Bài tập 2 Giải các phương trình đệ quy bằng phương pháp truy hồi

1. $T(n) = \begin{cases} T(n-1) + 5 \; khi \; n > 2 \\ 0 \qquad\quad khi \; n = 2 \end{cases}$

   Ta có: $T(n) = T(n-1) + 5 = [T(n-2) + 5] + 5 = T(n-2) + 2*5$

   $\qquad\qquad = [T(n-3) + 5] + 2*10 = T(n-3) + 3*5$

   $\qquad\qquad ....$

   $\qquad\qquad = T(n-i) + 5i$

   Quá trình kết thúc khi: $n - i = 1 \Leftrightarrow i = n - 1$

   Khi đó: $T(n) \; = T(n-i) + 5i$

   $\qquad\qquad = T(1) + 5(n-1)$

   $\qquad\qquad = 0 + 5(n-1)$

   $\qquad\qquad = 5n - 5$

2. $T(n) = T(n-1) + n \qquad\qquad T(1) = 1$

   Ta có: $T(n) \; = T(n-1) + n = [T(n-2) + n - 1] + n = T(n-2) + 2n - 1$

   $\qquad\qquad = [T(n-3) + n - 2] + 2n - 1 = T(n-3) + 3n - 1 - 2$

   $\qquad\qquad = [T(n-4) + n - 3] + 3n - 1 - 2$

   $\qquad\qquad = T(n-4) + 3n - 1 - 2 - 3$

   $\qquad\qquad ....$

   $\qquad\qquad = T(n-i) + in - \sum_{k=0}^{i-1} k$

   Quá trình kết thúc khi: $n - i = 1 \Leftrightarrow i = n - 1$

   Khi đó: $T(n) \; = T(n-i) + in - \sum_{k=0}^{i-1} k$

   $\qquad\qquad = T(1) + n(n-1) - \sum_{k=0}^{n-2} k$

   $\qquad\qquad = 1 + n(n-1) - \frac{(n-2+1)(0+n-2)}{2}$

   $\qquad\qquad = 1 + n(n-1) - \frac{(n-1)(n-2)}{2}$

   $\qquad\qquad = 1 + \frac{(n-1)(n+2)}{2}$

3. $T(n) = 3T(n-1) + 1 \qquad\qquad T(1) = 4$

   Ta có: $T(n) \quad = 3T(n-1) + 1 = 3[3T(n-2) + 1] + 1$

$$= 3^2 T(n-2) + 3 * 1 + 1$$

$$= 3^2[3T(n-3) + 1] + 3^1 + 3^0$$

$$= 3^3 T(n-3) + 3^2 + 3^1 + 3^0$$

$$\ldots\ldots$$

$$= 3^i T(n-i) + \sum_{k=0}^{i-1} 3^k$$

Quá trình kết thúc khi: $n - i = 1 \iff i = n - 1$

Khi đó: $T(n) = 3^i T(n-i) + \sum_{k=0}^{i-1} 3^k$

$$= 3^{n-1} T(1) + \sum_{k=0}^{n-2} 3^k$$

$$= 3^{n-1} * 4 + \frac{3^{n-2+1} - 1}{3-1}$$

$$= 3^{n-1} * 4 + \frac{3^{n-1} - 1}{2}$$

4. $T(n) = 2T\left(\frac{n}{2}\right) + 1 \qquad\qquad T(1) = 1$

Ta có: $T(n) = 2T\left(\frac{n}{2}\right) + 1 = 2\left[2T\left(\frac{n}{4}\right) + 1\right] + 1$

$$= 4T\left(\frac{n}{4}\right) + 2 + 1 = 4\left[2T\left(\frac{n}{8}\right) + 1\right] + 2 + 1$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 2^2 + 2^1 + 2^0$$

$$\ldots\ldots$$

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = log_2 n$

Khi đó: $T(n) = 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k$

$$= 2^{log_2 n} T\left(\frac{n}{2^{log_2 n}}\right) + \sum_{k=0}^{log_2 n - 1} 2^k$$

$$= nT(1) + 2^{log_2 n - 1 + 1} - 1$$

$$= n * 1 + n - 1 = 2n - 1$$

5. $T(n) = 2T\left(\frac{n}{2}\right) + n \qquad\qquad T(1) = 1$

Ta có: $T(n) = 2T\left(\frac{n}{2}\right) + n = 2\left[2T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$

$$= 4T\left(\frac{n}{4}\right) + 2n = 4\left[2T\left(\frac{n}{8}\right) + \frac{n}{4}\right] + 2n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

$$\ldots..$$

$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = log_2 n$

Khi đó: $T(n) = 2^i T\left(\frac{n}{2^i}\right) + in$

$$= 2^{log_2 n} T\left(\frac{n}{2^{log_2 n}}\right) + n log_2 n$$

$$= nT(1) + n log_2 n = n + log_2 n$$

6. $T(n) = 2T\left(\frac{n}{2}\right) + n^2 \qquad T(1) = 1$

Ta có: $T(n) = 2T\left(\frac{n}{2}\right) + n^2 = 2\left[2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$

$$= 4T\left(\frac{n}{4}\right) + 2 * \frac{n^2}{2^2} + n^2$$

$$= 4\left[2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 + \frac{n^2}{2} = 2^3 T\left(\frac{n}{2^3}\right) + \frac{n^2}{2^2} + \frac{n^2}{2^1} + \frac{n^2}{2^0}$$

$$\ldots..$$

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} \frac{n^2}{2^k}$$

Quá trình dừng khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = log_2 n$

Khi đó: $T(n) = 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} \frac{n^2}{2^k}$

$$= 2^{log_2 n} T(1) + n^2 \sum_{k=0}^{log_2 n - 1} \left(\frac{1}{2}\right)^k$$

$$= n * 1 + n^2 \left(\frac{0.5^{log_2 n - 1 + 1} - 1}{0.5 - 1}\right)$$

$$= n - 2n^2 (2^{-log_2 n} - 1) = n - 2n^2 \left(\frac{1}{n} - 1\right)$$

$$= 2n^2 - n$$

7. $T(n) = 2T\left(\frac{n}{2}\right) + log n \qquad T(1) = 1$

Ta có: $T(n) = 2T\left(\frac{n}{2}\right) + log n = 2\left[2T\left(\frac{n}{4}\right) + log \frac{n}{2}\right] + log n$

$$= 4T\left(\frac{n}{4}\right) + 2log\frac{n}{2} + logn$$

$$= 4\left[2T\left(\frac{n}{8}\right) + log\frac{n}{4}\right] + 2log\frac{n}{2} + logn$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 2^2 log\frac{n}{2^2} + 2^1 log\frac{n}{2^1} + 2^0 log\frac{n}{2^0}$$

....

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k log\frac{n}{2^k}$$

$$= 2^i T\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} 2^k logn - \sum_{k=0}^{i-1} 2^k log2^k$$

$$= 2^i T\left(\frac{n}{2^i}\right) + log(n)\sum_{k=0}^{i-1} 2^k - log(2)\sum_{k=0}^{i-1} k2^k$$

Quá trình kết thúc khi: $\frac{n}{2^i} = 1 \Leftrightarrow 2^i = n \Leftrightarrow i = log_2 n$

Khi đó: T(n) $= 2^i T\left(\frac{n}{2^i}\right) + log(n)\sum_{k=0}^{i-1} 2^k - log(2)\sum_{k=0}^{i-1} k2^k$

$$= 2^{log_2 n} T(1) + log(n)\sum_{k=0}^{log_2 n-1} 2^k - log(2)\sum_{k=0}^{log_2 n-1} k2^k$$

$$= n * 1 + log(n)(2^{log_2 n-1+1} - 1) - log(2)\left(0*2^0 + \sum_{=1}^{log_2 n-1} k2^k\right)$$

$$= n + (n-1)log(n) - (log_2(n) - 1 - 1)(2^{log_2 n-1+1} - 1)log2$$

$$= n + (n-1)logn - (log_2 n - 2)(n-1)log2$$

8. $T(n) = T(n-1) + T(n-2)$ $\qquad$ $T(0) = 1$ $\qquad$ $T(1) = 1$

Ta có: $T(n-1) + T(n-2) \le T(n-1) + T(n-1)$

$\qquad \Leftrightarrow T(n-1) + T(n-2) \le 2T(n-1)$

Ta có: $T(n) = T(n-1) + T(n-2)$

$\qquad = 2T(n-1)$

$\qquad = 2[2T(n-2)] = 4T(n-2)$

$\qquad = 4[2T(n-3)] = 2^3 T(n-3)$

$\qquad$ ... ...

$\qquad = 2^i T(n-i)$

Quá trình kết thúc khi: $n - i = 0$ hoặc $n - i = 1 \Leftrightarrow i = n$ hoặc $i = n - 1$

Khi đó ta chọn i lớn nhất ta được: i=n

Khi đó: $T(n) = 2^i T(n-i) = 2^n T(0) = 2^n$