

Bài tập 2 Giải các phương trình đệ quy bằng phương pháp truy hồi

$$1. \quad T(n) = \begin{cases} T(n-1) + 5 & \text{khi } n > 2 \\ 0 & \text{khi } n = 2 \end{cases}$$

$$\begin{aligned} \text{Ta có: } T(n) &= T(n-1) + 5 = [T(n-2) + 5] + 5 = T(n-2) + 2 * 5 \\ &= [T(n-3) + 5] + 2 * 10 = T(n-3) + 3 * 5 \\ &\dots\dots\dots \\ &= T(n-i) + 5i \end{aligned}$$

Quá trình kết thúc khi: $n-i = 1 \Leftrightarrow i = n-1$

$$\begin{aligned} \text{Khi đó: } T(n) &= T(n-i) + 5i \\ &= T(1) + 5(n-1) \\ &= 0 + 5(n-1) \\ &= 5n - 5 \end{aligned}$$

$$2. \quad T(n) = T(n-1) + n \quad T(1) = 1$$

$$\begin{aligned} \text{Ta có: } T(n) &= T(n-1) + n = [T(n-2) + n-1] + n = T(n-2) + 2n-1 \\ &= [T(n-3) + n-2] + 2n-1 = T(n-3) + 3n-1-2 \\ &= [T(n-4) + n-3] + 3n-1-2 \\ &= T(n-4) + 3n-1-2-3 \\ &\dots\dots\dots \\ &= T(n-i) + in - \sum_{k=0}^{i-1} k \end{aligned}$$

Quá trình kết thúc khi: $n-i = 1 \Leftrightarrow i = n-1$

$$\begin{aligned} \text{Khi đó: } T(n) &= T(n-i) + in - \sum_{k=0}^{i-1} k \\ &= T(1) + n(n-1) - \sum_{k=0}^{n-2} k \\ &= 1 + n(n-1) - \frac{(n-2+1)(0+n-2)}{2} \\ &= 1 + n(n-1) - \frac{(n-1)(n-2)}{2} \\ &= 1 + \frac{(n-1)(n+2)}{2} \end{aligned}$$

$$3. \quad T(n) = 3T(n-1) + 1 \quad T(1) = 4$$

$$\text{Ta có: } T(n) = 3T(n-1) + 1 = 3[3T(n-2) + 1] + 1$$

$$\begin{aligned}
&= 3^2 T(n-2) + 3 * 1 + 1 \\
&= 3^2 [3T(n-3) + 1] + 3^1 + 3^0 \\
&= 3^3 T(n-3) + 3^2 + 3^1 + 3^0
\end{aligned}$$

.....

$$= 3^i T(n-i) + \sum_{k=0}^{i-1} 3^k$$

Quá trình kết thúc khi: $n-i=1 \Leftrightarrow i=n-1$

$$\begin{aligned}
\text{Khi đó: } T(n) &= 3^i T(n-i) + \sum_{k=0}^{i-1} 3^k \\
&= 3^{n-1} T(1) + \sum_{k=0}^{n-2} 3^k \\
&= 3^{n-1} * 1 + \frac{3^{n-2+1}-1}{3-1} \\
&= 3^{n-1} * 1 + \frac{3^{n-1}-1}{2}
\end{aligned}$$

$$4. T(n) = 2T\left(\frac{n}{2}\right) + 1 \quad T(1) = 1$$

$$\begin{aligned}
\text{Ta có: } T(n) &= 2T\left(\frac{n}{2}\right) + 1 = 2\left[2T\left(\frac{n}{4}\right) + 1\right] + 1 \\
&= 4T\left(\frac{n}{4}\right) + 2 + 1 = 4\left[2T\left(\frac{n}{8}\right) + 1\right] + 2 + 1 \\
&= 2^3 T\left(\frac{n}{2^3}\right) + 2^2 + 2^1 + 2^0
\end{aligned}$$

.....

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = \log_2 n$

$$\begin{aligned}
\text{Khi đó: } T(n) &= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k \\
&= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \sum_{k=0}^{\log_2 n-1} 2^k \\
&= nT(1) + 2^{\log_2 n-1+1} - 1 \\
&= n * 1 + n - 1 = 2n - 1
\end{aligned}$$

$$5. T(n) = 2T\left(\frac{n}{2}\right) + n \quad T(1) = 1$$

$$\begin{aligned}
\text{Ta có: } T(n) &= 2T\left(\frac{n}{2}\right) + n = 2\left[2T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n \\
&= 4T\left(\frac{n}{4}\right) + 2n = 4\left[2T\left(\frac{n}{8}\right) + \frac{n}{4}\right] + 2n
\end{aligned}$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

.....

$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = \log_2 n$

$$\text{Khi đó: } T(n) = 2^i T\left(\frac{n}{2^i}\right) + in$$

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + n \log_2 n$$

$$= nT(1) + n \log_2 n = n + \log_2 n$$

$$6. T(n) = 2T\left(\frac{n}{2}\right) + n^2 \quad T(1) = 1$$

$$\text{Ta có: } T(n) = 2T\left(\frac{n}{2}\right) + n^2 = 2\left[2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$= 4T\left(\frac{n}{4}\right) + 2 * \frac{n^2}{2^2} + n^2$$

$$= 4\left[2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 + \frac{n^2}{2} = 2^3 T\left(\frac{n}{2^3}\right) + \frac{n^2}{2^2} + \frac{n^2}{2^1} + \frac{n^2}{2^0}$$

.....

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} \frac{n^2}{2^k}$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = \log_2 n$

$$\text{Khi đó: } T(n) = 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} \frac{n^2}{2^k}$$

$$= 2^{\log_2 n} T(1) + n^2 \sum_{k=0}^{\log_2 n - 1} \left(\frac{1}{2}\right)^k$$

$$= n * 1 + n^2 \left(\frac{0.5^{\log_2 n - 1 + 1} - 1}{0.5 - 1}\right)$$

$$= n - 2n^2(2^{-\log_2 n} - 1) = n - 2n^2\left(\frac{1}{n} - 1\right)$$

$$= 2n^2 - n$$

$$7. T(n) = 2T\left(\frac{n}{2}\right) + \log n \quad T(1) = 1$$

$$\text{Ta có: } T(n) = 2T\left(\frac{n}{2}\right) + \log n = 2\left[2T\left(\frac{n}{4}\right) + \log \frac{n}{2}\right] + \log n$$

$$\begin{aligned}
&= 4T\left(\frac{n}{4}\right) + 2\log\frac{n}{2} + \log n \\
&= 4\left[2T\left(\frac{n}{8}\right) + \log\frac{n}{4}\right] + 2\log\frac{n}{2} + \log n \\
&= 2^3T\left(\frac{n}{2^3}\right) + 2^2\log\frac{n}{2^2} + 2^1\log\frac{n}{2^1} + 2^0\log\frac{n}{2^0} \\
&\dots \\
&= 2^iT\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k \log\frac{n}{2^k} \\
&= 2^iT\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k \log n - \sum_{k=0}^{i-1} 2^k \log 2^k \\
&= 2^iT\left(\frac{n}{2^i}\right) + \log(n) \sum_{k=0}^{i-1} 2^k - \log(2) \sum_{k=0}^{i-1} k 2^k
\end{aligned}$$

Quá trình kết thúc khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = \log_2 n$

$$\begin{aligned}
\text{Khi đó: } T(n) &= 2^iT\left(\frac{n}{2^i}\right) + \log(n) \sum_{k=0}^{i-1} 2^k - \log(2) \sum_{k=0}^{i-1} k 2^k \\
&= 2^{\log_2 n} T(1) + \log(n) \sum_{k=0}^{\log_2 n - 1} 2^k - \log(2) \sum_{k=0}^{\log_2 n - 1} k 2^k \\
&= n * 1 + \log(n)(2^{\log_2 n - 1 + 1} - 1) - \log(2)(0 * 2^0 + \sum_{k=1}^{\log_2 n - 1} k 2^k) \\
&= n + (n - 1)\log(n) - (\log_2(n) - 1 - 1)(2^{\log_2 n - 1 + 1} - 1)\log 2 \\
&= n + (n - 1)\log n - (\log_2 n - 2)(n - 1)\log 2
\end{aligned}$$

$$8. T(n) = T(n - 1) + T(n - 2) \quad T(0) = 1 \quad T(1) = 1$$

$$\begin{aligned}
\text{Ta có: } T(n) &= T(n - 1) + T(n - 2) = [T(n - 2) + T(n - 3)] + T(n - 2) \\
&= 2T(n - 2) + T(n - 3) \\
&= 2[T(n - 3) + T(n - 4)] + T(n - 3) \\
&= 3T(n - 3) + 2T(n - 4) \\
&= 3[T(n - 4) + T(n - 5)] + 2T(n - 4) \\
&= 5T(n - 4) + 3T(n - 5)
\end{aligned}$$

