]$, which is a translation of $P(t)$.
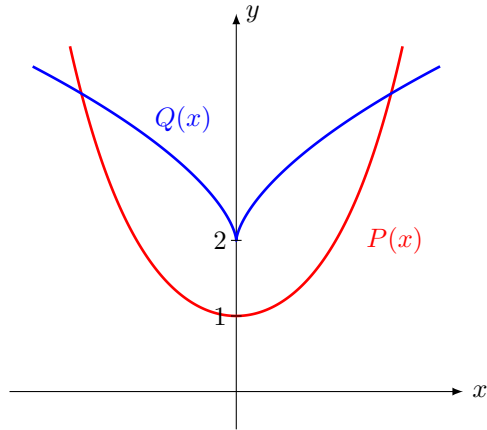
1.2.4 Example evolute catenary

Figure 2: The catenary $P(x) = [x, a \cosh(x/a)]$ and its evolute $Q(x) = \left[x - \frac{a}{2} \sinh(2x/a), 2a \cosh(x/a)\right]$.
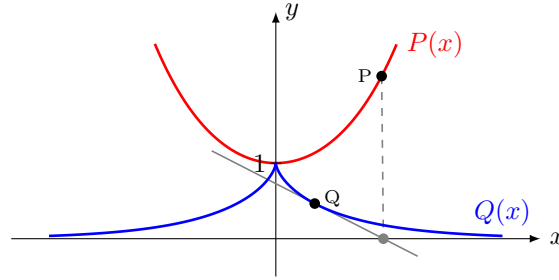
1.2.5 Example involute catenary (tractrix)

Figure 3: The catenary $P(x) = [x, a\cosh(x/a)]$ and its involute: the tractrix $Q(x) = \left[x - a\tanh(x/a), \frac{a}{\cosh(x/a)}\right]$. For a point Q on the tractrix, the intersection of the tangent to Q with the X-axis coincides with the orthogonal projection of the corresponding point on the catenary P.

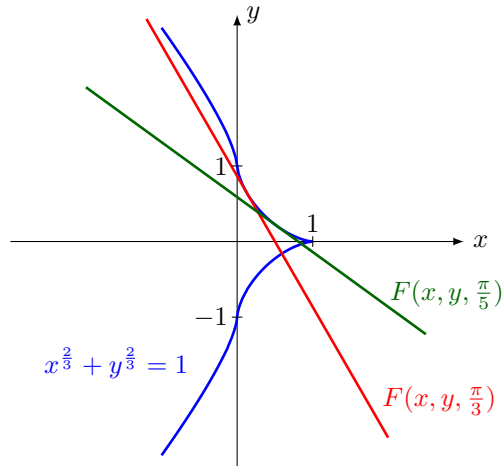## 1.2.8 Example envelope family of straight lines

Figure 4: Some examples from the family of lines $F(x, y, a) = \frac{x}{\cos(a)} + \frac{y}{\sin(a)} = 1$, and the corresponding astroid: $x^{2/3} + y^{2/3} = 1$.
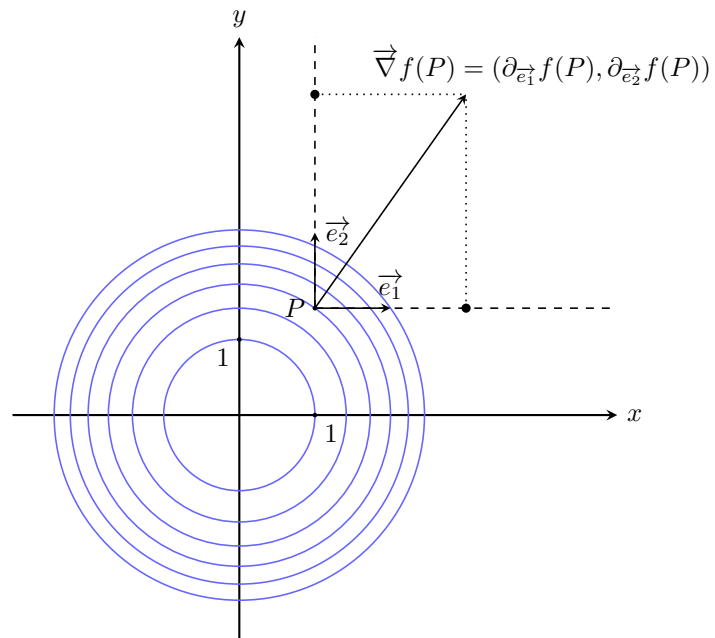
2.3 Gradient of scalar field

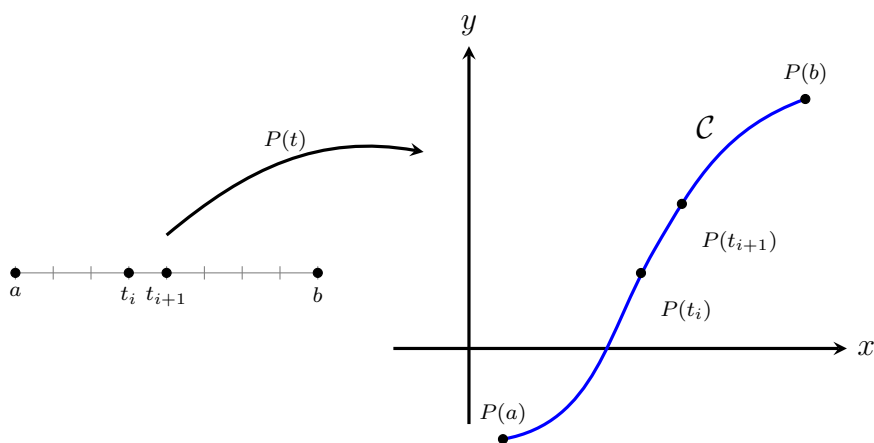Figure 5:

3.1 Line integral of a scalar field

Figure 6: The line integral of a scalar field ϕ over a smooth curve $P(t)$ is given by: $\int_{\mathcal{C}} \phi \; ds := \int_a^b \phi\left(P(t)\right)\left\|\frac{\overrightarrow{dP}}{dt}\right\| dt$. In the figure, the displacement between different points along the curve, $P(t_{i+1}) - P(t_i)$, is shown. This relates to the infinitesimal displacement $\left\|\frac{\overrightarrow{dP}}{dt}\right\| dt$.
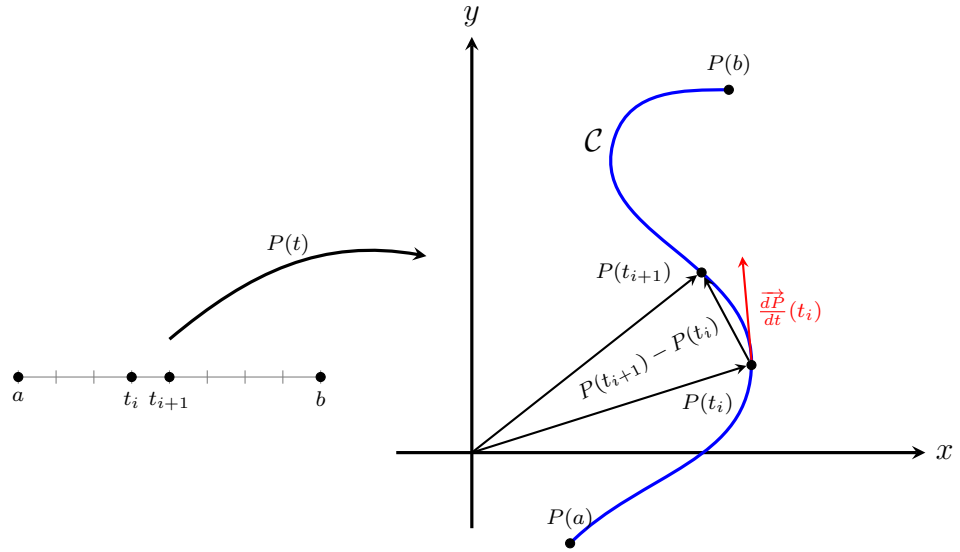
3.2 Line integral of a vector field

Figure 7: The line integral of a vector field $\overrightarrow{F}$ over a smooth curve $P(t)$ is given by: $\int_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{dP} := \int_a^b \overrightarrow{F}\left(P(t)\right) \cdot \frac{\overrightarrow{dP}}{dt}\ dt$. In the figure, the displacement between different points along the curve, $P(t_{i+1}) - P(t_i)$, is shown. This relates to the infinitesimal displacement vector $\frac{\overrightarrow{dP}}{dt}dt$.
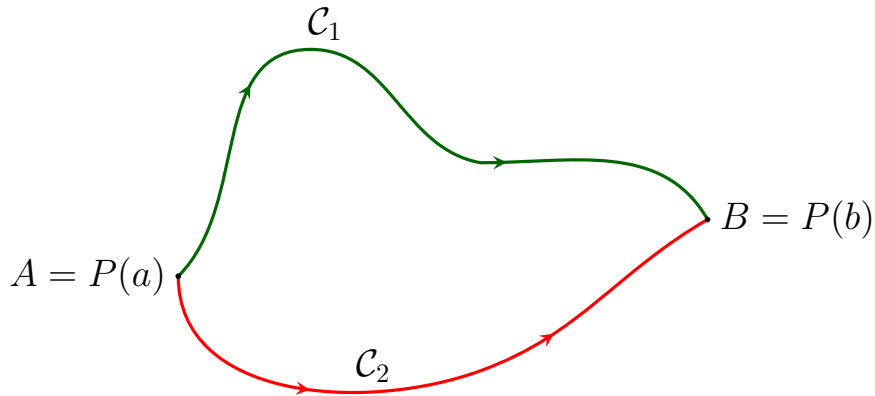
3.4.2 Conservative field along a curve

Figure 8: The points A and B lie on a closed curve $\mathcal{C}$. We can choose two arbitrary paths, $\mathcal{C}_1$ and $\mathcal{C}_2$, from A to B. If we evaluate the line integral of a conservative field ϕ along both paths, the result will be the same, since the line integral of a conservative field is independent of the travelled path.
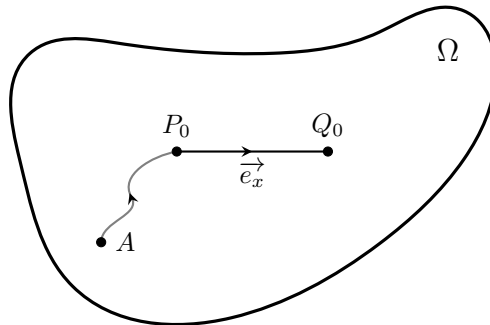
3.4.3 Proof conservative field

Figure 9: Proof that path-independence implies conservativeness in fields. Consider a path-independent vector field $\overrightarrow{F}$ in an open, connected region Ω. A potential function $\phi(P_0)$ is constructed by integrating $\overrightarrow{F}$ along an arbitrary curve $\widehat{AP_0}$ from a fixed point A to P_0. The path independence ensures that ϕ is well-defined, and its gradient reproduces $\overrightarrow{F}$.
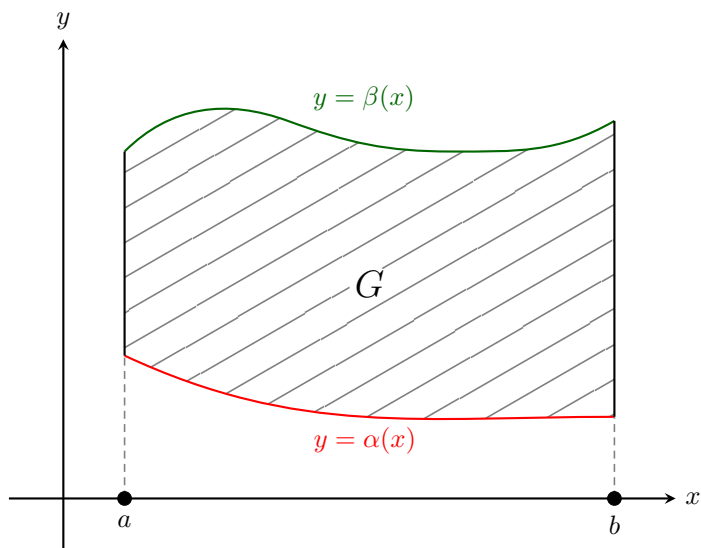
3.5.1 Proof Greens theorem

Figure 10: The normal region G can be projected onto the x-axis, with its upper and lower boundaries parametrized by $y = \alpha(x)$ and $y = \beta(x)$.
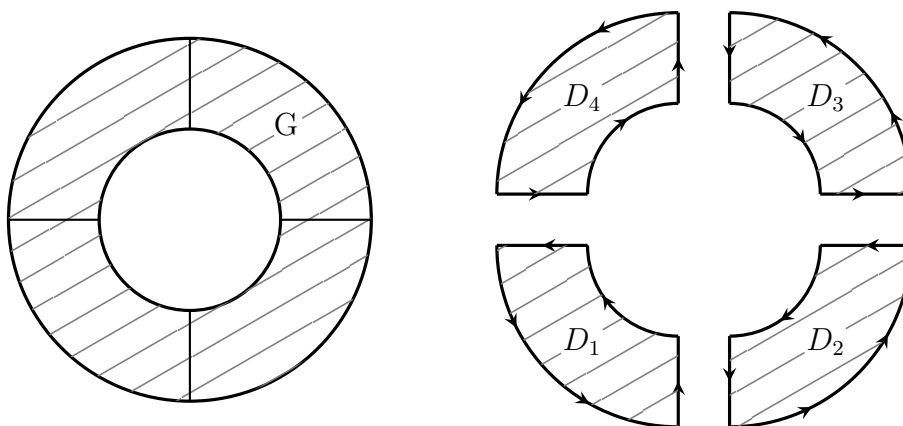
3.5.2 Union of normal spaces

Figure 11: (Left) The annulus G is not a normal region, as it cannot be projected onto the X- or Y-axis. (Right) However, G is a union of normal regions: it can be split into four regions (D_i) each of which can be projected onto the X- and Y-axis.
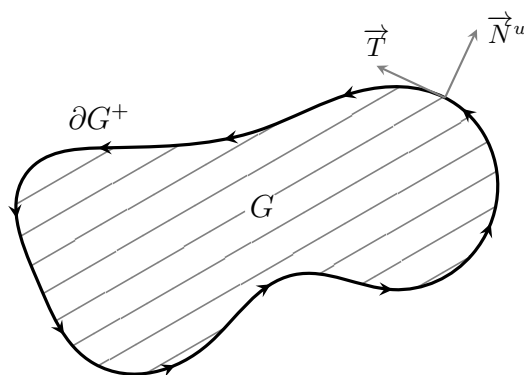
3.5.4 Alternative formulation Greens theorem

Figure 12: The alternative representation of Green's theorem: $\int_{\partial G^+} \overrightarrow{F} \cdot \overrightarrow{N}^u \, ds = \int_G \overrightarrow{\nabla} \cdot \overrightarrow{F} \, dA$, where $\overrightarrow{T}$ is the tangent unit vector and $\overrightarrow{N}^u$ the outward normal vector.
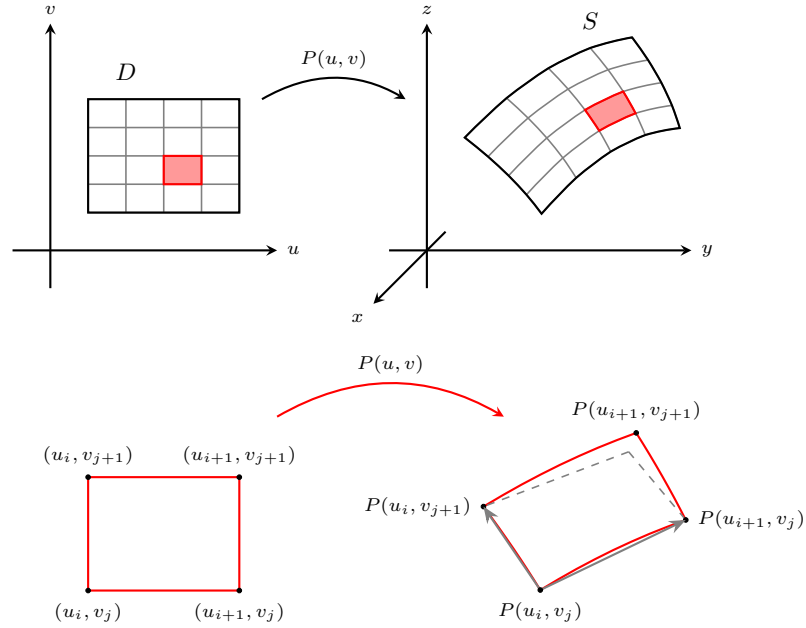
4.1 Surface integral of a scalar field

Figure 13: Construction of the surface integral of a scalar field ϕ over a smooth surface $S = P(u,v)$: $\int_S \phi \ d\sigma := \int_D \phi \left(P(u,v) \right) \|\overrightarrow{\partial_u P} \times \overrightarrow{\partial_v P}\| \ du \ dv$ (Below) An infinitesimal rectangle on D with area $du_i \ dv_j$ is projected onto a curvilinear quadrilateral on S, which we can approximate by a parallelogram, with area: $|\left(P(u_{i+1}, v_j) - P(u_i, v_j) \right) \left(P(u_i, v_{j+1}) - P(u_i, v_j) \right)| = \|\overrightarrow{\partial_u P(u_i, v_j)} \times \overrightarrow{\partial_v P(u_i, v_j)}\| \ du_i \ dv_j$. This is represented in the integral by the absolute value of the Jacobian determinant associated with the parameterization: $\|\overrightarrow{\partial_u P} \times \overrightarrow{\partial_v P}\|$.
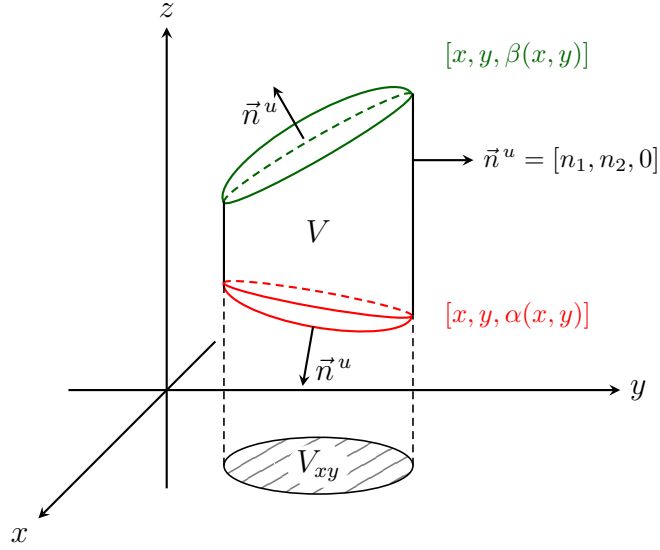
4.4.1 The divergence theorem

Figure 14: Proof of the divergence theorem. The normal region V is projected onto the XY-plane, with projection V_{xy}. The upper and lower boundaries of V are represented by a parametrization.
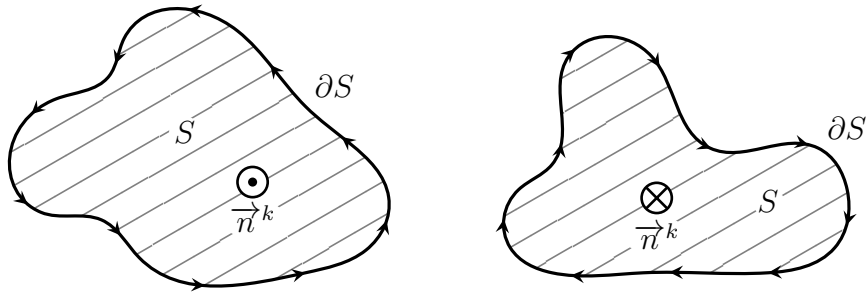
4.6.0 The corkscrew rule

Figure 15: The unit vector along the surface normal, $\overrightarrow{n}^k$, is oriented such that its direction agrees with the orientation of ∂S according to the corkscrew rule (or right-hand rule).

4.6.1 Stokes theorem
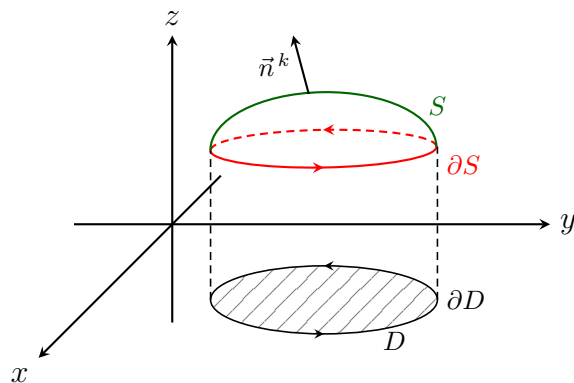
Figure 16: Proof Stokes' theorem. S is a smooth surface that can be projected onto the XY-plane, with projection D.
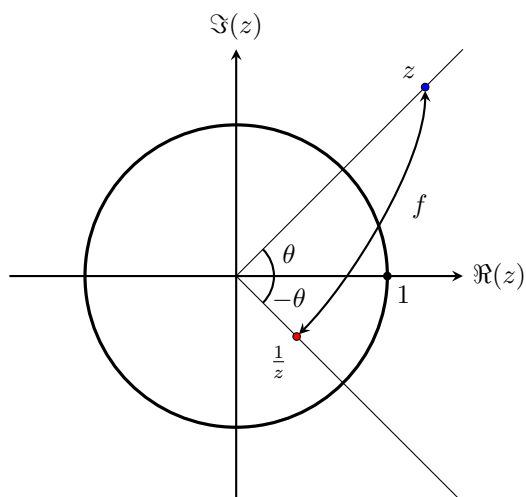
5.1 Inverse function

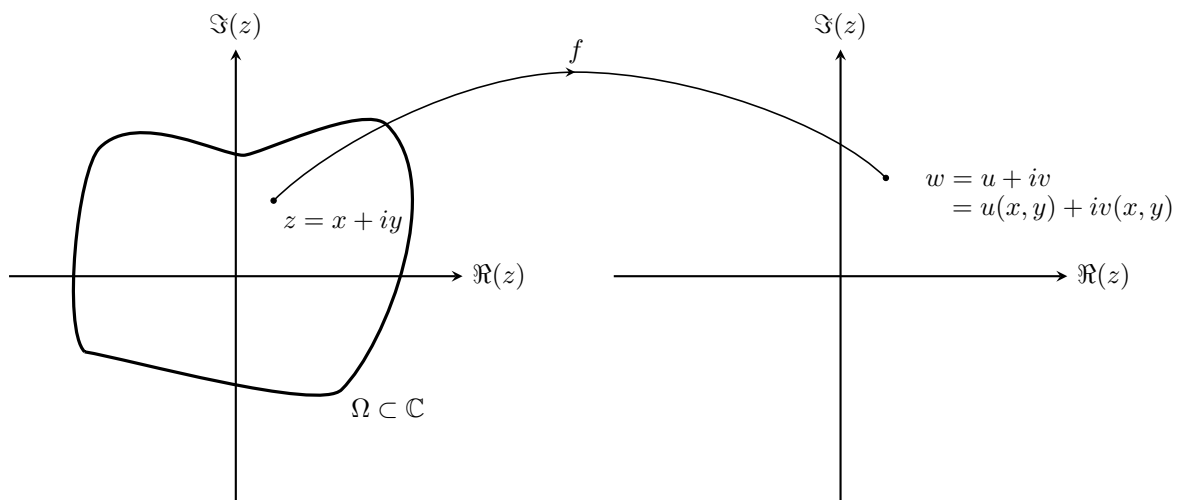
Figure 17:

5.1 Complex function

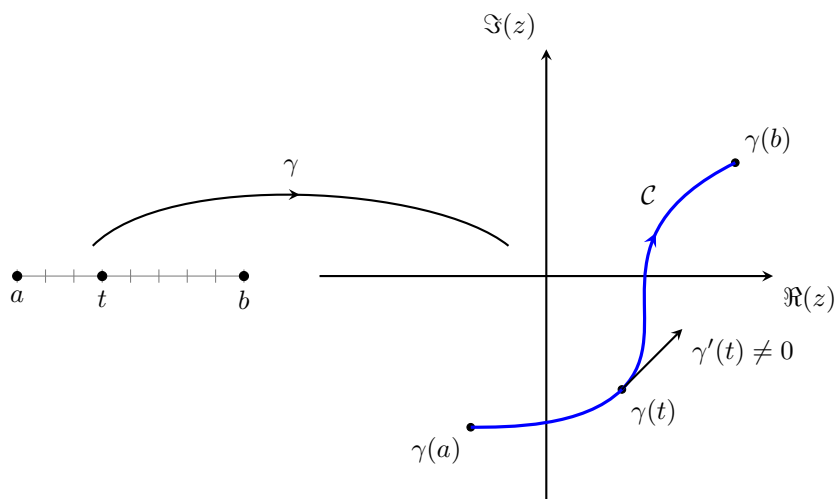Figure 18:

5.2 Complex line integral

Figure 19:

6.2.1 Complex derivative

Figure 20:

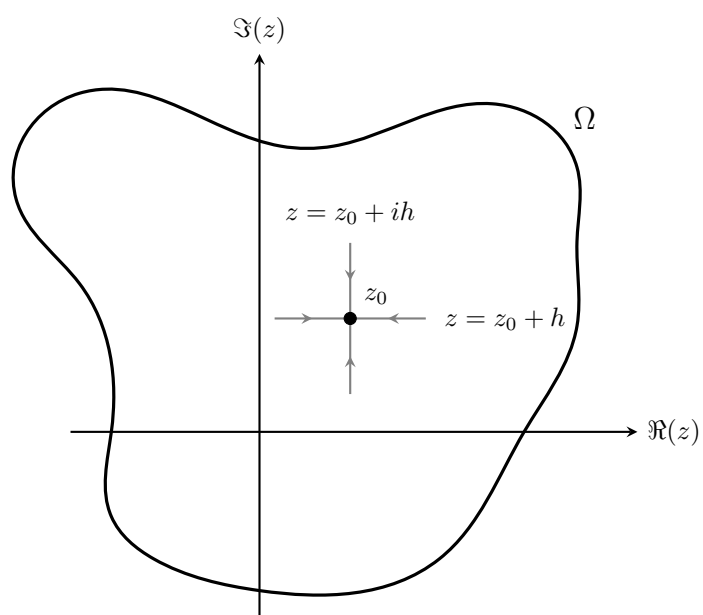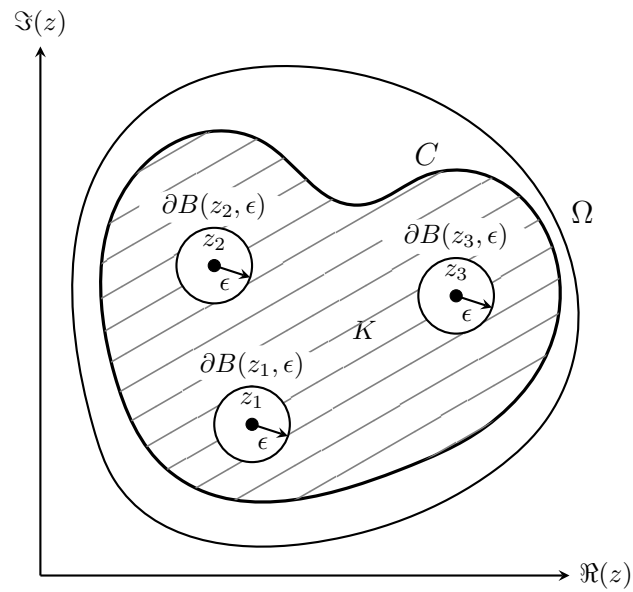## 6.3 Cauchy Goursat theorem for multiply connected domains
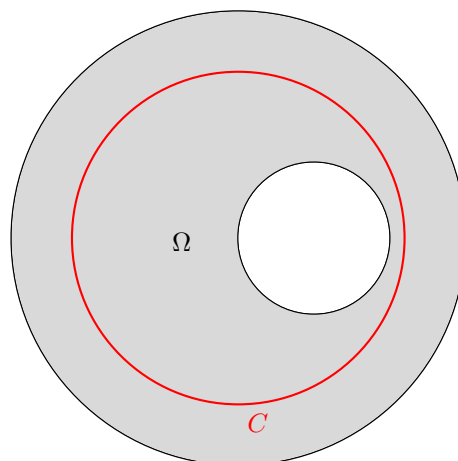

Figure 21:

6.3 Contour non simply connected
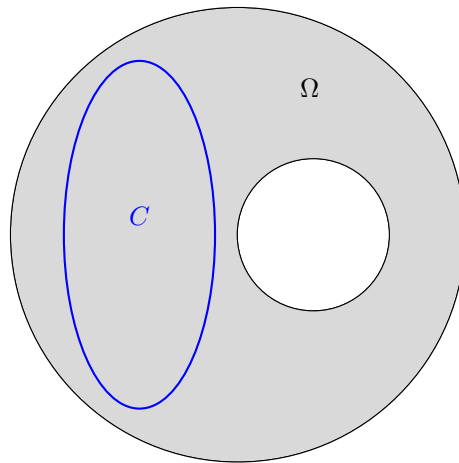

Figure 22:

6.3 Contour simply connected

Figure 23:

6.3.3 Proof integral formula Cauchy
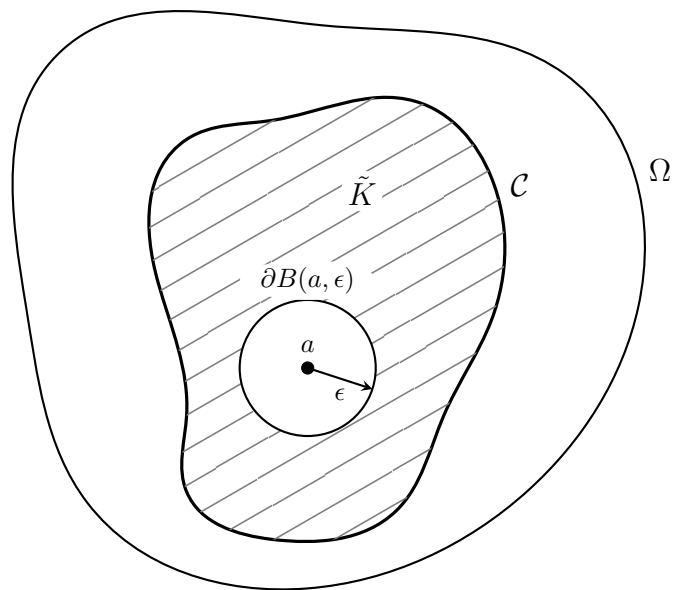

Figure 24: $\mathcal{C}$ is a bounded contour in Ω which encloses a compact set K lying completely within Ω. For a point $a \in \mathcal{C} \setminus K$, we parametrize the circle $\partial B(a, \epsilon)$, which lies entirely in the interior of $\mathcal{C}$.

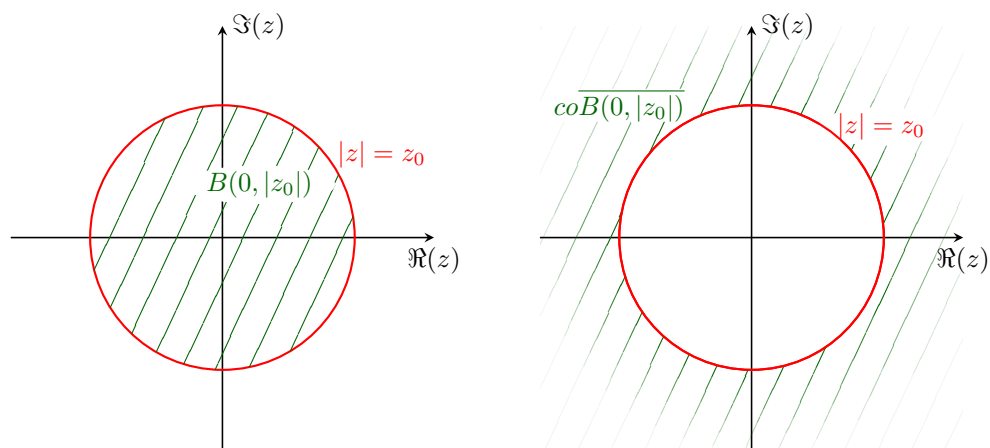## 7.2.4 Theorem convergence regions positive and negative power series

Figure 25: (Left) Region of convergence of a positive power series. (Right) Region of convergence of a negative power series.
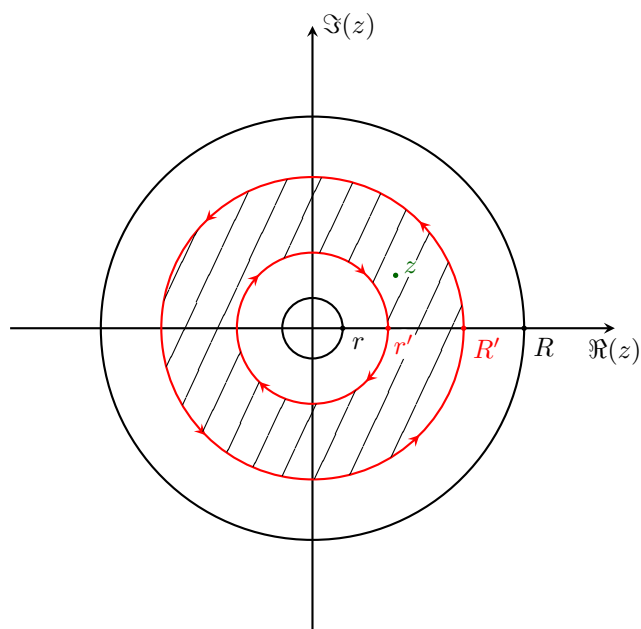
8.2.1 Proof theorem Laurent series

Figure 26: The region of convergence of the Laurent series is the annular region $B(0, R) \setminus \overline{B(0, r)}$. For every point z, it is possible to find R' and r' such that $0 < r < r' < |z| < R' < R$.

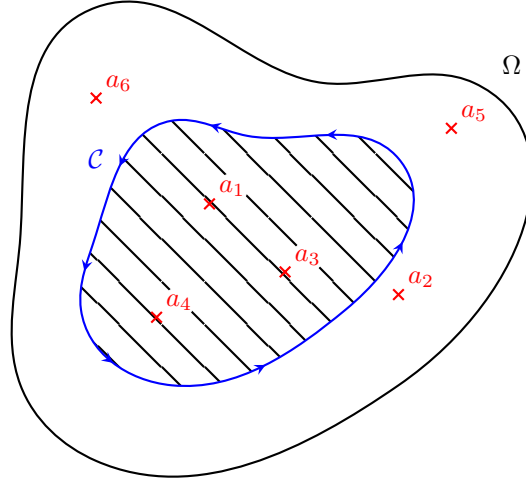### 8.5.6 Residue theorem for region with multiple singularities

Figure 27: The contour $\mathcal{C}$ passes through none of the singularities a_i of the complex function f and encloses a compact set that lies entirely within Ω. $\mathcal{C}$ contains a finite number of singularities of f.
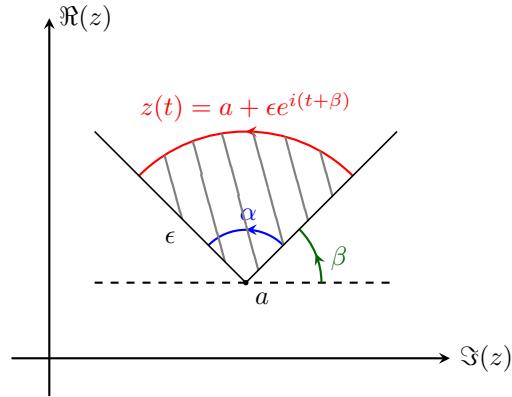
9.3 Estimation lemmas

Figure 28: Small limit theorem: The circular arc C_ε^t with center a, radius $\varepsilon > 0$, and central angle α can be described by the parametric equation $z(t) = a + \varepsilon e^{i(t+\beta)}$, $t : 0 \to \alpha$.
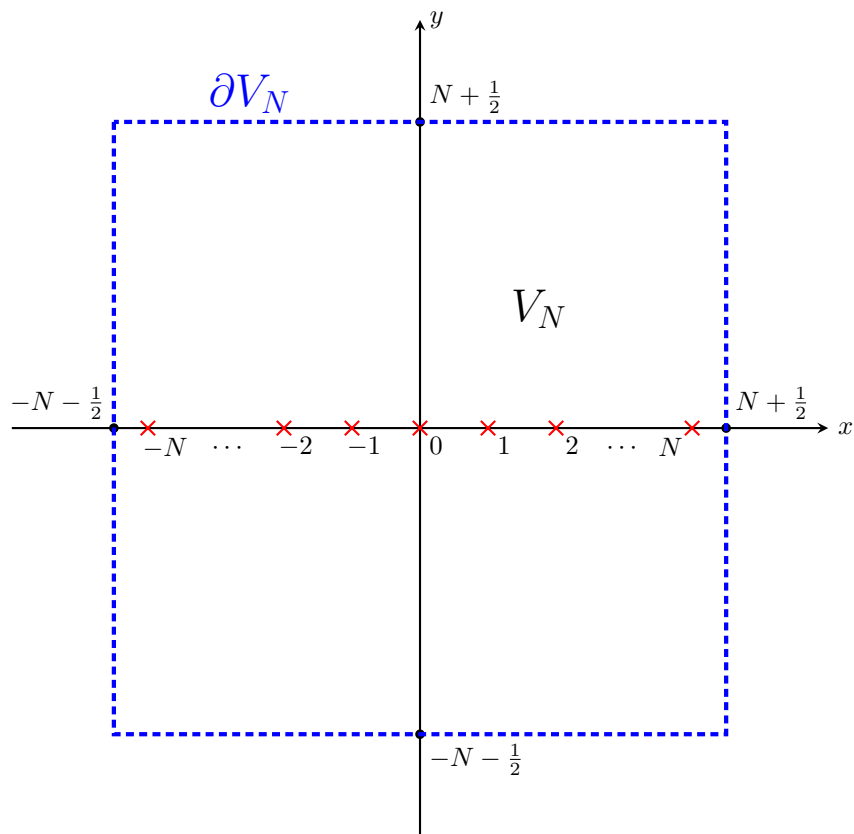
9.5 Summation of series

Figure 29: The function $g(z) = \cot(\pi z)$ has simple poles at $z = 0, \pm 1, \pm 2, \ldots$. If we consider the square V_N with center at the origin and side length $2N + 1$, where $N \in \mathbb{N}$, then we can compute the contour integral over ∂V_N^+ using the residue theorem.