CHAPTER 4

Integration

OVERVIEW This chapter examines two processes and their relation to one another. One is the process by which we determine functions from their derivatives. The other is the process by which we arrive at exact formulas for such things as volume and area through successive approximations. Both processes are called integration.

Integration and differentiation are intimately connected. The nature of the connection is one of the most important ideas in all mathematics, and its independent discovery by Leibniz and Newton still constitutes one of the greatest technical advances of modern times.

4.1 Indefinite Integrals

One of the early accomplishments of calculus was predicting the future position of a moving body from one of its known locations and a formula for its velocity function. Today we view this as one of a number of occasions on which we determine a function from one of its known values and a formula for its rate of change. It is a routine process today, thanks to calculus, to calculate how fast a space vehicle needs to be going at a certain point to escape the earth's gravitational field or to predict the useful life of a sample of radioactive polonium-210 from its present level of activity and its rate of decay.

The process of determining a function from one of its known values and its derivative $f(x)$ has two steps. The first is to find a formula that gives us all the functions that could possibly have f as a derivative. These functions are the so-called antiderivatives of f, and the formula that gives them all is called the indefinite integral of f. The second step is to use the known function value to select the particular antiderivative we want from the indefinite integral. The first step is the subject of the present section; the second is the subject of the next.

Finding a formula that gives all of a function's antiderivatives might seem like an impossible task, or at least to require a little magic. But this is not the case at all. If we can find even one of a function's antiderivatives we can find them all, because of the first two corollaries of the Mean Value Theorem of Section 3.2.

Finding Antiderivatives—Indefinite Integrals

Definitions

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral and x is the **variable of integration.**

According to Corollary 2 of the Mean Value Theorem (Section 3.2), once we have found one antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way:

$$\boxed{\int f(x)\,dx = F(x) + C. \qquad (1)}$$

The constant C is the **constant of integration** or **arbitrary constant.** Equation (1) is read, "The indefinite integral of f with respect to x is $F(x) + C$." When we find $F(x) + C$, we say that we have **integrated** f and **evaluated** the integral.

EXAMPLE 1 Evaluate $\int 2x\,dx$.

Solution

$$\int 2x\,dx = \overbrace{x^2}^{\text{an antiderivative of } 2x} + \underbrace{C}_{\text{the arbitrary constant}}$$

The formula $x^2 + C$ generates all the antiderivatives of the function $2x$. The functions $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function $2x$, as you can check by differentiation. ❑

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas. You will see what we mean if you look at Table 4.1, which lists a number of standard integral forms side by side with their derivative-formula sources.

In case you are wondering why the integrals of the tangent, cotangent, secant, and cosecant do not appear in the table, the answer is that the usual formulas for them require logarithms. In Section 4.7, we will see that these functions do have antiderivatives, but we will have to wait until Chapters 6 and 7 to see what they are.

Table 4.1 Integral formulas

Indefinite integral	Reversed derivative formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\quad n \neq -1,\ n$ rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1\, dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin kx\, dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx\, dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x\, dx = \tan x + C$	$\frac{d}{dx} \tan x = \sec^2 x$
5. $\int \csc^2 x\, dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6. $\int \sec x \tan x\, dx = \sec x + C$	$\frac{d}{dx} \sec x = \sec x \tan x$
7. $\int \csc x \cot x\, dx = -\csc x + C$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

EXAMPLE 2 *Selected integrals from Table 4.1*

a) $\int x^5\, dx = \frac{x^6}{6} + C$ — Formula 1 with $n = 5$

b) $\int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx = 2x^{1/2} + C = 2\sqrt{x} + C$ — Formula 1 with $n = -1/2$

c) $\int \sin 2x\, dx = -\frac{\cos 2x}{2} + C$ — Formula 2 with $k = 2$

d) $\int \cos \frac{x}{2}\, dx = \int \cos \frac{1}{2}x\, dx = \frac{\sin (1/2)x}{1/2} + C = 2 \sin \frac{x}{2} + C$ — Formula 3 with $k = 1/2$

❑

Finding an integral formula can sometimes be difficult, but checking it, once found, is relatively easy: differentiate the right-hand side. The derivative should be the integrand.

EXAMPLE 3

Right: $\int x \cos x\, dx = x \sin x + \cos x + C$

Reason: The derivative of the right-hand side is the integrand:

$$\frac{d}{dx}(x \sin x + \cos x + C) = x \cos x + \sin x - \sin x + 0 = x \cos x.$$

Wrong: $\displaystyle\int x \cos x \, dx = x \sin x + C$

Reason: The derivative of the right-hand side is not the integrand:

$$\frac{d}{dx}(x \sin x + C) = x \cos x + \sin x + 0 \neq x \cos x.$$

❑

Do not worry about how to derive the correct integral formula in Example 3. We will present a technique for doing so in Chapter 7.

Rules of Algebra for Antiderivatives

Among the things we know about antiderivatives are these:

1. A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.
2. In particular, a function is an antiderivative of $-f$ if and only if it is the negative of an antiderivative of f.
3. A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f and an antiderivative of g.

When we express these observations in integral notation, we get the standard arithmetic rules for indefinite integration (Table 4.2).

Table 4.2 Rules for indefinite integration

1. *Constant Multiple Rule:* $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$

(Does not work if k varies with x.)

2. *Rule for Negatives:* $\displaystyle\int -f(x)\,dx = -\int f(x)\,dx$

(Rule 1 with $k = -1$)

3. *Sum and Difference Rule:* $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

EXAMPLE 4 *Rewriting the constant of integration*

$\displaystyle\int 5 \sec x \tan x \, dx = 5\int \sec x \tan x \, dx$	Table 4.2, Rule 1
$= 5(\sec x + C)$	Table 4.1, Formula 6
$= 5 \sec x + 5C$	First form
$= 5 \sec x + C'$	Shorter form, where C' is $5C$
$= 5 \sec x + C$	Usual form—no prime. Since 5 times an arbitrary constant is an arbitrary constant, we rename C'.

❑

What about all the different forms in Example 4? Each one gives all the antiderivatives of $f(x) = 5 \sec x \tan x$, so each answer is correct. But the least complicated of the three, and the usual choice, is

$$\int 5 \sec x \tan x \, dx = 5 \sec x + C.$$

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions term by term, the Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

EXAMPLE 5 *Term-by-term integration*

Evaluate

$$\int (x^2 - 2x + 5)\, dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term by term with the Sum and Difference Rule:

$$\begin{aligned} \int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3. \end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, C_2, and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\begin{aligned} \int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} - x^2 + 5x + C. \end{aligned}$$

Find the simplest antiderivative you can for each part and add the constant at the end. ❑

The Integrals of $\sin^2 x$ and $\cos^2 x$

We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

EXAMPLE 6

a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$ As in part (a), but with a sign change

Exercises 4.1

Finding Antiderivatives

In Exercises 1–18, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a) $2x$ b) x^2 c) $x^2 - 2x + 1$
2. a) $6x$ b) x^7 c) $x^7 - 6x + 8$
3. a) $-3x^{-4}$ b) x^{-4} c) $x^{-4} + 2x + 3$
4. a) $2x^{-3}$ b) $\frac{x^{-3}}{2} + x^2$ c) $-x^{-3} + x - 1$
5. a) $\frac{1}{x^2}$ b) $\frac{5}{x^2}$ c) $2 - \frac{5}{x^2}$
6. a) $-\frac{2}{x^3}$ b) $\frac{1}{2x^3}$ c) $x^3 - \frac{1}{x^3}$
7. a) $\frac{3}{2}\sqrt{x}$ b) $\frac{1}{2\sqrt{x}}$ c) $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a) $\frac{4}{3}\sqrt[3]{x}$ b) $\frac{1}{3\sqrt[3]{x}}$ c) $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. a) $\frac{2}{3}x^{-1/3}$ b) $\frac{1}{3}x^{-2/3}$ c) $-\frac{1}{3}x^{-4/3}$
10. a) $\frac{1}{2}x^{-1/2}$ b) $-\frac{1}{2}x^{-3/2}$ c) $-\frac{3}{2}x^{-5/2}$
11. a) $-\pi \sin \pi x$ b) $3 \sin x$ c) $\sin \pi x - 3 \sin 3x$
12. a) $\pi \cos \pi x$ b) $\frac{\pi}{2}\cos\frac{\pi x}{2}$ c) $\cos\frac{\pi x}{2} + \pi \cos x$
13. a) $\sec^2 x$ b) $\frac{2}{3}\sec^2\frac{x}{3}$ c) $-\sec^2\frac{3x}{2}$
14. a) $\csc^2 x$ b) $-\frac{3}{2}\csc^2\frac{3x}{2}$ c) $1 - 8\csc^2 2x$
15. a) $\csc x \cot x$ b) $-\csc 5x \cot 5x$ c) $-\pi \csc\frac{\pi x}{2}\cot\frac{\pi x}{2}$
16. a) $\sec x \tan x$ b) $4 \sec 3x \tan 3x$ c) $\sec\frac{\pi x}{2}\tan\frac{\pi x}{2}$
17. $(\sin x - \cos x)^2$
18. $(1 + 2\cos x)^2$

Evaluating Integrals

Evaluate the integrals in Exercises 19–58. Check your answers by differentiation.

19. $\int (x+1)\,dx$
20. $\int (5-6x)\,dx$
21. $\int \left(3t^2 + \frac{t}{2}\right) dt$
22. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
23. $\int (2x^3 - 5x + 7)\,dx$
24. $\int (1 - x^2 - 3x^5)\,dx$
25. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
26. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
27. $\int x^{-1/3}\,dx$
28. $\int x^{-5/4}\,dx$
29. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
30. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
31. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
32. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

33. $\int 2x\left(1-x^{-3}\right)\,dx$

34. $\int x^{-3}(x+1)\,dx$

35. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt$

36. $\int \frac{4+\sqrt{t}}{t^3}\,dt$

37. $\int (-2\cos t)\,dt$

38. $\int (-5\sin t)\,dt$

39. $\int 7\sin\frac{\theta}{3}\,d\theta$

40. $\int 3\cos 5\theta\,d\theta$

41. $\int (-3\csc^2 x)\,dx$

42. $\int \left(-\frac{\sec^2 x}{3}\right)\,dx$

43. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

44. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

45. $\int (4\sec x\tan x - 2\sec^2 x)\,dx$

46. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx$

47. $\int (\sin 2x - \csc^2 x)\,dx$

48. $\int (2\cos 2x - 3\sin 3x)\,dx$

49. $\int 4\sin^2 y\,dy$

50. $\int \frac{\cos^2 y}{7}\,dy$

51. $\int \frac{1+\cos 4t}{2}\,dt$

52. $\int \frac{1-\cos 6t}{2}\,dt$

53. $\int (1+\tan^2\theta)\,d\theta$
(*Hint:* $1+\tan^2\theta = \sec^2\theta$)

54. $\int (2+\tan^2\theta)\,d\theta$

55. $\int \cot^2 x\,dx$
(*Hint:* $1+\cot^2 x = \csc^2 x$)

56. $\int (1-\cot^2 x)\,dx$

57. $\int \cos\theta\,(\tan\theta + \sec\theta)\,d\theta$

58. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

Checking Integration Formulas

Verify the integral formulas in Exercises 59–64 by differentiation. In Section 4.3, we will see where formulas like these come from.

59. $\int (7x-2)^3\,dx = \frac{(7x-2)^4}{28} + C$

60. $\int (3x+5)^{-2}\,dx = -\frac{(3x+5)^{-1}}{3} + C$

61. $\int \sec^2(5x-1)\,dx = \frac{1}{5}\tan(5x-1) + C$

62. $\int \csc^2\left(\frac{x-1}{3}\right)\,dx = -3\cot\left(\frac{x-1}{3}\right) + C$

63. $\int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$

64. $\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$

65. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int x\sin x\,dx = \frac{x^2}{2}\sin x + C$

b) $\int x\sin x\,dx = -x\cos x + C$

c) $\int x\sin x\,dx = -x\cos x + \sin x + C$

66. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \tan\theta\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C$

b) $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\tan^2\theta + C$

c) $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\sec^2\theta + C$

67. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int (2x+1)^2\,dx = \frac{(2x+1)^3}{3} + C$

b) $\int 3(2x+1)^2\,dx = (2x+1)^3 + C$

c) $\int 6(2x+1)^2\,dx = (2x+1)^3 + C$

68. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \sqrt{2x+1}\,dx = \sqrt{x^2+x+C}$

b) $\int \sqrt{2x+1}\,dx = \sqrt{x^2+x} + C$

c) $\int \sqrt{2x+1}\,dx = \frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C$

Theory and Examples

69. Suppose that

$$f(x) = \frac{d}{dx}(1-\sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x+2).$$

Find:

a) $\int f(x)\,dx$

b) $\int g(x)\,dx$

c) $\int [-f(x)]\,dx$

d) $\int [-g(x)]\,dx$

e) $\int [f(x)+g(x)]\,dx$

f) $\int [f(x)-g(x)]\,dx$

g) $\int [x + f(x)]\,dx$

h) $\int [g(x) - 4]\,dx$

70. Repeat Exercise 69, assuming that

$$f(x) = \frac{d}{dx}e^x \quad \text{and} \quad g(x) = \frac{d}{dx}(x \sin x).$$

4.2

Differential Equations, Initial Value Problems, and Mathematical Modeling

This section shows how to use a known value of a function to select a particular antiderivative from the functions in an indefinite integral. The ability to do this is important in mathematical modeling, the process by which we, as scientists, use mathematics to learn about reality.

Initial Value Problems

An equation like

$$\frac{dy}{dx} = f(x)$$

that has a derivative in it is called a **differential equation.** The problem of finding a function y of x when we know its derivative and its value y_0 at a particular point x_0 is called an **initial value problem.** We solve such a problem in two steps, as demonstrated in Example 1.

EXAMPLE 1 *Finding a body's velocity from its acceleration and initial velocity*

The acceleration of gravity near the surface of the earth is 9.8 m/sec^2. This means that the velocity v of a body falling freely in a vacuum changes at the rate of

$$\frac{dv}{dt} = 9.8 \text{ m/sec}^2.$$

If the body is dropped from rest, what will its velocity be t seconds after it is released?

Solution In mathematical terms, we want to solve the initial value problem that consists of

The differential equation: $\quad \dfrac{dv}{dt} = 9.8$

The initial condition: $\quad v = 0$ when $t = 0$ (abbreviated as $v(0) = 0$)

We first solve the differential equation by integrating both sides with respect to t:

$$\frac{dv}{dt} = 9.8 \qquad \text{The differential equation}$$

$$\int \frac{dv}{dt}\,dt = \int 9.8\,dt \qquad \text{Integrate with respect to } t.$$

$$v + C_1 = 9.8t + C_2 \qquad \text{Integrals evaluated}$$

$$v = 9.8t + C. \qquad \text{Constants combined as one}$$

This last equation tells us that the body's velocity t seconds into the fall is $9.8t + C$ m/sec for some value of C. What value? We find out from the initial condition:

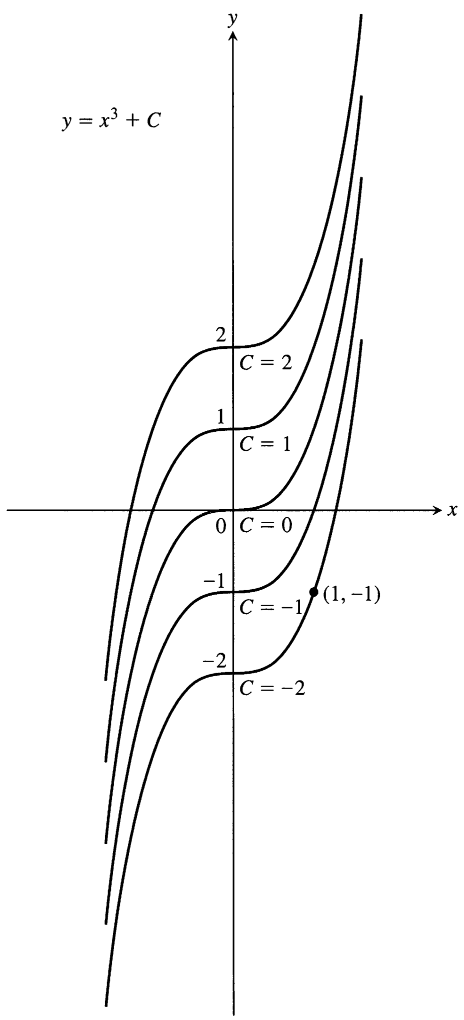
$$\begin{aligned} v &= 9.8t + C \\ 0 &= 9.8(0) + C \qquad v(0) = 0 \\ C &= 0. \end{aligned}$$

Conclusion: The body's velocity t seconds into the fall is

$$v = 9.8t + 0 = 9.8t \text{ m/sec.}$$

❑

The indefinite integral $F(x) + C$ of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$ (y has the value y_0 when $x = x_0$).



4.1 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2 we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

EXAMPLE 2 *Finding a curve from its slope function and a point*

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

Solution In mathematical language, we are asked to solve the initial value problem that consists of

The differential equation: $\dfrac{dy}{dx} = 3x^2$ The curve's slope is $3x^2$.

The initial condition: $y(1) = -1$.

To solve it we first solve the differential equation:

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \\ \int \frac{dy}{dx}\,dx &= \int 3x^2\,dx \\ y &= x^3 + C. \end{aligned}$$

Constants of integration combined, giving the general solution

This tells us that y equals $x^3 + C$ for some value of C. We find that value from the condition $y(1) = -1$:

$$\begin{aligned} y &= x^3 + C \\ -1 &= (1)^3 + C \\ C &= -2. \end{aligned}$$

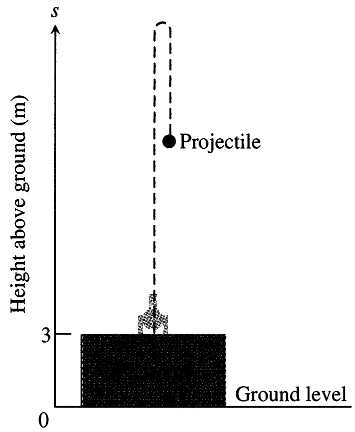
❑

The curve we want is $y = x^3 - 2$ (Fig. 4.1).

In the next example, we have to integrate a second derivative twice to find the function we are looking for. The first integration,

$$\int \frac{d^2s}{dt^2}dt = \frac{ds}{dt} + C,$$

gives the function's first derivative. The second integration gives the function.



4.2 The sketch for modeling the projectile motion in Example 3.

EXAMPLE 3 *Finding a projectile's height from its acceleration, initial velocity, and initial position*

A heavy projectile is fired straight up from a platform 3 m above the ground, with an initial velocity of 160 m/sec. Assume that the only force affecting the projectile during its flight is from gravity, which produces a downward acceleration of 9.8 m/sec^2. Find an equation for the projectile's height above the ground as a function of time t if $t = 0$ when the projectile is fired. How high above the ground is the projectile 3 sec after firing?

Solution To model the problem, we draw a figure (Fig. 4.2) and let s denote the projectile's height above the ground at time t. We assume s to be a twice-differentiable function of t and represent the projectile's velocity and acceleration with the derivatives

$$v = \frac{ds}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Since gravity acts in the direction of *decreasing* s in our model, the initial value problem to solve is the following:

$$\text{The differential equation:} \qquad \frac{d^2s}{dt^2} = -9.8$$

$$\text{The initial conditions:} \qquad \frac{ds}{dt}(0) = 160 \qquad \text{and} \qquad s(0) = 3.$$

We integrate the differential equation with respect to t to find ds/dt:

$$\int \frac{d^2s}{dt^2}\,dt = \int (-9.8)\,dt$$

$$\frac{ds}{dt} = -9.8t + C_1.$$

We apply the first initial condition to find C_1:

$$160 = -9.8(0) + C_1 \qquad \frac{ds}{dt}(0) = 160$$

$$C_1 = 160.$$

This completes the formula for ds/dt:

$$\frac{ds}{dt} = -9.8t + 160.$$

We integrate ds/dt with respect to t to find s:

$$\int \frac{ds}{dt}\,dt = \int (-9.8t + 160)\,dt$$

$$s = -4.9t^2 + 160t + C_2.$$

We apply the second initial condition to find C_2:

$$3 = -4.9(0)^2 + 160(0) + C_2 \qquad s(0) = 3$$

$$C_2 = 3.$$

This completes the formula for s as a function of t:

$$s = -4.9t^2 + 160t + 3.$$

To find the projectile's height 3 sec into the flight, we set $t = 3$ in the formula for s. The height is

$$s = -4.9(3)^2 + 160(3) + 3 = 438.9 \text{ m}. \qquad \square$$

When we find a function from its first derivative, we have one arbitrary constant, as in Examples 1 and 2. When we find a function from its second derivative, we have to deal with two constants, one from each antidifferentiation, as in Example 3. To find a function from its third derivative would require us to find the values of three constants, and so on. In each case, the values of the constants are determined by the problem's initial conditions. Each time we find an antiderivative, we need an initial condition to tell us the value of C.

Sketching Solution Curves

The graph of a solution of a differential equation is called a **solution curve (integral curve).** The curves $y = x^3 + C$ in Fig. 4.1 are solution curves of the differential equation $dy/dx = 3x^2$. When we cannot find explicit formulas for the solution curves of an equation $dy/dx = f(x)$ (that is, we cannot find an antiderivative of f), we may still be able to find their general shape by examining derivatives.

EXAMPLE 4 Sketch the solutions of the differential equation

$$y' = \frac{1}{x^2+1}.$$

Solution

Step 1: y' and y''. As in Section 3.4, the curve's general shape is determined by y' and y''. We already know y':

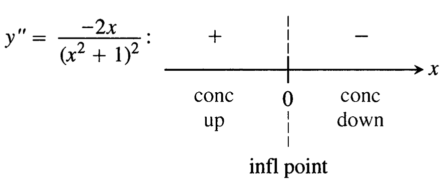
$$y' = \frac{1}{x^2+1}.$$

We find y'' by differentiation, in the usual way:

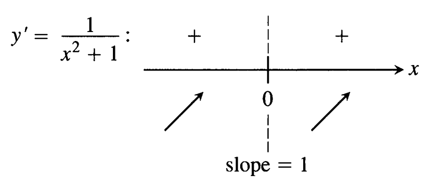
$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x^2+1}\right)$$

$$= \frac{-2x}{(x^2+1)^2}.$$

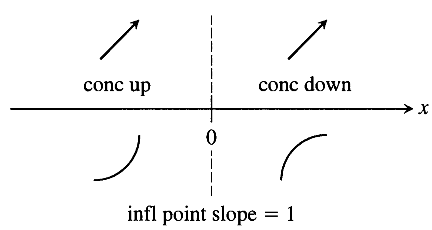
Step 3: *Concavity.* The second derivative changes from $(+)$ to $(-)$ at $x = 0$, so the curves all have an inflection point at $x = 0$.



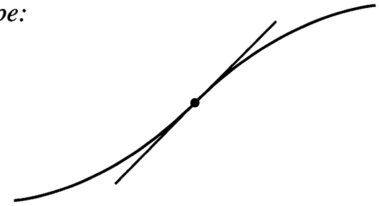
Step 2: *Rise and fall.* The domain of y' is $(-\infty, \infty)$. There are no critical points, so the solution curves have no cusps or extrema. The curves rise from left to right because $y' > 0$. At $x = 0$, the curves have slope 1.



Step 4: *Summary:*



General shape:

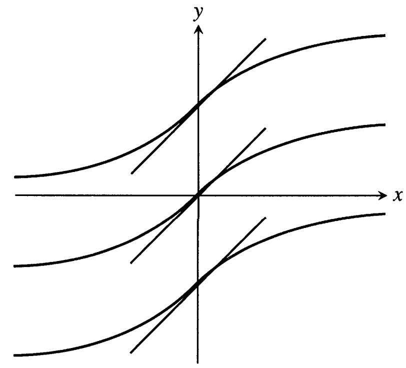


The first derivative tells us still more:

$$\lim_{x\to\pm\infty} y' = \lim_{x\to\pm\infty} \frac{1}{x^2+1} = 0,$$

so the curves level off as $x \to \pm\infty$.

***Step 5:** Specific points and solution curves.* We plot an assortment of points on the y-axis where we know the curves' slope (it is 1 at $x = 0$), mark tangents with that slope for guidance, and sketch "parallel" curves of the right general shape.



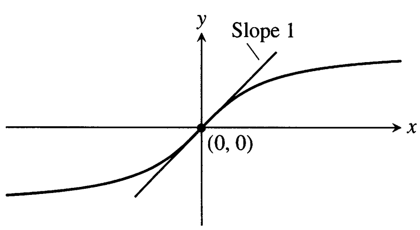
❑

EXAMPLE 5 Sketch the solution of the initial value problem

Differential equation: $y' = \dfrac{1}{x^2+1}$

Initial condition: $y = 0$ when $x = 0$.

Solution We find the solution's general shape (Example 4) and sketch the solution curve that passes through the point (0, 0) (Fig. 4.3). ❑



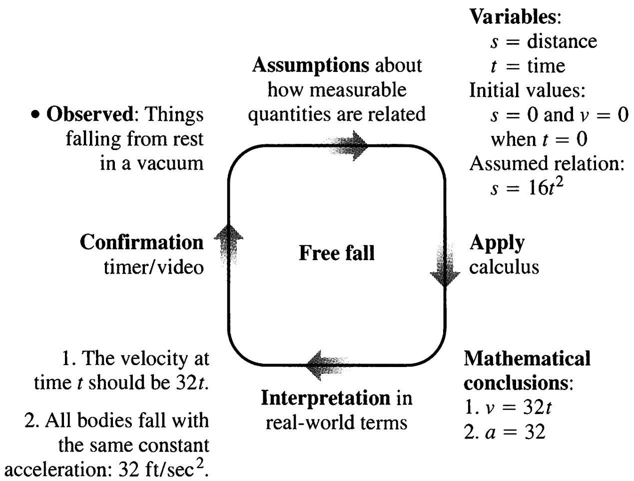
4.3 The solution curve in Example 5.

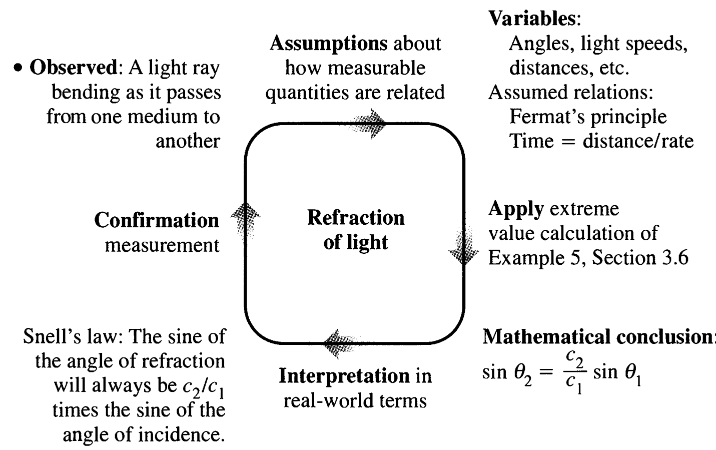
The technique we have learned for sketching solutions is particularly helpful when we are faced with an equation $dy/dx = f(x)$ that involves a function whose antiderivatives have no elementary formula. The antiderivatives of the function $f(x) = 1/(x^2+1)$ in Example 4 do have an elementary formula, as we will see in Chapter 6, but the antiderivatives of $g(x) = \sqrt{1+x^4}$ do not. To solve the equation $dy/dx = \sqrt{1+x^4}$, we must proceed either graphically or numerically.

Mathematical Modeling

The development of a mathematical model usually takes four steps: First we observe something in the real world (a ball bearing falling from rest or the trachea contracting during a cough, for example) and construct a system of mathematical variables and relationships that imitate some of its important features. We build a mathematical metaphor for what we see. Next we apply (usually) existing mathematics to the variables and relationships in the model to draw conclusions about them. After that we translate the mathematical conclusions into information about the system under study. Finally we check the information against observation to see if the model has predictive value. We also investigate the possibility that the model applies to other systems. The really good models are the ones that lead to conclusions that are consistent with observation, that have predictive value and broad application, and that are not too hard to use.

The natural cycle of mathematical imitation, deduction, interpretation, and confirmation is shown in the diagrams on the following page.



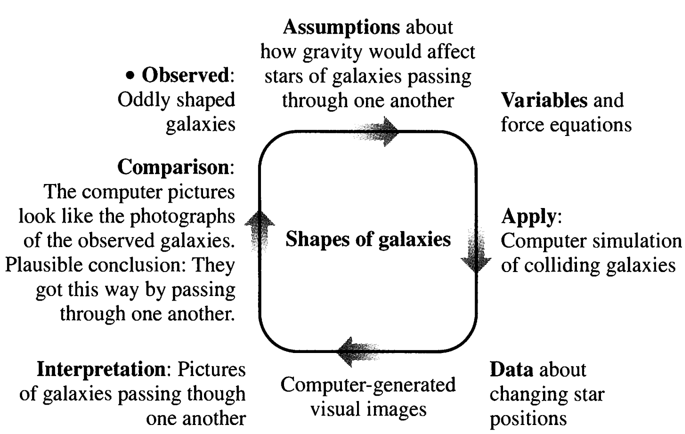


Computer Simulation

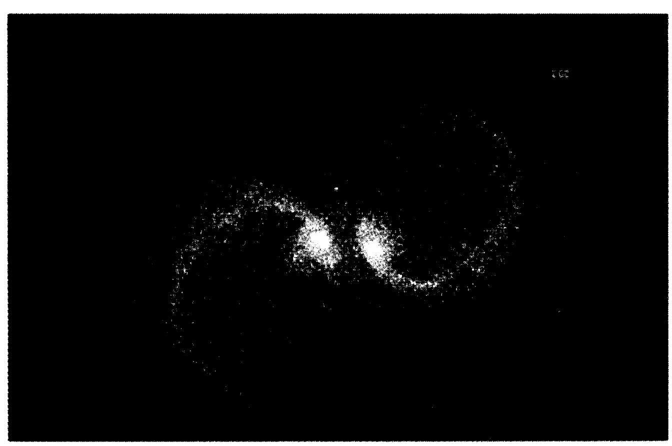
When a system we want to study is complicated, we can sometimes experiment first to see how the system behaves under different circumstances. But if this is not possible (the experiments might be expensive, time-consuming, or dangerous), we might run a series of simulated experiments on a computer—experiments that behave like the real thing, without the disadvantages. Thus we might model the effects of atomic war, the effect of waiting a year longer to harvest trees, the effect of crossing particular breeds of cattle, or the effect of reducing atmospheric ozone by 1%, all without having to pay the consequences or wait to see how things work out naturally.

We also bring computers in when the model we want to use has too many calculations to be practical any other way. NASA's space flight models are run on computers—they have to be to generate course corrections on time. If you want to model the behavior of galaxies that contain billions and billions of stars, a computer offers the only possible way. One of the most spectacular computer simulations in recent years, carried out by Alar Toomre at MIT, explained a peculiar galactic shape that was not consistent with our previous ideas about how galaxies are formed. The galaxies had acquired their odd shapes, Toomre concluded, by passing through one another (Fig. 4.4).

4.4 (a) The modeling cycle for the shapes of colliding galaxies. (b) The computer's image of how galaxies are reshaped by the collision.



(a)



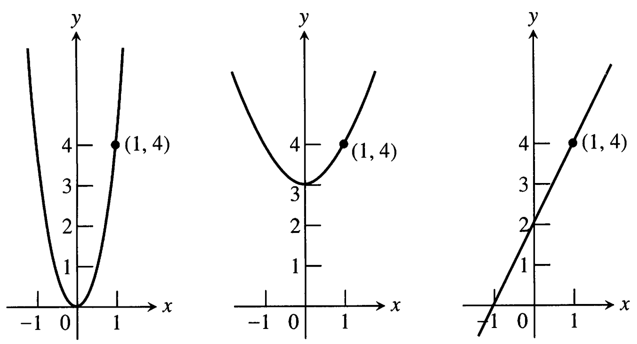
(b)

Exercises 4.2

Initial Value Problems

1. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

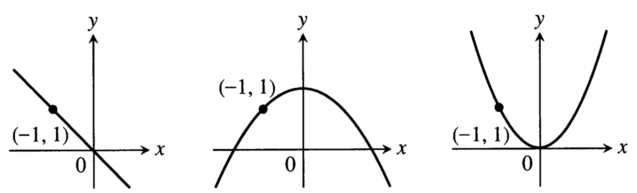


(a) (b) (c)

Give reasons for your answer.

2. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 3–22.

3. $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$

4. $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

5. $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

6. $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

7. $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

8. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

9. $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

10. $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

11. $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

12. $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

13. $\dfrac{dv}{dt} = \dfrac{1}{2} \sec t \tan t, \quad v(0) = 1$

14. $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

15. $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

16. $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

17. $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

18. $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

19. $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

20. $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

21. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

22. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

Finding Position from Velocity

Exercises 23–26 give the velocity $v = ds/dt$ and initial position of a body moving along a coordinate line. Find the body's position at time t.

23. $v = 9.8t + 5, \quad s(0) = 10$

24. $v = 32t - 2, \quad s(1/2) = 4$

25. $v = \sin \pi t, \quad s(0) = 0$

26. $v = \dfrac{2}{\pi} \cos \dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Finding Position from Acceleration

Exercises 27–30 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of a body moving on a coordinate line. Find the body's position at time t.

27. $a = 32; \quad v(0) = 20, \quad s(0) = 5$

28. $a = 9.8;\quad v(0) = -3,\quad s(0) = 0$

29. $a = -4\sin 2t;\quad v(0) = 2,\quad s(0) = -3$

30. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi};\quad v(0) = 0,\quad s(0) = -1$

Finding Curves

31. Find the curve $y = f(x)$ in the xy-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

32. **a)** Find a curve $y = f(x)$ with the following properties:

   i) $\dfrac{d^2y}{dx^2} = 6x$

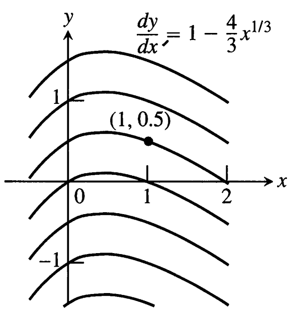
   ii) Its graph passes through the point (0, 1) and has a horizontal tangent there.

   b) How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 33–36 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

33.



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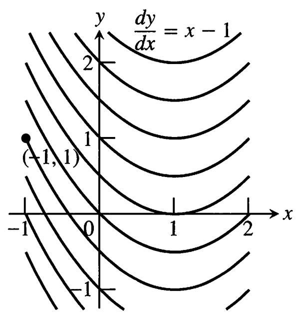

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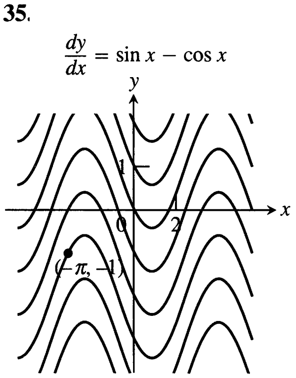

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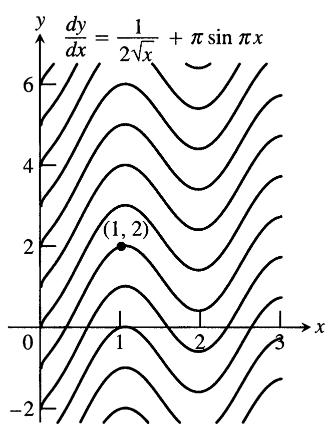


Use the technique described in Example 4 to sketch some of the solutions of the differential equations in Exercises 37–40. Then solve the equations to check on how well you did.

37. $\dfrac{dy}{dx} = 2x$

38. $\dfrac{dy}{dx} = -2x + 2$

39. $\dfrac{dy}{dx} = 1 - 3x^2$

40. $\dfrac{dy}{dx} = x^2$

Use the technique described in Examples 4 and 5 to sketch the solutions of the initial value problems in Exercises 41–44.

41. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}},\quad -1 < x < 1;\quad y(0) = 0$

42. $\dfrac{dy}{dx} = \sqrt{1+x^4},\quad y(0) = 1$

43. $\dfrac{dy}{dx} = \dfrac{1}{x^2+1} - 1,\quad y(0) = 1$

44. $\dfrac{dy}{dx} = \dfrac{x}{x^2+1},\quad y(0) = 0$

Applications

45. On the moon the acceleration of gravity is 1.6 m/sec^2. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

46. A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

47. With approximately what velocity do you enter the water if you dive from a 10-m platform? (Use $g = 9.8$ m/sec^2.)

48. CALCULATOR The acceleration of gravity near the surface of Mars is 3.72 m/sec^2. If a rock is blasted straight up from the surface with an initial velocity of 93 m/sec (about 208 mph), how high does it go? (*Hint:* When is the velocity zero?)

49. *Stopping a car in time.* You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

   Step 1: Solve the initial value problem

   Differential equation: $\dfrac{d^2s}{dt^2} = -k \qquad (k \text{ constant})$

   Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

   Measuring time and distance from when the brakes are applied

   Step 2: Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

   Step 3: Find the value of k that makes $s = 242$ for the value of t you found in step 2.

50. *Stopping a motorcycle.* The State of Illinois Cycle Rider Safety Program requires riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

51. *Motion along a coordinate line.* A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t})$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

   a) the velocity $v = ds/dt$ in terms of t,

   b) the position s in terms of t.

52. *The hammer and the feather.* When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\dfrac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$

Initial conditions: $\dfrac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

53. *Motion with constant acceleration.* The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \qquad (1)$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = a$

Initial conditions: $\dfrac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$

54. (*Continuation of Exercise 53.*) *Free fall near the surface of a planet.* For free fall near the surface of a planet where the acceleration of gravity has a constant magnitude of g length-units/sec^2, Eq. (1) takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad (2)$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$, and negative if the object is falling.

Instead of using the result of Exercise 53, you can derive Eq. (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

Theory and Examples

55. *Finding displacement from an antiderivative of velocity*

a) Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

1) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

2) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

3) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b) Suppose the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

56. *Uniqueness of solutions.* If differentiable functions $y = F(x)$ and $y = G(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

4.3 Integration by Substitution—Running the Chain Rule Backward

A change of variable can often turn an unfamiliar integral into one we can evaluate. The method for doing this is called the substitution method of integration. It is one of the principal methods for evaluating integrals. This section shows how and why the method works.

The Generalized Power Rule in Integral Form

When u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

This same equation, from another point of view, says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n du,$$

obtained by treating the dx's as differentials that cancel. Combining the last two equations gives the following rule.

Equation (1) actually holds for any real exponent $n \neq -1$, as we will see in Chapter 6.

> If u is any differentiable function,
>
> $$\int u^n du = \frac{u^{n+1}}{n+1} + C. \qquad (n \neq -1,\ n \text{ rational}) \qquad (1)$$

In deriving Eq. (1) we assumed u to be a differentiable function of the variable x, but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with θ, t, y, or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n du, \qquad (n \neq -1)$$

with u a differentiable function and du its differential, we can evaluate the integral as $\left[u^{n+1}/(n+1)\right] + C$.

EXAMPLE 1 Evaluate $\int (x+2)^5\, dx$.

Solution We can put the integral in the form

$$\int u^n du$$

by substituting

$$u = x + 2, \qquad du = d(x+2) = \frac{d}{dx}(x+2) \cdot dx$$
$$= 1 \cdot dx = dx.$$

Then

$$\int (x+2)^5 dx = \int u^5 du \qquad u = x+2, \quad du = dx$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate, using Eq. (1) with } n = 5.$$

$$= \frac{(x+2)^6}{6} + C. \qquad \text{Replace } u \text{ by } x+2.$$

❑

EXAMPLE 2

$$\int \sqrt{1+y^2} \cdot 2y\,dy = \int u^{1/2}\,du \qquad \text{Let } u = 1+y^2,\ du = 2y\,dy.$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \qquad \text{Replace } u \text{ by } 1+y^2.$$

EXAMPLE 3 *Adjusting the integrand by a constant*

$$\int \sqrt{4t-1}\,dt = \int u^{1/2} \cdot \frac{1}{4}du \qquad \text{Let } u = 4t-1,\ du = 4\,dt,\ (1/4)\,du = dt.$$

$$= \frac{1}{4}\int u^{1/2}du \qquad \text{With the } 1/4 \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

Trigonometric Functions

If u is a differentiable function of x, then $\sin u$ is a differentiable function of x. The Chain Rule gives the derivative of $\sin u$ as

$$\frac{d}{dx}\sin u = \cos u\,\frac{du}{dx}.$$

From another point of view, however, this same equation says that $\sin u$ is one of the antiderivatives of the product $\cos u \cdot (du/dx)$. Therefore,

$$\int \left(\cos u\,\frac{du}{dx}\right) dx = \sin u + C.$$

A formal cancellation of the dx's in the integral on the left leads to the following rule.

> If u is a differentiable function, then
>
> $$\int \cos u\,du = \sin u + C. \qquad (2)$$

Equation (2) says that whenever we can cast an integral in the form

$$\int \cos u \, du,$$

we can integrate with respect to u to evaluate the integral as $\sin u + C$.

EXAMPLE 4

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7} du \qquad \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta.$$

$$= \frac{1}{7}\int \cos u \, du \qquad \text{With } (1/7) \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

❑

The companion formula for the integral of $\sin u$ when u is a differentiable function is

$$\int \sin u \, du = -\cos u + C. \tag{3}$$

EXAMPLE 5

$$\int x^2 \sin(x^3)\, dx = \int \sin(x^3) \cdot x^2 dx$$

$$= \int \sin u \cdot \frac{1}{3} du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)du = x^2\,dx.$$

$$= \frac{1}{3}\int \sin u \, du$$

$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

❑

The Chain Rule formulas for the derivatives of the tangent, cotangent, secant, and cosecant of a differentiable function u lead to the following integrals.

$$\int \sec^2 u \, du = \tan u + C \tag{4}$$

$$\int \csc^2 u \, du = -\cot u + C \tag{5}$$

$$\int \sec u \tan u \, du = \sec u + C \tag{6}$$

$$\int \csc u \cot u \, du = -\csc u + C \tag{7}$$

In each formula, u is a differentiable function of a real variable. Each formula can be checked by differentiating the right-hand side with respect to that variable. In each case, the Chain Rule applies to produce the integrand on the left.

EXAMPLE 6

$$\begin{aligned}
\int \frac{1}{\cos^2 2\theta}\,d\theta &= \int \sec^2 2\theta\,d\theta && \sec 2\theta = \frac{1}{\cos 2\theta} \\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && \text{Let } u = 2\theta,\ du = 2\,d\theta,\ d\theta = (1/2)\,du. \\
&= \frac{1}{2}\int \sec^2 u\,du \\
&= \frac{1}{2}\tan u + C && \text{Integrate, using Eq. (4).} \\
&= \frac{1}{2}\tan 2\theta + C && \text{Replace } u \text{ by } 2\theta.
\end{aligned}$$

Check:

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{1}{2}\tan 2\theta + C\right) &= \frac{1}{2}\cdot\frac{d}{d\theta}(\tan 2\theta) + 0 \\
&= \frac{1}{2}\cdot\left(\sec^2 2\theta \cdot \frac{d}{d\theta}(2\theta)\right) && \text{Chain Rule} \\
&= \frac{1}{2}\cdot \sec^2 2\theta \cdot 2 = \frac{1}{\cos^2 2\theta}.
\end{aligned}$$

❑

The Substitution Method of Integration

The substitutions in the preceding examples are all instances of the following general rule.

$$\begin{aligned}
\int f(g(x))\cdot g'(x)\,dx &= \int f(u)\,du && \textbf{1. } \text{Substitute } u = g(x),\ du = g'(x)\,dx. \\
&= F(u) + C && \textbf{2. } \text{Evaluate by finding an antiderivative } F(u) \text{ of } f(u). \text{ (Any one will do.)} \\
&= F(g(x)) + C && \textbf{3. } \text{Replace } u \text{ by } g(x).
\end{aligned}$$

These three steps are the steps of the substitution method of integration. The method works because $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\begin{aligned}
\frac{d}{dx}F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\
&= f(g(x)) \cdot g'(x) && \text{Because } F' = f
\end{aligned}$$

Implicit in the substitution method is the assumption that we are replacing x by a function of u. Thus, the substitution $u = g(x)$ must be solvable for x to give x as a function $x = g^{-1}(u)$ ("g inverse of u"). The domains of u and x may need to be restricted on occasion to make this possible. You need not be concerned with this issue at the moment. We will discuss inverses in Section 6.1 and treat the theory of substitutions in greater detail in Sections 7.4 and 13.7.

The Substitution Method of Integration

Take these steps to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

Step 1: Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

Step 2: Integrate with respect to u.
Step 3: Replace u by $g(x)$ in the result.

EXAMPLE 7

$$\int (x^2+2x-3)^2(x+1)\,dx = \int u^2 \cdot \frac{1}{2}\,du \qquad \text{Let } u = x^2+2x-3,\ du = 2x\,dx + 2\,dx = 2(x+1)\,dx,\ (1/2)\,du = (x+1)\,dx.$$

$$= \frac{1}{2}\int u^2\,du$$

$$= \frac{1}{2}\cdot\frac{u^3}{3} + C = \frac{1}{6}u^3 + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{6}(x^2+2x-3)^3 + C \qquad \text{Replace } u.$$

❑

EXAMPLE 8

$$\int \sin^4 t \cos t\,dt = \int u^4\,du \qquad \text{Let } u = \sin t,\ du = \cos t\,dt.$$

$$= \frac{u^5}{5} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{\sin^5 t}{5} + C \qquad \text{Replace } u.$$

❑

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we can try to simplify the integrand further with an additional substitution or two. (You will see what we mean if you do Exercises 47 and 48.) Alternatively, we can start afresh. There can be more than one good way to start, as in the next example.

EXAMPLE 9 Evaluate

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$$

Solution We can use the substitution method of integration as an exploratory tool: substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2+1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1 Substitute $u = z^2+1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2+1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \textstyle\int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2+1.$$

Solution 2 Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2+1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

❑

Exercises 4.3

Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.

1. $\int \sin 3x\,dx, \quad u = 3x$

2. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$

3. $\int \sec 2t\tan 2t\,dt, \quad u = 2t$

4. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1-\cos\frac{t}{2}$

5. $\int 28(7x-2)^{-5}\,dx, \quad u = 7x-2$

6. $\int x^3(x^4-1)^2\,dx, \quad u = x^4-1$

7. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1-r^3$

8. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

9. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2}-1$

10. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta\cot 2\theta\,d\theta$

a) Using $u = \cot 2\theta$ **b)** Using $u = \csc 2\theta$

12. $\int \frac{dx}{\sqrt{5x+8}}$

a) Using $u = 5x+8$ **b)** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 13–46.

13. $\int \sqrt{3-2s}\,ds$

14. $\int (2x+1)^3\,dx$

15. $\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\int \frac{3\,dx}{(2-x)^2}$

17. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\int 3y\sqrt{7-3y^2}\,dy$

20. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\int \cos(3z+4)\,dz$

24. $\int \sin(8z-5)\,dz$

25. $\int \sec^2(3x+2)\,dx$

26. $\int \tan^2 x\sec^2 x\,dx$

27. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)\,dv$

34. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)\,dv$

35. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\int \sqrt{\cot y}\,\csc^2 y\,dy$

38. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

39. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)\,dt$

40. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

41. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

42. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

43. $\int (s^3+2s^2-5s+5)(3s^2+4s-5)\,ds$

44. $\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta$

45. $\int t^3(1+t^4)^3\,dt$

46. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 47 and 48.

47. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a) $u=\tan x$, followed by $v=u^3$, then by $w=2+v$
b) $u=\tan^3 x$, followed by $v=2+u$
c) $u=2+\tan^3 x$

48. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

a) $u=x-1$, followed by $v=\sin u$, then by $w=1+v^2$
b) $u=\sin(x-1)$, followed by $v=1+u^2$
c) $u=1+\sin^2(x-1)$

Evaluate the integrals in Exercises 49 and 50.

49. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

50. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 51–56.

51. $\frac{ds}{dt}=12t(3t^2-1)^3,\quad s(1)=3$

52. $\frac{dy}{dx}=4x(x^2+8)^{-1/3},\quad y(0)=0$

53. $\frac{ds}{dt}=8\sin^2\left(t+\frac{\pi}{12}\right),\quad s(0)=8$

54. $\frac{dr}{d\theta}=3\cos^2\left(\frac{\pi}{4}-\theta\right),\quad r(0)=\frac{\pi}{8}$

55. $\frac{d^2s}{dt^2}=-4\sin\left(2t-\frac{\pi}{2}\right),\quad s'(0)=100,\ s(0)=0$

56. $\frac{d^2y}{dx^2}=4\sec^2 2x\tan 2x,\quad y'(0)=4,\ y(0)=-1$

57. The velocity of a particle moving back and forth on a line is $v=ds/dt=6\sin 2t$ m/sec for all t. If $s=0$ when $t=0$, find the value of s when $t=\pi/2$ sec.

58. The acceleration of a particle moving back and forth on a line is $a=d^2s/dt^2=\pi^2\cos\pi t$ m/sec^2 for all t. If $s=0$ and $v=8$ m/sec when $t=0$, find s when $t=1$ sec.

Theory and Examples

59. It looks as if we can integrate $2\sin x\cos x$ with respect to x in three different ways:

a) $$\int 2\sin x\cos x\,dx=\int 2u\,du \qquad u=\sin x,$$
$$=u^2+C_1=\sin^2 x+C_1$$

b) $$\int 2\sin x\cos x\,dx=\int -2u\,du \qquad u=\cos x,$$
$$=-u^2+C_2=-\cos^2 x+C_2$$

c) $$\int 2\sin x\cos x\,dx=\int \sin 2x\,dx \qquad 2\sin x\cos x=\sin 2x$$
$$=-\frac{\cos 2x}{2}+C_3.$$

Can all three integrations be correct? Give reasons for your answer.

60. The substitution $u=\tan x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\tan^2 x}{2}+C.$$
The substitution $u=\sec x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\sec^2 x}{2}+C.$$
Can both integrations be correct? Give reasons for your answer.

4.4

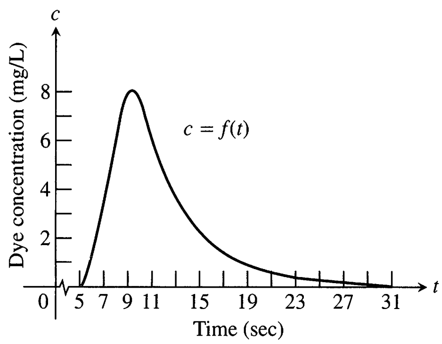
Estimating with Finite Sums

This section shows how practical questions can lead in natural ways to approximations by finite sums.

Area and Cardiac Output

The number of liters of blood your heart pumps in a minute is called your *cardiac output.* For a person at rest, the rate might be 5 or 6 liters per minute. During strenuous exercise the rate might be as high as 30 liters per minute. It might also be altered significantly by disease.

Instead of measuring a patient's cardiac output with exhaled carbon dioxide, as in Exercise 25 in Section 2.7, a doctor may prefer to use the dye-dilution technique described here. You inject 5 to 10 mg of dye in a main vein near the heart. The dye is drawn into the right side of the heart and pumped through the lungs and out the left side of the heart into the aorta, where its concentration can be measured every few seconds as the blood flows past. The data in Table 4.3 and the plot in Fig. 4.5 show the response of a healthy, resting patient to an injection of 5.6 mg of dye.



4.5 The dye concentrations from Table 4.3, plotted and fitted with a smooth curve. Time is measured with $t = 0$ at the time of injection. The dye concentrations are zero at the beginning, while the dye passes through the lungs. They then rise to a maximum at about $t = 9$ sec and taper to zero by $t = 31$ sec.

To calculate the patient's cardiac output, we divide the amount of dye by the area under the dye concentration curve and multiply the result by 60:

$$\text{Cardiac output} = \frac{\text{amount of dye}}{\text{area under curve}} \times 60. \tag{1}$$

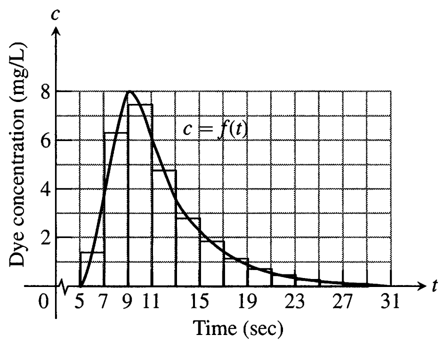
You can see why the formula works if you check the units in which the various quantities are measured. The amount of dye is in milligrams and the area is in (milligrams/liter) × seconds, which gives cardiac output in liters/minute:

$$\frac{\text{mg}}{\dfrac{\text{mg}}{\text{L}} \cdot \text{sec}} \cdot \frac{\text{sec}}{\text{min}} = \text{mg} \cdot \frac{\text{L}}{\text{mg} \cdot \text{sec}} \cdot \frac{\text{sec}}{\text{min}} = \frac{\text{L}}{\text{min}}.$$

In the example that follows, we estimate the area under the concentration curve in Fig. 4.5 and find the patient's cardiac output.

Table 4.3 Dye-dilution data

Seconds after injection t	**Dye concentration (adjusted for recirculation)** c	**Seconds after injection** t	**Dye concentration (adjusted for recirculation)** c
5	0	19	0.91
7	3.8	21	0.57
9	8.0	23	0.36
11	6.1	25	0.23
13	3.6	27	0.14
15	2.3	29	0.09
17	1.45	31	0



4.6 The region under the concentration curve of Fig. 4.5 is approximated with rectangles. We ignore the portion from $t = 29$ to $t = 31$; its concentration is negligible.

EXAMPLE 1 Find the cardiac output of the patient whose data appear in Table 4.3 and Fig. 4.5.

Solution We know the amount of dye to use in Eq. (1) (it is 5.6 mg), so all we need is the area under the concentration curve. None of the area formulas we know can be used for this irregularly shaped region. But we can get a good estimate of this area by approximating the region between the curve and the t-axis with rectangles and adding the areas of the rectangles (Fig. 4.6). Each rectangle omits some of the area under the curve but includes area from outside the curve, which compensates. In Fig. 4.6 each rectangle has a base 2 units long and a height that is equal to the height of the curve above the midpoint of the base. The rectangle's height acts as a sort of average value of the function over the time interval on which the rectangle stands. After reading rectangle heights from the curve, we multiply each rectangle's height and base to find its area, and then get the following estimate:

$$\begin{aligned}\text{Area under curve} &\approx \text{sum of rectangle areas}\\ &\approx f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + \cdots + f(28)\cdot 2\\ &\approx (1.4)(2) + (6.3)(2) + (7.5)(2) + \cdots + (0.1)(2)\\ &\approx (28.8)(2) = 57.6 \text{ mg}\cdot\text{sec/L}. \qquad (2)\end{aligned}$$

Dividing this figure into the amount of dye and multiplying by 60 gives a corresponding estimate of the cardiac output:

$$\text{Cardiac output} \approx \frac{\text{amount of dye}}{\text{area estimate}} \times 60 = \frac{5.6}{57.6} \times 60 \approx 5.8 \text{ L/min}.$$

The patient's cardiac output is about 5.8 L/min. ❑

Technology *Using a Grapher to Calculate Finite Sums* If your graphing utility has a method for evaluating sums, you might want to use it in this section. Later in the chapter, you will find it useful for approximating "definite" integrals. There will be other uses still later in your study of calculus.

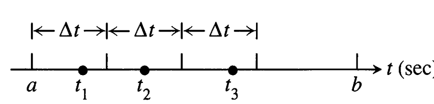
Distance Traveled

Suppose we know the velocity function $v = ds/dt = f(t)$ m/sec of a car moving down a highway and want to know how far the car will travel in the time interval $a \le t \le b$. If we know an antiderivative F of f, we can find the car's position function $s = F(t) + C$ and calculate the distance traveled as the difference between the car's positions at times $t = a$ and $t = b$ (as in Section 4.2, Exercise 55).

If we do not know an antiderivative of $v = f(t)$, we can approximate the answer with a sum in the following way. We partition $[a, b]$ into short time intervals *on each of which v is fairly constant.* Since velocity is the rate at which the car is traveling, we approximate the distance traveled on each time interval with the formula

$$\text{Distance} = \text{rate} \times \text{time} = f(t)\cdot\Delta t$$

and add the results across $[a, b]$. To be specific, suppose the partitioned interval looks like this



with the subintervals all of length Δt. Let t_1 be a point in the first subinterval. If the interval is short enough so the rate is almost constant, the car will move about $f(t_1)\Delta t$ m during that interval. If t_2 is a point in the second interval, the car will move an additional $f(t_2)\Delta t$ m during that interval, and so on. The sum of these products approximates the total distance D traveled from $t = a$ to $t = b$. If we use n subintervals, then

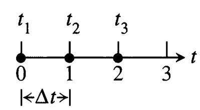
$$D \approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t. \tag{3}$$

Let's try this on the projectile in Example 3, Section 4.2. The projectile was fired straight into the air. Its velocity t sec into the flight was $v = f(t) = 160 - 9.8t$ and it rose 435.9 m from a height of 3 m to a height of 438.9 m during the first 3 sec of flight.

EXAMPLE 2 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact figure of 435.9 m?

Solution We explore the results for different numbers of intervals and different choices of evaluation points.

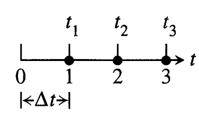
3 *subintervals of length* 1, *with f evaluated at left-hand endpoints:*



With f evaluated at $t = 0$, 1, and 2, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t && \text{Eq. (1)} \\ &\approx [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &\approx 450.6. \end{aligned}$$

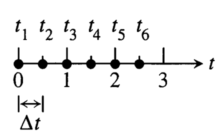
3 *subintervals of length* 1, *with f evaluated at right-hand endpoints:*

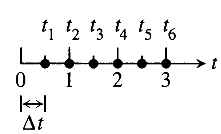


With f evaluated at $t = 1$, 2, and 3, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t && \text{Eq. (1)} \\ &\approx [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &\approx 421.2. \end{aligned}$$

With 6 *subintervals of length* 1/2, *we get*





Using left-hand endpoints: $D \approx 443.25$.

Using right-hand endpoints: $D \approx 428.55$.

These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

Table 4.4 Travel-distance estimates

Number of subintervals	Length of each subinterval	Left-endpoint sum	Right-endpoint sum
3	1	450.6	421.2
6	0.5	443.25	428.55
12	0.25	439.58	432.23
24	0.125	437.74	434.06
48	0.0625	436.82	434.98
96	0.03125	436.36	435.44
192	0.015625	436.13	435.67

Error magnitude = |true value − calculated value|

As we can see in Table 4.4, the left-endpoint sums approach the true value 435.9 from above while the right-endpoint sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entries is 0.23, a small percentage of the true value.

$$\text{Error percentage} = \frac{0.23}{435.9} \approx 0.05\%.$$

It would be safe to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ❑

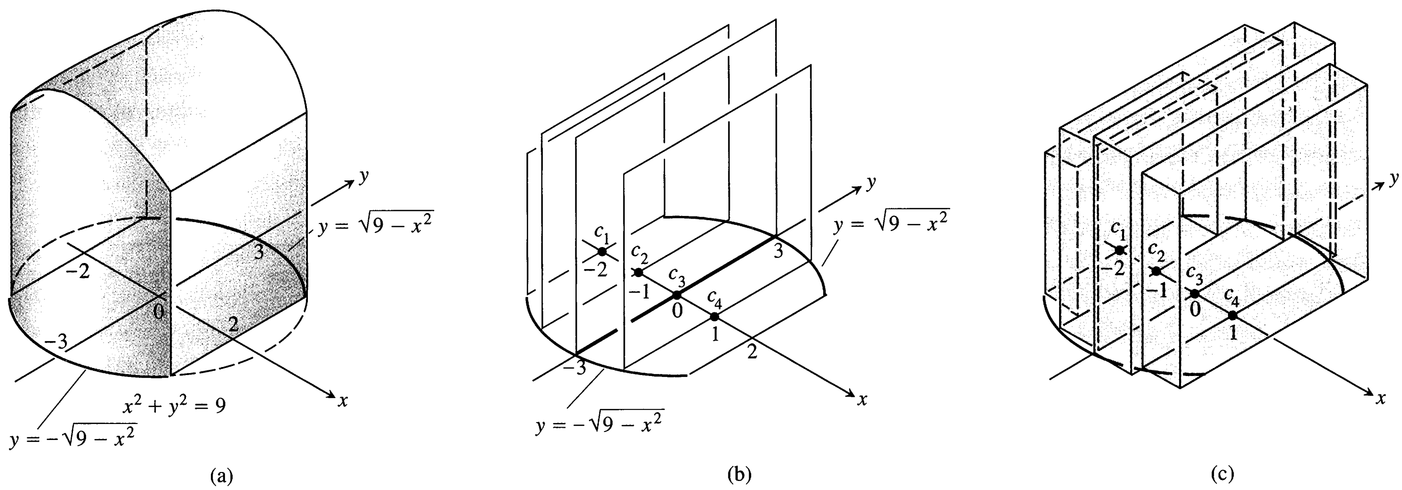
Notice the mathematical similarity between Examples 1 and 2. In each case, we have a function f defined on a closed interval and estimate what we want to know with a sum of function values multiplied by interval lengths. We can use similar sums to estimate volumes.

Volume

Here are two examples using finite sums to estimate volumes.

EXAMPLE 3 A solid lies between planes perpendicular to the x-axis at $x = -2$ and $x = 2$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{9 - x^2}$ to the semicircle $y = \sqrt{9 - x^2}$ (Fig. 4.7a, on the following page). The height of the square at x is $2\sqrt{9 - x^2}$. Estimate the volume of the solid.

Solution We partition the interval $[-2, 2]$ on the x-axis into four subintervals of length $\Delta x = 1$. The solid's cross section at the left-hand endpoint of each subinterval is a square (Fig. 4.7b). On each of these squares we construct a right cylinder (square slab) of height 1 extending to the right (Fig. 4.7c). We add the cylinders' volumes to estimate the volume of the solid.



4.7 (a) The solid in Example 3. (b) Square cross sections of the solid at $x = -2, -1,$ 0, and 1. (c) Rectangular cylinders (slabs) based on the cross sections to approximate the solid.

We calculate the volume of each cylinder with the formula $V = Ah$ (base area × height). The area of the solid's cross section at x is $A(x) = (\text{side})^2 = (2\sqrt{9 - x^2})^2 = 4(9 - x^2)$, so the sum of the volumes of the cylinders is

$$\begin{aligned} S_4 &= A(c_1)\,\Delta x + A(c_2)\,\Delta x + A(c_3)\,\Delta x + A(c_4)\,\Delta x \\ &= 4(9 - {c_1}^2)(1) + 4(9 - {c_2}^2)(1) + 4(9 - {c_3}^2)(1) + 4(9 - {c_4}^2)(1) \\ &= 4\left[(9 - (-2)^2) + (9 - (-1)^2) + (9 - (0)^2) + (9 - (1)^2)\right] \\ &= 4\left[(9 - 4) + (9 - 1) + (9 - 0) + (9 - 1)\right] \\ &= 4(36 - 6) = 120. \end{aligned}$$

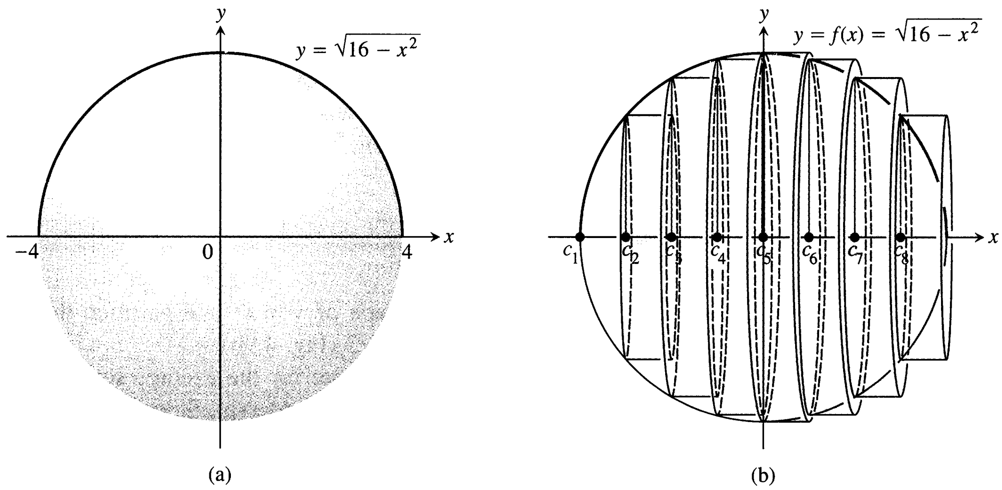
This compares favorably with the solid's true volume $V = 368/3 \approx 122.67$ (we will see how to calculate V in Section 4.7). The difference between S and V is a small percentage of V:

$$\begin{aligned} \text{Error percentage} &= \frac{|V - S_4|}{V} = \frac{(368/3) - 120}{(368/3)} \\ &= \frac{8}{368} \approx 2.2\%. \end{aligned}$$

With a finer partition (more subintervals) the approximation would be even better. ❑

EXAMPLE 4 Estimate the volume of a solid sphere of radius 4.

Solution We picture the sphere as if its surface were generated by revolving the graph of the function $f(x) = \sqrt{16 - x^2}$ about the x-axis (Fig. 4.8a). We partition the interval $-4 \le x \le 4$ into 8 subintervals of length $\Delta x = 1$. We then approximate the solid with right circular cylinders based on cross sections of the solid by planes perpendicular to the x-axis at the subintervals' left-hand endpoints (Fig. 4.8b). (The cylinder at $x = -4$ is degenerate because the cross section there is just a point.) We add the cylinders' volumes to estimate the volume of a sphere.



4.8 (a) The semicircle $y = \sqrt{16 - x^2}$ revolved about the x-axis to outline a sphere. (b) The solid sphere approximated with cross-section-based cylinders.

We calculate the volume of each cylinder with the formula $V = \pi r^2 h$. The sum of the eight cylinders' volumes is

$$\begin{aligned} S_8 &= \pi\,[f(c_1)]^2\,\Delta x + \pi\,[f(c_2)]^2\,\Delta x + \pi\,[f(c_3)]^2\,\Delta x + \cdots + \pi\,[f(c_8)]^2\,\Delta x \\ &= \pi\left[\sqrt{16 - c_1{}^2}\right]^2 \Delta x + \pi\left[\sqrt{16 - c_2{}^2}\right]^2 \Delta x + \pi\left[\sqrt{16 - c_3{}^2}\right]^2 \Delta x \\ &\quad + \cdots + \pi\left[\sqrt{16 - c_8{}^2}\right]^2 \Delta x \\ &= \pi\left[(16 - (-4)^2) + (16 - (-3)^2) + (16 - (-2)^2) + \cdots + (16 - (3)^2)\right] \\ &= \pi\,[0 + 7 + 12 + 15 + 16 + 15 + 12 + 7] \\ &= 84\pi. \end{aligned}$$

This compares favorably with the sphere's true volume,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\,(4)^3 = \frac{256\pi}{3}.$$

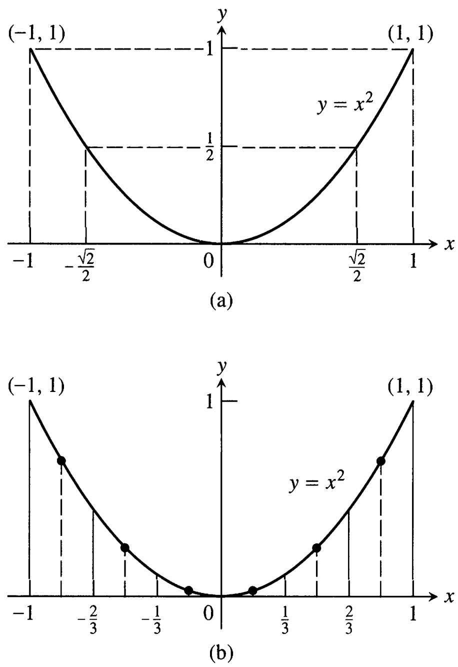
The difference between S_8 and V is a small percentage of V:

$$\begin{aligned} \text{Error percentage} &= \frac{|V - S_8|}{V} = \frac{(256/3)\pi - 84\pi}{(256/3)\pi} \\ &= \frac{256 - 252}{256} = \frac{1}{64} \approx 1.6\%. \end{aligned}$$

❑

The Average Value of a Nonnegative Function

To find the average of a finite set of values, we add them and divide by the number of values added. But what happens if we want to find the average of an infinite number of values? For example, what is the average value of the function $f(x) = x^2$ on the interval $[-1,\ 1]$? To see what this kind of "continuous" average might mean, imagine that we are pollsters sampling the function. We pick random x's between -1 and 1, square them, and average the squares. As we take larger samples, we expect this average to approach some number, which seems reasonable to call the *average of f over* $[-1,\ 1]$.



4.9 (a) The graph of $f(x) = x^2$, $-1 \leq x \leq 1$. (b) Values of f sampled at regular intervals.

The graph in Fig. 4.9(a) suggests that the average square should be less than 1/2, because numbers with squares less than 1/2 make up more than 70% of the interval $[-1,\ 1]$. If we had a computer to generate random numbers, we could carry out the sampling experiment described above, but it is much easier to estimate the average value with a finite sum.

EXAMPLE 5 Estimate the average value of the function $f(x) = x^2$ on the interval $[-1,\ 1]$.

Solution We look at the graph of $y = x^2$ and partition the interval [0, 1] into 6 subintervals of length $\Delta x = 1/3$ (Fig. 4.9b).

It appears that a good estimate for the average square on each subinterval is the square of the midpoint of the subinterval. Since the subintervals have the same length, we can average these six estimates to get a final estimate for the average value over $[-1,\ 1]$.

$$\text{Average value} \approx \frac{\left(-\frac{5}{6}\right)^2+\left(-\frac{3}{6}\right)^2+\left(-\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{3}{6}\right)^2+\left(\frac{5}{6}\right)^2}{6}$$

$$\approx \frac{1}{6}\cdot\frac{25+9+1+1+9+25}{36} = \frac{70}{216} \approx 0.324$$

We will be able to show later that the average value is 1/3.

Notice that

$$\frac{\left(-\frac{5}{6}\right)^2+\left(-\frac{3}{6}\right)^2+\left(-\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{3}{6}\right)^2+\left(\frac{5}{6}\right)^2}{6}$$

$$= \frac{1}{2}\left[\left(-\frac{5}{6}\right)^2\cdot\frac{1}{3}+\left(-\frac{3}{6}\right)^2\cdot\frac{1}{3}+\cdots+\left(\frac{5}{6}\right)^2\cdot\frac{1}{3}\right]$$

$$= \frac{1}{\text{length of }[-1,1]}\cdot\left[f\left(-\frac{5}{6}\right)\cdot\frac{1}{3}+f\left(-\frac{3}{6}\right)\cdot\frac{1}{3}+\cdots+f\left(\frac{5}{6}\right)\cdot\frac{1}{3}\right]$$

$$= \frac{1}{\text{length of }[-1,1]}\cdot\left[\begin{array}{l}\text{a sum of function values}\\ \text{multiplied by interval lengths}\end{array}\right].$$

Once again our estimate has been achieved by multiplying function values by interval lengths and summing the results for all the intervals. ❑

Conclusion

The examples in this section describe instances in which sums of function values multiplied by interval lengths provide approximations that are good enough to answer practical questions. You will find additional examples in the exercises.

The distance approximations in Example 2 improved as the intervals involved became shorter and more numerous. We knew this because we had already found the exact answer with antiderivatives in Section 4.2. If we had made our partitions of the time interval still finer, would the sums have approached the exact answer as a limit? Is the connection between the sums and the antiderivative in this case just a coincidence? Could we have calculated the area in Example 1, the volumes in

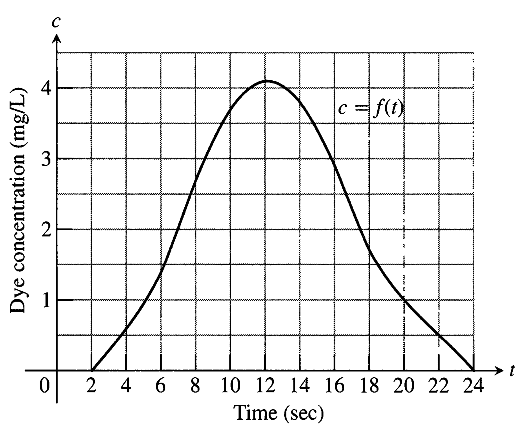
Examples 3 and 4, and the average value in Example 5 with antiderivatives as well? As we will see, the answers are "Yes, they would have," "No, it is not a coincidence," and "Yes, we could have."

Exercises 4.4

Cardiac Output

1. The table below gives dye concentrations for a dye-dilution cardiac-output determination like the one in Example 1. The amount of dye injected in this case was 5 mg instead of 5.6 mg. Use rectangles to estimate the area under the dye concentration curve and then go on to estimate the patient's cardiac output.

Seconds after injection t	Dye concentration (adjusted for recirculation) c
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0



2. The accompanying table gives dye concentrations for a cardiac-output determination like the one in Example 1. The amount of dye injected in this case was 10 mg. Plot the data and connect the data points with a smooth curve. Estimate the area under the curve and calculate the cardiac output from this estimate.

Seconds after injection t	Dye concentration (adjusted for recirculation) c	Seconds after injection t	Dye concentration (adjusted for recirculation) c
0	0	16	7.9
2	0	18	7.8
4	0.1	20	6.1
6	0.6	22	4.7
8	2.0	24	3.5
10	4.2	26	2.1
12	6.3	28	0.7
14	7.5	30	0

Distance

3. The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with (a) left-endpoint values and (b) right-endpoint values.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

4. You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every five minutes for an hour, with the results shown in the table on the following page. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with (a) left-endpoint values and (b) right-endpoint values.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

5. You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the table below. Estimate the length of the road (a) using left-endpoint values and (b) using right-endpoint values.

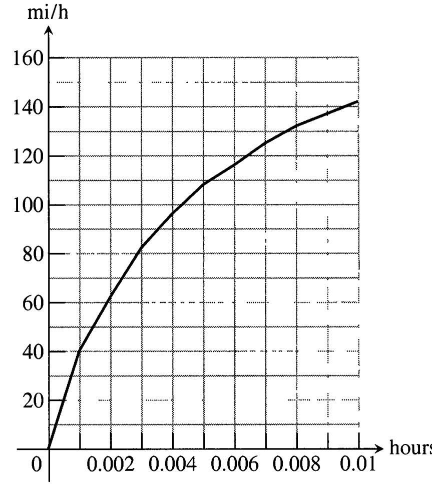
Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

6. The table below gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

a) Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

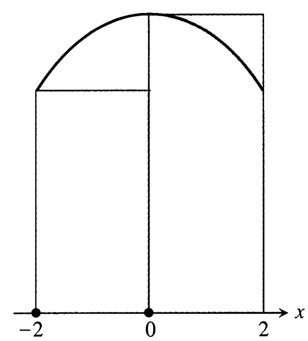


Volume

7. *(Continuation of Example 3.)* Suppose we use only two square cylinders to estimate the volume V of the solid in Example 3, as shown in profile in the figure here.

a) Find the sum S_2 of the volumes of the cylinders.

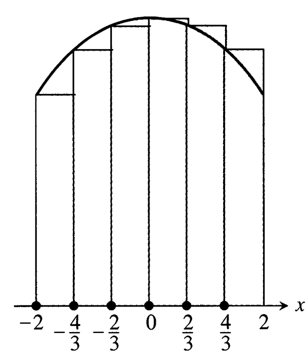
b) Express $|V - S_2|$ as a percentage of V to the nearest percent.



8. *(Continuation of Example 3.)* Suppose we use six square cylinders to estimate the volume V of the solid in Example 3, as shown in the accompanying profile view.

a) Find the sum S_6 of the volumes of the cylinders.

b) Express $|V - S_6|$ as a percentage of V to the nearest percent.



9. *(Continuation of Example 4.)* Suppose we approximate the volume V of the sphere in Example 4 by partitioning the interval $-4 \leq x \leq 4$ into four subintervals of length 2 and using cylinders based on the cross sections at the subintervals' left-hand endpoints. (As in Example 4, the leftmost cylinder will have a zero radius.)

a) Find the sum S_4 of the volumes of the cylinders.

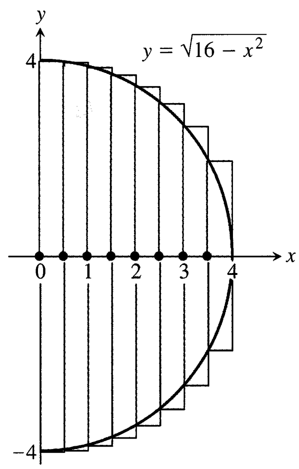
b) Express $|V - S_4|$ as a percentage of V to the nearest percent.

10. To estimate the volume V of a solid sphere of radius 5 you partition its diameter into five subintervals of length 2. You then slice the sphere with planes perpendicular to the diameter at the subintervals' left-hand endpoints and add the volumes of cylinders of height 2 based on the cross sections of the sphere determined by these planes.

a) Find the sum S_5 of the volumes of the cylinders.

b) Express $|V - S_5|$ as a percentage of V to the nearest percent.

11. To estimate the volume V of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval [0, 4] on the x-axis. Partition [0, 4] into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' left-hand endpoints. (See the accompanying profile view.)

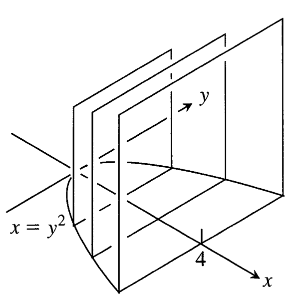


a) Find the sum S_8 of the volumes of the cylinders. Do you expect S_8 to overestimate V, or to underestimate V? Give reasons for your answer.

b) Express $|V - S_8|$ as a percentage of V to the nearest percent.

12. Repeat Exercise 11 using cylinders based on cross sections at the *right-hand* endpoints of the subintervals.

13. *Estimates with large error.* A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the parabolic curve $y = -\sqrt{x}$ to the parabolic curve $y = \sqrt{x}$.

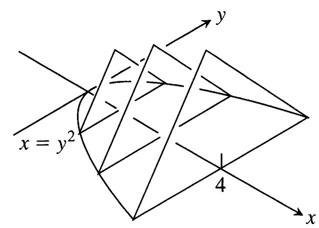


a) Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's right-hand endpoints.

b) The true volume is $V = 32$. Express $|V - S_4|$ as a percentage of V to the nearest percent.

c) Repeat parts (a) and (b) for the sum S_8.

14. *Estimates with large error.* A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the axis between these planes are vertical equilateral triangles whose base edges run from the parabolic curve $y = -\sqrt{x}$ to the parabolic curve $y = \sqrt{x}$.



a) Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's left-hand endpoints.

b) The true volume is $V = 8\sqrt{3}$. Express $|V - S_4|$ as a percentage of V to the nearest percent.

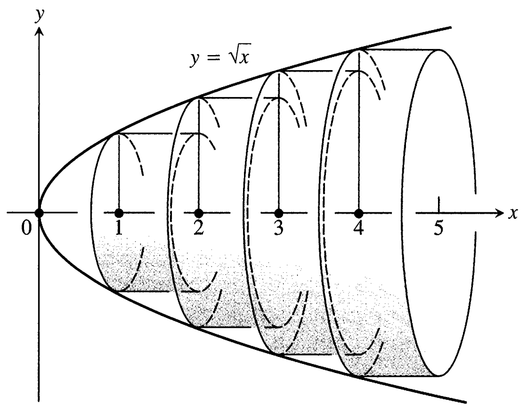
c) CALCULATOR Repeat parts (a) and (b) for the sum S_8.

15. A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m. (a) Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders. (b) As you will see in Section 4.7, Exercise 71, the water's volume is $V = 320\pi/3$ m^3. Find the error $|V - S|$ as a percentage of V to the nearest percent.

16. A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using (a) left-endpoint values of h; (b) right-endpoint values of h.

Position x ft	Depth $h(x)$ ft	Position x ft	Depth $h(x)$ ft
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

17. The nose "cone" of a rocket is a paraboloid obtained by revolving the curve $y = \sqrt{x}$, $0 \le x \le 5$, about the x-axis, where x is measured in feet. To estimate the volume V of the nose cone, we partition [0, 5] into five subintervals of equal length, slice the cone with planes perpendicular to the x-axis at the subintervals' left-hand endpoints, and construct cylinders of height 1 based on cross sections at these points. (See the accompanying figure.)



a) Find the sum S_5 of the volumes of the cylinders. Do you expect S_5 to overestimate V, or to underestimate V? Give reasons for your answer.

b) As you will see in Section 4.7, Exercise 72, the volume of the nose cone is $V = 25\pi/2$ ft^3. Express $|V - S_5|$ as a percentage of V to the nearest percent.

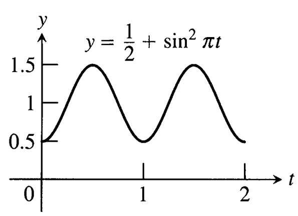
18. Repeat Exercise 17 using cylinders based on cross sections at the *right-hand* endpoints of the subintervals.

Average Value of a Function

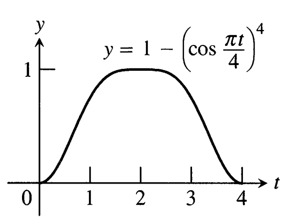
In Exercises 19–22, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

19. $f(x) = x^3$ on $[0, 2]$ **20.** $f(x) = 1/x$ on $[1, 9]$

21. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



22. $f(t) = 1 - \left(\cos \dfrac{\pi t}{4}\right)^4$ on $[0, 4]$



Velocity and Distance

23. An object is dropped straight down from an airplane. The object falls faster and faster but the acceleration is decreasing over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown in the following table.

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

a) Find an upper estimate for the speed when $t = 5$.

b) Find a lower estimate for the speed when $t = 5$.

c) Find an upper estimate for the distance fallen when $t = 3$.

24. An object is shot straight upward from sea level with an initial velocity of 400 ft/sec. Assuming gravity is the only force acting on the object, give an upper estimate for its speed after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational constant. Find a lower estimate for the height attained after 5 sec.

Pollution Control

25. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (hours)	0	1	2	3	4
Leakage (gal/hr)	50	70	97	136	190
Time (hours)	5	6	7	8	
Leakage (gal/hr)	265	369	516	720	

a) Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b) Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all of the oil has spilled? in the best case?

26. A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smoke stacks. Over time the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant release rate (tons/day)	0.2	0.25	0.27	0.34	0.45	0.52	0.63	0.70	0.81	0.85	0.89	0.95

a) Assuming a 30-day month and that new scrubbers allow only 0.05 tons/day released, give an upper estimate of the total tonnage of pollutant released by the end of June. What is a lower estimate?

b) In the best case, approximately when will a total of 125 tons of pollutant have been released into the atmosphere?

CAS Explorations and Projects

In Exercises 27–30, use a CAS to perform the following steps:

a) Plot the functions over the given interval.

b) Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c) Compute the average value of the function values generated in part (b).

d) Solve the equation $f(x) =$ (average value) for x using the average value calculated in (c) for the $n = 1000$ partitioning.

27. $f(x) = \sin x$ on $[0, \pi]$ **28.** $f(x) = \sin^2 x$ on $[0, \pi]$

29. $f(x) = x \sin \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

30. $f(x) = x \sin^2 \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

4.5 Riemann Sums and Definite Integrals

In the preceding section, we estimated distances, areas, volumes, and average values with finite sums. The terms in the sums were obtained by multiplying selected function values by the lengths of intervals. In this section, we say what it means for sums like these to approach a limit as the intervals involved become more numerous and shorter. We begin by introducing a compact notation for sums that contain large numbers of terms.

Sigma Notation for Finite Sums

We use the capital Greek letter Σ ("sigma") to write an abbreviation for the sum

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t$$

as $\sum_{k=1}^{n} f(t_k)\,\Delta t$, "the sum from k equals 1 to n of f of t_k times delta t." When we write a sum this way, we say that we have written it in sigma notation.

Definitions

Sigma Notation for Finite Sums

The symbol $\sum_{k=1}^{n} a_k$ denotes the sum $a_1 + a_2 + \cdots + a_n$. The a's are the **terms** of the sum: a_1 is the first term, a_2 is the second term, a_k is the **kth term,** and a_n is the nth and last term. The variable k is the **index of summation.** The values of k run through the integers from 1 to n. The number 1 is the **lower limit of summation;** the number n is the **upper limit of summation.**

EXAMPLE 1

The sum in sigma notation	The sum written out—one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1)+(-1)^2(2)+(-1)^3(3)$	$-1+2-3=-2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1}+\frac{2}{2+1}$	$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$

❑

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 2 Express the sum $1+3+5+7+9$ in sigma notation.

Solution

Starting with $k=2$: $\quad 1+3+5+7+9=\sum_{k=2}^{6}(2k-3)$

Starting with $k=-3$: $\quad 1+3+5+7+9=\sum_{k=-3}^{1}(2k+7)$

The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k=0$ or $k=1$.

Starting with $k=0$: $\quad 1+3+5+7+9=\sum_{k=0}^{4}(2k+1)$

Starting with $k=1$: $\quad 1+3+5+7+9=\sum_{k=1}^{5}(2k-1)$ ❑

Algebra with Finite Sums

We can use the following rules whenever we work with finite sums.

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$

2. *Difference Rule:* $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$

3. *Constant Multiple Rule:* $\sum_{k=1}^{n}ca_k=c\cdot\sum_{k=1}^{n}a_k$ (Any number c)

4. *Constant Value Rule:* $\sum_{k=1}^{n}c=n\cdot c$ (c is any constant value.)

There are no surprises in this list. The formal proofs can be done by mathematical induction (Appendix 1).

EXAMPLE 3

a) $\sum_{k=1}^{n}(3k-k^2)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$ — Difference Rule and Constant Multiple Rule

b) $\sum_{k=1}^{n}(-a_k)=\sum_{k=1}^{n}(-1)\cdot a_k=-1\cdot\sum_{k=1}^{n}a_k=-\sum_{k=1}^{n}a_k$ — Constant Multiple Rule

c) $\sum_{k=1}^{3}(k+4)=\sum_{k=1}^{3}k+\sum_{k=1}^{3}4$ — Sum Rule

$=(1+2+3)+(3\cdot 4)$ — Constant Value Rule

$=6+12=18$ ❑

Sum Formulas for Positive Integers

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n integers (Gauss discovered it at age 5) and the formulas for the sums of the squares and cubes of the first n integers.

The first n integers:	$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$	(1)
The first n squares:	$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$	(2)
The first n cubes:	$\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2$	(3)

EXAMPLE 4 Evaluate $\sum_{k=1}^{4}(k^2-3k)$.

Solution We can use the algebra rules and known formulas to evaluate the sum without writing out the terms.

$$\sum_{k=1}^{4}(k^2-3k)=\sum_{k=1}^{4}k^2-3\sum_{k=1}^{4}k$$

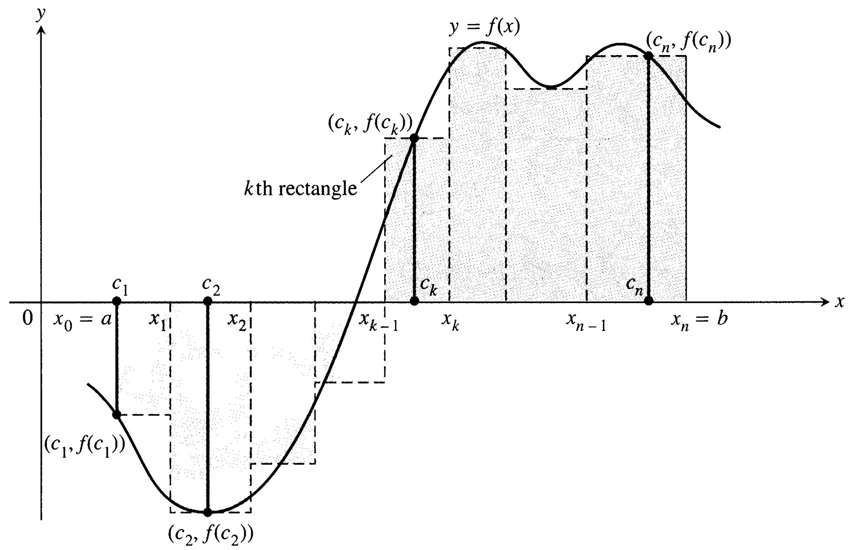
Difference Rule and Constant Multiple Rule

$$=\frac{4(4+1)(8+1)}{6}-3\left(\frac{4(4+1)}{2}\right)$$

Eqs. (2) and (1) with $n=4$

$$=30-30=0$$

❑



4.10 The graph of a typical function $y = f(x)$ over a closed interval $[a, b]$. The rectangles approximate the region between the graph of the function and the x-axis.

Riemann Sums

The approximating sums in Section 4.4 are examples of a more general kind of sum called a *Riemann* ("*ree*-mahn") *sum.* The functions in the examples had nonnegative values, but the more general notion has no such restriction. Given an arbitrary continuous function $y = f(x)$ on an interval $[a, b]$ (Fig. 4.10), we partition the interval into n subintervals by choosing $n - 1$ points, say $x_1, x_2, \dots, x_{n-1}$, between a and b subject only to the condition that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we usually denote a by x_0 and b by x_n. The set

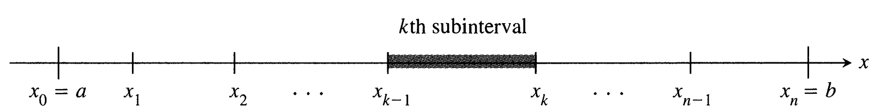
$$P = \{x_0, x_1, \dots, x_n\}$$

is called a **partition** of $[a, b]$.

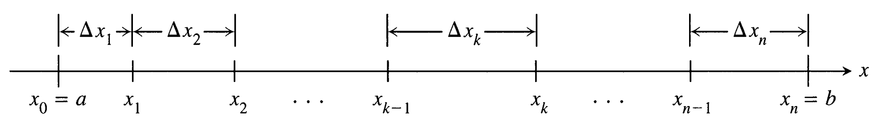
The partition P defines n closed **subintervals**

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The typical closed subinterval $[x_{k-1}, x_k]$ is called the ***k*th subinterval** of P.

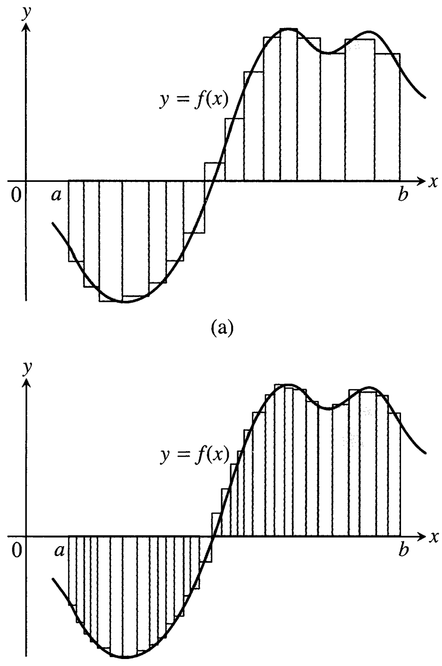


The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.



In each subinterval $[x_{k-1}, x_k]$, we select a point c_k and construct a vertical rectangle from the subinterval to the point $(c_k, f(c_k))$ on the curve $y = f(x)$. The choice of c_k does not matter as long as it lies in $[x_{k-1}, x_k]$. See Fig. 4.10 again.

If $f(c_k)$ is positive, the number $f(c_k)\,\Delta x_k = \text{height} \times \text{base}$ is the area of the



4.11 The curve of Fig. 4.10 with rectangles from finer partitions of $[a, b]$. Finer partitions create more rectangles with shorter bases.

rectangle. If $f(c_k)$ is negative, then $f(c_k)\,\Delta x_k$ is the negative of the area. In any case, we add the n products $f(c_k)\,\Delta x_k$ to form the sum

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

This sum, which depends on P and the choice of the numbers c_k, is called a **Riemann sum for f on the interval $[a, b]$,** after German mathematician Georg Friedrich Bernhard Riemann (1826–1866), who studied the limits of such sums.

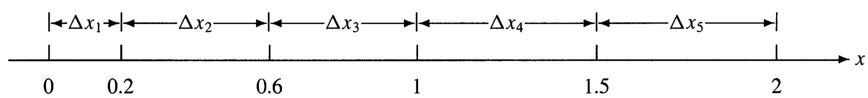
As the partitions of $[a, b]$ become finer, the rectangles defined by the partition approximate the region between the x-axis and the graph of f with increasing accuracy (Fig. 4.11). So we expect the associated Riemann sums to have a limiting value. To test this expectation, we need to develop a numerical way to say that partitions become finer and to determine whether the corresponding sums have a limit. We accomplish this with the following definitions.

The **norm** of a partition P is the partition's longest subinterval length. It is denoted by

$$\|P\| \qquad \text{(read "the norm of } P\text{")}.$$

The way to say that successive partitions of an interval become finer is to say that the norms of these partitions approach zero. As the norms go to zero, the subintervals become shorter and their number approaches infinity.

EXAMPLE 5 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$.



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ❑

Definition

The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that the **limit** of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\,\Delta x_k$ on $[a, b]$ as $\|P\| \to 0$ is the number I if the following condition is satisfied:

Given any number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for every partition P of $[a, b]$

$$\|P\| < \delta \qquad \Rightarrow \qquad \left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon$$

for any choice of the numbers c_k in the subintervals $[x_{k-1}, x_k]$.

If the limit exists, we write

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = I.$$

We call I the **definite integral** of f over $[a, b]$, we say that f is **integrable** over $[a, b]$, and we say that the Riemann sums of f on $[a, b]$ **converge** to the number I.

We usually write I as $\int_a^b f(x)\,dx$, which is read "integral of f from a to b." Thus, if the limit exists,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx.$$

The amazing fact is that despite the variety in the Riemann sums $\Sigma f(c_k)\,\Delta x_k$ as the partitions change and the arbitrary choice of c_k's in the intervals of each new partition, the sums always have the same limit as $\|P\| \to 0$ as long as f is continuous. The need to establish the existence of this limit became clear as the nineteenth century progressed, and it was finally established when Riemann proved the following theorem in 1854. You can find a current version of Riemann's proof in most advanced calculus books.

Theorem 1

The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Why should we expect such a theorem to hold? Imagine a typical partition P of the interval $[a, b]$. The function f, being continuous, has a minimum value $\min_k$ ("min kay") and a maximum value $\max_k$ ("max kay") on each subinterval. The products $\min_k \Delta x_k$ associated with the minimum values (Fig. 4.12a) add up to what we call the **lower sum** for f on P:

$$L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \cdots + \min_n \Delta x_n.$$

The products $\max_k \Delta x_k$ obtained from the maximum values (Fig. 4.12b) add up to the **upper sum** for f on P:

$$U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \cdots + \max_n \Delta x_n.$$

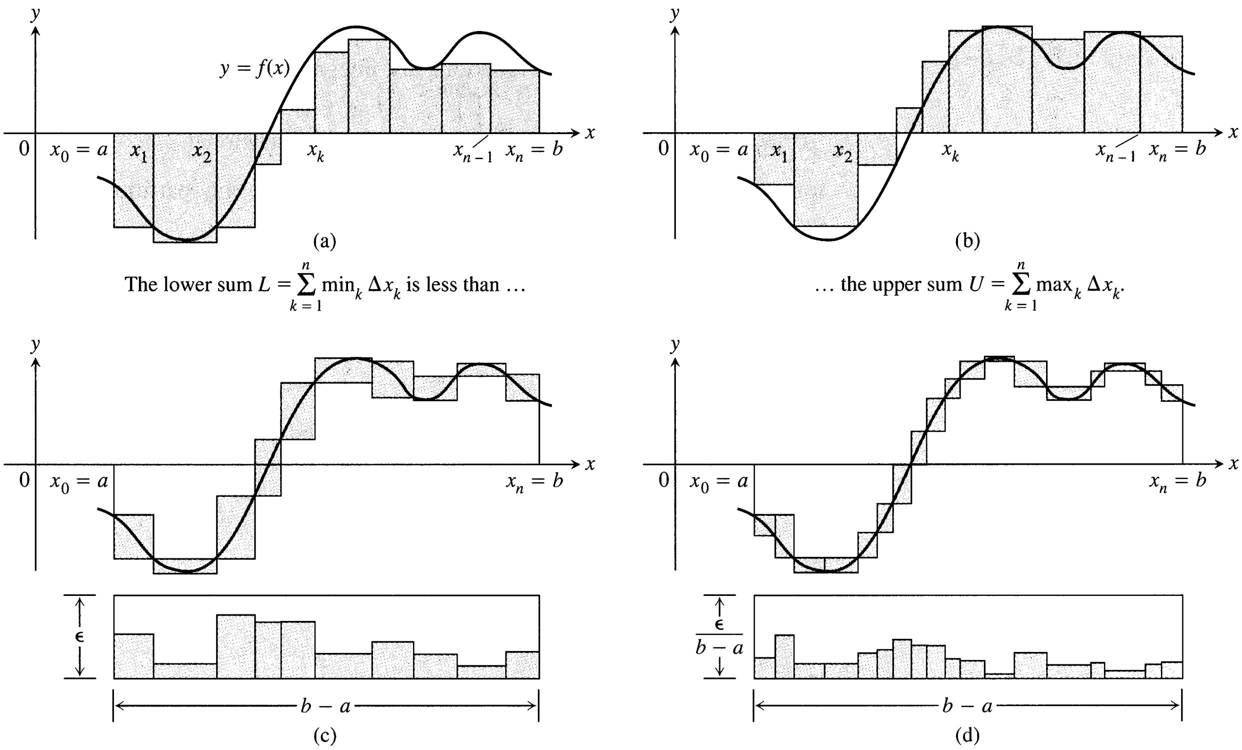
The difference $U - L$ between the upper and lower sums is the sum of the areas of the shaded blocks in Fig. 4.12(c). As $\|P\| \to 0$, the blocks in Fig. 4.12(c) become more numerous, narrower, and shorter. As Fig. 4.12(d) suggests, we can make the nonnegative number $U - L$ less than any prescribed positive ϵ by taking $\|P\|$ close enough to zero. In other words,

$$\lim_{\|P\|\to 0} (U - L) = 0, \tag{4}$$

and, as shown in more advanced texts,

$$\lim_{\|P\|\to 0} L = \lim_{\|P\|\to 0} U. \tag{5}$$

The fact that Eqs. (4) and (5) hold for any continuous function is a consequence of a special property, called *uniform continuity,* that continuous functions have on closed intervals. This property guarantees that as $\|P\| \to 0$ the blocks that make up the difference between U and L in Fig. 4.12(c) become less tall as they become less wide and that we can make them all as short as we please by making them narrow enough. Passing over the ϵ - δ arguments associated with uniform continuity



The lower sum $L = \sum_{k=1}^{n} \min_k \Delta x_k$ is less than ...

... the upper sum $U = \sum_{k=1}^{n} \max_k \Delta x_k$.

The difference $U - L$ can be made very small: less than $\epsilon \cdot (b - a)$.

We can make $U - L$ smaller than any given positive ϵ by making $\|P\|$ small enough.

4.12 The difference between upper and lower sums.

keeps our derivation of Eq. (5) from being a proof. But the argument is right in spirit and gives a faithful portrait of the proof.

Assuming that Eq. (5) holds for any continuous function f on $[a, b]$, suppose we choose a point c_k from each subinterval $[x_{k-1}, x_k]$ of P and form the Riemann sum $\Sigma_{k=1}^{n} f(c_k)\, \Delta x_k$. Then $\min_k \leq f(c_k) \leq \max_k$ for each k, so

$$L \leq \sum_{k=1}^{n} f(c_k)\, \Delta x_k \leq U.$$

The Riemann sum for f is sandwiched between L and U. By a modified version of the Sandwich Theorem of Section 1.2, the limit of the Riemann sums as $\|P\| \to 0$ exists and equals the common limit of U and L:

$$\lim_{\|P\|\to 0} L = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\, \Delta x_k = \lim_{\|P\|\to 0} U.$$

Pause for a moment to see how remarkable this conclusion really is. It says that no matter how we choose the points c_k to form the Riemann sums as $\|P\| \to 0$, the limit is always the same. We can take every $f(c_k)$ to be the minimum value of f on $[x_{k-1}, x_k]$. The limit is the same. We can take every $f(c_k)$ to be the maximum value of f on $[x_{k-1}, x_k]$. The limit is the same. We can choose every c_k at random. The limit is the same.

Although we stated the integral existence theorem specifically for continuous functions, many discontinuous functions are integrable as well. We treat the integration of bounded piecewise continuous functions in Additional Exercises 11–18 at the end of this chapter. We explore the integration of unbounded functions in Section 7.6.

Functions with No Riemann Integral

While some discontinuous functions are integrable, others are not. The function

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ 0 & \text{when } x \text{ is irrational,} \end{cases}$$

for example, has no Riemann integral over [0, 1]. For any partition P of [0, 1], the upper and lower sums are

$$U = \sum \max_k \Delta x_k = \sum 1 \cdot \Delta x_k = \sum \Delta x_k = 1,$$

Every subinterval contains a rational number

$$L = \sum \min_k \Delta x_k = \sum 0 \cdot \Delta x_k = 0.$$

Every subinterval contains an irrational number.

For the integral of f to exist over [0, 1], U and L would have to have the same limit as $\|P\| \to 0$. But they do not:

$$\lim_{\|P\| \to 0} L = 0 \qquad \text{while} \qquad \lim_{\|P\| \to 0} U = 1.$$

Therefore, f has no integral on [0, 1]. No constant multiple kf has an integral either, unless k is zero.

Terminology

There is a fair amount of terminology associated with the symbol $\int_a^b f(x)\,dx$.

Upper limit of integration

The function is the **integrand**.

x is the **variable of integration**.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

Integral of f from a to b

When you find the value of the integral, you have **evaluated** the integral.

The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number, defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a **dummy variable.**

EXAMPLE 6 Express the limit of Riemann sums

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} (3c_k{}^2 - 2c_k + 5)\,\Delta x_k$$

as an integral if P denotes a partition of the interval $[-1, 3]$.

Solution The function being evaluated at c_k in each term of the sum is $f(x) = 3x^2 - 2x + 5$. The interval being partitioned is $[-1, 3]$. The limit is therefore the integral of f from -1 to 3:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (3c_k{}^2 - 2c_k + 5)\,\Delta x_k = \int_{-1}^{3} (3x^2 - 2x + 5)\,dx.$$ ❑

Constant Functions

Theorem 1 says nothing about how to *calculate* definite integrals. Except for a few special cases, that takes another theorem (Section 4.7). Among the exceptions are constant functions. Suppose that f has the constant value $f(x) = c$ over $[a, b]$. Then, no matter how the c_k's are chosen,

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\,\Delta x_k &= \sum_{k=1}^{n} c \cdot \Delta x_k && f(c_k) \text{ always equals } c.\\
&= c \cdot \sum_{k=1}^{n} \Delta x_k && \text{Constant Multiple Rule for Sums}\\
&= c(b-a). && \textstyle\sum_{k=1}^{n} \Delta x_k = \text{length of interval } [a, b] = b - a
\end{aligned}$$

Since the sums all have the value $c(b - a)$, their limit, the integral, does too.

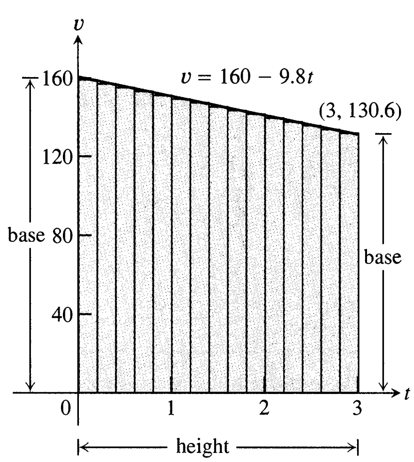
If $f(x)$ has the constant value c on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

EXAMPLE 7

a) $\displaystyle\int_{-1}^{4} 3\,dx = 3(4 - (-1)) = (3)(5) = 15$

b) $\displaystyle\int_{-1}^{4} (-3)\,dx = -3(4 - (-1)) = (-3)(5) = -15$ ❑



4.13 Rectangles for a Riemann sum of the velocity function $f(t) = 160 - 9.8t$ over the interval $[0, 3]$.

The Area Under the Graph of a Nonnegative Function

The sums we used to estimate the height of the projectile in Section 4.4, Example 2, were Riemann sums for the projectile's velocity function

$$v = f(t) = 160 - 9.8t$$

on the interval $[0, 3]$. We can see from Fig. 4.13 how the associated rectangles approximate the trapezoid between the t-axis and the curve $v = 160 - 9.8t$. As the norm of the partition goes to zero, the rectangles fit the trapezoid with increasing accuracy and the sum of the areas they enclose approaches the trapezoid's area, which is

$$\text{Trapezoid area} = h \cdot \frac{b_1 + b_2}{2} = 3 \cdot \frac{160 + 130.6}{2} = 435.9.$$

This confirms our suspicion that the sums we were constructing in Section 4.4, Example 2, approached a limit of 435.9. Since the limit of these sums is also the integral of f from 0 to 3, we now know the value of the integral as well:

$$\int_0^3 (160 - 9.8t)\,dt = \text{trapezoid area} = 435.9.$$

We can exploit the connection between integrals and area in two ways. When we know a formula for the area of the region between the x-axis and the graph of a continuous nonnegative function $y = f(x)$, we can use it to evaluate the function's integral. When we do not know the region's area, we can use the function's integral to define and calculate the area.

Definition

Let $f(x) \geq 0$ be continuous on $[a, b]$. The **area** of the region between the graph of f and the x-axis is

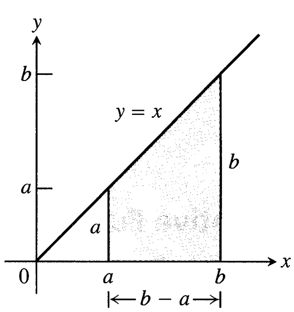
$$A = \int_a^b f(x)\,dx.$$

Whenever we make a new definition, as we have here, consistency becomes an issue. Does the definition that we have just developed for nonstandard shapes give correct results for standard shapes? The answer is yes, but the proof is complicated and we will not go into it.

EXAMPLE 8 *Using an area to evaluate a definite integral*

Evaluate

$$\int_a^b x\,dx, \qquad 0 < a < b.$$



4.14 The region in Example 8.

Solution We sketch the region under the curve $y = x, a \leq x \leq b$ (Fig. 4.14), and see that it is a trapezoid with height $(b - a)$ and bases a and b. The value of the integral is the area of this trapezoid:

$$\int_a^b x\,dx = (b - a) \cdot \frac{a + b}{2} = \frac{b^2}{2} - \frac{a^2}{2}.$$

Thus,

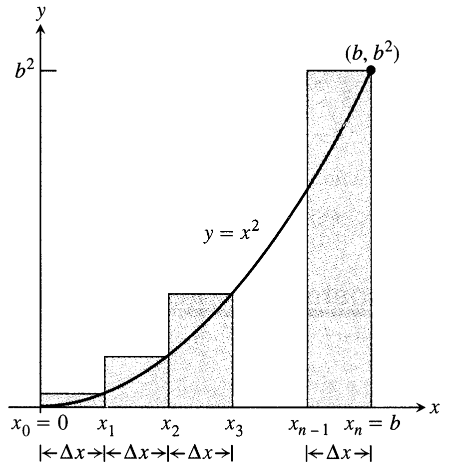
$$\int_1^{\sqrt{5}} x\,dx = \frac{(\sqrt{5})^2}{2} - \frac{(1)^2}{2} = 2$$

and so on.

Notice that $x^2/2$ is an antiderivative of x, further evidence of a connection between antiderivatives and summation. ❑

EXAMPLE 9 *Using a definite integral to find an area*

Find the area of the region between the parabola $y = x^2$ and the x-axis on the interval $[0, b]$.



4.15 The rectangles of the Riemann sums in Example 9.

Solution We evaluate the integral for the area as a limit of Riemann sums.

We sketch the region (a nonstandard shape) (Fig. 4.15) and partition $[0, b]$ into n subintervals of length $\Delta x = (b - 0)/n = b/n$. The points of the partition are

$$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \quad \cdots, \quad x_{n-1} = (n-1)\,\Delta x, \quad x_n = n\Delta x = b.$$

We are free to choose the c_k's any way we please. We choose each c_k to be the right-hand endpoint of its subinterval, a choice that leads to manageable arithmetic. Thus, $c_1 = x_1, c_2 = x_2$, and so on. The rectangles defined by these choices have areas

$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = (\Delta x)^2\,\Delta x = (1^2)(\Delta x)^3$$

$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = (2\Delta x)^2\,\Delta x = (2^2)(\Delta x)^3$$

$$\vdots$$

$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = (n\Delta x)^2\,\Delta x = (n^2)(\Delta x)^3.$$

The sum of these areas is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f(c_k)\,\Delta x \\ &= \sum_{k=1}^{n} k^2(\Delta x)^3 \\ &= (\Delta x)^3 \sum_{k=1}^{n} k^2 && (\Delta x)^3 \text{ is a constant.} \\ &= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && \Delta x = b/n, \text{ and Eq. (2)} \\ &= \frac{b^3}{6} \cdot \frac{(n+1)(2n+1)}{n^2} \\ &= \frac{b^3}{6} \cdot \frac{2n^2+3n+1}{n^2} \\ &= \frac{b^3}{6} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right). \end{aligned} \tag{6}$$

We can now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

to find the area under the parabola from $x = 0$ to $x = b$ as

Notice that $x^3/3$ is an antiderivative of x^2.

$$\begin{aligned} \int_0^b x^2\,dx &= \lim_{n\to\infty} S_n && \text{In this example, } \|P\| \to 0 \text{ is equivalent to } n \to \infty. \\ &= \lim_{n\to\infty} \frac{b^3}{6} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) && \text{Eq. (6)} \\ &= \frac{b^3}{6} \cdot (2+0+0) = \frac{b^3}{3}. \end{aligned}$$

With different values of b, we get

$$\int_0^1 x^2\,dx = \frac{1^3}{3} = \frac{1}{3}, \qquad \int_0^{1.5} x^2\,dx = \frac{(1.5)^3}{3} = \frac{3.375}{3} = 1.125,$$

and so on. ❑

Exercises 4.5

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$ **4.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$

5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\displaystyle\sum_{k=1}^{4} (-1)^k \cos k\pi$

7. Which of the following express $1+2+4+8+16+32$ in sigma notation?

a) $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ b) $\displaystyle\sum_{k=0}^{5} 2^k$ c) $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1-2+4-8+16-32$ in sigma notation?

a) $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ b) $\displaystyle\sum_{k=0}^{5} (-1)^k 2^k$

c) $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a) $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ b) $\displaystyle\sum_{k=0}^{2} \frac{(-1)^k}{k+1}$ c) $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^k}{k+2}$

10. Which formula is not equivalent to the other two?

a) $\displaystyle\sum_{k=1}^{4} (k-1)^2$ b) $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ c) $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

11. $1+2+3+4+5+6$ **12.** $1+4+9+16$

13. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ **14.** $2+4+6+8+10$

15. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ **16.** $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

a) $\displaystyle\sum_{k=1}^{n} 3a_k$ b) $\displaystyle\sum_{k=1}^{n} \frac{b_k}{6}$ c) $\displaystyle\sum_{k=1}^{n} (a_k+b_k)$

d) $\displaystyle\sum_{k=1}^{n} (a_k-b_k)$ e) $\displaystyle\sum_{k=1}^{n} (b_k-2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

a) $\displaystyle\sum_{k=1}^{n} 8a_k$ b) $\displaystyle\sum_{k=1}^{n} 250b_k$

c) $\displaystyle\sum_{k=1}^{n} (a_k+1)$ d) $\displaystyle\sum_{k=1}^{n} (b_k-1)$

Use the algebra rules on p. 310 and the formulas in Eqs. (1)–(3) to evaluate the sums in Exercises 19–28.

19. a) $\displaystyle\sum_{k=1}^{10} k$ b) $\displaystyle\sum_{k=1}^{10} k^2$ c) $\displaystyle\sum_{k=1}^{10} k^3$

20. a) $\displaystyle\sum_{k=1}^{13} k$ b) $\displaystyle\sum_{k=1}^{13} k^2$ c) $\displaystyle\sum_{k=1}^{13} k^3$

21. $\displaystyle\sum_{k=1}^{7} (-2k)$ **22.** $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\displaystyle\sum_{k=1}^{6} (3-k^2)$ **24.** $\displaystyle\sum_{k=1}^{6} (k^2-5)$

25. $\displaystyle\sum_{k=1}^{5} k(3k+5)$ **26.** $\displaystyle\sum_{k=1}^{7} k(2k+1)$

27. $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\displaystyle\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

Rectangles for Riemann Sums

In Exercises 29–32, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

29. $f(x) = x^2 - 1, \quad [0, 2]$

30. $f(x) = -x^2, \quad [0, 1]$

31. $f(x) = \sin x, \quad [-\pi, \pi]$

32. $f(x) = \sin x + 1, \quad [-\pi, \pi]$

33. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

34. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Expressing Limits as Integrals

Express the limits in Exercises 35–42 as definite integrals.

35. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k{}^2 \Delta x_k$, where P is a partition of $[0, 2]$

36. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k{}^3 \Delta x_k$, where P is a partition of $[-1, 0]$

37. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k{}^2 - 3c_k)\, \Delta x_k$, where P is a partition of $[-7, 5]$

38. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$

39. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \Delta x_k$, where P is a partition of $[2, 3]$

40. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k{}^2}\, \Delta x_k$, where P is a partition of $[0, 1]$

41. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sec c_k)\, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

42. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Constant Functions

Evaluate the integrals in Exercises 43–48.

43. $\int_{-2}^{1} 5\, dx$ **44.** $\int_{3}^{7} (-20)\, dx$

45. $\int_{0}^{3} (-160)\, dt$ **46.** $\int_{-4}^{-1} \frac{\pi}{2}\, d\theta$

47. $\int_{-2.1}^{3.4} 0.5\, ds$ **48.** $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\, dr$

Using Area to Evaluate Integrals

In Exercises 49–56, graph the integrands and use areas to evaluate the integrals.

49. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$ **50.** $\int_{1/2}^{3/2} (-2x + 4)\, dx$

51. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$ **52.** $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$

53. $\int_{-2}^{1} |x|\, dx$ **54.** $\int_{-1}^{1} (1 - |x|)\, dx$

55. $\int_{-1}^{1} (2 - |x|)\, dx$ **56.** $\int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx$

Use areas to evaluate the integrals in Exercises 57–60.

57. $\int_{0}^{b} x\, dx, \quad b > 0$ **58.** $\int_{0}^{b} 4x\, dx, \quad b > 0$

59. $\int_{a}^{b} 2s\, ds, \quad 0 < a < b$ **60.** $\int_{a}^{b} 3t\, dt, \quad 0 < a < b$

Evaluations

Use the results of Examples 8 and 9 to evaluate the integrals in Exercises 61–72.

61. $\int_{1}^{\sqrt{2}} x\, dx$ **62.** $\int_{0.5}^{2.5} x\, dx$ **63.** $\int_{\pi}^{2\pi} \theta\, d\theta$

64. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$ **65.** $\int_{0}^{\sqrt[3]{7}} x^2\, dx$ **66.** $\int_{0}^{0.3} s^2\, ds$

67. $\int_{0}^{1/2} t^2\, dt$ **68.** $\int_{0}^{\pi/2} \theta^2\, d\theta$ **69.** $\int_{a}^{2a} x\, dx$

70. $\int_{a}^{\sqrt{3}a} x\, dx$ **71.** $\int_{0}^{\sqrt[3]{b}} x^2\, dx$ **72.** $\int_{0}^{3b} x^2\, dx$

Finding Area

In Exercises 73–76, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$, as in Example 9.

73. $y = 3x^2$ **74.** $y = \pi x^2$

75. $y = 2x$ **76.** $y = \frac{x}{2} + 1$

Theory and Examples

77. What values of a and b maximize the value of

$$\int_{a}^{b} (x - x^2)\, dx?$$

(*Hint:* Where is the integrand positive?)

78. What values of a and b minimize the value of

$$\int_{a}^{b} (x^4 - 2x^2)\, dx?$$

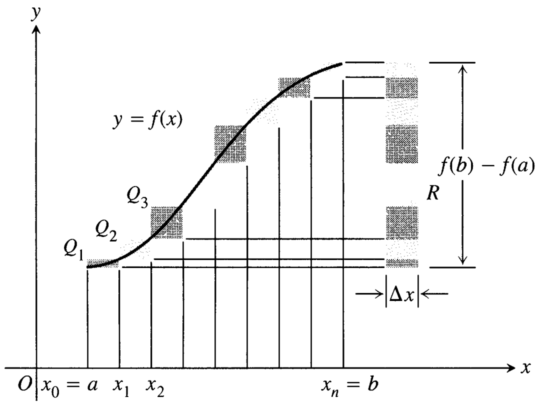
79. *Upper and lower sums for increasing functions*

a) Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b) Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \leq |f(b) - f(a)|\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.



80. *Upper and lower sums for decreasing functions (Continuation of Exercise 79)*

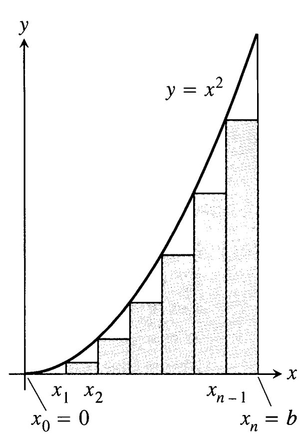
a) Draw a figure like the one in Exercise 79 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 79(a).

b) Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)|\Delta x_{\max}$$

of Exercise 79(b) still holds and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.

81. Evaluate $\int_0^b x^2\,dx$, $b > 0$, by carrying out the calculations of Example 9 with inscribed rectangles, as shown here, instead of circumscribed rectangles.



82. Let

$$S_n = \frac{1}{n}\left[\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n-1}{n}\right].$$

Calculate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximating sum of the integral

$$\int_0^1 x\,dx,$$

whose value we know from Example 8. (*Hint:* Partition [0, 1] into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

83. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n\to\infty} S_n$, show that

$$S_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right]$$

and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2\,dx,$$

whose value we know from Example 9. (*Hint:* Partition [0, 1] into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

84. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$, in two steps:

a) Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b) Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

CAS Explorations and Projects

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 85–90. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

85. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

86. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

87. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

88. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

89. $\int_{-1}^{1} |x|\,dx = 1$

90. $\int_{1}^{2} \frac{1}{x}\,dx$ (The integral's value is ln 2.)

91. a) Write the sum S_n in Exercise 82 in sigma notation and use your CAS to find $\lim_{n\to\infty} S_n$.

b) Do the same for the sum S_n in Exercise 83.

92. Write the sum $\sin h + \sin 2h + \cdots + \sin mh$ in Exercise 84 in sigma notation and use your CAS to find $\lim_{n\to\infty} S_n$.

93. *(Continuation of Section 4.4, Example 3.)* In sigma notation, the left-endpoint sum in Example 3, Section 4.4, is

$$S_4 = \sum_{k=1}^{4} 4\left[9 - (-2 + (k-1))^2\right].$$

a) Use sigma notation to write the analogous left-endpoint sums S_8 for eight subintervals of length 4/8 and S_{25} for 25 subintervals of length $4/25$.

b) Use sigma notation to write the left-endpoint sum S_n for n subintervals of length $4/n$.

c) Find $\lim_{n\to\infty} S_n$. How does this limit appear to be related to the volume of the solid?

94. (*Continuation of Section 4.4 Example 4.*) In sigma notation, the left-endpoint sum in Example 4, Section 4.4, is

$$S_8 = \sum_{k=1}^{8} \pi\left[16 - (-4 + (k-1))^2\right].$$

a) Use sigma notation to write the analogous left-endpoint sums S_{16} for 16 subintervals of length $1/2$ and S_{80} for 80 subintervals of length $1/10$.

b) Use sigma notation to write the left-endpoint sum S_n for n subintervals of length $8/n$.

c) Find $\lim_{n\to\infty} S_n$. How does this limit appear to be related to the volume of the sphere?

4.6 Properties, Area, and the Mean Value Theorem

This section describes working rules for integrals, examines the relationship between the integral of an arbitrary continuous function and area, and takes a fresh look at average value.

Properties of Definite Integrals

We often want to add and subtract definite integrals, multiply their integrands by constants, and compare them with other definite integrals. We do this with the rules in Table 4.5 (on the following page). All the rules except the first two follow from the way integrals are defined with Riemann sums. You might think that this would make them relatively easy to prove. After all, we might argue, sums have these properties so their limits should have them, too. But when we get down to the details we find that most of the proofs require complicated ϵ-δ arguments with norms of subdivisions and are not easy at all. We omit all but two of the proofs. The remaining proofs can be found in more advanced texts.

Notice that Rule 1 is a definition. We want every integral over an interval of zero length to be zero. Rule 1 extends the definition of definite integral to allow for the case $a = b$. Rule 2, also a definition, extends the definition of definite integral to allow for the case $b < a$. Rules 3 and 4 are like the analogous rules for limits and indefinite integrals. Once we know the integrals of two functions, we automatically know the integrals of all constant multiples of these functions and their sums and differences. We can also use Rules 3 and 4 repeatedly to evaluate integrals of arbitrary finite linear combinations of integrable functions term by term. For any

Table 4.5 Rules for definite integrals

1. *Zero:* $\int_a^a f(x)\,dx = 0$ (A definition)
2. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (Also a definition)
3. *Constant Multiples:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ (Any number k)

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ $(k = -1)$
4. *Sums and Differences:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

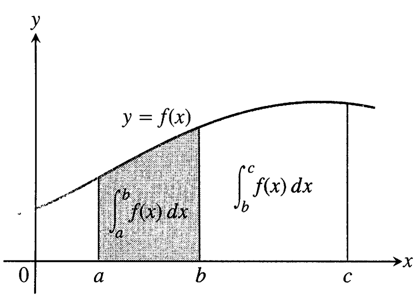
   $f(x) \ge 0$ on $[a, b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \ge 0$ (Special case)

constants $c_1, \ldots, c_n$, regardless of sign, and functions $f_1(x), \ldots, f_n(x)$, integrable on $[a, b]$,

$$\int_a^b (c_1 f_1(x) + \cdots + c_n f_n(x))\,dx = c_1\int_a^b f_1(x)\,dx + \cdots + c_n\int_a^b f_n(x)\,dx.$$

The proof, omitted, comes from mathematical induction.

Figure 4.16 illustrates Rule 5 with a positive function, but the rule applies to any integrable function.



4.16 Additivity for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx.$$

Proof of Rule 3 Rule 3 says that the integral of k times a function is k times the integral of the function. This is true because

$$\begin{aligned}\int_a^b kf(x)\,dx &= \lim_{\|P\|\to 0} \sum_{i=1}^n kf(c_i)\Delta x_i \\ &= \lim_{\|P\|\to 0} k \sum_{i=1}^n f(c_i)\Delta x_i \\ &= k \lim_{\|P\|\to 0} \sum_{i=1}^n f(c_i)\Delta x_i = k\int_a^b f(x)\,dx.\end{aligned}$$

❑

Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k,

$$
\begin{aligned}
\min f \cdot (b-a) &= \min f \cdot \sum_{k=1}^{n} \Delta x_k && \textstyle\sum_{k=1}^{n} \Delta x_k = b-a \\
&= \sum_{k=1}^{n} \min f \cdot \Delta x_k \\
&\le \sum_{k=1}^{n} f(c_k)\, \Delta x_k && \min f \le f(c_k) \\
&\le \sum_{k=1}^{n} \max f \cdot \Delta x_k && f(c_k) \le \max f \\
&= \max f \cdot \sum_{k=1}^{n} \Delta x_k \\
&= \max f \cdot (b-a).
\end{aligned}
$$

In short, all Riemann sums for f on $[a, b]$ satisfy the inequality

$$\min f \cdot (b-a) \le \sum_{k=1}^{n} f(c_k)\, \Delta x_k \le \max f \cdot (b-a).$$

Hence their limit, the integral, does too. ❑

EXAMPLE 1 Suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \qquad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ Rule 2

2. $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$
$\qquad = 2\,(5) + 3\,(7) = 31$ Rules 3 and 4

3. $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ Rule 5 ❑

In Section 4.5 we learned to evaluate three general integrals:

$$\int_{a}^{b} c\,dx = c(b-a) \qquad \text{(Any constant } c\text{)} \tag{1}$$

$$\int_{a}^{b} x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad (0 < a < b) \tag{2}$$

$$\int_{0}^{b} x^2\,dx = \frac{b^3}{3} \qquad (b > 0). \tag{3}$$

The rules in Table 4.5 enable us to build on these results.

EXAMPLE 2 Evaluate $\int_0^2 \left(\frac{t^2}{4} - 7t + 5\right) dt.$

Solution

$$\begin{aligned}
\int_0^2 \left(\frac{t^2}{4} - 7t + 5\right) dt &= \frac{1}{4}\int_0^2 t^2\,dt - 7\int_0^2 t\,dt + \int_0^2 5\,dt && \text{Rules 3 and 4} \\
&= \frac{1}{4}\left(\frac{(2)^3}{3}\right) - 7\left(\frac{(2)^2}{2} - \frac{(0)^2}{2}\right) + 5(2-0) && \text{Eqs. (1)–(3)} \\
&= \frac{2}{3} - 14 + 10 = -\frac{10}{3}
\end{aligned}$$

❑

EXAMPLE 3 Evaluate $\int_2^3 x^2\,dx.$

Solution We cannot apply Eq. (3) directly because the lower limit of integration is different from 0. We can, however, use the Additivity Rule to express $\int_2^3 x^2\,dx$ as a difference of two integrals that *can* be evaluated with Eq. (3):

$$\begin{aligned}
\int_0^2 x^2\,dx + \int_2^3 x^2\,dx &= \int_0^3 x^2\,dx && \text{Rule 5} \\
\int_2^3 x^2\,dx &= \int_0^3 x^2\,dx - \int_0^2 x^2\,dx && \text{Solve for } \textstyle\int_2^3 x^2 dx. \\
&= \frac{(3)^3}{3} - \frac{(2)^3}{3} && \text{Eq. (3) now applies.} \\
&= \frac{27}{3} - \frac{8}{3} = \frac{19}{3}.
\end{aligned}$$

In Section 4.7, we will see how to evaluate $\int_2^3 x^2\,dx$ in a more direct way. ❑

The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b-a)$ is a **lower bound** for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b-a)$ is an **upper bound.**

EXAMPLE 4 Show that the value of

$$\int_0^1 \sqrt{1+\cos x}\,dx$$

cannot possibly be 2.

Solution The maximum value of $\sqrt{1+\cos x}$ on [0, 1] is $\sqrt{1+1} = \sqrt{2}$, so

$$\begin{aligned}
\int_0^1 \sqrt{1+\cos x}\,dx &\le \max \sqrt{1+\cos x} \cdot (1-0) && \text{Table 4.5, Rule 6} \\
&\le \sqrt{2} \cdot 1 = \sqrt{2}.
\end{aligned}$$

The integral cannot exceed $\sqrt{2}$, so it cannot possibly equal 2. ❑

EXAMPLE 5 Use the inequality $\cos x \ge (1 - x^2/2)$, which holds for all x, to find a lower bound for the value of $\int_0^1 \cos x\,dx$.

Solution

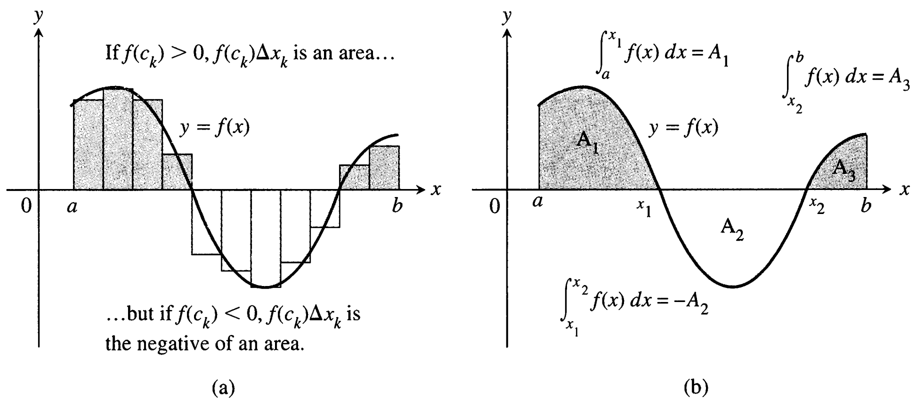
$$\begin{aligned}
\int_0^1 \cos x\,dx &\geq \int_0^1 \left(1 - \frac{x^2}{2}\right) dx && \text{Rule 7}\\
&\geq \int_0^1 1\,dx - \frac{1}{2}\int_0^1 x^2\,dx && \text{Rules 3 and 4}\\
&\geq 1 \cdot (1-0) - \frac{1}{2}\cdot\frac{(1)^3}{3} = \frac{5}{6} \approx 0.83.
\end{aligned}$$

The value of the integral is at least 5/6. ❑

Integrals and Total Area

If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a, b]$, then the Riemann sums for f on $[a, b]$ add the areas of the rectangles that lie above the x-axis to the negatives of the areas of the rectangles that lie below it (Fig. 4.17). The resulting cancellation reduces the sums, so their limiting value is a number whose magnitude is less than the total area between the curve and the x-axis. The value of the integral is the area above the axis minus the area below the axis.

This means that we must take special care in finding areas by integration.

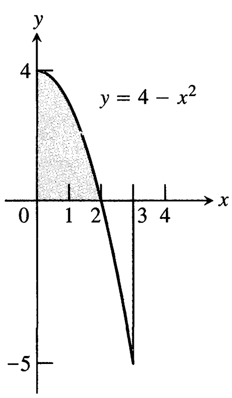


4.17 (a) The Riemann sums are algebraic sums of areas and so is the integral to which they converge. (b) The value of the integral of f from a to b is

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{b} f(x)\,dx = A_1 - A_2 + A_3.$$

EXAMPLE 6 Find the area of the region between the curve $y = 4 - x^2$, $0 \leq x \leq 3$, and the x-axis.

Solution The x-intercept of the curve partitions [0, 3] into subintervals on which $f(x) = 4 - x^2$ has the same sign (Fig. 4.18). To find the area of the region between the graph of f and the x-axis, we integrate f over each subinterval and add the absolute values of the results.



4.18 Part of the region in Example 6 lies below the x-axis.

Integral over [0, 2]:

$$\begin{aligned}
\int_0^2 (4 - x^2)\,dx &= \int_0^2 4\,dx - \int_0^2 x^2\,dx\\
&= 4(2-0) - \frac{(2)^3}{3} && \text{Eqs. (1) and (3)}\\
&= 8 - \frac{8}{3} = \frac{16}{3}
\end{aligned}$$

How to Find the Area of the Region Between a Curve $y = f(x)$, $a \le x \le b$, and the x-axis

1. Partition $[a, b]$ with the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Integral over [2, 3]:

$$\int_2^3 (4 - x^2)\,dx = \int_2^3 4\,dx - \int_2^3 x^2\,dx$$

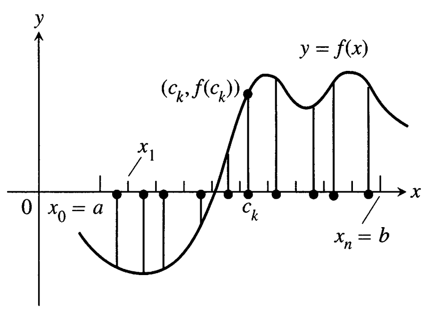
$$= 4(3 - 2) - \left(\frac{(3)^3}{3} - \frac{(2)^3}{3}\right) \qquad \text{Eq. (1) and Example 3}$$

$$= 4 - \frac{19}{3} = -\frac{7}{3}$$

The region's area: $\quad \text{Area} = \frac{16}{3} + \left|-\frac{7}{3}\right| = \frac{23}{3}.$ ❑

The Average Value of an Arbitrary Continuous Function

In Section 4.4, Example 5, we discussed the average value of a nonnegative continuous function. We are now ready to define average value without requiring f to be nonnegative, and to show that every continuous function assumes its average value at least once.



4.19 A sample of values of a function on an interval $[a, b]$.

We start once again with the idea from arithmetic that the average of n numbers is the sum of the numbers divided by n. For a continuous function f on a closed interval $[a, b]$ there may be infinitely many values to consider, but we can sample them in an orderly way. We partition $[a, b]$ into n subintervals of equal length (the length is $\Delta x = (b - a)/n$) and evaluate f at a point c_k in each subinterval (Fig. 4.19). The average of the n sampled values is

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} f(c_k) \qquad \text{The sum in sigma notation}$$

$$= \frac{\Delta x}{b - a} \cdot \sum_{k=1}^{n} f(c_k) \qquad \Delta x = \frac{b - a}{n}$$

$$= \frac{1}{b - a} \cdot \underbrace{\sum_{k=1}^{n} f(c_k)\,\Delta x}_{\text{a Riemann sum for } f \text{ on } [a,\, b]}$$

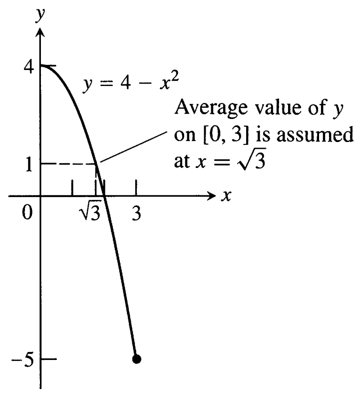
Thus, the average of the sampled values is always $1/(b - a)$ times a Riemann sum for f on $[a, b]$. As we increase the size of the sample and let the norm of the partition approach zero, the average must approach $(1/(b - a)) \int_a^b f(x)\,dx$. We are led by this remarkable fact to the following definition.

Definition

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

EXAMPLE 7 Find the average value of $f(x) = 4 - x^2$ on [0, 3]. Does f actually take on this value at some point in the given domain?



4.20 The average value of $f(x) = 4 - x^2$ on $[0, 3]$ occurs at $x = \sqrt{3}$ (Example 7).

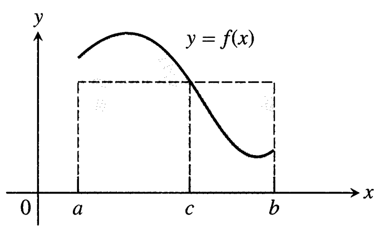
Solution

$$\begin{aligned}
\operatorname{av}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{3-0}\int_0^3 (4-x^2)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x^2\,dx\right) \\
&= \frac{1}{3}\left(4(3-0) - \frac{(3)^3}{3}\right) = \frac{1}{3}(12-9) = 1
\end{aligned}$$

The average value of $f(x) = 4 - x^2$ over the interval [0, 3] is 1. The function assumes this value when $4 - x^2 = 1$ or $x = \pm\sqrt{3}$. Since one of these points, $x = \sqrt{3}$, lies in [0, 3], the function does assume its average value in the given domain (Fig. 4.20). ❑

The Mean Value Theorem for Definite Integrals

The statement that a continuous function on a closed interval assumes its average value at least once in the interval is known as the Mean Value Theorem for Definite Integrals.



4.21 Theorem 2 for a positive function: At some point c in $[a, b]$,

$$f(c) \cdot (b-a) = \int_a^b f(x)\,dx.$$

Theorem 2
The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

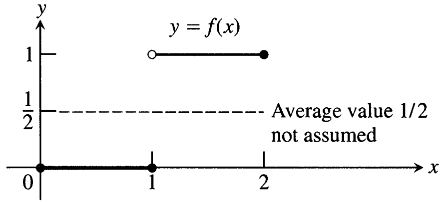
(Fig. 4.21).

In Example 7, we found a point where f assumed its average value by setting $f(x)$ equal to the calculated average value and solving for x. But this does not prove that such a point will always exist. It proves only that it existed in Example 7. To prove Theorem 2, we need a more general argument.

Proof of Theorem 2 If we divide both sides of the Max-Min Inequality (Rule 6) by $(b-a)$, we obtain

$$\min f \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 1.5) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b-a))\int_a^b f(x)\,dx$ at some point c in $[a, b]$. ❑



4.22 A discontinuous function need not assume its average value.

The continuity of f is important here. A discontinuous function can step over its average value (Fig. 4.22).

What else can we learn from Theorem 2? Here is an example.

EXAMPLE 8 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By Theorem 2, f assumes this value at some point c in $[a, b]$. ❑

Exercises 4.6

Using Properties and Known Values to Find Other Integrals

1. Suppose that f and g are continuous and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 4.5 to find

a) $\displaystyle\int_2^2 g(x)\,dx$ **b)** $\displaystyle\int_5^1 g(x)\,dx$

c) $\displaystyle\int_1^2 3f(x)\,dx$ **d)** $\displaystyle\int_2^5 f(x)\,dx$

e) $\displaystyle\int_1^5 [f(x) - g(x)]\,dx$ **f)** $\displaystyle\int_1^5 [4f(x) - g(x)]\,dx$

2. Suppose that f and h are continuous and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 4.5 to find

a) $\displaystyle\int_1^9 -2f(x)\,dx$ **b)** $\displaystyle\int_7^9 [f(x) + h(x)]\,dx$

c) $\displaystyle\int_7^9 [2f(x) - 3h(x)]\,dx$ **d)** $\displaystyle\int_9^1 f(x)\,dx$

e) $\displaystyle\int_1^7 f(x)\,dx$ **f)** $\displaystyle\int_9^7 [h(x) - f(x)]\,dx$

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a) $\displaystyle\int_1^2 f(u)\,du$ **b)** $\displaystyle\int_1^2 \sqrt{3}f(z)\,dz$

c) $\displaystyle\int_2^1 f(t)\,dt$ **d)** $\displaystyle\int_1^2 [-f(x)]\,dx$

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a) $\displaystyle\int_0^{-3} g(t)\,dt$ **b)** $\displaystyle\int_{-3}^0 g(u)\,du$

c) $\displaystyle\int_{-3}^0 [-g(x)]\,dx$ **d)** $\displaystyle\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

5. Suppose that f is continuous and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a) $\displaystyle\int_3^4 f(z)\,dz$ **b)** $\displaystyle\int_4^3 f(t)\,dt$

6. Suppose that h is continuous and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a) $\displaystyle\int_1^3 h(r)\,dr$ **b)** $\displaystyle -\int_3^1 h(u)\,du$

Evaluate the integrals in Exercises 7–18.

7. $\displaystyle\int_3^1 7\,dx$ **8.** $\displaystyle\int_0^{-2} \sqrt{2}\,dx$

9. $\displaystyle\int_0^2 5x\,dx$ **10.** $\displaystyle\int_3^5 \frac{x}{8}\,dx$

11. $\displaystyle\int_0^2 (2t - 3)\,dt$ **12.** $\displaystyle\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right)\,dt$

13. $\displaystyle\int_2^1 \left(1 + \frac{z}{2}\right)\,dz$ **14.** $\displaystyle\int_3^0 (2z - 3)\,dz$

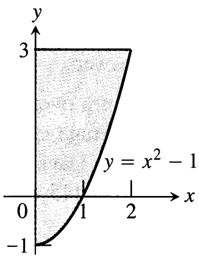
15. $\displaystyle\int_1^2 3u^2\,du$ **16.** $\displaystyle\int_{1/2}^1 24u^2\,du$

17. $\displaystyle\int_0^2 (3x^2 + x - 5)\,dx$ **18.** $\displaystyle\int_1^0 (3x^2 + x - 5)\,dx$

Area

In Exercises 19–22, find the total shaded area.

19.



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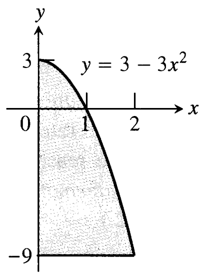

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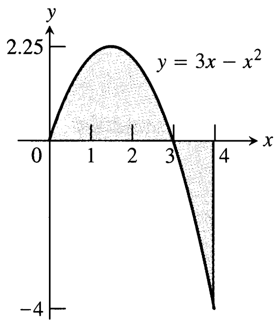

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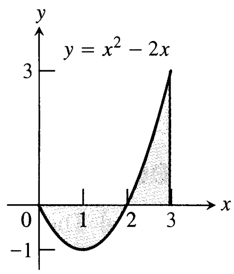


In Exercises 23–26, graph the function over the given interval. Then (a) integrate the function over the interval and (b) find the area of the region between the graph and the x-axis.

23. $y = x^2 - 6x + 8, \quad [0, 3]$

24. $y = -x^2 + 5x - 4, \quad [0, 2]$

25. $y = 2x - x^2, \quad [0, 3]$

26. $y = x^2 - 4x, \quad [0, 5]$

Average Value

In Exercises 27–34, graph the function and find its average value over the given interval. At what point or points in the given interval does the function assume its average value?

27. $f(x) = x^2 - 1 \quad \text{on} \quad \left[0, \sqrt{3}\right]$

28. $f(x) = -\dfrac{x^2}{2} \quad \text{on} \quad [0, 3]$

29. $f(x) = -3x^2 - 1 \quad \text{on} \quad [0, 1]$

30. $f(x) = 3x^2 - 3 \quad \text{on} \quad [0, 1]$

31. $f(t) = (t - 1)^2 \quad \text{on} \quad [0, 3]$

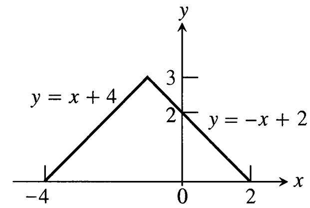
32. $f(t) = t^2 - t \quad \text{on} \quad [-2, 1]$

33. $g(x) = |x| - 1 \quad \text{on} \quad \text{(a)}\,[-1, 1], \text{(b)}\,[1, 3], \text{and (c)}\,[-1, 3]$

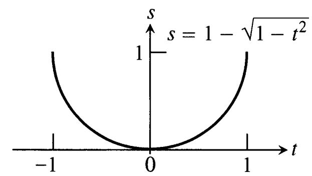
34. $h(x) = -|x| \quad \text{on} \quad \text{(a)}\,[-1, 0], \text{(b)}\,[0, 1], \text{and (c)}\,[-1, 1]$

In Exercises 35–38, find the average value of the function over the given interval from the graph of f (without integrating).

35. $f(x) = \begin{cases} x + 4, & -4 \le x \le -1 \\ -x + 2, & -1 < x \le 2 \end{cases} \quad \text{on} \quad [-4, 2]$



36. $f(t) = 1 - \sqrt{1 - t^2} \quad \text{on} \quad [-1, 1]$



37. $f(t) = \sin t \quad \text{on} \quad [0, 2\pi]$

38. $f(\theta) = \tan\theta \quad \text{on} \quad \left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

Theory and Examples

39. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

40. (*Continuation of Exercise* 39.) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\, dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\, dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

41. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

42. Show that the value of $\int_0^1 \sqrt{x + 8}\, dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

43. Suppose that f is continuous and that $\int_1^2 f(x)\, dx = 4$. Show that $f(x) = 4$ at least once on $[1, 2]$.

44. Suppose that f and g are continuous on $[a, b]$, $a \ne b$, and that $\int_a^b (f(x) - g(x))\, dx = 0$. Show that $f(x) = g(x)$ at least once in $[a, b]$.

45. *Integrals of nonnegative functions.* Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \ge 0.$$

46. *Integrals of nonpositive functions.* Show that if f is integrable then

$$f(x) \leq 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \leq 0.$$

47. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

48. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

49. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\text{av}(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b \text{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

50. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$:

a) $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$
b) $\text{av}(kf) = k\,\text{av}(f)$ (any number k)
c) $\text{av}(f) \leq \text{av}(g)$ if $f(x) \leq g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

51. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer. (Source: David H. Pleacher, *The Mathematics Teacher,* Vol. 85, No. 6, pp. 445–446, September 1992.)

52. A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

4.7 The Fundamental Theorem

This section presents the Fundamental Theorem of Integral Calculus. The independent discovery by Leibniz and Newton of this astonishing connection between integration and differentiation started the mathematical developments that fueled the scientific revolution for the next two hundred years and constitutes what is still regarded as the most important computational discovery in the history of the world.

The Fundamental Theorem, Part 1

If $f(t)$ is an integrable function, the integral from any fixed number a to another number x defines a function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

For example, if f is nonnegative and x lies to the right of a, $F(x)$ is the area under the graph from a to x. The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x there is a well-defined numerical output, in this case the integral of f from a to x.

Equation (1) gives an important way to define new functions and to describe solutions of differential equations (more about this later). The reason for mentioning Eq. (1) now, however, is the connection it makes between integrals and derivatives. For if f is any continuous function whatever, then F is a differentiable function of x whose derivative is f itself. At every value of x,

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

This idea is so important that it is the first part of the Fundamental Theorem of Calculus.

Theorem 3

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point of $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad a \le x \le b. \tag{2}$$

This conclusion is beautiful, powerful, deep, and surprising, and Eq. (2) may well be the most important equation in mathematics. It says that the differential equation $dF/dx = f$ has a solution for every continuous function f. It says that every continuous function f is the derivative of some other function, namely $\int_a^x f(t)\,dt$. It says that every continuous function has an antiderivative. And it says that the processes of integration and differentiation are inverses of one another.

Proof of Theorem 3 We prove Theorem 3 by applying the definition of derivative directly to the function $F(x)$. This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$.

When we replace $F(x+h)$ and $F(x)$ by their defining integrals, the numerator in Eq. (3) becomes

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

The Additivity Rule for integrals (Table 4.5 in Section 4.6) simplifies the right-hand side to

$$\int_x^{x+h} f(t)\,dt,$$

so that Eq. (3) becomes

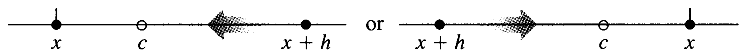
$$\begin{aligned} \frac{F(x+h) - F(x)}{h} &= \frac{1}{h}[F(x+h) - F(x)] \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned} \tag{4}$$

According to the Mean Value Theorem for Definite Integrals (Theorem 2 in the preceding section), the value of the last expression in Eq. (4) is one of the values taken on by f in the interval joining x and $x+h$. That is, for some number c in this interval,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c). \tag{5}$$

We can therefore find out what happens to $(1/h)$ times the integral as $h \to 0$ by watching what happens to $f(c)$ as $h \to 0$.

What does happen to $f(c)$ as $h \to 0$? As $h \to 0$, the endpoint $x + h$ approaches x, pushing c ahead of it like a bead on a wire:



So c approaches x, and, since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x). \tag{6}$$

Going back to the beginning, then, we have

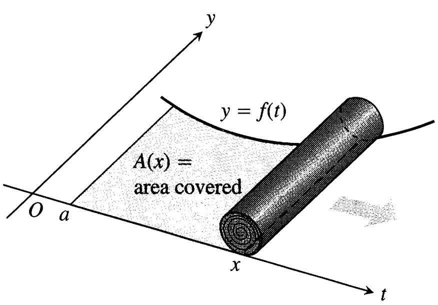
$$\begin{aligned}
\frac{dF}{dx} &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} && \text{Definition of derivative} \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt && \text{Eq. (4)} \\
&= \lim_{h \to 0} f(c) && \text{Eq. (5)} \\
&= f(x). && \text{Eq. (6)}
\end{aligned}$$

This concludes the proof. ❑

If the values of f are positive, the equation

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

has a nice geometric interpretation. For then the integral of f from a to x is the area $A(x)$ of the region between the graph of f and the x-axis from a to x. Imagine covering this region from left to right by unrolling a carpet of variable width $f(t)$ (Fig. 4.23). As the carpet rolls past x, the rate at which the floor is being covered is $f(x)$.



4.23 The rate at which the carpet covers the floor at the point x is the width of the carpet's leading edge as it rolls past x. In symbols, $dA/dx = f(x)$.

EXAMPLE 1

$$\frac{d}{dx} \int_{-\pi}^x \cos t\,dt = \cos x \qquad \text{Eq. (2) with } f(t) = \cos t$$

$$\frac{d}{dx} \int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2} \qquad \text{Eq. (2) with } f(t) = \frac{1}{1+t^2}$$

❑

EXAMPLE 2 Find dy/dx if

$$y = \int_1^{x^2} \cos t\,dt.$$

Solution Notice that the upper limit of integration is not x but x^2. To find dy/dx we must therefore treat y as the composite of

$$y = \int_1^u \cos t\,dt \qquad \text{and} \qquad u = x^2$$

and apply the Chain Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{Chain Rule}\\
&= \frac{d}{du}\int_1^u \cos t\,dt \cdot \frac{du}{dx} && \text{Substitute the formula for } y.\\
&= \cos u \cdot \frac{du}{dx} && \text{Eq. (2) with } f(t)=\cos t\\
&= \cos x^2 \cdot 2x && u=x^2\\
&= 2x\cos x^2. && \text{Usual form}
\end{aligned}$$

❑

EXAMPLE 3 Express the solution of the following initial value problem as an integral.

Differential equation: $\dfrac{dy}{dx} = \tan x$

Initial condition: $y(1) = 5$

Solution The function

$$F(x) = \int_1^x \tan t\,dt$$

is an antiderivative of $\tan x$. Hence the general solution of the equation is

$$y = \int_1^x \tan t\,dt + C.$$

As always, the initial condition determines the value of C:

$$\begin{aligned}
5 &= \int_1^1 \tan t\,dt + C && y(1)=5\\
5 &= 0 + C && (7)\\
C &= 5.
\end{aligned}$$

The solution of the initial value problem is

$$y = \int_1^x \tan t\,dt + 5.$$

How did we know where to start integrating when we constructed $F(x)$? We could have started anywhere, but the best value to start with is the initial value of x (in this case $x = 1$). Then the integral will be zero when we apply the initial condition (as it was in Eq. 7) and C will automatically be the initial value of y.

❑

The Evaluation of Definite Integrals

We now come to the second part of the Fundamental Theorem of Calculus, the part that describes how to evaluate definite integrals.

Theorem 4
The Fundamental Theorem of Calculus, Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{8}$$

How to Evaluate $\int_a^b f(x)\,dx$

1. Find an antiderivative F of f. Any antiderivative will do, so pick the simplest one you can.
2. Calculate the number $F(b) - F(a)$.

This number will be $\int_a^b f(x)\,dx$.

Theorem 4 says that to evaluate the definite integral of a continuous function f from a to b, all we need do is find an antiderivative F of f and calculate the number $F(b) - F(a)$. The existence of the antiderivative is assured by the first part of the Fundamental Theorem.

Proof of Theorem 4 To prove Theorem 4, we use the fact that functions with identical derivatives differ only by a constant. We already know one function whose derivative equals f, namely,

$$G(x) = \int_a^x f(t)\,dt.$$

Therefore, if F is any other such function, then

$$F(x) = G(x) + C \tag{9}$$

throughout $[a, b]$ for some constant C. When we use Eq. (9) to calculate $F(b) - F(a)$, we find that

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt. \end{aligned}$$

This establishes Eq. (8) and concludes the proof. ❑

Notation

The usual notation for the number $F(b) - F(a)$ is $F(x)]_a^b$ when $F(x)$ has a single term, or $[F(x)]_a^b$ for $F(b) - F(a)$ when $F(x)$ has more than one term.

EXAMPLE 4

a) $$\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi = \sin \pi - \sin 0 = 0 - 0 = 0$$

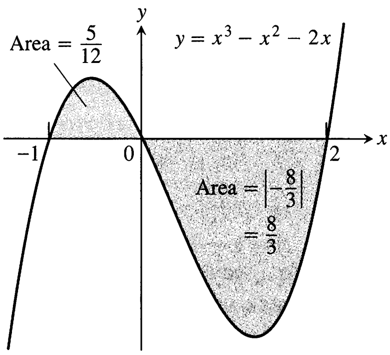
b) $$\int_{-\pi/4}^0 \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^0 = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

c) $$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\ &= [8 + 1] - [5] = 4. \end{aligned}$$ ❑

Theorem 4 explains the formulas we derived for the integrals of x and x^2 in Section 4.5. We can now see that without any restriction on the signs of a and b,

$$\int_a^b x\,dx = \frac{x^2}{2}\bigg]_a^b = \frac{b^2}{2} - \frac{a^2}{2} \qquad \text{Because } x^2/2 \text{ is an antiderivative of } x$$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\bigg]_a^b = \frac{b^3}{3} - \frac{a^3}{3} \qquad \text{Because } x^3/3 \text{ is an antiderivative of } x^2$$



4.24 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 5).

EXAMPLE 5 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2),$$

the zeros are $x = 0$, -1, and 2 (Fig. 4.24). The zeros partition $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$ and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated values.

Integral over $[-1, 0]$:
$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

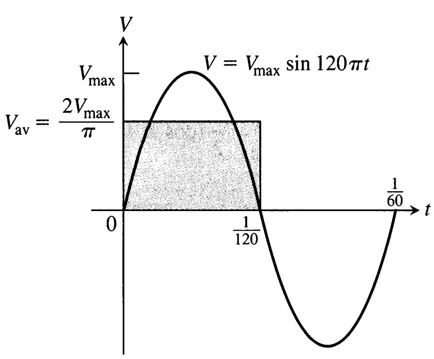
Integral over $[0, 2]$:
$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Enclosed area:
$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$ ❑

EXAMPLE 6 *Household electricity*

We model the voltage in our home wiring with the sine function

$$V = V_{\max} \sin 120\pi t,$$



4.25 The graph of the household voltage $V = V_{\max} \sin 120\pi t$ over a full cycle. Its average value over a half-cycle is $2V_{\max}/\pi$. Its average value over a full cycle is zero.

which expresses the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 hertz, or 60 Hz). The positive constant $V_{\max}$ ("vee max") is the **peak voltage.**

The average value of V over a half-cycle (duration 1/120 sec; see Fig. 4.25) is

$$\begin{aligned} V_{av} &= \frac{1}{(1/120) - 0} \int_0^{1/120} V_{\max} \sin 120\pi t\,dt \\ &= 120 V_{\max}\left[-\frac{1}{120\pi} \cos 120\pi t\right]_0^{1/120} \\ &= \frac{V_{\max}}{\pi}[-\cos\pi + \cos 0] \\ &= \frac{2V_{\max}}{\pi}. \end{aligned}$$

The average value of the voltage over a full cycle, as we can see from Fig. 4.25, is zero. (Also see Exercise 64.) If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{\text{rms}} = \sqrt{(V^2)_{\text{av}}}.$$

The subscript "rms" (read the letters separately) stands for "root mean square." Since the average value of $V^2 = (V_{\max})^2 \sin^2 120\pi t$ over a cycle is

$$(V^2)_{\text{av}} = \frac{1}{(1/60) - 0} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{2} \tag{10}$$

(Exercise 64c), the rms voltage is

$$V_{\text{rms}} = \sqrt{\frac{(V_{\max})^2}{2}} = \frac{V_{\max}}{\sqrt{2}}. \tag{11}$$

The values given for household currents and voltages are always rms values. Thus, "115 volts ac" means that the rms voltage is 115. The peak voltage,

$$V_{\max} = \sqrt{2}\, V_{\text{rms}} = \sqrt{2} \cdot 115 \approx 163 \text{ volts},$$

obtained from Eq. (11), is considerably higher. ❑

Exercises 4.7

Evaluating Integrals

Evaluate the integrals in Exercises 1–26.

1. $\displaystyle\int_{-2}^{0} (2x + 5)\, dx$

2. $\displaystyle\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx$

3. $\displaystyle\int_{0}^{4} \left(3x - \frac{x^3}{4}\right) dx$

4. $\displaystyle\int_{-2}^{2} (x^3 - 2x + 3)\, dx$

5. $\displaystyle\int_{0}^{1} (x^2 + \sqrt{x})\, dx$

6. $\displaystyle\int_{0}^{5} x^{3/2}\, dx$

7. $\displaystyle\int_{1}^{32} x^{-6/5}\, dx$

8. $\displaystyle\int_{-2}^{-1} \frac{2}{x^2}\, dx$

9. $\displaystyle\int_{0}^{\pi} \sin x\, dx$

10. $\displaystyle\int_{0}^{\pi} (1 + \cos x)\, dx$

11. $\displaystyle\int_{0}^{\pi/3} 2 \sec^2 x\, dx$

12. $\displaystyle\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$

13. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta\, d\theta$

14. $\displaystyle\int_{0}^{\pi/3} 4 \sec u \tan u\, du$

15. $\displaystyle\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\, dt$

16. $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2}\, dt$

17. $\displaystyle\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\, dy$

18. $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$

19. $\displaystyle\int_{1}^{-1} (r + 1)^2\, dr$

20. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4)\, dt$

21. $\displaystyle\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$

22. $\displaystyle\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv$

23. $\displaystyle\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$

24. $\displaystyle\int_{9}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}}\, du$

25. $\displaystyle\int_{-4}^{4} |x|\, dx$

26. $\displaystyle\int_{0}^{\pi} \frac{1}{2} (\cos x + |\cos x|)\, dx$

Evaluating Integrals Using Substitutions

In Exercises 27–34, use a substitution to find an antiderivative and then apply the Fundamental Theorem to evaluate the integral.

27. $\displaystyle\int_{0}^{1} (1 - 2x)^3\, dx$

28. $\displaystyle\int_{1}^{2} \sqrt{3x + 1}\, dx$

29. $\displaystyle\int_{0}^{1} t\sqrt{t^2 + 1}\, dt$

30. $\displaystyle\int_{-1}^{2} \frac{t\, dt}{\sqrt{2t^2 + 8}}$

31. $\displaystyle\int_0^{\pi} \sin^2\left(1+\frac{\theta}{2}\right) d\theta$

32. $\displaystyle\int_{3\pi/8}^{\pi/2} \sec^2(\pi - 2\theta)\, d\theta$

33. $\displaystyle\int_0^{\pi} \sin^2 \frac{x}{4} \cos \frac{x}{4}\, dx$

34. $\displaystyle\int_{2\pi/3}^{\pi} \tan^3 \frac{x}{4} \sec^2 \frac{x}{4}\, dx$

Area

In Exercises 35–40, find the total area between the region and the x-axis.

35. $y = -x^2 - 2x, \quad -3 \le x \le 2$

36. $y = 3x^2 - 3, \quad -2 \le x \le 2$

37. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

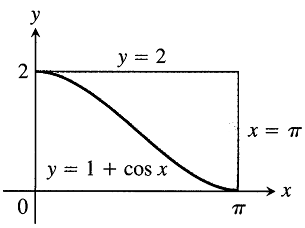
38. $y = x^3 - 4x, \quad -2 \le x \le 2$

39. $y = x^{1/3}, \quad -1 \le x \le 8$

40. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 41–44.

41.



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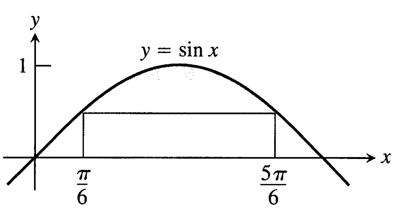

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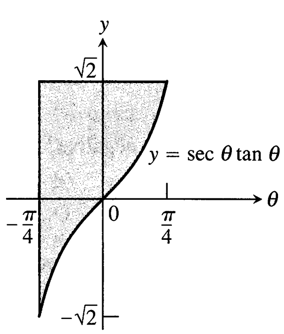

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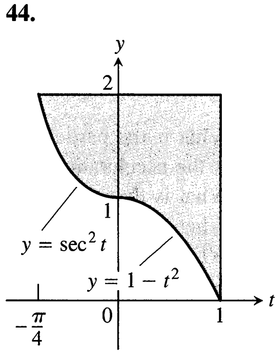


Derivatives of Integrals

Find the derivatives in Exercises 45–48 (a) by evaluating the integral and differentiating the result and (b) by differentiating the integral directly.

45. $\displaystyle\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\, dt$

46. $\displaystyle\frac{d}{dx}\int_1^{\sin x} 3t^2\, dt$

47. $\displaystyle\frac{d}{dt}\int_0^{t^4} \sqrt{u}\, du$

48. $\displaystyle\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy$

Find dy/dx in Exercises 49–54.

49. $\displaystyle y = \int_0^x \sqrt{1+t^2}\, dt$

50. $\displaystyle y = \int_1^x \frac{1}{t}\, dt, \quad x > 0$

51. $\displaystyle y = \int_0^{\sqrt{x}} \sin(t^2)\, dt$

52. $\displaystyle y = \int_0^{x^2} \cos\sqrt{t}\, dt$

53. $\displaystyle y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$

54. $\displaystyle y = \int_0^{\tan x} \frac{dt}{1+t^2}$

Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 55–58. Which function solves which problem? Give brief reasons for your answers.

a) $\displaystyle y = \int_1^x \frac{1}{t}\, dt - 3$

b) $\displaystyle y = \int_0^x \sec t\, dt + 4$

c) $\displaystyle y = \int_{-1}^x \sec t\, dt + 4$

d) $\displaystyle y = \int_\pi^x \frac{1}{t}\, dt - 3$

55. $\displaystyle\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$

56. $y' = \sec x, \quad y(-1) = 4$

57. $y' = \sec x, \quad y(0) = 4$

58. $\displaystyle y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 59–62 in terms of integrals.

59. $\displaystyle\frac{dy}{dx} = \sec x, \quad y(2) = 3$

60. $\displaystyle\frac{dy}{dx} = \sqrt{1+x^2}, \quad y(1) = -2$

61. $\displaystyle\frac{ds}{dt} = f(t), \quad s(t_0) = s_0$

62. $\displaystyle\frac{dv}{dt} = g(t), \quad v(t_0) = v_0$

Applications

63. *Archimedes' area formula for parabolas.* Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the western world, discovered that the area under a parabolic arch is two-thirds the base times the height.

a) Use an integral to find the area under the arch

$$y = 6 - x - x^2, \quad -3 \le x \le 2.$$

b) Find the height of the arch.
c) Show that the area is two-thirds the base b times the height h.
d) Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

64. (*Continuation of Example 6.*)

a) Show by evaluating the integral in the expression

$$\frac{1}{(1/60) - 0} \int_0^{1/60} V_{\max} \sin 120\pi t \, dt$$

that the average value of $V = V_{\max} \sin 120\pi t$ over a full cycle is zero.

b) The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?
c) Show that

$$\int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{120}.$$

65. *Cost from marginal cost.* The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find (a) $c(100) - c(1)$, the cost of printing posters 2–100; (b) $c(400) - c(100)$, the cost of printing posters 101–400.

66. *Revenue from marginal revenue.* Suppose that a company's marginal revenue from the manufacture and sale of egg beaters is

$$\frac{dr}{dx} = 2 - 2/(x+1)^2,$$

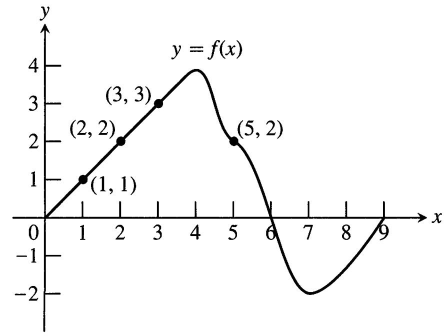
where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand egg beaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Drawing Conclusions about Motion from Graphs

67. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\, dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.

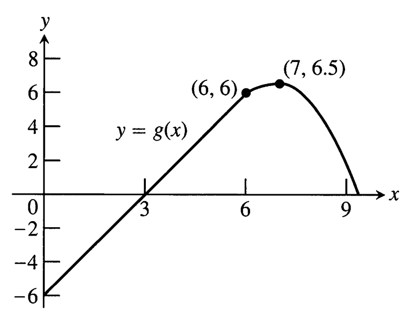


a) What is the particle's velocity at time $t = 5$?
b) Is the acceleration of the particle at time $t = 5$ positive, or negative?
c) What is the particle's position at time $t = 3$?
d) At what time during the first 9 sec does s have its largest value?
e) Approximately when is the acceleration zero?
f) When is the particle moving toward the origin? away from the origin?
g) On which side of the origin does the particle lie at time $t = 9$?

68. Suppose that g is the differentiable function graphed here and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t g(x)\, dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a) What is the particle's velocity at $t = 3$?
b) Is the acceleration at time $t = 3$ positive, or negative?
c) What is the particle's position at time $t = 3$?
d) When does the particle pass through the origin?
e) When is the acceleration zero?
f) When is the particle moving away from the origin? toward the origin?
g) On which side of the origin does the particle lie at $t = 9$?

Volumes from Section 4.4

69. (*Continuation of Section 4.4, Example 3.*) The approximating sum for the volume of the solid in Example 3, Section 4.4, was a Riemann sum for an integral. What integral? Evaluate it to find the volume.

70. (*Continuation of Section 4.4, Example 4.*) The approximating sum for the volume of the sphere in Example 4, Section 4.4, was a Riemann sum for an integral. What integral? Evaluate it to find the volume.

71. (*Continuation of Section 4.4, Exercise 15.*) The approximating sums for the volume of water in Exercise 15, Section 4.4, are Riemann sums for an integral. What integral? Evaluate it to find the volume.

72. (*Continuation of Section 4.4, Exercise 17.*) The approximating sums for the volume of the rocket nose cone in Exercise 17, Section 4.4, is a Riemann sum for an integral. What integral? Evaluate it to find the volume.

Theory and Examples

73. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

74. Find

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t^2}{t^4+1}\,dt.$$

75. Suppose $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

76. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos \pi x$.

77. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt$$

at $x = 1$.

78. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$$

at $x = -1$.

79. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a) g is a differentiable function of x.
b) g is a continuous function of x.
c) The graph of g has a horizontal tangent at $x = 1$.
d) g has a local maximum at $x = 1$.
e) g has a local minimum at $x = 1$.
f) The graph of g has an inflection point at $x = 1$.
g) The graph of dg/dx crosses the x-axis at $x = 1$.

80. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a) h is a twice-differentiable function of x.
b) h and dh/dx are both continuous.
c) The graph of h has a horizontal tangent at $x = 1$.
d) h has a local maximum at $x = 1$.
e) h has a local minimum at $x = 1$.
f) The graph of h has an inflection point at $x = 1$.
g) The graph of dh/dx crosses the x-axis at $x = 1$.

Grapher Explorations

81. *The Fundamental Theorem.* If f is continuous, we expect

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

to equal $f(x)$, as in the proof of Part 1 of the Fundamental Theorem. For instance, if $f(t) = \cos t$, then

$$\frac{1}{h}\int_x^{x+h}\cos t\,dt = \frac{\sin(x+h) - \sin x}{h}. \tag{12}$$

The right-hand side of Eq. (12) is the difference quotient for the derivative of the sine, and we expect its limit as $h \to 0$ to be $\cos x$.

Graph $\cos x$ for $-\pi \le x \le 2\pi$. Then, in a different color if possible, graph the right-hand side of Eq. (12) as a function of x for $h = 2$, 1, 0.5, and 0.1. Watch how the latter curves converge to the graph of the cosine as $h \to 0$.

82. Repeat Exercise 81 for $f(t) = 3t^2$. What is

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} 3t^2\,dt = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}?$$

Graph $f(x) = 3x^2$ for $-1 \le x \le 1$. Then graph the quotient $((x+h)^3 - x^3)/h$ as a function of x for $h = 1$, 0.5, 0.2, and 0.1. Watch how the latter curves converge to the graph of $3x^2$ as $h \to 0$.

CAS Explorations and Projects

In Exercises 83–86, let $F(x) = \int_a^x f(t)\,dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a) Plot the functions f and F together over $[a, b]$.
b) Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.
c) Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d) Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

83. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

84. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \frac{9}{2}\right]$

85. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

86. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 87–90, let $F(x) = \int_a^{u(x)} f(t)\,dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a) Find the domain of F.

b) Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? decreasing?

c) Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d) Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

87. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

88. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

89. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

90. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

91. Calculate $\frac{d}{dx} \int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

92. Calculate $\frac{d^2}{dx^2} \int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

4.8 Substitution in Definite Integrals

There are two methods for evaluating a definite integral by substitution, and they both work well. One is to find the corresponding indefinite integral by substitution and use one of the resulting antiderivatives to evaluate the definite integral by the Fundamental Theorem. The other is to use the following formula.

This formula first appeared in a book written by Isaac Barrow (1630–1677), Newton's teacher and predecessor at Cambridge University.

Substitution in Definite Integrals

THE FORMULA

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \tag{1}$$

HOW TO USE IT

Substitute $u = g(x)$, $du = g'(x)\,dx$, and integrate from $g(a)$ to $g(b)$.

To use the formula, make the same u-substitution you would use to evaluate the corresponding indefinite integral. Then integrate with respect to u from the value u has at $x = a$ to the value u has at $x = b$.

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral and evaluate the transformed integral with the transformed limits given by Eq. (1).

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_{0}^{2} \sqrt{u}\,du \qquad \begin{array}{l}\text{Let } u = x^3+1,\ du = 3x^2\,dx.\\ \text{When } x=-1,\ u=(-1)^3+1=0.\\ \text{When } x=1,\ u=(1)^3+1=2.\end{array}$$

$$= \frac{2}{3}u^{3/2}\Big]_{0}^{2} \qquad \text{Evaluate the new definite integral.}$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$ ❑

Which method is better—transforming the integral, integrating, and transforming back to use the original limits of integration, or evaluating the transformed integral with transformed limits? In Example 1, the second method seems easier, but that is not always the case. As a rule, it is best to know both methods and to use whichever one seems better at the time.

Here is another example of evaluating a transformed integral with transformed limits.

EXAMPLE 2

$$\int_{\pi/4}^{\pi/2} \cot\theta\,\csc^2\theta\,d\theta = \int_{1}^{0} u\cdot(-du) \qquad \begin{array}{l}\text{Let } u=\cot\theta,\ du=-\csc^2\theta\,d\theta\\ \qquad -du=\csc^2\theta\,d\theta.\\ \text{When } \theta=\pi/4,\ u=\cot(\pi/4)=1.\\ \text{When } \theta=\pi/2,\ u=\cot(\pi/2)=0.\end{array}$$

$$= -\int_{1}^{0} u\,du$$

$$= -\left[\frac{u^2}{2}\right]_{1}^{0}$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$ ❑

Technology *Visualizing Integrals with Elusive Antiderivatives* Many integrable functions, such as the important

$$f(x) = e^{-x^2}$$

from probability theory, *do not* have antiderivatives that can be expressed in terms of elementary functions. Nevertheless, we know the antiderivative of f exists by Part 1 of the Fundamental Theorem of Calculus. Use your graphing utility to visualize the integral function

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

What can you say about $F(x)$? Where is it increasing and decreasing? Where are its extreme values, if any? What can you say about the concavity of its graph?

Exercises 4.8

Evaluating Definite Integrals

Evaluate the integrals in Exercises 1–24.

1. a) $\displaystyle\int_0^3 \sqrt{y+1}\,dy$ **b)** $\displaystyle\int_{-1}^0 \sqrt{y+1}\,dy$

2. a) $\displaystyle\int_0^1 r\sqrt{1-r^2}\,dr$ **b)** $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a) $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b)** $\displaystyle\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a) $\displaystyle\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b)** $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a) $\displaystyle\int_0^1 t^3(1+t^4)^3\,dt$ **b)** $\displaystyle\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a) $\displaystyle\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b)** $\displaystyle\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a) $\displaystyle\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b)** $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a) $\displaystyle\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b)** $\displaystyle\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a) $\displaystyle\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b)** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b)** $\displaystyle\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a) $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$ **b)** $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

12. a) $\displaystyle\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$ **b)** $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

13. a) $\displaystyle\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b)** $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a) $\displaystyle\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2}\,dw$ **b)** $\displaystyle\int_0^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\,dw$

15. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$ **16.** $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

17. $\displaystyle\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\displaystyle\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\displaystyle\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\displaystyle\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\displaystyle\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

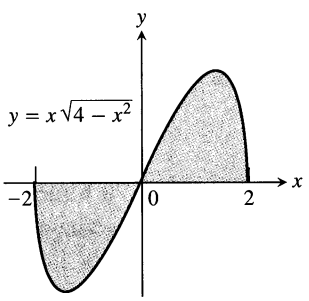
22. $\displaystyle\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2}(y^2 + 4y - 4)\,dy$

23. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$ **24.** $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1 + \frac{1}{t}\right)dt$

Area

Find the total areas of the shaded regions in Exercises 25–28.

25.



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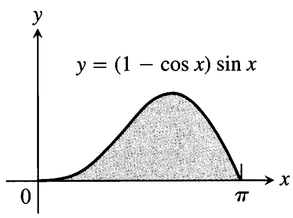

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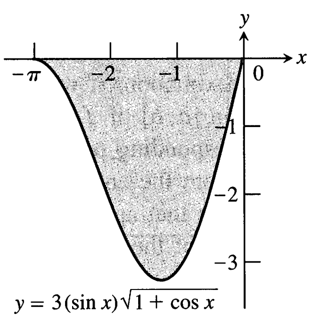

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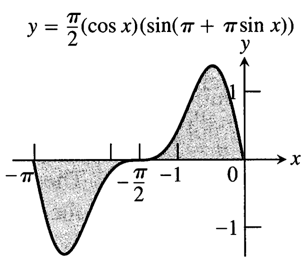


Theory and Examples

29. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\,dx$$

in terms of F.

30. Show that if f is continuous, then

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx.$$

31. Suppose that

$$\int_0^1 f(x)\,dx = 3.$$

Find

$$\int_{-1}^0 f(x)\,dx$$

if (a) f is odd, (b) f is even.

32. **a)** Show that

$$\int_{-a}^{a} h(x)\,dx = \begin{cases} 0 & \text{if } h \text{ is odd} \\ 2\displaystyle\int_0^a h(x)\,dx & \text{if } h \text{ is even.} \end{cases}$$

b) Test the result in part (a) with $h(x) = \sin x$ and with $h(x) = \cos x$, taking $a = \pi/2$ in each case.

33. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

34. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

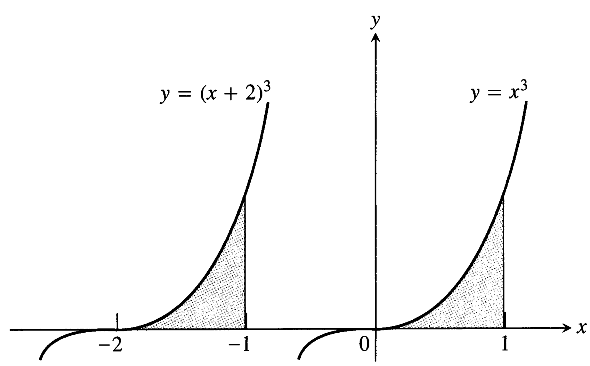
The Shift Property for Definite Integrals

A basic property of definite integrals is their invariance under translation, as expressed by the equation.

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x + c)\,dx. \qquad (2)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example (Fig. 4.26),

$$\int_{-2}^{-1} (x + 2)^3\,dx = \int_0^1 x^3\,dx. \qquad (3)$$



4.26 The integrations in Eq. (3). The shaded regions, being congruent, have equal areas.

35. Use a substitution to verify Eq. (2).

36. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x + c)$ over $[a - c, b - c]$ to convince yourself that Eq. (2) is reasonable.

a) $f(x) = x^2$, $a = 0$, $b = 1$, $c = 1$
b) $f(x) = \sin x$, $a = 0$, $b = \pi$, $c = \pi/2$
c) $f(x) = \sqrt{x - 4}$, $a = 4$, $b = 8$, $c = 5$

4.9

Numerical Integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$. But some antiderivatives are hard to find, and still others, like the antiderivatives of $(\sin x)/x$ and $\sqrt{1+x^4}$, have no elementary formulas. We do not mean merely that no one has yet succeeded in finding elementary formulas for the antiderivatives of $(\sin x)/x$ and $\sqrt{1+x^4}$. We mean it has been proved that no such formulas exist.

Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the trapezoidal rule and Simpson's rule, described in this section.

The Trapezoidal Rule

When we cannot find a workable antiderivative for a function f that we have to integrate, we partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f. The higher the degrees of the polynomials for a given partition, the better the results. For a given degree, the finer the partition, the better the results, until we reach limits imposed by round-off and truncation errors.

The length $h = (b-a)/n$ is called the **step size.** It is conventional to use h in this context instead of Δx.

The polynomials do not need to be of high degree to be effective. Even line segments (graphs of polynomials of degree 1) give good approximations if we use enough of them. To see why, suppose we partition the domain $[a, b]$ of f into n subintervals of length $\Delta x = h = (b-a)/n$ and join the corresponding points on the curve with line segments (Fig. 4.27). The vertical lines from the ends of the segments to the partition points create a collection of trapezoids that approximate the region between the curve and the x-axis. We add the areas of the trapezoids, counting area above the x-axis as positive and area below the axis as negative:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \cdots + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h \\ &= h\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

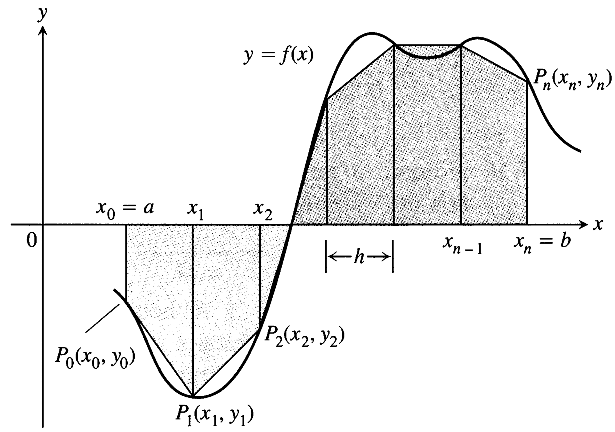
The trapezoidal rule says: Use T to estimate the integral of f from a to b.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \tag{1}$$

(for n subintervals of length $h = (b-a)/n$ and $y_k = f(x_k)$).



4.27 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To estimate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

Table 4.6

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Use the trapezoidal rule with $n = 4$ to estimate

$$\int_1^2 x^2\, dx.$$

Compare the estimate with the exact value of the integral.

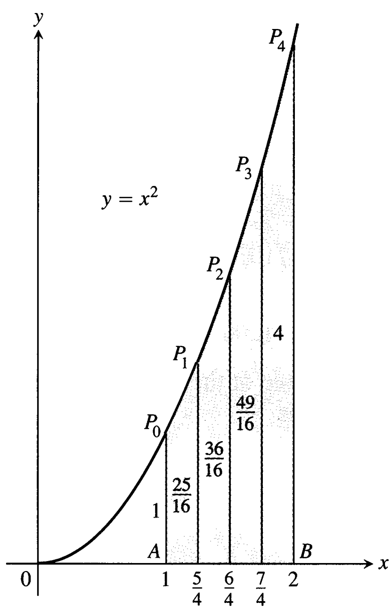
Solution To find the trapezoidal approximation, we divide the interval of integration into four subintervals of equal length and list the values of $y = x^2$ at the endpoints and partition points (see Table 4.6). We then evaluate Eq. (1) with $n = 4$ and $h = 1/4$:

$$\begin{aligned} T &= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) = \frac{75}{32} \\ &= 2.34375. \end{aligned}$$

The exact value of the integral is

$$\int_1^2 x^2\, dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.\overline{3}.$$

The approximation is a slight overestimate. Each trapezoid contains slightly more than the corresponding strip under the curve (Fig. 4.28). ❑



4.28 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate.

Controlling the Error in the Trapezoidal Approximation

Pictures suggest that the magnitude of the error

$$E_T = \int_a^b f(x)\, dx - T \tag{2}$$

in the trapezoidal approximation will decrease as the **step size** h decreases, because the trapezoids fit the curve better as their number increases. A theorem from advanced calculus assures us that this will be the case if f has a continuous second derivative.

The Error Estimate for the Trapezoidal Rule

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then

$$|E_T| \leq \frac{b-a}{12} h^2 M. \tag{3}$$

Although theory tells us there will always be a smallest safe value of M, in practice we can hardly ever find it. Instead, we find the best value we can and go on from there to estimate $|E_T|$. This may seem sloppy, but it works. To make $|E_T|$ small for a given M, we make h small.

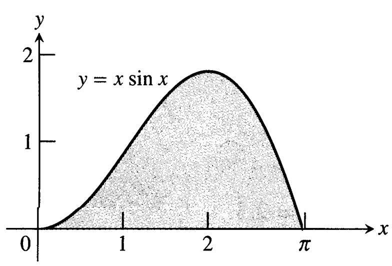
EXAMPLE 2 Find an upper bound for error in the approximation found in Example 1 for the value of

$$\int_1^2 x^2\,dx.$$

Solution We first find an upper bound M for the magnitude of the second derivative of $f(x) = x^2$ on the interval $1 \leq x \leq 2$. Since $f''(x) = 2$ for all x, we may safely take $M = 2$. With $b - a = 1$ and $h = 1/4$, Eq. (3) gives

$$|E_T| \leq \frac{b-a}{12} h^2 M = \frac{1}{12}\left(\frac{1}{4}\right)^2 (2) = \frac{1}{96}.$$

This is precisely what we find when we subtract $T = 75/32$ from $\int_1^2 x^2\,dx = 7/3$, since $|7/3 - 75/32| = |-1/96|$. Here our estimate gave the error's magnitude *exactly,* but this is exceptional. ❑



4.29 Graph of the integrand in Example 3.

EXAMPLE 3 Find an upper bound for the error incurred in estimating

$$\int_0^\pi x \sin x\,dx$$

with the trapezoidal rule with $n = 10$ steps (Fig. 4.29).

Solution With $a = 0$, $b = \pi$, and $h = (b - a)/n = \pi/10$, Eq. (3) gives

$$|E_T| \leq \frac{b-a}{12} h^2 M = \frac{\pi}{12}\left(\frac{\pi}{10}\right)^2 M = \frac{\pi^3}{1200} M.$$

The number M can be any upper bound for the magnitude of the second derivative of $f(x) = x \sin x$ on $[0, \pi]$. A routine calculation gives

$$f''(x) = 2\cos x - x \sin x,$$

so

$$\begin{aligned} |f''(x)| &= |2\cos x - x\sin x| \\ &\leq 2|\cos x| + |x||\sin x| && \text{Triangle inequality: } |a+b| \leq |a| + |b| \\ &\leq 2 \cdot 1 + \pi \cdot 1 = 2 + \pi. && |\cos x| \text{ and } |\sin x| \text{ never exceed 1, and } 0 \leq x \leq \pi. \end{aligned}$$

We can safely take $M = 2 + \pi$. Therefore,

$$|E_T| \le \frac{\pi^3}{1200} M = \frac{\pi^3(2+\pi)}{1200} < 0.133.$$

Rounded up to be safe

The absolute error is no greater than 0.133.

For greater accuracy, we would not try to improve M but would take more steps. With $n = 100$ steps, for example, $h = \pi/100$ and

$$|E_T| \le \frac{\pi}{12}\left(\frac{\pi}{100}\right)^2 M = \frac{\pi^3(2+\pi)}{120{,}000} < 0.00133 = 1.33 \times 10^{-3}.$$

❑

EXAMPLE 4 As we will see in Chapter 6, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

How many subintervals (steps) should be used in the trapezoidal rule to approximate the integral with an error of magnitude less than 10^{-4}?

Solution To determine n, the number of subintervals, we use Eq. (3) with

$$b - a = 2 - 1 = 1, \qquad h = \frac{b-a}{n} = \frac{1}{n},$$

$$f''(x) = \frac{d^2}{dx^2}(x^{-1}) = 2x^{-3} = \frac{2}{x^3}.$$

Then

$$\left|E_T\right| \le \frac{b-a}{12} h^2 \max\left|f''(x)\right| = \frac{1}{12}\left(\frac{1}{n}\right)^2 \max\left|\frac{2}{x^3}\right|,$$

where max refers to the interval [1, 2].

This is one of the rare cases where we can find the exact value of $\max|f''|$. On [1, 2], $y = 2/x^3$ decreases steadily from a maximum of $y = 2$ to a minimum of $y = 1/4$. Therefore,

$$|E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2 \cdot 2 = \frac{1}{6n^2}.$$

The error's absolute value will therefore be less than 10^{-4} if

$$\frac{1}{6n^2} < 10^{-4},$$

$$\frac{10^4}{6} < n^2, \qquad \text{Multiply both sides by } 10^4 n^2.$$

$$\frac{100}{\sqrt{6}} < |n|, \qquad \text{Square roots of both sides}$$

$$\frac{100}{\sqrt{6}} < n, \qquad n \text{ is positive.}$$

$$40.83 < n. \qquad \text{Rounded up, to be safe}$$

Simpson's one-third rule

The idea of using the formula

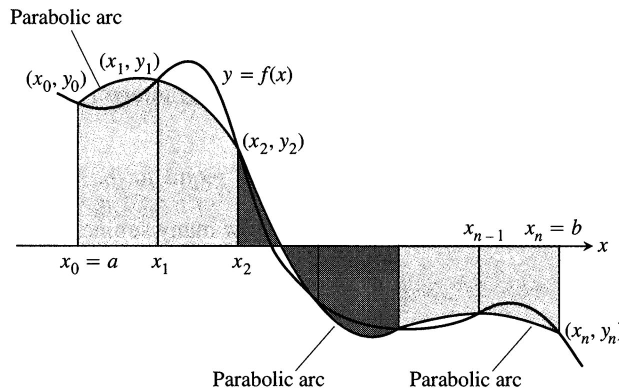
$$A = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

to estimate the area under a curve is known as Simpson's one-third rule. But the rule was in use long before Thomas Simpson (1720–1761) was born. It is another of history's beautiful quirks that one of the ablest mathematicians of eighteenth-century England is remembered not for his successful texts and his contributions to mathematical analysis but for a rule that was never his, that he never laid claim to, and that bears his name only because he happened to mention it in a book he wrote.

The first integer beyond 40.83 is $n = 41$. With $n = 41$ subintervals we can guarantee calculating $\ln 2$ with an error of magnitude less than 10^{-4}. Any larger n will work, too. ❑

Simpson's Rule

Simpson's rule for approximating $\int_a^b f(x)\,dx$ is based on approximating f with quadratic polynomials instead of linear polynomials. We approximate the graph with parabolic arcs instead of line segments (Fig. 4.30).

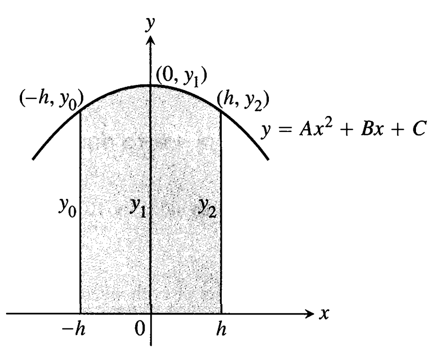


4.30 Simpson's rule approximates short stretches of curve with parabolic arcs.

The integral of the quadratic polynomial $y = Ax^2 + Bx + C$ in Fig. 4.31 from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) \tag{4}$$

(Appendix 4). Simpson's rule follows from partitioning $[a, b]$ into an even number of subintervals of equal length h, applying Eq. (4) to successive interval pairs, and adding the results.



4.31 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \tag{5}$$

The y's are the values of f at the partition points

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \ldots,\ x_{n-1} = a + (n-1)h,\ x_n = b.$$

The number n is even, and $h = (b - a)/n$.

Error Control for Simpson's Rule

The magnitude of the Simpson's rule error,

$$E_S = \int_a^b f(x)\,dx - S, \tag{6}$$

decreases with the step size, as we would expect from our experience with the trapezoidal rule. The inequality for controlling the Simpson's rule error, however, assumes f to have a continuous fourth derivative instead of merely a continuous second derivative. The formula, once again from advanced calculus, is this:

The Error Estimate for Simpson's Rule

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \le \frac{b-a}{180} h^4 M. \tag{7}$$

As with the trapezoidal rule, we can almost never find the smallest possible value of M. We just find the best value we can and go on from there to estimate $|E_S|$.

EXAMPLE 5 Use Simpson's rule with $n = 4$ to approximate

$$\int_0^1 5x^4\,dx.$$

What estimate does Eq. (7) give for the error in the approximation?

Solution Again we have chosen an integral whose exact value we can calculate directly:

$$\int_0^1 5x^4\,dx = x^5\Big]_0^1 = 1.$$

To find the Simpson approximation, we partition the interval of integration into four subintervals and evaluate $f(x) = 5x^4$ at the partition points (Table 4.7).

We then evaluate Eq. (5) with $n = 4$ and $h = 1/4$:

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$
$$= \frac{1}{12}\left(0 + 4\left(\frac{5}{256}\right) + 2\left(\frac{80}{256}\right) + 4\left(\frac{405}{256}\right) + 5\right) \approx 1.00260.$$

Table 4.7

x	$y = 5x^4$
0	0
$\frac{1}{4}$	$\frac{5}{256}$
$\frac{2}{4}$	$\frac{80}{256}$
$\frac{3}{4}$	$\frac{405}{256}$
1	5

To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \le x \le 1$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we may safely take $M = 120$. With $b - a = 1$ and $h = 1/4$, Eq. (7) gives

$$|E_S| \le \frac{b-a}{180} h^4 M = \frac{1}{180}\left(\frac{1}{4}\right)^4 (120) = \frac{1}{384} < 0.00261. \quad \square$$

Which Rule Gives Better Results?

The answer lies in the error-control formulas

$$|E_T| \le \frac{b-a}{12} h^2 M, \qquad |E_S| \le \frac{b-a}{180} h^4 M.$$

Trapezoidal vs. Simpson

If Simpson's rule is more accurate, why bother with the trapezoidal rule? There are two reasons. First, the trapezoidal rule is useful in a number of specific applications because it leads to much simpler expressions. Second, the trapezoidal rule is the basis for *Rhomberg integration,* one of the most satisfactory machine methods when high precision is required.

The M's of course mean different things, the first being an upper bound on $|f''|$ and the second an upper bound on $|f^{(4)}|$. But there is more. The factor $(b-a)/180$ in the Simpson formula is one-fifteenth of the factor $(b-a)/12$ in the trapezoidal formula. More important still, the Simpson formula has an h^4 while the trapezoidal formula has only an h^2. If h is one-tenth, then h^2 is one-hundredth but h^4 is only one ten-thousandth. If both M's are 1, for example, and $b-a=1$, then, with $h=1/10$,

$$|E_T| \le \frac{1}{12}\left(\frac{1}{10}\right)^2 \cdot 1 = \frac{1}{1200},$$

while

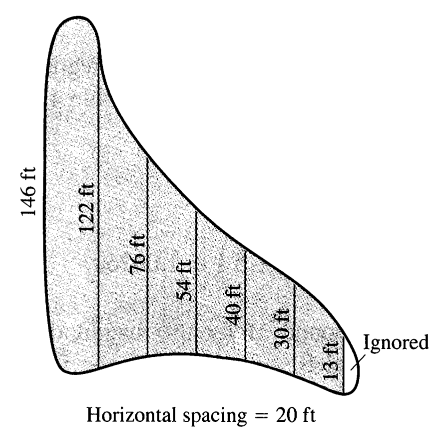
$$|E_S| \le \frac{1}{180}\left(\frac{1}{10}\right)^4 \cdot 1 = \frac{1}{1{,}800{,}000} = \frac{1}{1500} \cdot \frac{1}{1200}.$$

For roughly the same amount of computational effort, we get better accuracy with Simpson's rule—at least in this case.

The h^2 versus h^4 is the key. If h is less than 1, then h^4 can be significantly smaller than h^2. On the other hand, if h equals 1, there is no difference between h^2 and h^4. If h is greater than 1, the value of h^4 may be significantly larger than the value of h^2. In the latter two cases, the error-control formulas offer little help. We have to go back to the geometry of the curve $y=f(x)$ to see whether trapezoids or parabolas, if either, are going to give the results we want.

Working with Numerical Data

The next example shows how we can use Simpson's rule to estimate the integral of a function from values measured in the laboratory or in the field even when we have no formula for the function. We can use the trapezoidal rule the same way.



4.32 The swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Fig. 4.32). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's rule with $h=20$ ft and the y's equal to the distances measured across the swamp, as shown in Fig. 4.32.

$$\begin{aligned} S &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100. \end{aligned}$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3. ❑

Round-off Errors

Although decreasing the step size h reduces the error in the Simpson and trapezoidal approximations in theory, it may fail to do so in practice. When h is very small, say $h=10^{-5}$, the round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking h below a certain size can actually make things worse. While this will not be an issue in the present book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

Exercises 4.9

Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the trapezoidal rule and one for Simpson's rule.

I. *Using the trapezoidal rule*

a) Estimate the integral with $n = 4$ steps and use Eq. (3) to find an upper bound for $|E_T|$.

b) Evaluate the integral directly, and use Eq. (2) to find $|E_T|$.

c) CALCULATOR Use the formula ($|E_T|$/true value) $\times$ 100 to express $|E_T|$ as a percentage of the integral's true value.

II. *Using Simpson's rule*

a) Estimate the integral with $n = 4$ steps and use Eq. (7) to find an upper bound for $|E_S|$.

b) Evaluate the integral directly, and use Eq. (6) to find $|E_S|$.

c) CALCULATOR Use the formula ($|E_S|$/true value) $\times$ 100 to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x - 1)\,dx$

3. $\int_{-1}^1 (x^2 + 1)\,dx$

4. $\int_{-2}^0 (x^2 - 1)\,dx$

5. $\int_0^2 (t^3 + t)\,dt$

6. $\int_{-1}^1 (t^3 + 1)\,dt$

7. $\int_1^2 \frac{1}{s^2}\,ds$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

9. $\int_0^\pi \sin t\,dt$

10. $\int_0^1 \sin \pi t\,dt$

In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with (a) the trapezoidal rule and (b) Simpson's rule with $n = 8$ steps. Round your answers to 5 decimal places. Then (c) find the integral's exact value and the approximation error E_T or E_S, as appropriate, from Eqs. (2) and (6).

11. $\int_0^1 x\sqrt{1 - x^2}\,dx$

x	$x\sqrt{1-x^2}$
0	0.0
0.125	0.12402
0.25	0.24206
0.375	0.34763
0.5	0.43301
0.625	0.48789
0 75	0.49608
0.875	0.42361
1.0	0

12. $\int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}}\,d\theta$

θ	$\theta/\sqrt{16+\theta^2}$
0	0.0
0.375	0.09334
0.75	0.18429
1.125	0.27075
1.5	0.35112
1.875	0.42443
2.25	0.49026
2.625	0.58466
3.0	0.6

13. $\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2 + \sin t)^2}\,dt$

t	$(3\cos t)/(2+\sin t)^2$
−1.57080	0.0
−1.17810	0.99138
−0.78540	1.26906
−0.39270	1.05961
0	0.75
0.39270	0.48821
0.78540	0.28946
1.17810	0.13429
1.57080	0

14. $\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

y	$(\csc^2 y)\sqrt{\cot y}$
0.78540	2.0
0.88357	1.51606
0.98175	1.18237
1.07992	0.93998
1.17810	0.75402
1.27627	0.60145
1.37445	0.46364
1.47262	0.31688
1.57080	0

The Minimum Number of Subintervals

In Exercises 15–26, use Eqs. (3) and (7), as appropriate, to estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by (a) the trapezoidal rule and (b) Simpson's rule. (The integrals in Exercises 15–22 are the integrals from Exercises 1–8.)

15. $\int_1^2 x\,dx$

16. $\int_1^3 (2x - 1)\,dx$

17. $\int_{-1}^1 (x^2 + 1)\,dx$

18. $\int_{-2}^0 (x^2 - 1)\,dx$

19. $\int_0^2 (t^3 + t)\, dt$

20. $\int_{-1}^1 (t^3 + 1)\, dt$

21. $\int_1^2 \frac{1}{s^2}\, ds$

22. $\int_2^4 \frac{1}{(s-1)^2}\, ds$

23. $\int_0^3 \sqrt{x+1}\, dx$

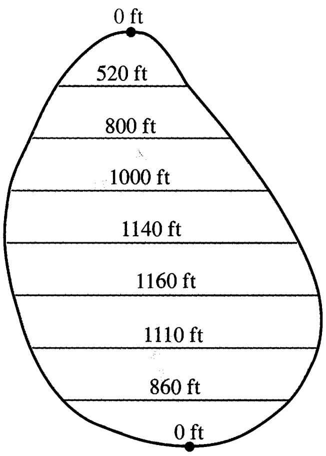
24. $\int_0^3 \frac{1}{\sqrt{x+1}}\, dx$

25. $\int_0^2 \sin(x+1)\, dx$

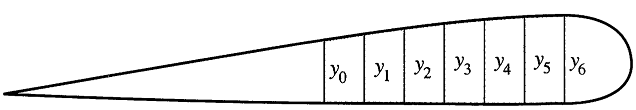
26. $\int_{-1}^1 \cos(x+\pi)\, dx$

Applications

27. As the fish-and-game warden of your township, you are responsible for stocking the town pond with fish before fishing season. The average depth of the pond is 20 ft. You plan to start the season with one fish per 1000 ft^3. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?

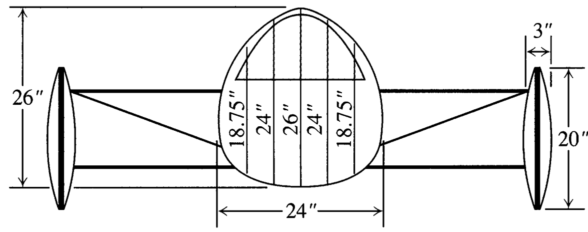


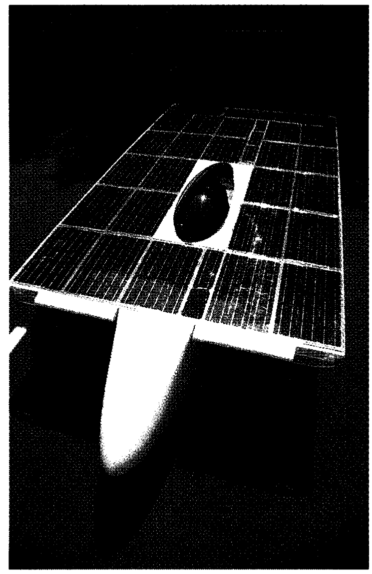
28. CALCULATOR The design of a new airplane requires a gasoline tank of constant cross-section area in each wing. A scale drawing of a cross section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

29. CALCULATOR A vehicle's aerodynamic drag is determined in part by its cross-section area and, all other things being equal, engineers try to make this area as small as possible. Use Simpson's rule to estimate the cross-section area of James Worden's solar-powered Solectria car at MIT (Fig. 4.33).





4.33 Solectria cars are produced by Selectron Corp., Arlington, MA (Exercise 29).

30. The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed?

Speed change	Seconds
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

Source: *Car and Driver,* April 1994.

Theory and Examples

31. *Polynomials of low degree.* The magnitude of the error in the trapezoidal approximation of $\int_a^b f(x)\,dx$ is

$$|E_T| = \frac{b-a}{12}h^2\,|f''(c)|,$$

where c is some point (usually unidentified) in (a, b). If f is a linear function of x, then $f''(c) = 0$, so $E_T = 0$ and T gives the exact value of the integral for any value of h. This is no surprise, really, for if f is linear, the line segments approximating the graph of f fit the graph exactly. The surprise comes with Simpson's rule. The magnitude of the error in Simpson's rule is

$$|E_S| = \frac{b-a}{180}h^4\,|f^{(4)}(c)|,$$

where once again c lies in (a, b). If f is a polynomial of degree less than 4, then $f^{(4)} = 0$ no matter what c is, so $E_S = 0$ and S gives the integral's exact value—even if we use only two steps. As a case in point, use Simpson's rule with $n = 2$ to estimate

$$\int_0^2 x^3\,dx.$$

Compare your answer with the integral's exact value.

32. *Usable values of the sine-integral function.* The sine-integral function,

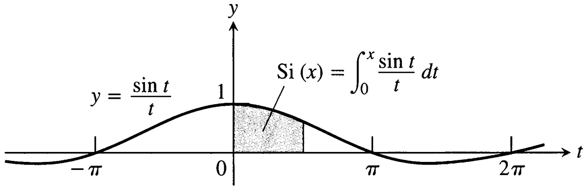
$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\operatorname{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth (Fig. 4.34) and you can expect good results from Simpson's rule.



4.34 The continuous extension of $y = (\sin t)/t$. The sine-integral function Si(x) is the subject of Exercise 32.

a) Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\operatorname{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's rule with $n = 4$.

b) Estimate $\operatorname{Si}(\pi/2)$ by Simpson's rule with $n = 4$.

c) Express the error bound you found in (a) as a percentage of the value you found in (b).

33. (*Continuation of Example 3.*) The error bounds in Eqs. (3) and (7) are "worst case" estimates, and the trapezoidal and Simpson rules are often more accurate than the bounds suggest. The trapezoidal rule estimate of

$$\int_0^{\pi} x \sin x\,dx$$

in Example 3 is a case in point.

a) Use the trapezoidal rule with $n = 10$ to approximate the value of the integral. The table to the right gives the necessary y-values.

x	$x \sin x$
0	0
$(0.1)\pi$	0.09708
$(0.2)\pi$	0.36932
$(0.3)\pi$	0.76248
$(0.4)\pi$	1.19513
$(0.5)\pi$	1.57080
$(0.6)\pi$	1.79270
$(0.7)\pi$	1.77912
$(0.8)\pi$	1.47727
$(0.9)\pi$	0.87372
π	0

b) Find the magnitude of the difference between π, the integral's value, and your approximation in (a). You will find the difference to be considerably less than the upper bound of 0.133 calculated with $n = 10$ in Example 3.

c) GRAPHER The upper bound of 0.133 for $|E_T|$ in Example 3 could have been improved somewhat by having a better bound for

$$|f''(x)| = |2\cos x - x \sin x|$$

on $[0, \pi]$. The upper bound we used was $2 + \pi$. Graph f'' over $[0, \pi]$ and use TRACE or ZOOM to improve this upper bound.

Use the improved upper bound as M in Eq. (3) to make an improved estimate of $|E_T|$. Notice that the trapezoidal rule approximation in (a) is also better than this improved estimate would suggest.

34. CALCULATOR (*Continuation of Exercise 33*)

a) GRAPHER Show that the fourth derivative of $f(x) = x \sin x$ is

$$f^{(4)}(x) = -4\cos x + x \sin x.$$

Use TRACE or ZOOM to find an upper bound M for the values of $|f^{(4)}|$ on $[0, \pi]$.

b) Use the value of M from (a) together with Eq. (7) to obtain an upper bound for the magnitude of the error in estimating the value of

$$\int_0^{\pi} x \sin x\,dx$$

with Simpson's rule with $n = 10$ steps.

c) Use the data in the table in Exercise 33 to estimate $\int_0^{\pi} x \sin x \, dx$ with Simpson's rule with $n = 10$ steps.

d) To 6 decimal places, find the magnitude of the difference between your estimate in (c) and the integral's true value, π. You will find the error estimate obtained in (b) to be quite good.

You are planning to use Simpson's rule to estimate the values of the integrals in Exercises 35 and 36. Before proceeding, you turn to Eq. (7) to determine the step size h needed to assure the accuracy you want. What happens? Can this be avoided by using the trapezoidal rule and Eq. (3) instead? Give reasons for your answers.

35. $\displaystyle\int_0^4 x^{3/2}\,dx$

36. $\displaystyle\int_0^1 x^{5/2}\,dx$

Numerical Integrator

As we mentioned at the beginning of the section, the definite integrals of many continuous functions cannot be evaluated with the Fundamental Theorem of Calculus because their antiderivatives lack elementary formulas. Numerical integration offers a practical way to estimate the values of these so-called *nonelementary integrals*. If your calculator or computer has a numerical integration routine, try it on the integrals in Exercises 37–40.

37. $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$ — A nonelementary integral that came up in Newton's research

38. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx$ — The integral from Exercise 32. To avoid division by zero, you may have to start the integration at a small positive number like 10^{-6} instead of 0.

39. $\displaystyle\int_0^{\pi/2} \sin(x^2)\,dx$ — An integral associated with the diffraction of light

40. $\displaystyle\int_0^{\pi/2} 40\sqrt{1-0.64\cos^2 t}\,dt$ — The length of the ellipse $(x^2/25)+(y^2/9)=1$

Chapter 4 Questions to Guide Your Review

1. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
2. What is an indefinite integral? How do you evaluate one? What general formulas do you know for evaluating indefinite integrals?
3. How can you sometimes use a trigonometric identity to transform an unfamiliar intregal into one you know how to evaluate?
4. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
5. What is an initial value problem? How do you solve one? Give an example.
6. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.
7. How do you sketch the solutions of a differential equation $dy/dx = f(x)$ when you do not know an antiderivative of f? How would you sketch the solution of an initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$ under these circumstances?
8. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
9. How can you sometimes estimate quantities like distance traveled, area, volume, and average value with finite sums? Why might you want to do so?
10. What is sigma notation? What advantage does it offer? Give examples.
11. What rules are available for calculating with sigma notation?
12. What is a Riemann sum? Why might you want to consider such a sum?
13. What is the norm of a partition of a closed interval?
14. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
15. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
16. Describe the rules for working with definite integrals (Table 4.5). Give examples.
17. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
18. What does a function's average value have to do with sampling a function's values?
19. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
20. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
21. How does the method of substitution work for definite integrals? Give examples.
22. How is integration by substitution related to the Chain Rule?
23. You are collaborating to produce a short "how-to" manual for

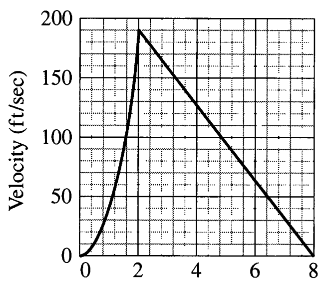
numerical integration, and you are writing about the trapezoidal rule. (a) What would you say about the rule itself and how to use it? how to achieve accuracy? (b) What would you say if you were writing about Simpson's rule instead?

24. How would you compare the relative merits of Simpson's rule and the trapezoidal rule?

CHAPTER 4 PRACTICE EXERCISES

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

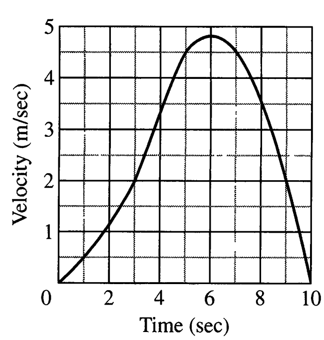


a) Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 2.3, Exercise 19, but you do not need to do Exercise 19 to do the exercise here.)

b) Sketch a graph of the rocket's height aboveground as a function of time for $0 \le t \le 8$.

2. a) The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

b) Sketch a graph of s as a function of t for $0 \le t \le 10$ assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

a) $\sum_{k=1}^{10} \frac{a_k}{4}$ b) $\sum_{k=1}^{10} (b_k - 3a_k)$

c) $\sum_{k=1}^{10} (a_k + b_k - 1)$ d) $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

a) $\sum_{k=1}^{20} 3a_k$ b) $\sum_{k=1}^{20} (a_k + b_k)$

c) $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ d) $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2}\, \Delta x_k$, where P is a partition of $[1, 5]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k(c_k^2 - 1)^{1/3}\, \Delta x_k$, where P is a partition of $[1, 3]$

7. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a) $\int_{-2}^{2} f(x)\,dx$ b) $\int_{2}^{5} f(x)\,dx$

c) $\int_{5}^{-2} g(x)\,dx$ d) $\int_{-2}^{5} (-\pi g(x))\,dx$

e) $\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_0^2 f(x)\,dx = \pi$, $\int_0^2 7g(x)\,dx = 7$, and $\int_0^1 g(x)\,dx = 2$, find the values of the following.

a) $\displaystyle\int_0^2 g(x)\,dx$

b) $\displaystyle\int_1^2 g(x)\,dx$

c) $\displaystyle\int_2^0 f(x)\,dx$

d) $\displaystyle\int_0^2 \sqrt{2}\,f(x)\,dx$

e) $\displaystyle\int_0^2 (g(x) - 3f(x))\,dx$

Area

In Exercises 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Initial Value Problems

Solve the initial value problems in Exercises 15–18.

15. $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2}, \quad y(1) = -1$

16. $\dfrac{dy}{dx} = \left(x + \dfrac{1}{x}\right)^2, \quad y(1) = 1$

17. $\dfrac{d^2r}{dt^2} = 15\sqrt{t} + \dfrac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

18. $\dfrac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

19. Show that $y = x^2 + \displaystyle\int_1^x \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

20. Show that $y = \int_0^x (1 + 2\sqrt{\sec t})\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x}\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 21 and 22 in terms of integrals.

21. $\dfrac{dy}{dx} = \dfrac{\sin x}{x}, \quad y(5) = -3$

22. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 23–44.

23. $\displaystyle\int (x^3 + 5x - 7)\,dx$

24. $\displaystyle\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

25. $\displaystyle\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

26. $\displaystyle\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

27. $\displaystyle\int \frac{r\,dr}{(r^2+5)^2}$

28. $\displaystyle\int \frac{6r^2dr}{(r^3 - \sqrt{2})^3}$

29. $\displaystyle\int 3\theta\sqrt{2 - \theta^2}\,d\theta$

30. $\displaystyle\int \frac{\theta^2}{9\sqrt{73 + \theta^3}}\,d\theta$

31. $\displaystyle\int x^3(1 + x^4)^{-1/4}\,dx$

32. $\displaystyle\int (2 - x)^{3/5}\,dx$

33. $\displaystyle\int \sec^2 \frac{s}{10}\,ds$

34. $\displaystyle\int \csc^2 \pi s\,ds$

35. $\displaystyle\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\,d\theta$

36. $\displaystyle\int \sec \frac{\theta}{3} \tan \frac{\theta}{3}\,d\theta$

37. $\displaystyle\int \sin^2 \frac{x}{4}\,dx$

38. $\displaystyle\int \cos^2 \frac{x}{2}\,dx$

39. $\displaystyle\int 2(\cos x)^{-1/2} \sin x\,dx$

40. $\displaystyle\int (\tan x)^{-3/2} \sec^2 x\,dx$

41. $\displaystyle\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

42. $\displaystyle\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

43. $\displaystyle\int \left(t - \frac{2}{t}\right)\left(t + \frac{2}{t}\right) dt$

44. $\displaystyle\int \frac{(t+1)^2 - 1}{t^4}\,dt$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 45–70.

45. $\displaystyle\int_{-1}^1 (3x^2 - 4x + 7)\,dx$

46. $\displaystyle\int_0^1 (8s^3 - 12s^2 + 5)\,ds$

47. $\displaystyle\int_1^2 \frac{4}{v^2}\,dv$

48. $\displaystyle\int_1^{27} x^{-4/3}\,dx$

49. $\displaystyle\int_1^4 \frac{dt}{t\sqrt{t}}$

50. $\displaystyle\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}}\,du$

51. $\displaystyle\int_0^1 \frac{36\,dx}{(2x+1)^3}$

52. $\displaystyle\int_0^1 \frac{dr}{\sqrt[3]{(7 - 5r)^2}}$

53. $\displaystyle\int_{1/8}^1 x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$

54. $\displaystyle\int_0^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

55. $\displaystyle\int_0^{\pi} \sin^2 5r\,dr$

56. $\displaystyle\int_0^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right) dt$

57. $\displaystyle\int_0^{\pi/3} \sec^2 \theta\,d\theta$

58. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2 x\,dx$

59. $\int_{\pi}^{3\pi} \cot^2 \frac{x}{6}\, dx$

60. $\int_{0}^{\pi} \tan^2 \frac{\theta}{3}\, d\theta$

61. $\int_{-\pi/3}^{0} \sec x \tan x\, dx$

62. $\int_{\pi/4}^{3\pi/4} \csc z \cot z\, dz$

63. $\int_{0}^{\pi/2} 5(\sin x)^{3/2} \cos x\, dx$

64. $\int_{-1}^{1} 2x \sin(1 - x^2)\, dx$

65. $\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x\, dx$

66. $\int_{0}^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$

67. $\int_{0}^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3 \sin^2 x}}\, dx$

68. $\int_{0}^{\pi/4} \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}}\, dx$

69. $\int_{0}^{\pi/3} \frac{\tan \theta}{\sqrt{2 \sec \theta}}\, d\theta$

70. $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}}\, dt$

Average Values

71. Find the average value of $f(x) = mx + b$

a) over $[-1, 1]$
b) over $[-k, k]$

72. Find the average value of

a) $y = \sqrt{3x}$ over $[0, 3]$
b) $y = \sqrt{ax}$ over $[0, a]$

73. Let f be a function that is differentiable on $[a, b]$. In Chapter 1 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

74. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

Numerical Integration

75. CALCULATOR According to the error-bound formula for Simpson's rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_{1}^{3} \frac{1}{x}\, dx$$

by Simpson's rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's rule, the number of subintervals has to be even.)

76. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the trapezoidal rule?

77. A direct calculation shows that

$$\int_{0}^{\pi} 2 \sin^2 x\, dx = \pi.$$

How close do you come to this value by using the trapezoidal rule with $n = 6$? Simpson's rule with $n = 6$? Try them and find out.

78. You are planning to use Simpson's rule to estimate the value of the integral

$$\int_{1}^{2} f(x)\, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's rule the number has to be even.)

79. CALCULATOR Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$. Figure 2.42 shows why.

80. *Specific heat of a gas.* Specific heat C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mole (calories per degree gram molecule). The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675$°C and the temperature at which it is attained.

Theory and Examples

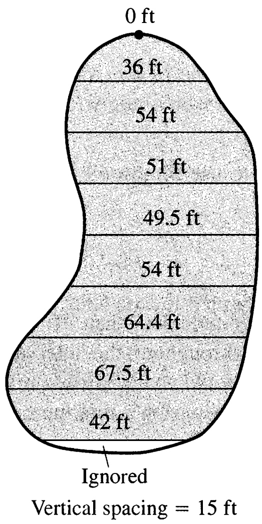
81. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

82. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

83. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

84. Find dy/dx if $y = \int_{\cos x}^{0} (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

85. *A new parking lot.* To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$11,000?



86. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening her parachute. Both skydivers descend at 16 ft/sec with parachute open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a) At what altitude does A's parachute open?
b) At what altitude does B's parachute open?
c) Which skydiver lands first?

Average Daily Inventory

Average value is used in economics to study such things as average daily inventory. If $I(t)$ is the number of radios, tires, shoes, or whatever product a firm has on hand on day t (we call I an **inventory function**), the average value of I over a time period $[0, T]$ is called the firm's average daily inventory for the period.

$$\textbf{Average daily inventory} = \text{av}(I) = \frac{1}{T}\int_0^T I(t)\,dt.$$

If h is the dollar cost of holding one item per day, the product $\text{av}(I)\cdot h$ is the **average daily holding cost** for the period.

87. As a wholesaler, Tracey Burr Distributors receives a shipment of 1200 cases of chocolate bars every 30 days. TBD sells the chocolate to retailers at a steady rate, and t days after a shipment arrives, its inventory of cases on hand is $I(t) = 1200 - 40t$, $0 \le t \le 30$. What is TBD's average daily inventory for the 30-day period? What is its average daily holding cost if the cost of holding one case is 3¢ a day?

88. Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) = 600 + 600t$, $0 \le t \le 14$. The daily holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost.

89. Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(t) = 450 - t^2/2$. Find the average daily inventory. If the holding cost for one drum is 2¢ per day, find the average daily holding cost.

90. Mitchell Mailorder receives a shipment of 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the average daily inventory. If the holding cost for one case is $1/2$¢ per day, find the average daily holding cost.

CHAPTER 4 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Theory and Examples

1. a) If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

b) If $\int_0^1 f(x)\,dx = 4$ and $f(x) \ge 0$, does $\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2$?

Give reasons for your answers.

2. Suppose $\int_{-2}^{2} f(x)\,dx = 4$, $\int_{2}^{5} f(x)\,dx = 3$, $\int_{-2}^{5} g(x)\,dx = 2$.

Which, if any, of the following statements are true?

a) $\int_5^2 f(x)\,dx = -3$ **b)** $\int_{-2}^{5} (f(x) + g(x)) = 9$

c) $f(x) \le g(x)$ on the interval $-2 \le x \le 5$

3. Show that

$$y = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \quad \frac{dy}{dx} = 0 \quad \text{and} \quad y = 0 \text{ when } x = 0.$$

(*Hint:* $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. Suppose x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

a) $\displaystyle\int_0^{x^2} f(t)\,dt = x\cos\pi x,$

b) $\displaystyle\int_0^{f(x)} t^2\,dt = x\cos\pi x.$

6. Find $f(\pi/2)$ from the following information.

i) f is positive and continuous.
ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a.$$

7. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \ge 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2+1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt\right) du = \int_0^x f(u)(x-u)\,du.$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

9. Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

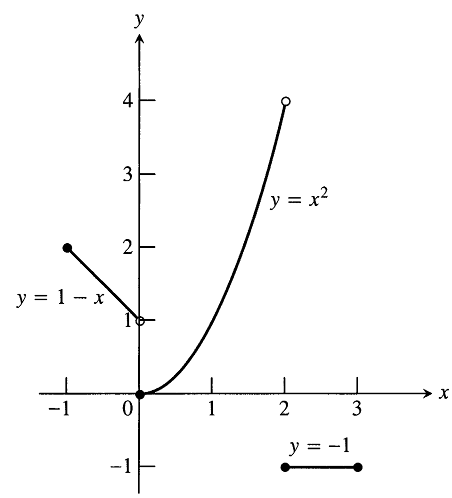
Bounded Piecewise Continuous Functions

Although we are mainly interested in continous functions, many functions in applications are piecewise continuous. All bounded piecewise continuous functions are integrable (as are many unbounded functions, as we will see in Chapter 7). **Bounded** on an interval I means that for some finite constant M, $|f(x)| \le M$ for all x in I. **Piecewise continuous** on I means that I can be partitioned into open or half open subintervals on which f is continuous. To integrate a bounded piecewise continuous function that has a continuous extension to each closed subinterval of the partition, we integrate the individual extensions and add the results. The integral of the function

$$f(x) = \begin{cases} 1-x, & -1 \le x < 0 \\ x^2, & 0 \le x < 2 \\ -1, & 2 \le x \le 3, \end{cases}$$

(Fig. 4.35) over $[-1, 3]$ is

$$\int_{-1}^3 f(x)\,dx = \int_{-1}^0 (1-x)\,dx + \int_0^2 x^2\,dx + \int_2^3 (-1)\,dx$$
$$= \left[x - \frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^3}{3}\right]_0^2 + \left[-x\right]_2^3$$
$$= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}.$$



4.35 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to bounded piecewise continuous functions with the restriction that $(d/dx)\int_a^x f(t)\,dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's rule below.

Graph the functions in Exercises 11–16 and integrate them over their domains.

11. $f(x) = \begin{cases} x^{2/3}, & -8 \le x < 0 \\ -4, & 0 \le x \le 3, \end{cases}$

12. $f(x) = \begin{cases} \sqrt{-x}, & -4 \le x < 0 \\ x^2 - 4, & 0 \le x \le 3 \end{cases}$

13. $g(t) = \begin{cases} t, & 0 \le t < 1 \\ \sin \pi t, & 1 \le t \le 2 \end{cases}$

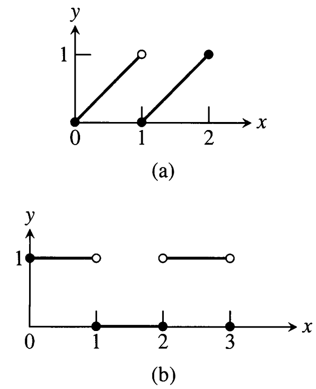
14. $h(z) = \begin{cases} \sqrt{1-z}, & 0 \le z < 1 \\ (7z-6)^{-1/3}, & 1 \le z \le 2 \end{cases}$

15. $f(x) = \begin{cases} 1, & -2 \le x < -1 \\ 1 - x^2, & -1 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

16. $h(r) = \begin{cases} r, & -1 \le r < 0 \\ 1 - r^2, & 0 \le r < 1 \\ 1, & 1 \le r \le 2 \end{cases}$

17. Find the average value of the function graphed in Fig. 4.36(a).

18. Find the average value of the function graphed in Fig. 4.36(b).



4.36 The graphs for Exercises 17 and 18.

Leibniz's Rule

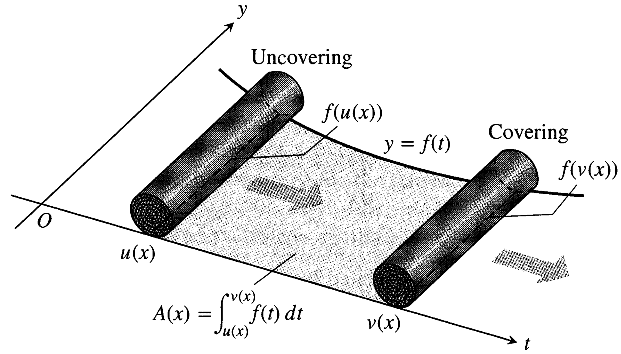
In applications, we sometimes encounter functions like

$$f(x) = \int_{\sin x}^{x^2} (1 + t)\,dt \qquad \text{and} \qquad g(x) = \int_{\sqrt{x}}^{2\sqrt{x}} \sin t^2\,dt,$$

defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. The first integral can be evaluated directly but the second cannot. We may find the derivative of either integral, however, by a formula called **Leibniz's rule:**

> **Leibniz's Rule**
>
> If f is continuous on $[a, b]$, and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then
>
> $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Figure 4.37 gives a geometric interpretation of Leibniz's rule. It shows a carpet of variable width $f(t)$ that is being rolled up at the left at the same time x as it is being unrolled at the right. (In this interpretation time is x, not t.) At time x, the floor is covered from $u(x)$ to $v(x)$. The rate du/dx at which the carpet is being rolled up need not be the same as the rate dv/dx at which the carpet is being laid down. At any given time x, the area covered by carpet is



4.37 Rolling and unrolling a carpet: a geometric interpretation of Leibniz's rule:

$$\frac{dA}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

$$A(x) = \int_{u(x)}^{v(x)} f(t)\,dt.$$

At what rate is the covered area changing? At the instant x, $A(x)$ is increasing by the width $f(v(x))$ of the unrolling carpet times the rate dv/dx at which the carpet is being unrolled. That is, $A(x)$ is being increased at the rate

$$f(v(x))\frac{dv}{dx}.$$

At the same time, A is being decreased at the rate

$$f(u(x))\frac{du}{dx},$$

the width at the end that is being rolled up times the rate du/dx. The net rate of change in A is

$$\frac{dA}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx},$$

which is precisely Leibniz's rule.

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x)). \tag{1}$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned} \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt &= \frac{d}{dx}\Big[F(v(x)) - F(u(x))\Big] \\ &= F'(v(x))\frac{dv}{dx} - F'(u(x))\frac{du}{dx} \qquad \text{Chain Rule} \\ &= f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}. \end{aligned}$$

You will see another way to derive the rule in Chapter 12, Additional Exercise 3.

Use Leibniz's rule to find the derivatives of the functions in Exercises 19–21.

19. $f(x) = \int_{1/x}^{x} \frac{1}{t}\,dt$

20. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2}\,dt$

21. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\,dt$

22. Use Leibniz's rule to find the value of x that maximizes the value of the integral

$$\int_x^{x+3} t(5-t)\,dt.$$

Problems like this arise in the mathematical theory of political elections. See "The Entry Problem in a Political Race," by Steven J. Brams and Philip D. Straffin, Jr., in *Political Equilibrium,* Peter Ordeshook and Kenneth Shepfle, Editors, Kluwer-Nijhoff, Boston, 1982, pp. 181–195.

Approximating Finite Sums with Integrals

In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals. Here is an example.

EXAMPLE 7 Estimate the sum of the square roots of the first n positive integers, $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$.

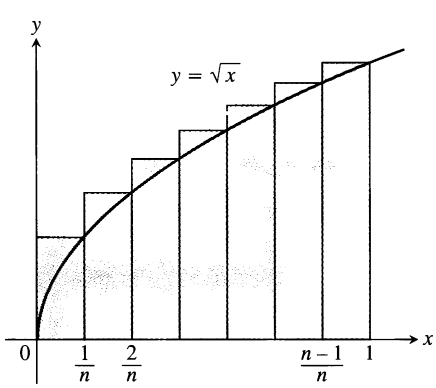
Solution See Fig. 4.38. The integral

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg]_0^1 = \frac{2}{3}$$

is the limit of the sums

$$S_n = \sqrt{\frac{1}{n}}\cdot\frac{1}{n} + \sqrt{\frac{2}{n}}\cdot\frac{1}{n} + \cdots + \sqrt{\frac{n}{n}}\cdot\frac{1}{n}$$

$$= \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n^{3/2}}.$$



4.38 The relation of the circumscribed rectangles to the integral $\int_0^1 \sqrt{x}\,dx$ leads to an estimate of the sum $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$.

Therefore, when n is large, S_n will be close to 2/3 and we will have

$$\text{Root sum} = \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3}n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	1.386/22.468≈ 6%
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

❑

23. Evaluate

$$\lim_{n\to\infty} \frac{1^5+2^5+3^5+\cdots+n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5\,dx$$

and evaluating the integral.

24. See Exercise 23. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4}(1^3+2^3+3^3+\cdots+n^3).$$

25. Let $f(x)$ be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]$$

as a definite integral.

26. Use the result of Exercise 25 to evaluate

a) $\lim_{n\to\infty} \frac{1}{n^2}(2+4+6+\cdots+2n)$,

b) $\lim_{n\to\infty} \frac{1}{n^{16}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$,

c) $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$.

What can be said about the following limits?

d) $\lim_{n\to\infty} \frac{1}{n^{17}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$

e) $\lim_{n\to\infty} \frac{1}{n^{15}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$

27. a) Show that the area A_n of an n-sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}.$$

b) Find the limit of A_n as $n\to\infty$. Is this answer consistent with what you know about the area of a circle?

28. *The error function.* The error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a) Use Simpson's rule with $n = 10$ to estimate erf(1).

b) In [0, 1],

$$\left| \frac{d^4}{dt^4} \left(e^{-t^2} \right) \right| \le 12.$$

Give an upper bound for the magnitude of the error of the estimate in (a).

CHAPTER 5

Applications of Integrals

OVERVIEW Many things we want to know can be calculated with integrals: the areas between curves, the volumes and surface areas of solids, the lengths of curves, the amount of work it takes to pump liquids from belowground, the forces against floodgates, the coordinates of the points where solid objects will balance. We define all of these as limits of Riemann sums of continuous functions on closed intervals, that is, as integrals, and evaluate these limits with calculus.

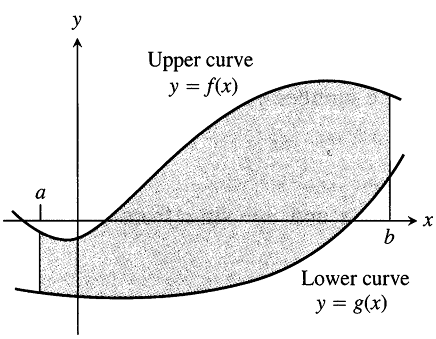
There is a pattern to how we define the integrals in applications, a pattern that, once learned, enables us to define new integrals when we need them. We look at specific applications first, then examine the pattern and show how it leads to integrals in new situations.

5.1 Areas Between Curves

This section shows how to find the areas of regions in the coordinate plane by integrating the functions that define the regions' boundaries.

The Basic Formula as a Limit of Riemann Sums

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Fig. 5.1). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions we usually have to find the area with an integral.



5.1 The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

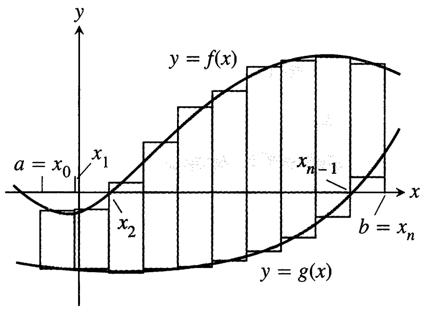
To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Fig. 5.2, on the following page). The area of the kth rectangle (Fig. 5.3, on the following page) is

$$\Delta A_k = \text{height } \times \text{width } = [f(c_k) - g(c_k)]\,\Delta x_k.$$

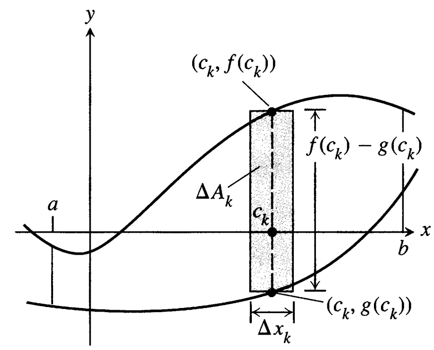
We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\| P \| \to 0$ the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because



5.2 We approximate the region with rectangles perpendicular to the *x*-axis.



5.3 ΔA_k = area of kth rectangle, $f(c_k) - g(c_k)$ = height, Δx_k = width

f and g are continuous. We take the area of the region to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\, \Delta x_k = \int_a^b [f(x) - g(x)]\, dx.$$

Definition

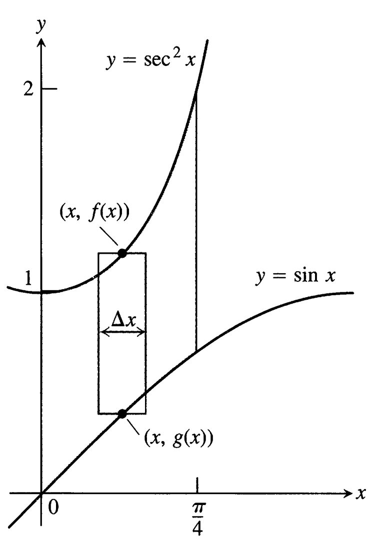
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area** of the region between the curves $y = f(x)$ and $y = g(x)$ from **a** to **b** is the integral of $[f - g]$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx. \qquad \textbf{(1)}$$

To apply Eq. (1) we take the following steps.

How to Find the Area Between Two Curves

1. *Graph the curves and draw a representative rectangle.* This reveals which curve is f (upper curve) and which is g (lower curve). It also helps find the limits of integration if you do not already know them.
2. *Find the limits of integration.*
3. *Write a formula for* $f(x) - g(x)$. Simplify it if you can.
4. *Integrate* $[f(x) - g(x)]$ *from a to b.* The number you get is the area.



5.4 The region in Example 1 with a typical approximating rectangle.

EXAMPLE 1 Find the area between $y = \sec^2 x$ and $y = \sin x$ from 0 to $\pi/4$.

Solution

Step 1: We sketch the curves and a vertical rectangle (Fig. 5.4). The upper curve is the graph of $f(x) = \sec^2 x$; the lower is the graph of $g(x) = \sin x$.

Step 2: The limits of integration are already given: $a = 0, b = \pi/4$.

Step 3: $f(x) - g(x) = \sec^2 x - \sin x$

Step 4:

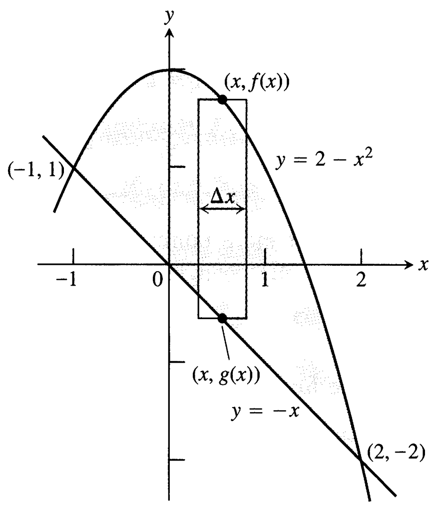
$$A = \int_0^{\pi/4} (\sec^2 x - \sin x)\,dx = \Big[\tan x + \cos x\Big]_0^{\pi/4}$$
$$= \left[1 + \frac{\sqrt{2}}{2}\right] - [0 + 1] = \frac{\sqrt{2}}{2}$$

❑

Curves That Intersect

When a region is determined by curves that intersect, the intersection points give the limits of integration.

EXAMPLE 2 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



5.5 The region in Example 2 with a typical approximating rectangle.

Solution

Step 1: Sketch the curves and a vertical rectangle (Fig. 5.5). Identifying the upper and the lower curves, we take $f(x) = 2 - x^2$ and $g(x) = -x$. The x-coordinates of the intersection points are the limits of integration.

Step 2: We find the limits of integration by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x:

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x+1)(x-2) &= 0 && \text{Factor.} \\ x = -1, \qquad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

Step 3:

$$\begin{aligned} f(x) - g(x) = (2 - x^2) - (-x) &= 2 - x^2 + x && \text{Rearrangement} \\ &= 2 + x - x^2 && \text{a matter of taste} \end{aligned}$$

Step 4:

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$
$$= 6 + \frac{3}{2} - \frac{9}{3} = \frac{9}{2}$$

❑

Technology *The Intersection of Two Graphs* One of the difficult and sometimes frustrating parts of integration applications is finding the limits of integration. To do this you often have to find the zeroes of a function or the intersection points of two curves.

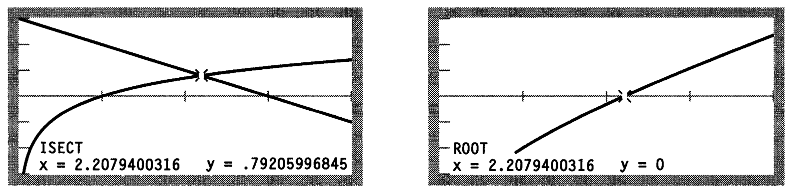
To solve the equation $f(x) = g(x)$ using a graphing utility, you enter

$$y_1 = f(x) \quad \text{and} \quad y_2 = g(x)$$

and use the grapher routine to find the points of intersection. Alternatively, you can solve the equation $f(x) - g(x) = 0$ with a root finder. Try both procedures with

$$f(x) = \ln x \quad \text{and} \quad g(x) = 3 - x.$$

When points of intersection are not clearly revealed or you suspect hidden behavior, additional work with the graphing utility or further use of calculus may be necessary.



a) The intersecting curves $y_1 = \ln x$ and $y_2 = 3 - x$, using a built-in function to find the intersection
b) Using a built-in root finder to find the zero of $f(x) = \ln x - 3 + x$

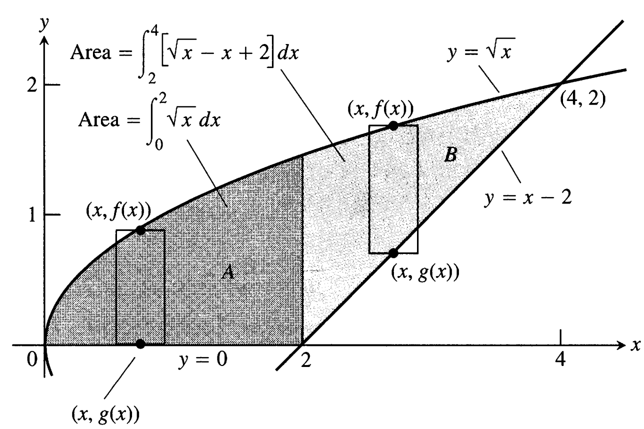
Boundaries with Changing Formulas

If the formula for a bounding curve changes at one or more points, we partition the region into subregions that correspond to the formula changes and apply Eq. (1) to each subregion.

EXAMPLE 3 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution

Step 1: The sketch (Fig. 5.6) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (there is agreement at $x = 2$). We partition the region at $x = 2$ into subregions A and B and sketch a representative rectangle for each subregion.



5.6 When the formula for a bounding curve changes, the area integral changes to match (Example 3).

Step 2: The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations

$y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\sqrt{x} = x - 2 \qquad \text{Equate } f(x) \text{ and } g(x).$$
$$x = (x-2)^2 = x^2 - 4x + 4 \qquad \text{Square both sides.}$$
$$x^2 - 5x + 4 = 0 \qquad \text{Rewrite.}$$
$$(x-1)(x-4) = 0 \qquad \text{Factor.}$$
$$x = 1, \qquad x = 4. \qquad \text{Solve.}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

Step 3: For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

Step 4: We add the area of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

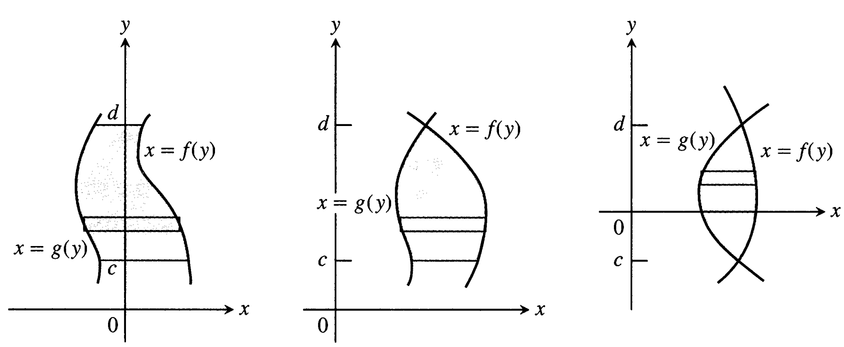
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$ ❑

Integration with Respect to *y*

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

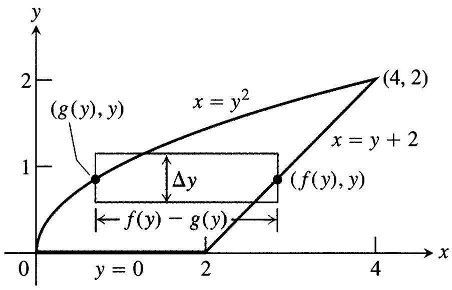
For regions like these



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy. \tag{2}$$

In Eq. (2), f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.



5.7 It takes two integrations to find the area of this region if we integrate with respect to *x*. It takes only one if we integrate with respect to *y* (Example 4).

EXAMPLE 4 Find the area of the region in Example 3 by integrating with respect to y.

Solution

Step 1: We sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Fig. 5.7). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$.

Step 2: The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y+1)(y-2) &= 0 && \text{Factor.} \\ y = -1, \qquad y &= 2 && \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

Step 3:

$$f(y) - g(y) = y + 2 - y^2 = 2 + y - y^2 \qquad \text{Rearrangement a matter of taste}$$

Step 4:

$$\begin{aligned} A = \int_a^b [f(y) - g(y)]\,dy &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3} \end{aligned}$$

This is the result of Example 3, found with less work. ❑

Combining Integrals with Formulas from Geometry

The fastest way to find an area may be to combine calculus and geometry.

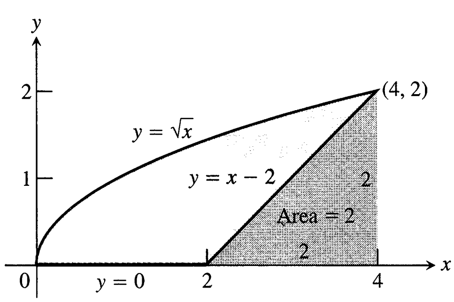
EXAMPLE 5 *The Area of the Region in Example 3 Found the Fastest Way*

Find the area of the region in Example 3.

Solution The area we want is the area between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis, *minus* the area of a triangle with base 2 and height 2 (Fig. 5.8):

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

❑



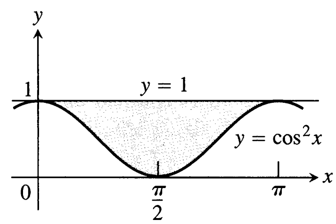
5.8 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Moral of Examples 3–5 It is sometimes easier to find the area between two curves by integrating with respect to y instead of x. Also, it may help to combine geometry and calculus. After sketching the region, take a moment to determine the best way to proceed.

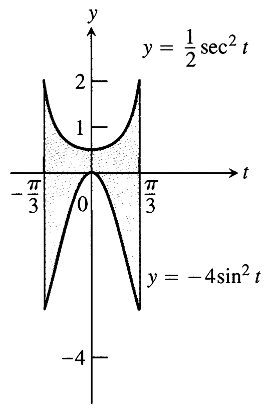
Exercises 5.1

Find the areas of the shaded regions in Exercises 1–8.

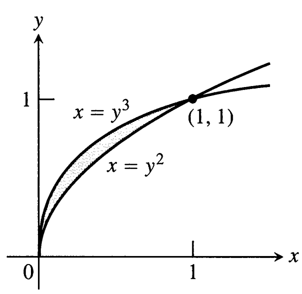
1.



2.



3.



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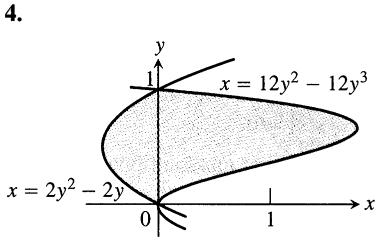
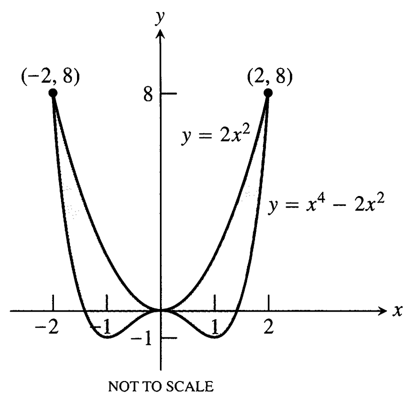
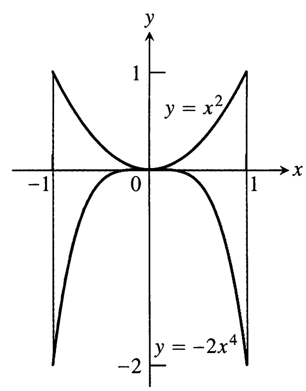


5.



6.



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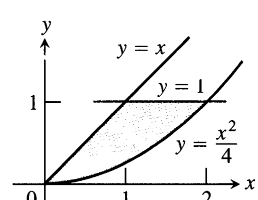
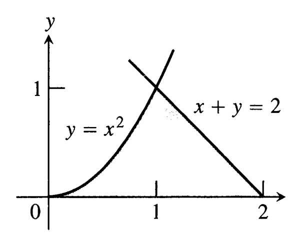
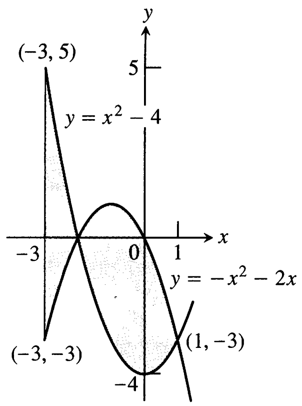


8.

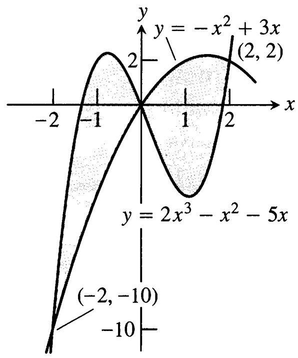


In Exercises 9–12, find the total shaded area.

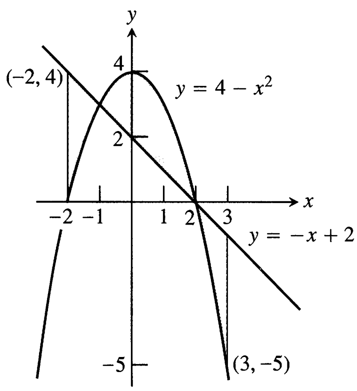
9.



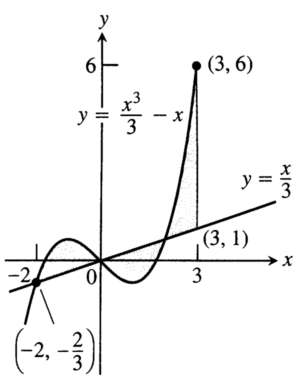
10.



11.



12.



Find the areas of the regions enclosed by the lines and curves in Exercises 13–22.

13. $y = x^2 - 2$ and $y = 2$

14. $y = 2x - x^2$ and $y = -3$

15. $y = x^4$ and $y = 8x$

16. $y = x^2 - 2x$ and $y = x$

17. $y = x^2$ and $y = -x^2 + 4x$

18. $y = 7 - 2x^2$ and $y = x^2 + 4$

19. $y = x^4 - 4x^2 + 4$ and $y = x^2$

20. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

21. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

22. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 23–30.

23. $x = 2y^2$, $x = 0$, and $y = 3$

24. $x = y^2$ and $x = y + 2$

25. $y^2 - 4x = 4$ and $4x - y = 16$

26. $x - y^2 = 0$ and $x + 2y^2 = 3$

27. $x + y^2 = 0$ and $x + 3y^2 = 2$

28. $x - y^{2/3} = 0$ and $x + y^4 = 2$

29. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

30. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 31–34.

31. $4x^2 + y = 4$ and $x^4 - y = 1$

32. $x^3 - y = 0$ and $3x^2 - y = 4$

33. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

34. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 35–42.

35. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

36. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

37. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

38. $y = \sin(\pi x/2)$ and $y = x$

39. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

40. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

41. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

42. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

43. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

44. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

45. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

46. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

47. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

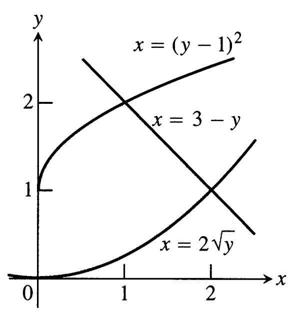
b) Find c by integrating with respect to y. (This puts c in the limits of integration.)

c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

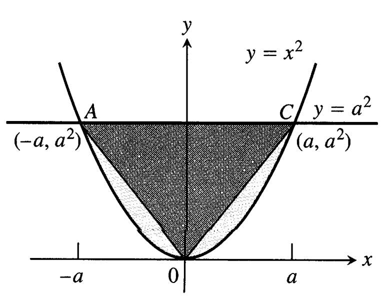
48. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x, (b) y.

49. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

50. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.



51. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

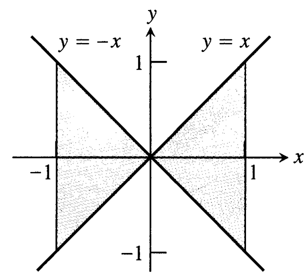


52. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

53. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a) $\displaystyle\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

b) $\displaystyle\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



54. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer.

CAS Explorations and Projects

In Exercises 55–58, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a) Plot the curves together to see what they look like and how many points of intersection they have.

b) Use the numerical equation solver in your CAS to find all the points of intersection.

c) Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d) Sum together the integrals found in part (c).

55. $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}, \quad g(x) = x - 1$

56. $f(x) = \dfrac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

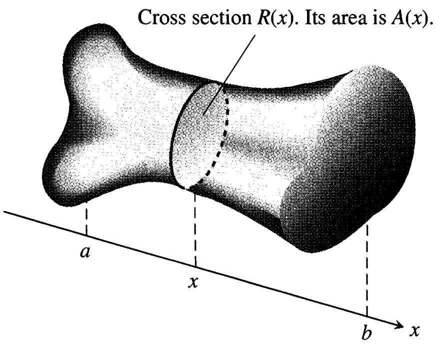
57. $f(x) = x + \sin(2x), \quad g(x) = x^3$

58. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

5.2 Finding Volumes by Slicing

From the areas of regions with curved boundaries, we can calculate the volumes of cylinders with curved bases by multiplying base area by height. From the volumes of such cylinders, we can calculate the volumes of other solids.

Slicing



5.9 If the area $A(x)$ of the cross section $R(x)$ is a continuous function of x, we can find the volume of the solid by integrating $A(x)$ from a to b.

Suppose we want to find the volume of a solid like the one shown in Fig. 5.9. At each point x in the closed interval $[a, b]$ the cross section of the solid is a region $R(x)$ whose area is $A(x)$. This makes A a real-valued function of x. If it is also a continuous function of x, we can use it to define and calculate the volume of the solid as an integral in the following way.

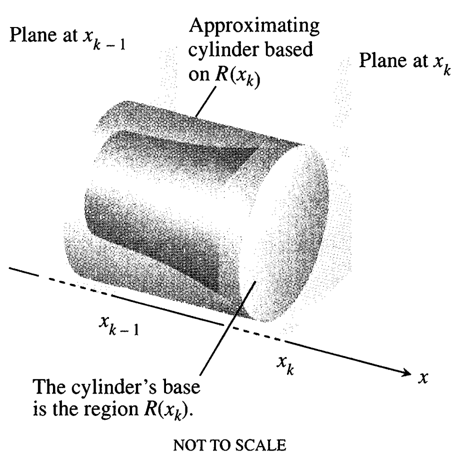
We partition the interval $[a, b]$ along the x-axis in the usual manner and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points. The kth slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between these two planes based on the region $R(x_k)$ (Fig. 5.10). The volume of this cylinder is

$$\begin{aligned} V_k &= \text{base area} \times \text{height} \\ &= A(x_k) \times (\text{distance between the planes at } x_{k-1} \text{ and } x_k) \\ &= A(x_k)\Delta x_k. \end{aligned}$$

The volume of the solid is therefore approximated by the cylinder volume sum

$$\sum_{k=1}^{n} A(x_k)\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting integral to be the volume of the solid.



5.10 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k and its approximating cylinder.

Definition

The **volume** of a solid of known integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b:

$$V = \int_a^b A(x)\,dx. \tag{1}$$

To apply Eq. (1), we take the following steps.

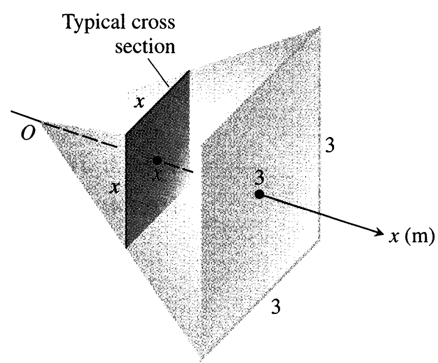
How to Find Volumes by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

Step 1: *A sketch.* We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross section (Fig. 5.11).



5.11 The cross sections of the pyramid in Example 1 are squares.

Step 2: *A formula for $A(x)$.* The cross section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

Step 3: *The limits of integration.* The squares go from $x = 0$ to $x = 3$.

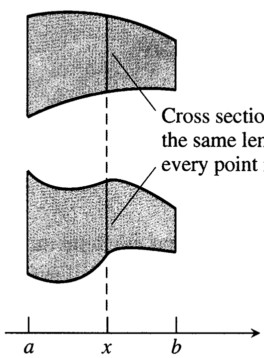
Step 4: *The volume.*

$$V = \int_a^b A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\bigg]_0^3 = 9.$$

The volume is 9 m^3. ❑

Bonaventura Cavalieri (1598–1647)

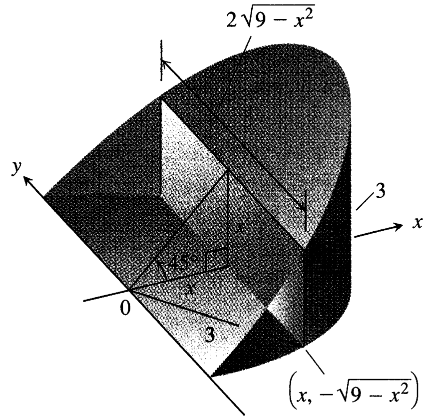
Cavalieri, a student of Galileo's, discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area. The theorem (and a letter of recommendation from Galileo) were enough to win Cavalieri a chair at the University of Bologna in 1629. The solid geometry version in Example 3, which Cavalieri never proved, was given his name by later geometers.



EXAMPLE 2 A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

Step 1: *A sketch.* We draw the wedge and sketch a typical cross section perpendicular to the x-axis (Fig. 5.12).



5.12 The wedge of Example 2, sliced perpendicular to the x-axis. The cross sections are rectangles.

Step 2: *The formula for $A(x)$.* The cross section at x is a rectangle of area

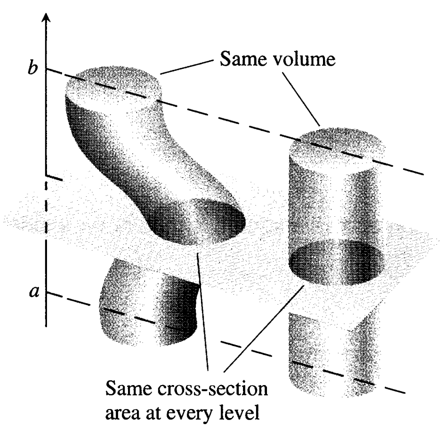
$$A(x) = (\text{height})(\text{width}) = (x)\left(2\sqrt{9-x^2}\right)$$
$$= 2x\sqrt{9-x^2}.$$

Step 3: *The limits of integration.* The rectangles run from $x = 0$ to $x = 3$.

Step 4: *The volume.*

$$V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx$$
$$= -\frac{2}{3}(9-x^2)^{3/2}\Big]_0^3$$
$$= 0 + \frac{2}{3}(9)^{3/2} \qquad \text{Let } u = 9 - x^2,\ du = -2x\,dx, \text{ integrate, and substitute back.}$$
$$= 18.$$

❑



5.13 Cavalieri's theorem: These solids have the same volume. You can illustrate this yourself with stacks of coins.

EXAMPLE 3 *Cavalieri's Theorem*

Cavalieri's theorem says that solids with equal altitudes and identical parallel cross-section areas have the same volume (Fig. 5.13). We can see this immediately from Eq. (1) because the cross-section area function $A(x)$ is the same in each case. ❑

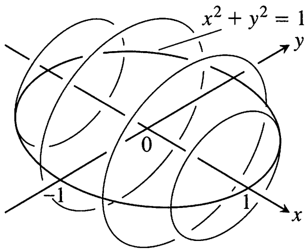
Exercises 5.2

Cross-Section Areas

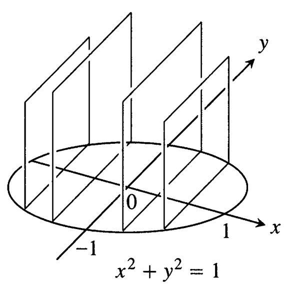
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

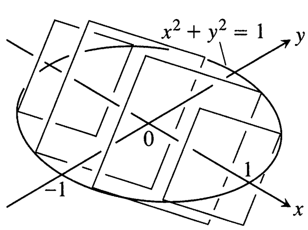
a) The cross sections are circular disks with diameters in the xy-plane.



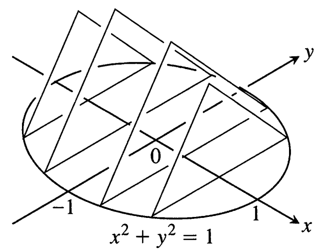
b) The cross sections are squares with bases in the xy-plane.



c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

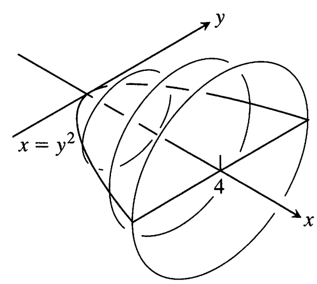


d) The cross sections are equilateral triangles with bases in the xy-plane.

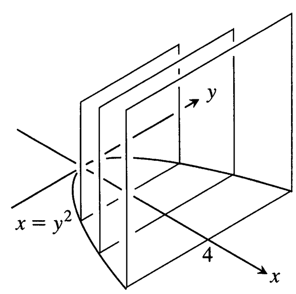


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

a) The cross sections are circular disks with diameters in the xy-plane.



b) The cross sections are squares with bases in the xy-plane.



c) The cross sections are squares with diagonals in the xy-plane.

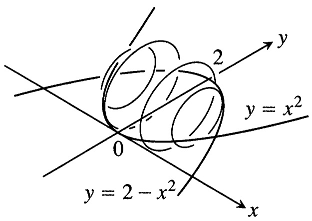
d) The cross sections are equilateral triangles with bases in the xy-plane.

Volumes by Slicing

Find the volumes of the solids in Exercises 3–12.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

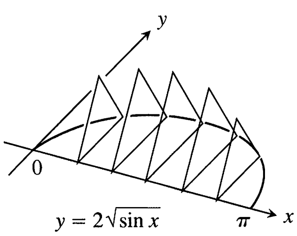
4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

7. The base of the solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross sections perpendicular to the x-axis are



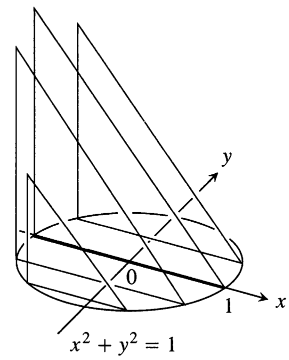
a) vertical equilateral triangles with bases running from the x-axis to the curve;
b) vertical squares with bases running from the x-axis to the curve.

8. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x-axis are

a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$;
b) vertical squares whose base edges run from the curve $y = \tan x$ to the curve $y = \sec x$.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

10. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

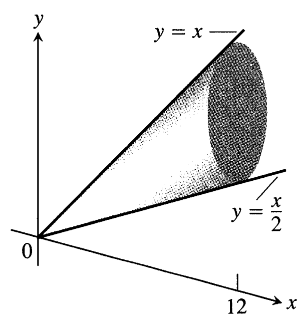


Cavalieri's Theorem

11. *A twisted solid.* A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross sections.

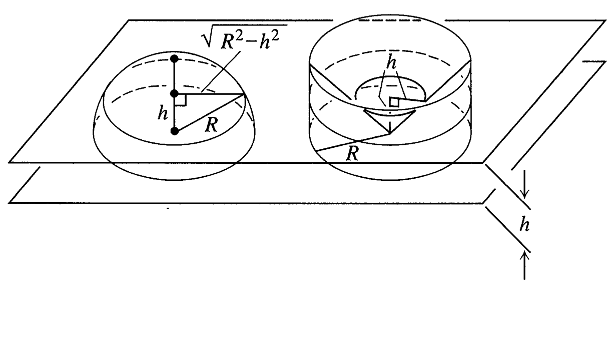
a) Find the volume of the column.
b) What will the volume be if the square turns twice instead of once? Give reasons for your answer.

12. A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



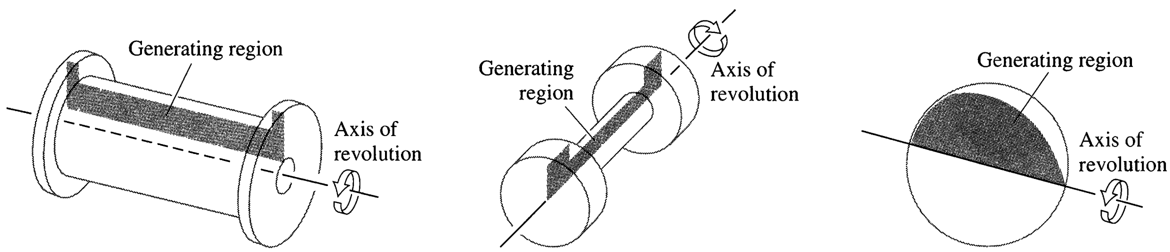
13. *Cavalieri's original theorem.* Prove Cavalieri's original theorem (marginal note, page 376), assuming that each region is bounded above and below by the graphs of continuous functions.

14. *The volume of a hemisphere (a classical application of Cavalieri's theorem).* Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross sections with the cross sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed.



5.3 Volumes of Solids of Revolution—Disks and Washers

The most common application of the method of slicing is to solids of revolution. **Solids of revolution** are solids whose shapes can be generated by revolving plane regions about axes. Thread spools are solids of revolution; so are hand weights and billiard balls. Solids of revolution sometimes have volumes we can find with formulas from geometry, as in the case of a billiard ball. But when we want to find the volume of a blimp or to predict the weight of a part we are going to have turned on a lathe, formulas from geometry are of little help and we turn to calculus for the answers.

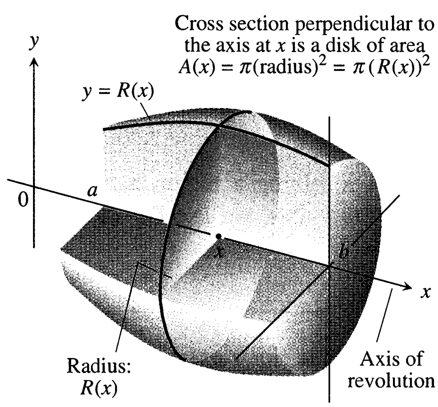


If we can arrange for the region to be the region between the graph of a continuous function $y = R(x)$, $a \le x \le b$, and the x-axis, and for the axis of revolution to be the x-axis (Fig. 5.14), we can find the solid's volume in the following way.

The typical cross section of the solid perpendicular to the axis of revolution is a disk of radius $R(x)$ and area

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

The solid's volume, being the integral of A from $x = a$ to $x = b$, is the integral of $\pi[R(x)]^2$ from a to b.



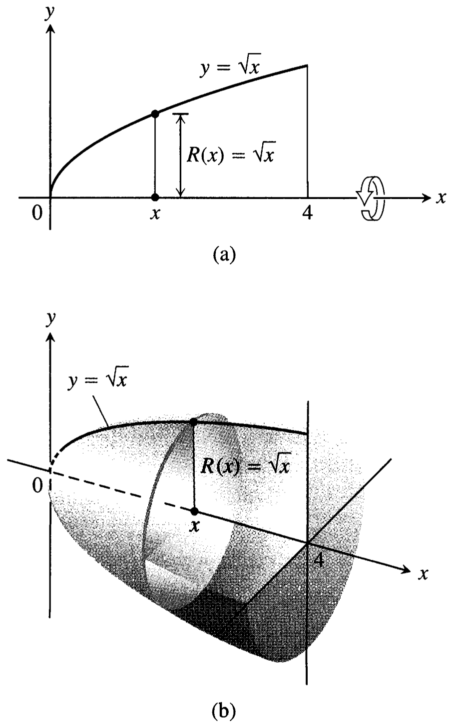
5.14 The solid generated by revolving the region between the curve $y = R(x)$ and the x-axis from a to b about the x-axis.

Volume of a Solid of Revolution (Rotation About the x-axis)

The volume of the solid generated by revolving about the x-axis the region between the x-axis and the graph of the continuous function $y = R(x)$, $a \le x \le b$, is

$$V = \int_a^b \pi[\text{radius}]^2\,dx = \int_a^b \pi[R(x)]^2\,dx. \tag{1}$$

EXAMPLE 1 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



5.15 The region (a) and solid (b) in Example 1.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.15). The volume is

$$
\begin{aligned}
V &= \int_a^b \pi[R(x)]^2\,dx && \text{Eq. (1)}\\
&= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx && R(x)=\sqrt{x}\\
&= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.
\end{aligned}
$$

❑

How to Find Volumes Using Eq. (1)

1. Draw the region and identify the radius function $R(x)$.
2. Square $R(x)$ and multiply by π.
3. Integrate to find the volume.

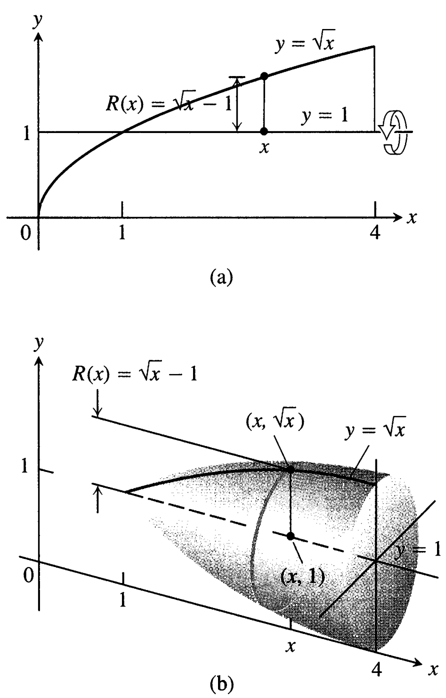
The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 2 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.16). The volume is

$$
\begin{aligned}
V &= \int_1^4 \pi[R(x)]^2\,dx && \text{Eq. (1)}\\
&= \int_1^4 \pi\left[\sqrt{x}-1\right]^2 dx && R(x)=\sqrt{x}-1\\
&= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right]dx\\
&= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.
\end{aligned}
$$

❑

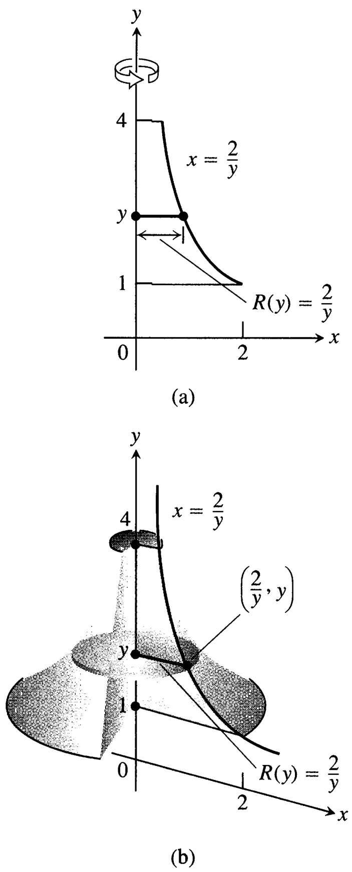


5.16 The region (a) and solid (b) in Example 2.

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use Eq. (1) with x replaced by y.

Volume of a Solid of Revolution (Rotation About the *y*-axis)

$$V = \int_c^d \pi(\text{radius})^2\,dy = \int_c^d \pi[R(y)]^2\,dy \tag{2}$$



5.17 The region (a) and solid (b) in Example 3.

EXAMPLE 3 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.17). The volume is

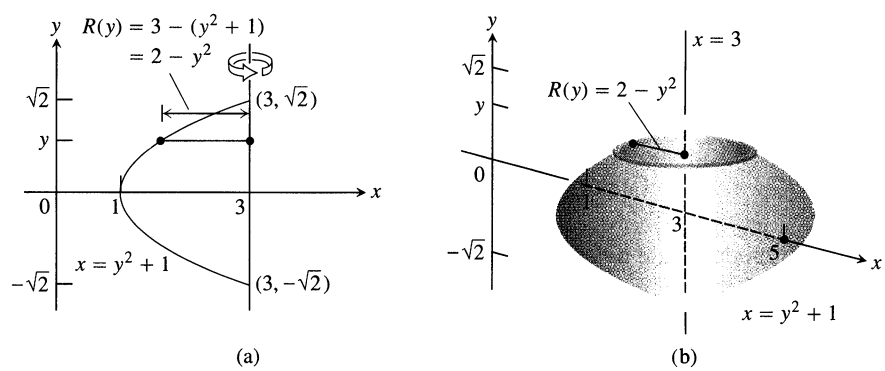
$$
\begin{aligned}
V &= \int_1^4 \pi [R(y)]^2\, dy && \text{Eq. (2)} \\
&= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy && R(y) = \frac{2}{y} \\
&= \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] \\
&= 3\pi.
\end{aligned}
$$

❑

EXAMPLE 4 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

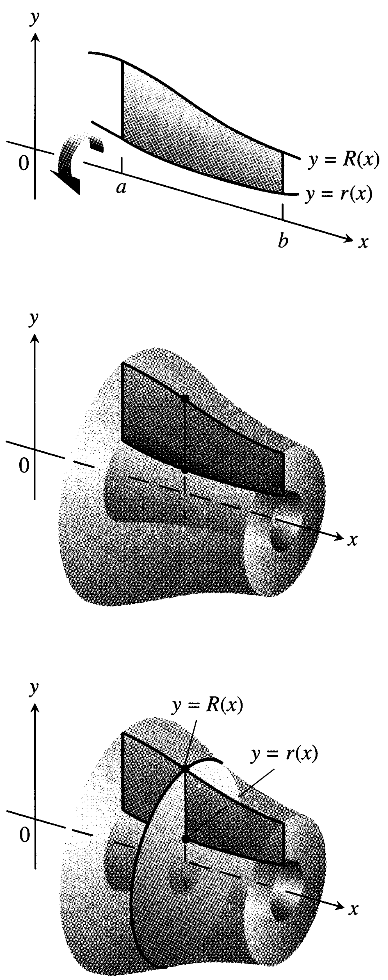
Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.18). The volume is

$$
\begin{aligned}
V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2\, dy && \text{Eq. (2)} \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2\, dy && R(y) = 3 - (y^2 + 1) = 2 - y^2 \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy \\
&= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{64\pi\sqrt{2}}{15}.
\end{aligned}
$$



5.18 The region (a) and solid (b) in Example 4.

❑



5.19 The cross sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Fig. 5.19). The cross sections perpendicular to the axis of revolution are washers instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$

$$\text{Inner radius:} \quad r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

The Washer Formula for Finding Volumes

$$V = \int_a^b \pi\left(\underbrace{[R(x)]^2}_{\substack{\text{outer}\\ \text{radius}\\ \text{squared}}} - \underbrace{[r(x)]^2}_{\substack{\text{inner}\\ \text{radius}\\ \text{squared}}}\right) dx \tag{3}$$

Notice that the function integrated in Eq. (3) is $\pi(R^2 - r^2)$, not $\pi(R-r)^2$. Also notice that Eq. (3) gives the disk method formula if $r(x)$ is zero throughout $[a, b]$. Thus, the disk method is a special case of the washer method.

EXAMPLE 5 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Fig. 5.20).

Step 2: Find the limits of integration by finding the x-coordinates of the intersection points.

$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

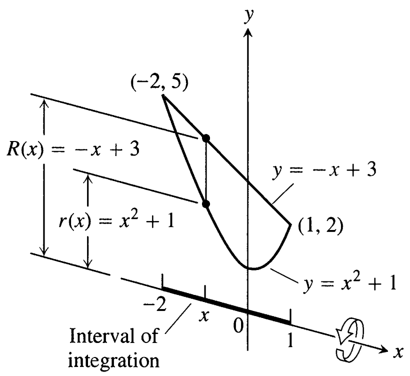
$$(x+2)(x-1) = 0$$

$$x = -2, \qquad x = 1$$

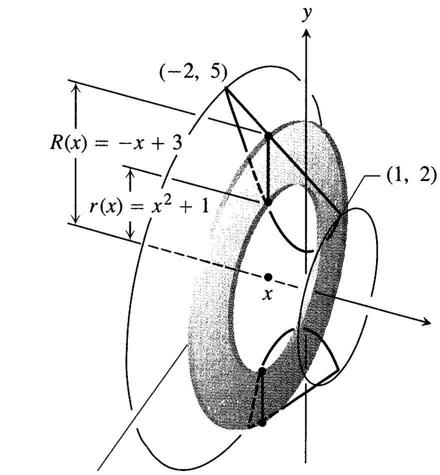
Step 3: Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region. (We drew the washer in Fig. 5.21, but in your own work you need not do that.) These radii are the distances of the ends of the line segment from the axis of revolution.

$$\text{Outer radius:} \quad R(x) = -x + 3$$

$$\text{Inner radius:} \quad r(x) = x^2 + 1$$



5.20 **The region in Example 5 spanned by a line segment perpendicular to the axis of revolution. When the region is revolved about the *x*-axis, the line segment will generate a washer.**



Washer cross section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

5.21 **The inner and outer radii of the washer swept out by the line segment in Fig. 5.20.**

Step 4: Evaluate the volume integral.

$$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx \qquad \text{Eq. (3)}$$

$$= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\, dx \qquad \text{Values from steps 2 and 3}$$

$$= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\, dx \qquad \text{Expressions squared and combined}$$

$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}$$

❑

How to Find Volumes by the Washer Method

1. *Draw the region and sketch a line segment across it perpendicular to the axis of revolution.* When the region is revolved, this segment will generate a typical washer cross section of the generated solid.
2. *Find the limits of integration.*
3. *Find the outer and inner radii* of the washer swept out by the line segment.
4. *Integrate* to find the volume.

To find the volume of a solid generated by revolving a region about the y-axis, we use the steps listed above but integrate with respect to y instead of x.

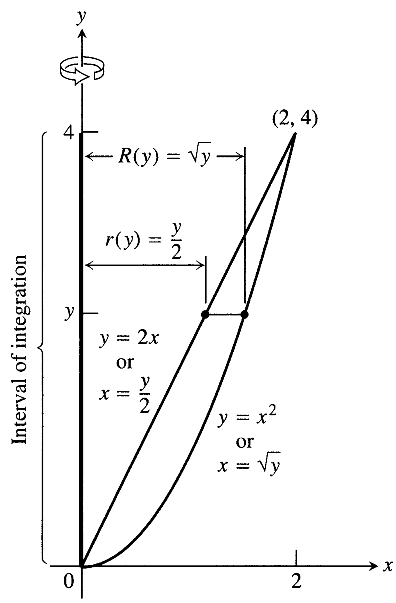
EXAMPLE 6 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

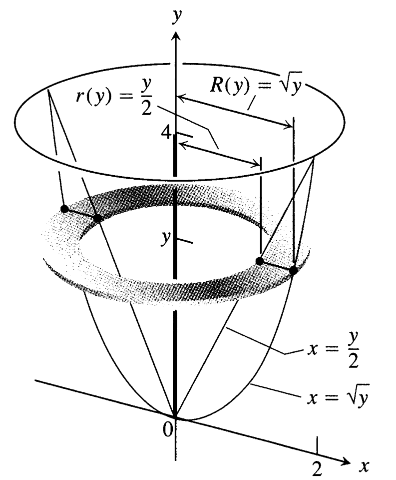
Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution, in this case the y-axis (Fig. 5.22).

Step 2: The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$.

Step 3: The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figs. 5.22 and 5.23).



5.22 The region, limits of integration, and radii in Example 6.



5.23 The washer swept out by the line segment in Fig. 5.22.

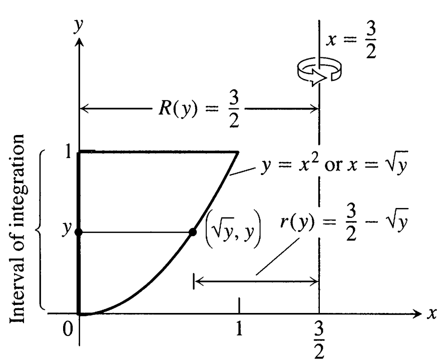
Step 4:

$$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy \qquad \text{Eq. (3) with } y \text{ in place of } x$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \qquad \text{Values from steps 2 and 3}$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$$

❑



5.24 The region, limits of integration, and radii in Example 7.

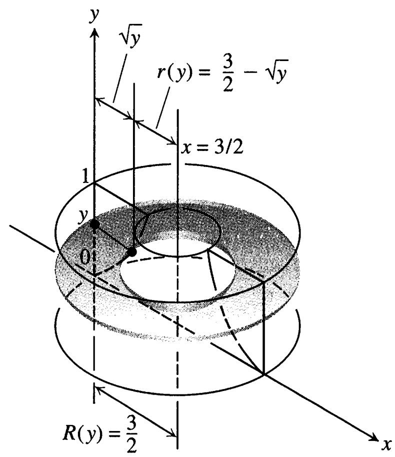
EXAMPLE 7 The region in the first quadrant enclosed by the parabola $y = x^2$, the y-axis, and the line $y = 1$ is revolved about the line $x = 3/2$ to generate a solid. Find the volume of the solid.

Solution

Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution, in this case the line $x = 3/2$ (Fig. 5.24).

Step 2: The limits of integration are $y = 0$ to $y = 1$.

Step 3: The radii of the washer swept out by the line segment are $R(y) = 3/2$, $r(y) = (3/2) - \sqrt{y}$ (Figs. 5.24 and 5.25).



5.25 The washer swept out by the line segment in Fig. 5.24.

Step 4:

$$V = \int_c^d \pi \left([R(y)]^2 - [r(y)]^2\right) dy$$

Eq. (3) with y in place of x

$$= \int_0^1 \pi \left(\left[\frac{3}{2}\right]^2 - \left[\frac{3}{2} - \sqrt{y}\right]^2\right) dy$$

$$= \pi \int_0^1 (3\sqrt{y} - y)\, dy = \pi \left[2y^{3/2} - \frac{y^2}{2}\right]_0^1 = \frac{3\pi}{2}$$

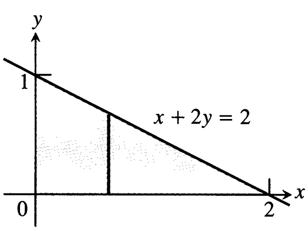
❑

Exercises 5.3

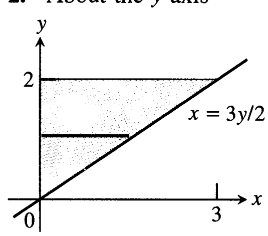
Volumes by the Disk Method

In Exercises 1–4, find the volume of the solid generated by revolving the shaded region about the given axis.

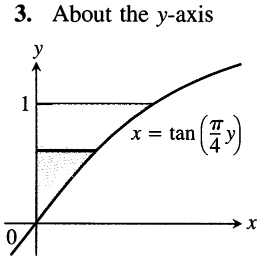
1. About the x-axis



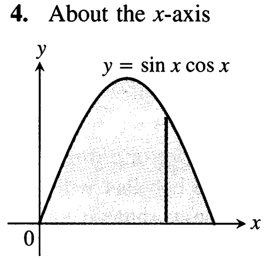
2. About the y-axis



3. About the y-axis



4. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 5–10 about the x-axis.

5. $y = x^2, \quad y = 0, \quad x = 2$

6. $y = x^3, \quad y = 0, \quad x = 2$

7. $y = \sqrt{9 - x^2}, \quad y = 0$

8. $y = x - x^2, \quad y = 0$

9. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

10. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 11 and 12, find the volume of the solid generated by revolving the region about the given line.

11. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

12. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x, 0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 13–18 about the y-axis.

13. $x = \sqrt{5}\, y^2, \quad x = 0, \quad y = -1, \quad y = 1$

14. $x = y^{3/2}, \quad x = 0, \quad y = 2$

15. $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

16. $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

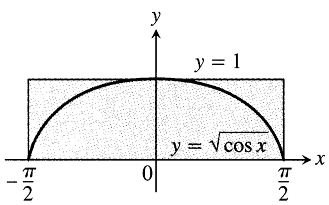
17. $x = 2/(y+1), \quad x = 0, \quad y = 0, \quad y = 3$

18. $x = \sqrt{2y}/(y^2+1), \quad x = 0, \quad y = 1$

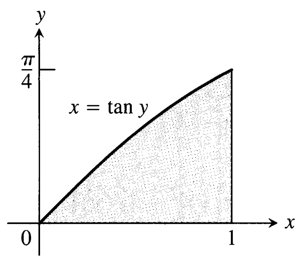
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 19 and 20 about the indicated axes.

19. The x-axis



20. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 21–28 about the x-axis.

21. $y = x, \quad y = 1, \quad x = 0$

22. $y = 2x, \quad y = x, \quad x = 1$

23. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

24. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$

25. $y = x^2 + 1, \quad y = x + 3$

26. $y = 4 - x^2, \quad y = 2 - x$

27. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

28. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 29–34, find the volume of the solid generated by revolving each region about the y-axis.

29. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

30. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

31. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

32. The region bounded above by the curve $y = \sqrt{x}$ and below by the line $y = x$

33. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

34. The region bounded on the left by the line $x = 4$ and on the right by the circle $x^2 + y^2 = 25$

In Exercises 35 and 36, find the volume of the solid generated by revolving each region about the given axis.

35. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

36. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

37. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a) the x-axis;
b) the y-axis;
c) the line $y = 2$;
d) the line $x = 4$.

38. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a) the line $x = 1$;
b) the line $x = 2$.

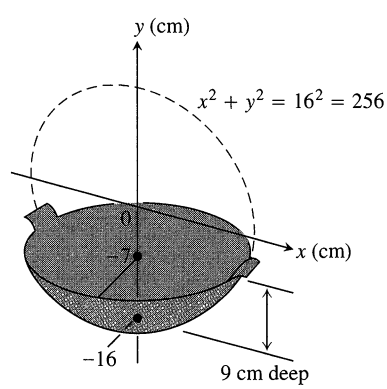
39. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a) the line $y = 1$;
b) the line $y = 2$;
c) the line $y = -1$.

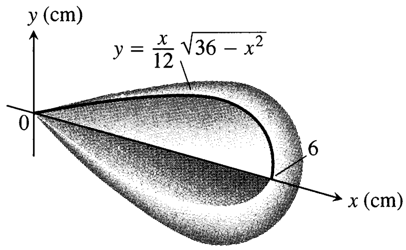
40. By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), $(b, 0)$, $(0, h)$ about

a) the x-axis;
b) the y-axis.

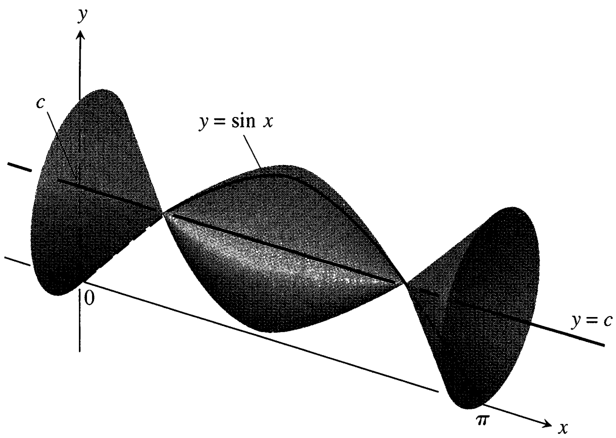
41. *Designing a wok.* You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3)



42. *Designing a plumb bob.* Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh (to the nearest gram)?



43. The arch $y = \sin x, 0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in Fig. 5.26.



5.26 Exercise 43 asks for the value of c that minimizes the volume of this solid.

a) Find the value of c that minimizes the volume of the solid. What is the minimum value?

b) What value of c in [0, 1] maximizes the volume of the solid?

c) GRAPHER Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from [0, 1]? Does this make sense physically? Give reasons for your answer.

44. *An auxiliary fuel tank.* You are designing an auxiliary fuel tank that will fit under a helicopter's fuselage to extend its range. After some experimentation at your drawing board, you decide to shape the tank like the surface generated by revolving the curve $y = 1 - (x^2/16), -4 \le x \le 4$, about the x-axis (dimensions in feet).

a) How many cubic feet of fuel will the tank hold (to the nearest cubic foot)?

b) A cubic foot holds 7.481 gal. If the helicopter gets 2 mi to the gallon, how many additional miles will the helicopter be able to fly once the tank is installed (to the nearest mile)?

45. *The volume of a torus.* The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

46. a) A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

b) (Related rates) Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep?

47. *Testing the consistency of the calculus definition of volume.* The volume formulas in this section are all consistent with the standard formulas from geometry.

a) As a case in point, show that if you revolve the region enclosed by the semicircle $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis to generate a solid sphere, the disk formula for volume (Eq. 1) will give $(4/3)\pi a^3$ just as it should.

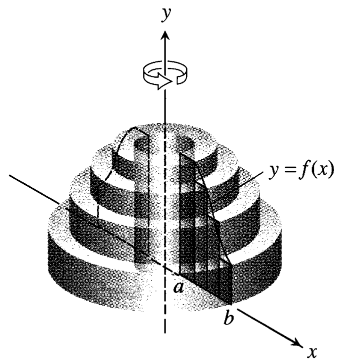
b) Use calculus to find the volume of a right circular cone of height h and base radius r.

5.4 Cylindrical Shells

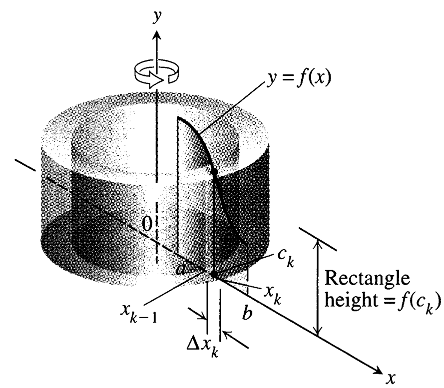
When we need to find the volume of a solid of revolution, cylindrical shells sometimes work better than washers (Fig. 5.27, on the following page). In part, the reason is that the formula they lead to does not require squaring.

The Shell Formula

Suppose we revolve the tinted region in Fig. 5.28 (on the following page) about the y-axis to generate a solid. To estimate the volume of the solid, we can approximate the region with rectangles based on a partition P of the interval $[a, b]$ over which the region stands. The typical approximating rectangle is Δx_k units wide by $f(c_k)$ units high, where c_k is the midpoint of the rectangle's base. A formula from geometry tells



5.27 A solid of revolution approximated by cylindrical shells.



5.28 The shell swept out by the *k*th rectangle.

us that the volume of the shell swept out by the rectangle is

$$\Delta V_k = 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness},$$

which in our case is

$$\Delta V_k = 2\pi \times c_k \times f(c_k) \times \Delta x_k.$$

We approximate the volume of the solid by adding the volumes of the shells swept out by the n rectangles based on P:

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi c_k\, f(c_k)\Delta x_k. \qquad \text{A Riemann sum}$$

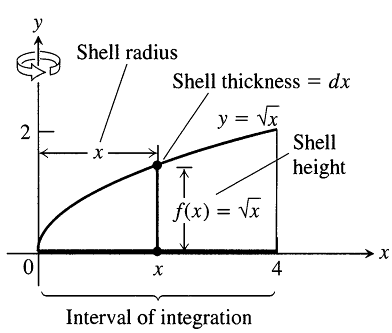
The limit of this sum as $\|P\| \to 0$ gives the volume of the solid:

$$V = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi c_k\, f(c_k)\Delta x_k = \int_a^b 2\pi x\, f(x)\, dx.$$

The Shell Formula for Revolution About the *y*-axis

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \ge 0$, $0 \le a \le x \le b$, about the y-axis is

$$V = \int_a^b 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_a^b 2\pi x\, f(x)\, dx. \qquad (1)$$



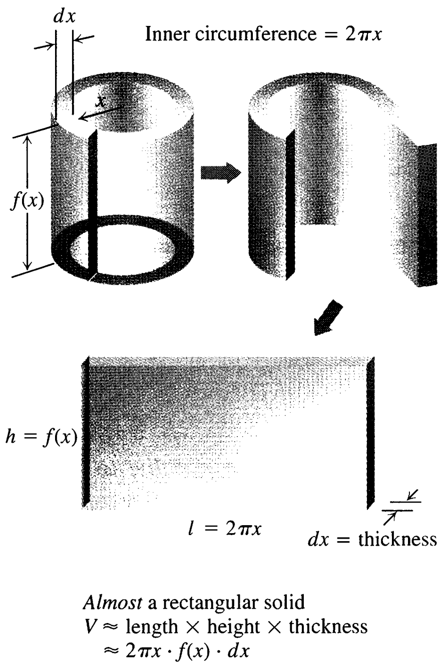
5.29 The region, shell dimensions, and interval of integration in Example 1.

EXAMPLE 1 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

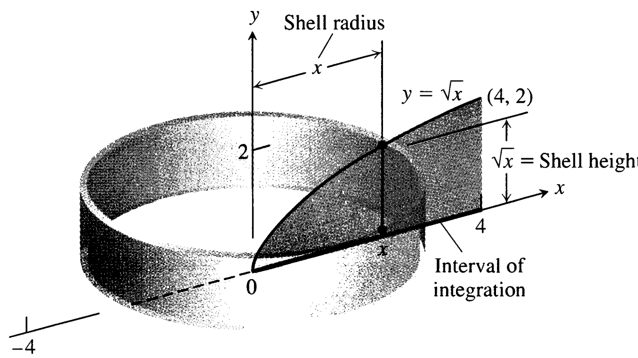
Solution

Step 1: Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Fig. 5.29). Label the segment's height (shell height) and distance from

One way to remember Eq. (1) is to imagine cutting and unrolling a cylindrical shell to get a (nearly) flat rectangular solid.



the axis of revolution (shell radius). The width of the segment is the shell thickness dx. (We drew the shell in Fig. 5.30, but you need not do that.)



5.30 The shell swept out by the line segment in Fig. 5.29.

Step 2: Find the limits of integration: x runs from $a = 0$ to $b = 4$.

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \qquad \text{Eq. (1)}$$

$$= \int_0^4 2\pi(x)(\sqrt{x})\,dx \qquad \text{Values from steps 1 and 2}$$

$$= 2\pi \int_0^4 x^{3/2}\,dx = 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

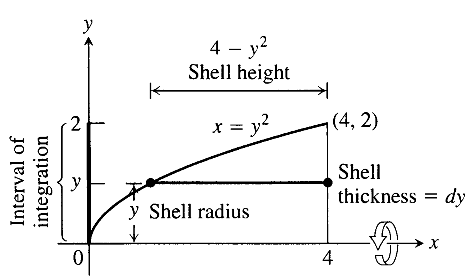
❑

Equation (1) is for vertical axes of revolution. For horizontal axes, we replace the x's with y's.

The Shell Formula for Revolution About the x-axis

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy = \int_c^d 2\pi y f(y)\,dy \qquad (2)$$

(for $f(y) \geq 0$ and $0 \leq c \leq y \leq d$)

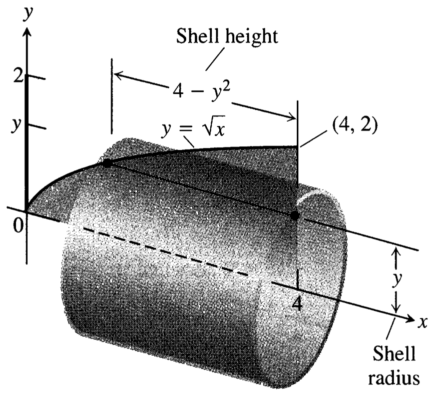


5.31 The region, shell dimensions, and interval of integration in Example 2.

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

Step 1: Sketch the region and draw a line segment across it parallel to the axis of revolution (Fig. 5.31). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. (We drew the shell in Fig. 5.32, shown on the following page, but you need not do that.)



5.32 The shell swept out by the line segment in Fig. 5.31.

Step 2: Identify the limits of integration: y runs from $c = 0$ to $d = 2$.

Step 3: Integrate to find the volume.

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \qquad \text{Eq. (2)}$$

$$= \int_0^2 2\pi(y)(4 - y^2)\,dy \qquad \text{Values from steps 1 and 2}$$

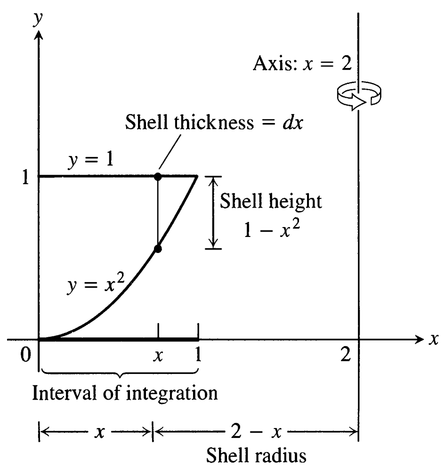
$$= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$$

This agrees with the disk method of calculation in Section 5.3, Example 1. ❑

How to Use the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these:

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height), distance from the axis of revolution (shell radius), and width (shell thickness).
2. *Find* the limits of integration.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the appropriate variable (x or y) to find the volume.



5.33 The region, shell dimensions, and interval of integration in Example 3.

In the next example, the axis of revolution is the vertical line $x = 2$.

EXAMPLE 3 The region in the first quadrant bounded by the parabola $y = x^2$, the y-axis, and the line $y = 1$ is revolved about the line $x = 2$ to generate a solid. Find the volume of the solid.

Solution

Step 1: Draw a line segment across the region parallel to the axis of revolution (the line $x = 2$) (Fig. 5.33). Label the segment's height (shell height), distance from the axis of revolution (shell radius), and width (in this case, dx). (We drew the shell in Fig. 5.34, but you need not do that.)

Step 2: The limits of integration: x runs from $a = 0$ to $b = 1$.

Step 3:

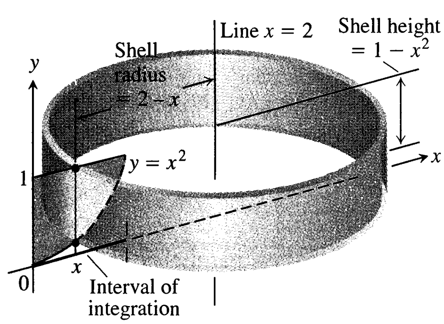
$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \qquad \text{Eq. (1)}$$

$$= \int_0^1 2\pi(2 - x)(1 - x^2)\,dx \qquad \text{Values from steps 1 and 2}$$

$$= 2\pi\int_0^1 (2 - x - 2x^2 + x^3)\,dx$$

$$= \frac{13\pi}{6}$$

❑

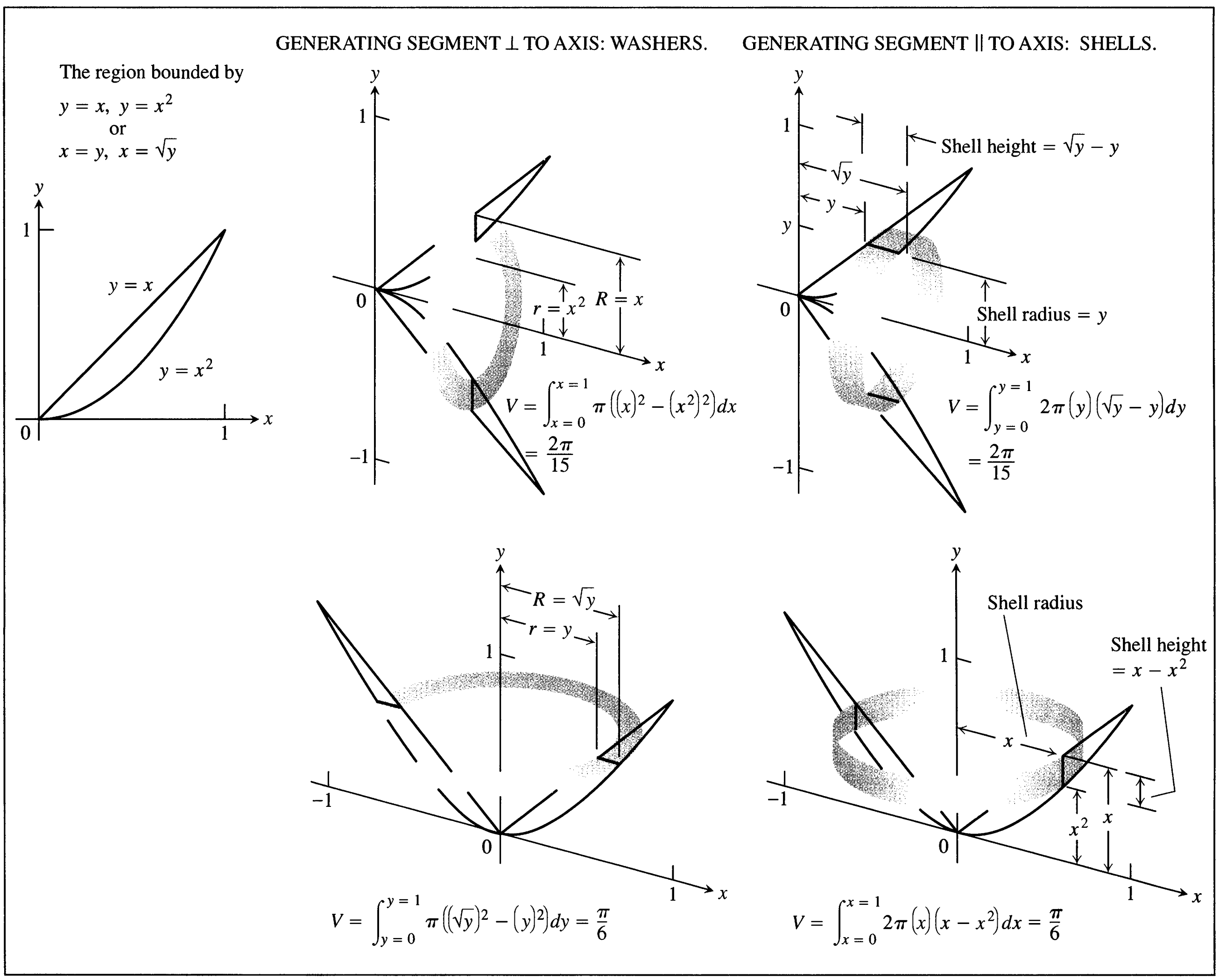


5.34 The shell swept out by the line segment in Fig. 5.33.

Table 5.1 summarizes the washer and shell methods for the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about the coordinate axes. For this particular region, both methods work well for both axes of revolution. But this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. However, it may not be possible to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead.

The washer and shell methods for calculating volumes of solids of revolution always agree. In Section 6.1 (Exercise 52), we will be able to prove the equivalence for a broad class of solids.

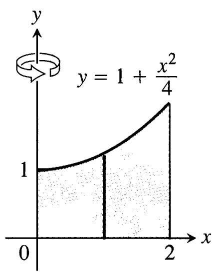
Table 5.1 Washers vs. shells



Exercises 5.4

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

1.



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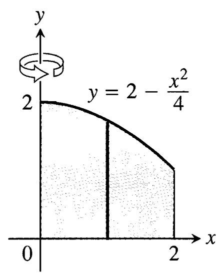

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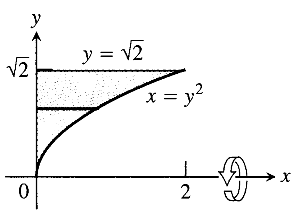

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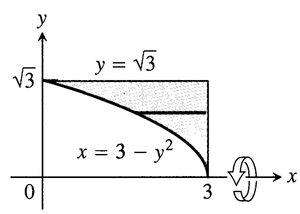
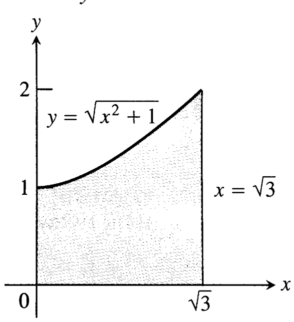
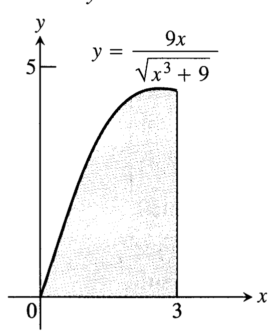


5. The y-axis



6. The y-axis



Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–14 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0$, for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

11. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

12. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

13. $y = 1/x, \quad y = 0, \quad x = 1/2, \quad x = 2$

14. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y}, \quad x = -y, \quad y = 2$

16. $x = y^2, \quad x = -y, \quad y = 2$

17. $x = 2y - y^2, \quad x = 0$

18. $x = 2y - y^2, \quad x = y$

19. $y = |x|, \quad y = 1$

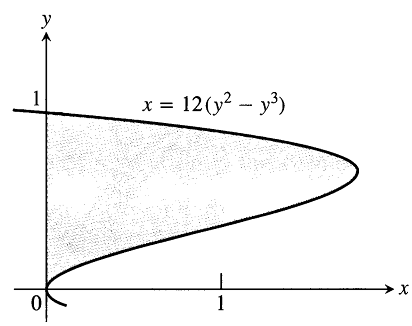
20. $y = x, \quad y = 2x, \quad y = 2$

21. $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

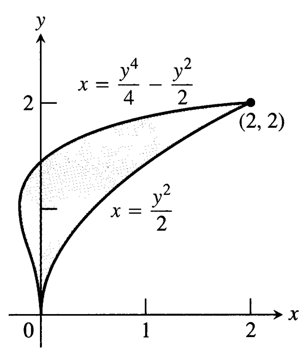
22. $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

In Exercises 23 and 24, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

23. **a)** The x-axis
b) The line $y = 1$
c) The line $y = 8/5$
d) The line $y = -2/5$



24. **a)** The x-axis
b) The line $y = 2$
c) The line $y = 5$
d) The line $y = -5/8$



In Exercises 25–32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use disks or washers in any given instance, feel free to do so.

25. The triangle with vertices (1, 1), (1, 2), and (2, 2) about (a) the x-axis; (b) the y-axis; (c) the line $x = 10/3$; (d) the line $y = 1$

26. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y-axis about (a) the x-axis; (b) the line $y = 1$

27. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 1$

28. The triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 8$

29. The region in the first quadrant bounded by $y = x^3$ and $y = 4x$ about (a) the x-axis; (b) the line $y = 8$

30. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about (a) the x-axis; (b) the y-axis

31. The region bounded by $y = 2x - x^2$ and $y = x$ about (a) the y-axis; (b) the line $x = 1$

32. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 2$

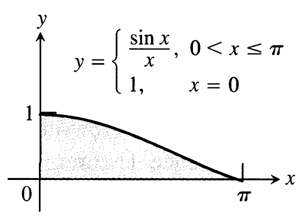
33. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$, is revolved about the x-axis to generate a solid. Find the volume of the solid by (a) the washer method; (b) the shell method.

34. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $y = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by (a) the washer method; (b) the shell method.

35. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0. \end{cases}$

a) Show that $xf(x) = \sin x, 0 \le x \le \pi$.

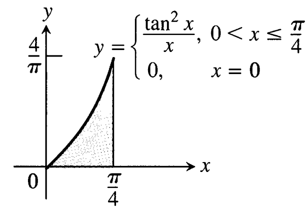
b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



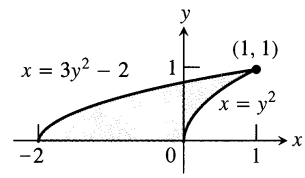
36. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \le \pi/4 \\ 0, & x = 0. \end{cases}$

a) Show that $xg(x) = (\tan x)^2, 0 \le x \le \pi/4$.

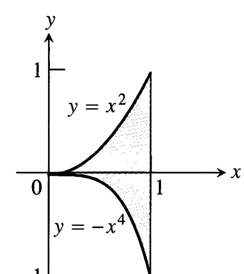
b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



37. The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



38. The region shown here is to be revolved about the y-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



39. Suppose that the function $f(x)$ is nonnegative and continuous for $x \ge 0$. Suppose also that, for every positive number b, revolving the region enclosed by the graph of f, the coordinate axes, and the line $x = b$ about the y-axis generates a solid of volume $2\pi b^3$. Find $f(x)$.

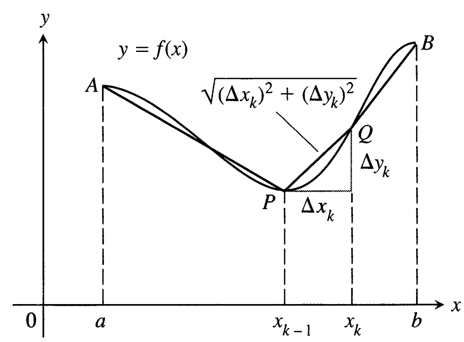
5.5 Lengths of Plane Curves

We approximate the length of a curved path in the plane the way we use a ruler to estimate the length of a curved road on a map, by measuring from point to point with straight-line segments and adding the results. There is a limit to the accuracy of such an estimate, however, imposed in part by how accurately we measure and in part by how many line segments we use.

With calculus we can usually do a better job because we can imagine using straight-line segments as short as we please, each set of segments making a polygonal path that fits the curve more tightly than before. When we proceed this way, with a smooth curve, the lengths of the polygonal paths approach a limit we can calculate with an integral.

The Basic Formula

Suppose we want to find the length of the curve $y = f(x)$ from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and connect the corresponding points on the curve with line segments to form a polygonal path that approximates the curve (Fig. 5.35). If we can find a formula for the length of the path, we will have a formula for approximating the length of the curve.

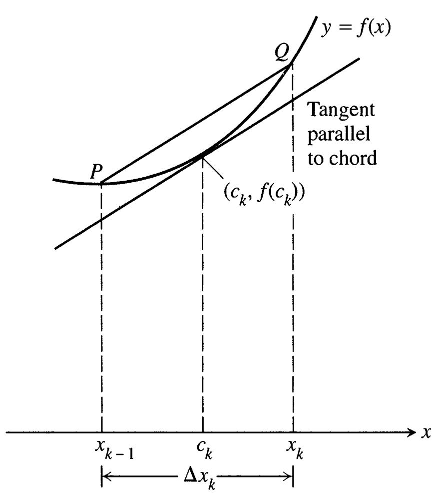


5.35 A typical segment *PQ* of a polygonal path approximating the curve *AB*.

The length of a typical line segment PQ (see the figure) is $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. The length of the curve is therefore approximated by the sum

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer, and we would like to show that the sums in (1) approach a calculable limit as the norm of the partition goes to zero. To show this, we rewrite the sum in (1) in a form to which we can apply the Integral Existence Theorem from Chapter 4. Our starting point is the Mean Value Theorem for derivatives.



5.36 Enlargement of the arc *PQ* in Fig. 5.35.

Definition

A function with a continuous first derivative is said to be **smooth** and its graph is called a **smooth curve.**

If f is smooth, by the Mean Value Theorem there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Fig. 5.36). At this point

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}, \qquad \text{or} \qquad \Delta y_k = f'(c_k)\Delta x_k.$$

With this substitution for Δy_k, the sums in (1) take the form

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\, \Delta x_k.$$

A Riemann sum

Because $\sqrt{1 + (f'(x))^2}$ is continuous on $[a, b]$, the limit of the sums on the right as the norm of the partition goes to zero is $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$. We define the length of the curve to be the value of this integral.

Definition

If f is smooth on $[a, b]$, the **length** of the curve $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dy. \tag{2}$$

EXAMPLE 1 Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Eq. (2) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2} x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (2) with $a = 0$, $b = 1$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}.$$

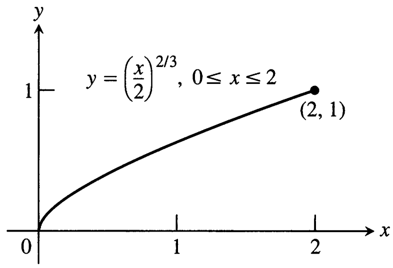
Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$. ❑

Dealing with Discontinuities in *dy/dx*

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Eq. (2):

Formula for the Length of a Smooth Curve $x = g(y)$, $c \le y \le d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy. \tag{3}$$



5.37 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$

EXAMPLE 2 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Eq. (2).

We therefore rewrite the equation to express x in terms of y:

$$\begin{aligned} y &= \left(\frac{x}{2}\right)^{2/3} \\ y^{3/2} &= \frac{x}{2} && \text{Raise both sides to the power 3/2.} \\ x &= 2y^{3/2}. && \text{Solve for } x. \end{aligned}$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Fig. 5.37).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Eq. (3) to find the curve's length:

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy && \text{Eq. (3) with } c = 0, d = 1 \\ &= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1 && \text{Let } u = 1 + 9y,\ du/9 = dy, \text{ integrate, and substitute back.} \\ &= \frac{2}{27}(10\sqrt{10} - 1) \approx 2.27. \end{aligned}$$

❑

The Short Differential Formula

The equations

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{and} \quad L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{4}$$

are often written with differentials instead of derivatives. This is done formally by thinking of the derivatives as quotients of differentials and bringing the dx and dy inside the radicals to cancel the denominators. In the first integral we have

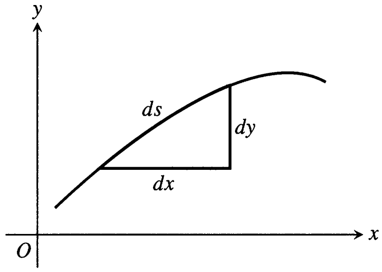
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{dy^2}{dx^2}}\, dx = \sqrt{dx^2 + \frac{dy^2}{dx^2}\, dx^2} = \sqrt{dx^2 + dy^2}.$$

In the second integral we have

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{dx^2}{dy^2}}\, dy = \sqrt{dy^2 + \frac{dx^2}{dy^2}\, dy^2} = \sqrt{dx^2 + dy^2}.$$

Thus the integrals in (4) reduce to the same differential formula:

$$L = \int_a^b \sqrt{dx^2 + dy^2}. \tag{5}$$



5.38 Diagram for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

Of course, dx and dy must be expressed in terms of a common variable, and appropriate limits of integration must be found before the integration in Eq. (5) is performed.

We can shorten Eq. (5) still further. Think of dx and dy as two sides of a small triangle whose "hypotenuse" is $ds = \sqrt{dx^2 + dy^2}$ (Fig. 5.38). The differential ds is then regarded as a differential of arc length that can be integrated between appropriate limits to give the length of the curve. With $\sqrt{dx^2 + dy^2}$ set equal to ds, the integral in Eq. (5) simply becomes the integral of ds.

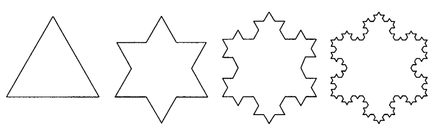
Definition

The Arc Length Differential and the Differential Formula for Arc Length

$$ds = \sqrt{dx^2 + dy^2} \qquad\qquad L = \int ds$$

arc length differential — differential formula for arc length

Curves with Infinite Length



5.39 The first four polygonal approximations in the construction of Helga von Koch's snowflake.

As you may recall from Section 2.6, Helga von Koch's snowflake curve K is the limit curve of an infinite sequence $C_1, C_2, \ldots, C_n, \ldots$ of "triangular" polygonal curves. Figure 5.39 shows the first four curves in the sequence. Each time we introduce a new vertex in the construction process, it remains as a vertex in all subsequent curves and becomes a point on the limit curve K. This means that each of the C's is itself a polygonal approximation of K—the endpoints of its sides all belonging to K. The length of K should therefore be the limit of the lengths of the curves C_n. At least, that is what it should be if we apply the definition of length we developed for smooth curves.

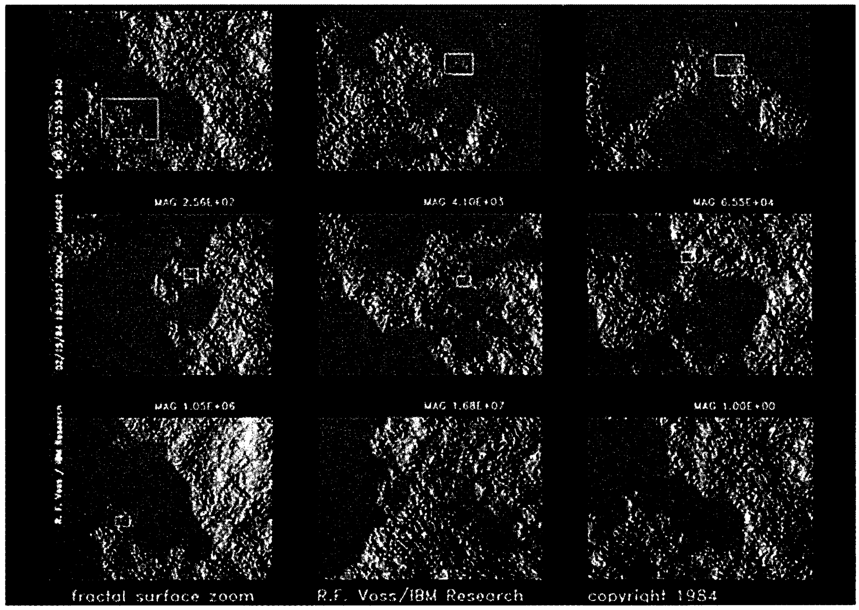
What, then, is the limit of the lengths of the curves C_n? If the original equilateral triangle C_1 has sides of length 1, the total length of C_1 is 3. To make C_2 from C_1, we replace each side of C_1 by four segments, each of which is one-third as long as the original side. The total length of C_2 is therefore 3(4/3). To get the length of C_3, we multiply by 4/3 again. We do so again to get the length of C_4. By the time we get out to C_n, we have a curve of length $3(4/3)^{n-1}$.

Curve Number	1	2	3	$\cdots$	n	$\cdots$
Length	3	$3\left(\frac{4}{3}\right)$	$3\left(\frac{4}{3}\right)^2$	$\cdots$	$3\left(\frac{4}{3}\right)^{n-1}$	$\cdots$

The length of C_{10} is nearly 40 and the length of C_{100} is greater than 7,000,000,000,000. The lengths grow too rapidly to have a finite limit. Therefore the snowflake curve has no length, or, if you prefer, infinite length.

What went wrong? Nothing. The formulas we derived for length are for the graphs of smooth functions, curves that are smooth enough to have a continuously turning tangent at every point. Helga von Koch's snowflake curve is too rough for that, and our derivative-based formulas do not apply.

Benoit Mandelbrot's theory of fractals has proved to be a rich source of curves with infinite length, curves that when magnified prove to be as rough and varied as they looked before magnification. Like coastlines on an ocean, such curves cannot be smoothed out by magnification (Fig. 5.40, on the following page).



5.40 Repeated magnifications of a fractal coastline. Like Helga Von Koch's snowflake curve, coasts like these are too rough to have a measurable length.

Exercises 5.5

Finding Integrals for Lengths of Curves

In Exercises 1–8:

a) Set up an integral for the length of the curve.

b) Graph the curve to see what it looks like.

c) Use your grapher's or computer's integral evaluator to find the curve's length numerically.

1. $y = x^2, \quad -1 \le x \le 2$

2. $y = \tan x, \quad -\pi/3 \le x \le 0$

3. $x = \sin y, \quad 0 \le y \le \pi$

4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

5. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

6. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$

7. $y = \displaystyle\int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$

8. $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 9–18. If you have a grapher, you may want to graph these curves to see what they look like.

9. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

10. $y = x^{3/2}$ from $x = 0$ to $x = 4$

11. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

12. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

13. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

14. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

15. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$

17. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$

18. $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$

19. a) Find a curve through the point (1, 1) whose length integral (Eq. 2) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

b) How many such curves are there? Give reasons for your answer.

20. a) Find a curve through the point (0, 1) whose length integral (Eq. 3) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

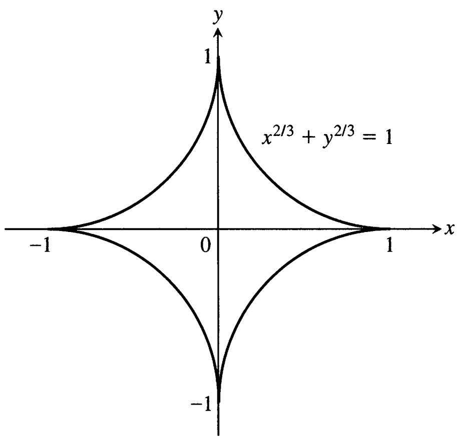
b) How many such curves are there? Give reasons for your answer.

21. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

22. *The length of an astroid.* The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



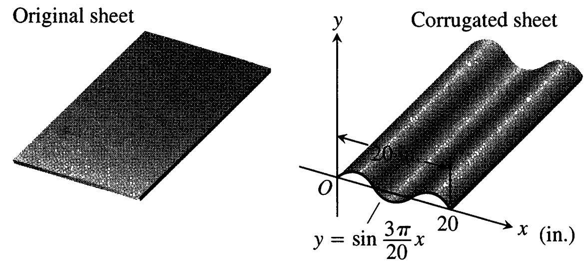
Numerical Integration

You may have wondered why so many of the curves we have been working with have unusual formulas. The reason is that the square root $\sqrt{1 + (dy/dx)^2}$ that appears in the integrals for length and surface area almost never leads to a function whose antiderivative we can find. In fact, the square root itself is a well-known source of nonelementary integrals. Most integrals for length and surface area have to be evaluated numerically, as in Exercises 23 and 24.

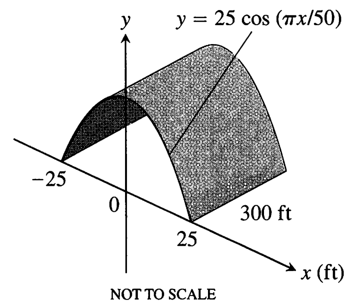
23. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20} x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to 2 decimal places.



24. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross section is shaped like one arch of the curve $y = 25 \cos (\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Theory and Examples

25. Is there a smooth curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

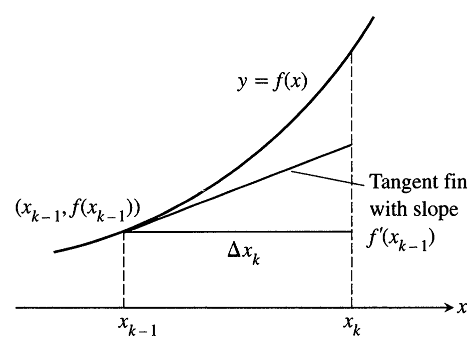
26. *Using tangent fins to derive the length formula for curves.* Assume f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$ construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, shown in the figure.

a) Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}$.

b) Show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length L of the curve $y = f(x)$ from a to b.



CAS Explorations and Projects

In Exercises 27–32, use a CAS to perform the following steps for the given curve over the closed interval.

a) Plot the curve together with the polygonal path approximations for $n = 2,\ 4,\ 8$ partition points over the interval. (See Fig. 5.35.)
b) Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.
c) Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ to the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

27. $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

28. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

29. $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

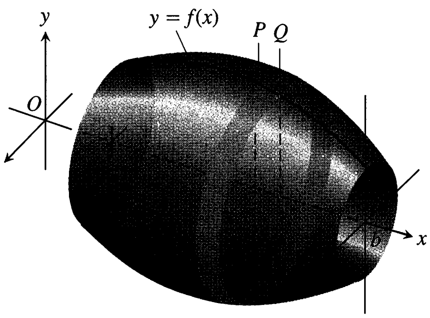
30. $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

31. $f(x) = \dfrac{x-1}{4x^2+1}, \quad -\dfrac{1}{2} \le x \le 1$

32. $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

5.6 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you, a surface called a surface of revolution. As you can imagine, the area of this surface depends on the rope's length and on how far away each segment of the rope swings. This section explores the relation between the area of a surface of revolution and the length and reach of the curve that generates it. The areas of more complicated surfaces will be treated in Chapter 14.



5.41 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

The Basic Formula

Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative function $y = f(x), a \le x \le b$, about the x-axis. We partition $[a, b]$ in the usual way and use the points in the partition to partition the graph into short arcs. Figure 5.41 shows a typical arc PQ and the band it sweeps out as part of the graph of f.

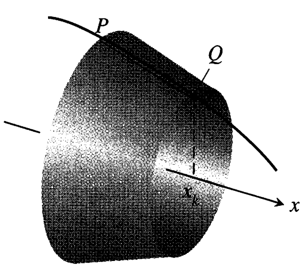
As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out part of a cone whose axis lies along the x-axis (magnified view in Fig. 5.42). A piece of a cone like this is called a *frustum* of the cone, *frustum* being Latin for "piece." The surface area of the frustum approximates the surface area of the band swept out by the arc PQ.

The surface area of the frustum of a cone (see Fig. 5.43) is 2π times the average of the base radii times the slant height:

$$\text{Frustum surface area } = 2\pi \cdot \frac{r_1 + r_2}{2} \cdot L = \pi(r_1 + r_2)L.$$

For the frustum swept out by the segment PQ (Fig. 5.44), this works out to be

$$\text{Frustum surface area } = \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

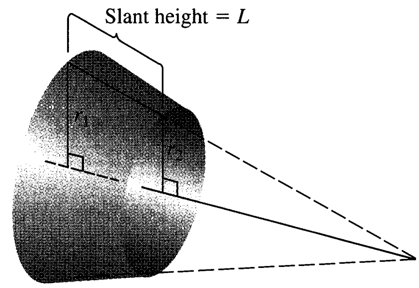


5.42 The line segment joining P and Q sweeps out a frustum of a cone.

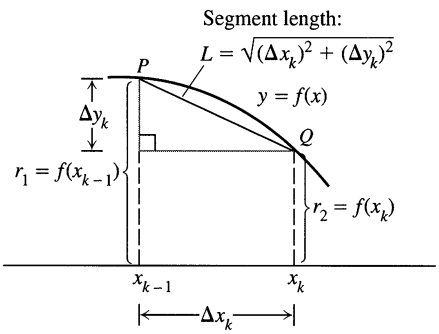
The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \qquad (1)$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer, and we would like to show that the sums in (1) approach a calculable limit as the norm of the partition goes to zero.



5.43 The important dimensions of the frustum in Fig. 5.42.



5.44 Dimensions associated with the arc and segment *PQ*.

To show this, we try to rewrite the sum in (1) as the Riemann sum of some function over the interval from a to b. As in the calculation of arc length, we begin by appealing to the Mean Value Theorem for derivatives.

If f is smooth, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Fig. 5.45). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\Delta x_k.$$

With this substitution for Δy_k, the sums in (1) take the form

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\ \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2}$$

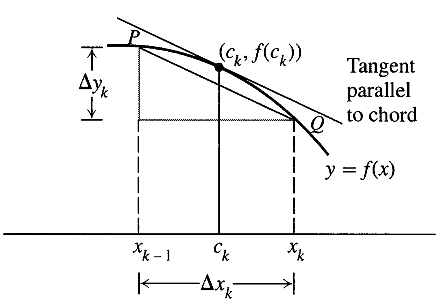
$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\ \Delta x_k. \quad (2)$$

At this point there is both good news and bad news.

The bad news is that the sums in (2) are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same and there is no way to make them the same. The good news is that this does not matter. A theorem called Bliss's theorem, from advanced calculus, assures us that as the norm of the partition of $[a, b]$ goes to zero, the sums in Eq. (2) converge to

$$\int_a^b 2\pi f(x)\ \sqrt{1 + (f'(x))^2}\ dx$$

just the way we want them to. We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.



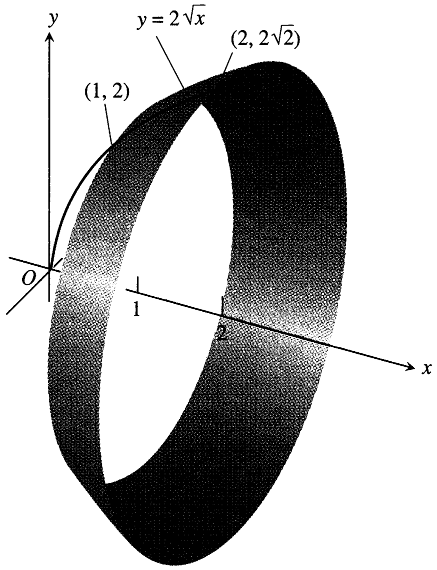
5.45 If f is smooth, the Mean Value Theorem guarantees the existence of a point on arc *PQ* where the tangent is parallel to segment *PQ*.

Definition

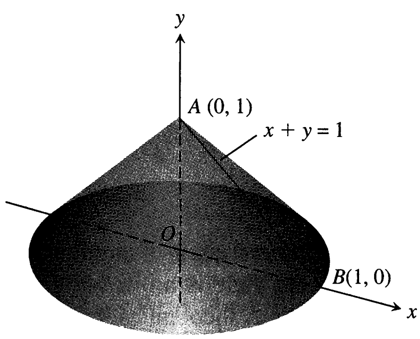
The Surface Area Formula for the Revolution About the *x*-axis

If the function $f(x) \geq 0$ is smooth on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \quad (3)$$



5.46 Example 1 calculates the area of this surface.



5.47 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

The square root in Eq. (3) is the same one that appears in the formula for the length of the generating curve.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Fig. 5.46).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\, \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

❑

Revolution About the *y*-axis

For revolution about the y-axis, we interchange x and y in Eq. (3).

> **Surface Area Formula for Revolution About the *y*-axis**
>
> If $x = g(y) \ge 0$ is smooth on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is
>
> $$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

EXAMPLE 2 The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone in Fig. 5.47. Find its lateral surface area.

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Eq. (4) gives the same result, we take

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi (1 - y)\sqrt{2}\, dy \\ &= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) \\ &= \pi\sqrt{2}. \end{aligned}$$

The results agree, as they should. ❑

The Short Differential Form

The equations

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{and} \qquad S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

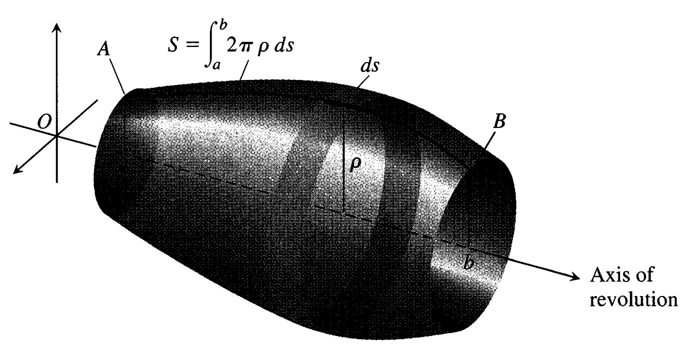
are often written in terms of the arc length differential $ds = \sqrt{dx^2 + dy^2}$ as

$$S = \int_a^b 2\pi y\, ds \qquad \text{and} \qquad S = \int_c^d 2\pi x\, ds.$$

In the first of these, y is the distance from the x-axis to an element of arc length ds. In the second, x is the distance from the y-axis to an element of arc length ds. Both integrals have the form

$$S = \int 2\pi (\text{radius})(\text{band width}) = \int 2\pi \rho\, ds,$$

where ρ is the radius from the axis of revolution to an element of arc length ds (Fig. 5.48).



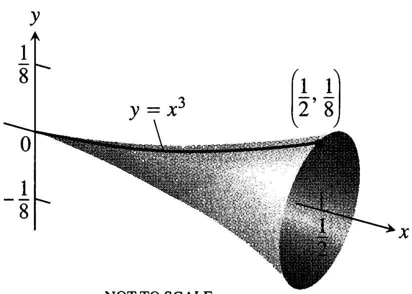
5.48 The area of the surface swept out by revolving arc AB about the axis shown here is $\int_a^b 2\pi\rho\, ds$. The exact expression depends on the formulas for ρ and ds.

If you wish to remember only one formula for surface area, you might make it the short differential form.

Short Differential Form

$$S = \int 2\pi\rho \, ds$$

In any particular problem, you would then express the radius function ρ and the arc length differential ds in terms of a common variable and supply limits of integration for that variable.



5.49 The surface generated by revolving the curve $y = x^3, 0 \le x \le 1/2$, about the x-axis could be the design for a champagne glass (Example 3).

EXAMPLE 3 Find the area of the surface generated by revolving the curve $y = x^3, 0 \le x \le 1/2$, about the x-axis (Fig. 5.49).

Solution We start with the short differential form:

$$S = \int 2\pi\rho \, ds$$

$$= \int 2\pi y \, ds \qquad \text{For revolution about the } x\text{-axis, the radius function is } \rho = y.$$

$$= \int 2\pi y \sqrt{dx^2 + dy^2}. \qquad ds = \sqrt{dx^2 + dy^2}$$

We then decide whether to express dy in terms of dx or dx in terms of dy. The original form of the equation, $y = x^3$, makes it easier to express dy in terms of dx, so we continue the calculation with

$$y = x^3, \qquad dy = 3x^2\,dx, \qquad \text{and} \qquad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\,dx)^2} = \sqrt{1 + 9x^4}\,dx.$$

With these substitutions, x becomes the variable of integration and

$$S = \int_{x=0}^{x=1/2} 2\pi y \sqrt{dx^2 + dy^2}$$

$$= \int_0^{1/2} 2\pi x^3 \sqrt{1 + 9x^4}\,dx$$

$$= 2\pi \left(\frac{1}{36}\right)\left(\frac{2}{3}\right)(1 + 9x^4)^{3/2}\Big]_0^{1/2} \qquad \text{Substitute } u = 1 + 9x^4,\ du/36 = x^3dx, \text{ integrate, and substitute back.}$$

$$= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right]$$

$$= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right] = \frac{\pi}{27}\left(\frac{125}{64} - 1\right)$$

$$= \frac{61\pi}{1728}.$$

As with arc length calculations, even the simplest curves can provide a workout. ❑

Exercises 5.6

Finding Integrals for Surface Area

In Exercises 1–8:

a) Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

b) Graph the curve to see what it looks like. If you can, graph the surface, too.

c) Use your grapher's or computer's integral evaluator to find the surface's area numerically.

1. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x$-axis
2. $y = x^2, \quad 0 \le x \le 2; \quad x$-axis
3. $xy = 1, \quad 1 \le y \le 2; \quad y$-axis
4. $x = \sin y, \quad 0 \le y \le \pi; \quad y$-axis
5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis
6. $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y$-axis
7. $x = \int_0^y \tan t \, dt, \quad 0 \le y \le \pi/3; \quad y$-axis
8. $y = \int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x$-axis

Finding Surface Areas

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$ about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

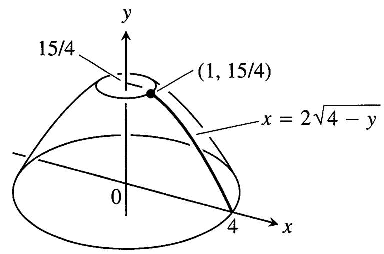
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

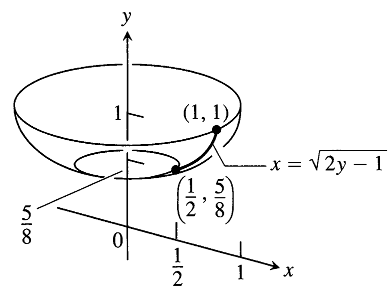
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–22 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x$-axis
14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x$-axis
15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x$-axis
16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x$-axis
17. $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis
18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis
19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis



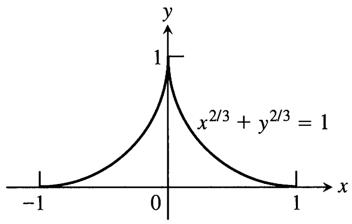
20. $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y$-axis



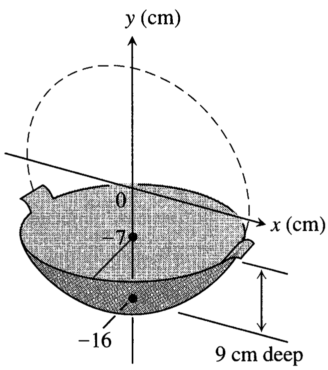
21. $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y \, ds$ with appropriate limits.)
22. $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x \, ds$ with appropriate limits.)
23. *Testing the new definition.* Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Eq. (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}$, $-a \le x \le a$, about the x-axis.
24. *Testing the new definition.* The lateral (side) surface area of a cone of height h and base radius r should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x, 0 \le x \le h$, about the x-axis.
25. a) Write an integral for the area of the surface generated by revolving the curve $y = \cos x, -\pi/2 \le x \le \pi/2$, about the x-axis. In Section 7.4 we will see how to evaluate such integrals.

    b) CALCULATOR Find the surface area numerically.

26. *The surface of an astroid.* Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown here. (*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$, about the x-axis and double your result.)

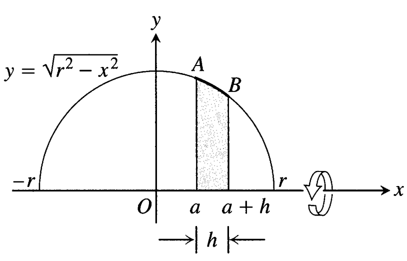


27. *Enameling woks.* Your company decided to put out a deluxe version of the successful wok you designed in Section 5.3, Exercise 41. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See diagram here.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that 1 cm^3 = 1 mL, so 1 L = 1000 cm^3.)

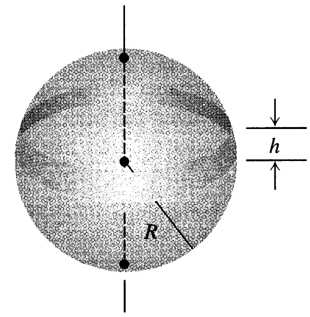


$x^2 + y^2 = 16^2 = 256$

28. *Slicing bread.* Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



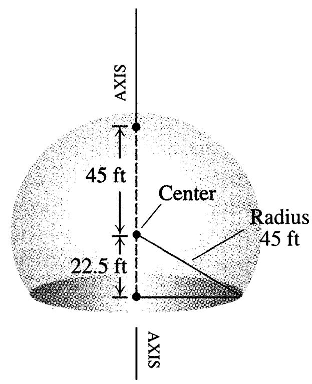
29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Mont.

a) How much outside surface is there to paint (not counting the bottom)?

b) CALCULATOR Express the answer to the nearest square foot.



31. *Surfaces generated by curves that cross the axis of revolution.* The surface area formula in Eq. (3) was developed under the assumption that the function f whose graph generated the surface was nonnegative over the interval $[a, b]$. For curves that cross the axis of revolution, we replace Eq. (3) with the absolute value formula

$$S = \int 2\pi\rho \, ds = \int 2\pi |f(x)| \, ds. \tag{5}$$

Use Eq. (5) to find the surface area of the double cone generated by revolving the line segment $y = x$, $-1 \le x \le 2$, about the x-axis.

32. *(Exercise 31, continued.)* Find the area of the surface generated by revolving the curve $y = x^3/9$, $-\sqrt{3} \le x \le \sqrt{3}$, about the x-axis. What do you think will happen if you drop the absolute value bars from Eq. (5) and attempt to find the surface area with the formula $S = \int 2\pi f(x)\, ds$ instead? Try it.

Numerical Integration

Find, to 2 decimal places, the areas of the surfaces generated by revolving the curves in Exercises 33–36 about the x-axis.

33. $y = \sin x, \quad 0 \le x \le \pi$

34. $y = x^2/4, \quad 0 \le x \le 2$

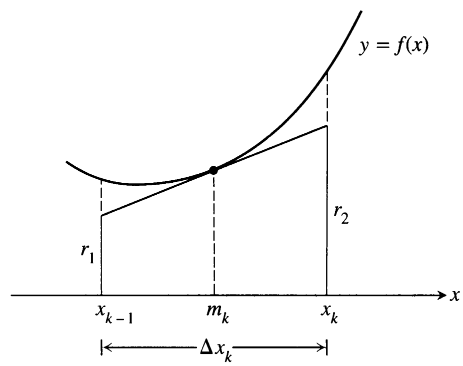
35. $y = x + \sin 2x, \quad -2\pi/3 \le x \le 2\pi/3$ (the curve in Section 3.4, Exercise 5)

36. $y = \frac{x}{12}\sqrt{36 - x^2}, 0 \le x \le 6$ (the surface of the plumb bob in Section 5.3, Exercise 44)

37. *An alternative derivation of the surface area formula.* Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the kth subinterval $[x_{k-1}, x_k]$ construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the figure here.

a) Show that $r_1 = f(m_k) - f'(m_k)\frac{\Delta x_k}{2}$ and $r_2 = f(m_k) + f'(m_k)\frac{\Delta x_k}{2}$.

b) Show that the length L_k of the tangent line segment in the kth subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\Delta x_k)^2}$.



c) Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k$.

d) Show that the area of the surface generated by revolving $y = f(x)$ about the x-axis over $[a, b]$ is

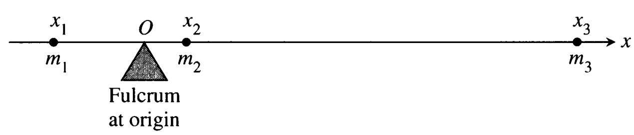
$$\lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix}\text{lateral surface area}\\ \text{of } k\text{th frustum}\end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

5.7 Moments and Centers of Mass

Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the center of mass (Fig. 5.50, on the following page). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. For the moment we deal with one- and two-dimensional objects. Three-dimensional objects are best done with the multiple integrals of Chapter 13.

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not. It depends on how large the masses are and how they are arranged.

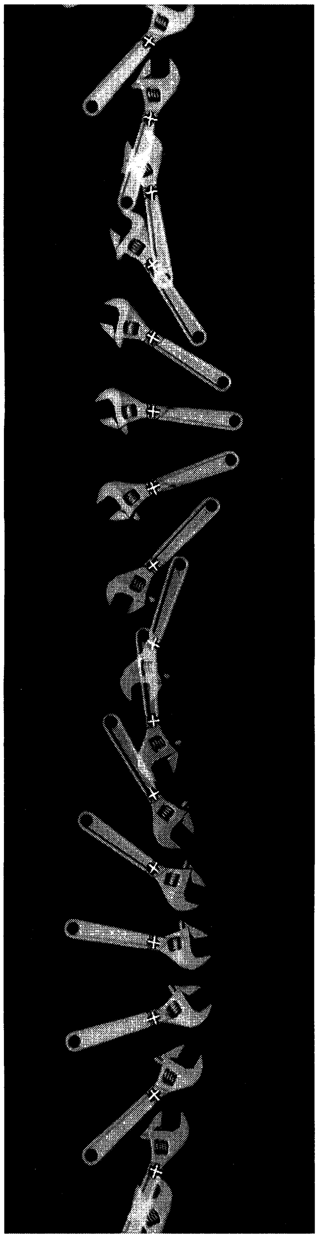
Each mass m_k exerts a downward force $m_k g$ equal to the magnitude of the mass times the acceleration of gravity. Each of these forces has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect, called a **torque,** is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

Mass vs. weight

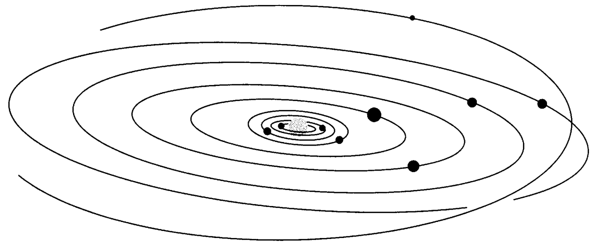
Weight is the force that results from gravity pulling on a mass. If an object of mass m is placed in a location where the acceleration of gravity is g, the object's weight there is

$$F = mg$$

(as in Newton's second law).



(a)



(b)

5.50 (a) The motion of this wrench gliding on ice seems haphazard until we notice that the wrench is simply turning about its center of mass as the center glides in a straight line. (b) The planets, asteroids, and comets of our solar system revolve about their collective center of mass. (It lies inside the sun.)

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque.**

$$\text{System torque} = m_1gx_1 + m_2gx_2 + m_3gx_3 \tag{1}$$

The system will balance if and only if its torque is zero.

If we factor out the g in Eq. (1), we see that the system torque is

$$\underbrace{g}_{\text{a feature of the environment}}\underbrace{(m_1x_1 + m_2x_2 + m_3x_3)}_{\text{a feature of the system}}.$$

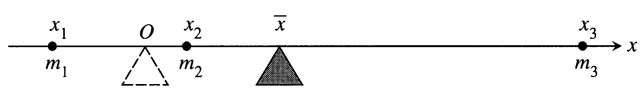
Thus the torque is the product of the gravitional acceleration g, which is a feature of the environment in which the system happens to reside, and the number $(m_1x_1 + m_2x_2 + m_3x_3)$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed.

The number $(m_1x_1 + m_2x_2 + m_3x_3)$ is called the **moment of the system about the origin.** It is the sum of the **moments** m_1x_1, m_2x_2, m_3x_3 of the individual masses.

$$M_O = \text{Moment of system about origin } = \sum m_kx_k$$

(We shift to sigma notation here to allow for sums with more terms. For $\sum m_kx_k$, read "summation m_k times x_k.")

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torque zero.



The torque of each mass about the fulcrum in this special location is

$$\begin{aligned}\text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}\\ &= (x_k - \bar{x})m_kg.\end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\sum (x_k - \bar{x}) m_k g = 0 \qquad \text{Sum of the torques equals zero}$$

$$g \sum (x_k - \bar{x}) m_k = 0 \qquad \text{Constant Multiple Rule for Sums}$$

$$\sum (m_k x_k - \bar{x} m_k) = 0 \qquad g \text{ divided out, } m_k \text{ distributed}$$

$$\sum m_k x_k - \sum \bar{x} m_k = 0 \qquad \text{Difference Rule for Sums}$$

$$\sum m_k x_k = \bar{x} \sum m_k \qquad \text{Rearranged, Constant Multiple Rule again}$$

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k}. \qquad \text{Solved for } \bar{x}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

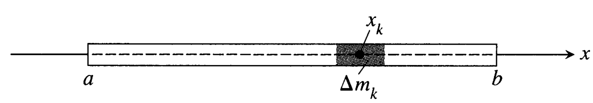
$$\bar{x} = \frac{\sum x_k m_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\bar{x}$ is called the system's **center of mass.**

Wires and Thin Rods

In many applications, we want to know the center of mass of a rod or a thin strip of metal. In cases like these where we can model the distribution of mass with a continuous function, the summation signs in our formulas become integrals in a manner we now describe.

Imagine a long, thin strip lying along the x-axis from $x = a$ to $x = b$ and cut into small pieces of mass Δm_k by a partition of the interval $[a, b]$.



The kth piece is Δx_k units long and lies approximately x_k units from the origin. Now observe three things.

First, the strip's center of mass $\bar{x}$ is nearly the same as that of the system of point masses we would get by attaching each mass Δm_k to the point x_k:

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}}.$$

Second, the moment of each piece of the strip about the origin is approximately $x_k \Delta m_k$, so the system moment is approximately the sum of the $x_k \Delta m_k$:

$$\text{System moment} \approx \sum x_k \Delta m_k.$$

Third, if the density of the strip at x_k is $\delta(x_k)$, expressed in terms of mass per unit length, and δ is continuous, then Δm_k is approximately equal to $\delta(x_k)\Delta x_k$ (mass per unit length times length):

$$\Delta m_k \approx \delta(x_k) \Delta x_k.$$

Combining these three observations gives

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}} \approx \frac{\sum x_k \Delta m_k}{\sum \Delta m_k} \approx \frac{\sum x_k \delta(x_k) \Delta x_k}{\sum \delta(x_k) \Delta x_k}. \qquad (2)$$

Density

A material's density is its mass per unit volume. In practice, however, we tend to use units we can conveniently measure. For wires, rods, and narrow strips we use mass per unit length. For flat sheets and plates we use mass per unit area.

The sum in the last numerator in Eq. (2) is a Riemann sum for the continuous function $x\delta(x)$ over the closed interval $[a, b]$. The sum in the denominator is a Riemann sum for the function $\delta(x)$ over this interval. We expect the approximations in (2) to improve as the strip is partitioned more finely, and we are led to the equation

$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx}.$$

This is the formula we use to find $\bar{x}$.

To find a center of mass, divide moment by mass.

Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the x-axis with Density Function $\delta(x)$

Moment about the origin: $$M_O = \int_a^b x\delta(x)\,dx \qquad \text{(3a)}$$

Mass: $$M = \int_a^b \delta(x)\,dx \qquad \text{(3b)}$$

Center of mass: $$\bar{x} = \frac{M_O}{M} \qquad \text{(3c)}$$

EXAMPLE 1 *Strips and rods of constant density*

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.

Solution We model the strip as a portion of the x-axis from $x = a$ to $x = b$ (Fig. 5.51). Our goal is to show that $\bar{x} = (a + b)/2$, the point halfway between a and b.

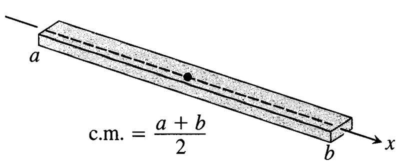
The key is the density's having a constant value. This enables us to regard the function $\delta(x)$ in the integrals in Eqs. (3) as a constant (call it δ), with the result that

$$M_O = \int_a^b \delta x\,dx = \delta \int_a^b x\,dx = \delta\left[\frac{1}{2}x^2\right]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

$$M = \int_a^b \delta\,dx = \delta \int_a^b dx = \delta\Big[x\Big]_a^b = \delta(b - a)$$

$$\bar{x} = \frac{M_O}{M} = \frac{\frac{\delta}{2}(b^2 - a^2)}{\delta(b - a)}$$

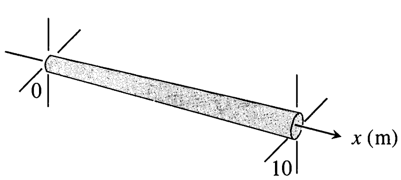
$$= \frac{a + b}{2}. \qquad \text{The } \delta\text{'s cancel in the formula for } \bar{x}.$$



5.51 **The center of mass of a straight, thin rod or strip of constant density lies halfway between its ends.**

EXAMPLE 2 *A variable density*

The 10-m-long rod in Fig. 5.52 thickens from left to right so that its density, instead of being constant, is $\delta(x) = 1 + (x/10)$ kg/m. Find the rod's center of mass.



5.52 **We can treat a rod of variable thickness as a rod of variable density. See Example 2.**

Solution The rod's moment about the origin (Eq. 3a) is

$$M_O = \int_0^{10} x\delta(x)\,dx = \int_0^{10} x\left(1 + \frac{x}{10}\right)dx = \int_0^{10}\left(x + \frac{x^2}{10}\right)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{30}\right]_0^{10} = 50 + \frac{100}{3} = \frac{250}{3}\ \text{kg} \cdot \text{m}.$$

The units of a moment are mass × length.

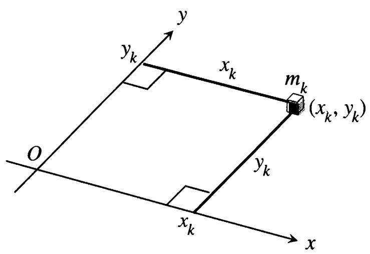
The rod's mass (Eq. 3b) is

$$M = \int_0^{10} \delta(x)\,dx = \int_0^{10}\left(1 + \frac{x}{10}\right)dx = \left[x + \frac{x^2}{20}\right]_0^{10} = 10 + 5 = 15\ \text{kg}.$$

The center of mass (Eq. 3c) is located at the point

$$\bar{x} = \frac{M_O}{M} = \frac{250}{3} \cdot \frac{1}{15} = \frac{50}{9} \approx 5.56\ \text{m}.$$

❑



5.53 Each mass m_k has a moment about each axis.

Masses Distributed over a Plane Region

Suppose we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Fig. 5.53). The mass of the system is

$$\text{System mass:} \qquad M = \sum m_k.$$

Each mass m_k has a moment about each axis. Its moment about the x-axis is $m_k y_k$, and its moment about the y-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k.$$

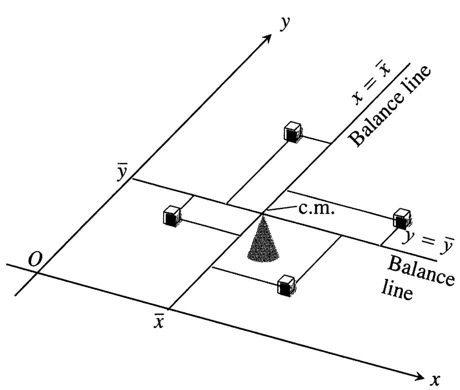
The x-coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \tag{4}$$

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Fig. 5.54).

The y-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{5}$$

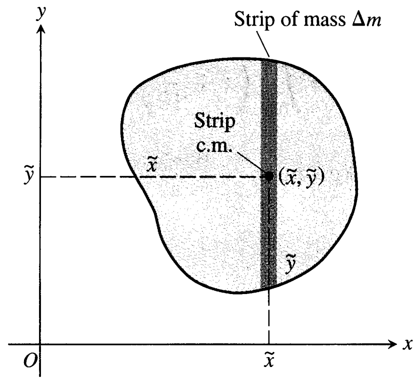


5.54 A two-dimensional array of masses balances on its center of mass.

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's *center of mass.*

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.



5.55 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

Imagine the plate occupying a region in the xy-plane, cut into thin strips parallel to one of the axes (in Fig. 5.55, the y-axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the y-axis is then $\tilde{x}\Delta m$. The moment of the strip about the x-axis is $\tilde{y}\Delta m$. Equations (4) and (5) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}.$$

As in the one-dimensional case, the sums are Riemann sums for integrals and approach these integrals as limiting values as the strips into which the plate is cut become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}.$$

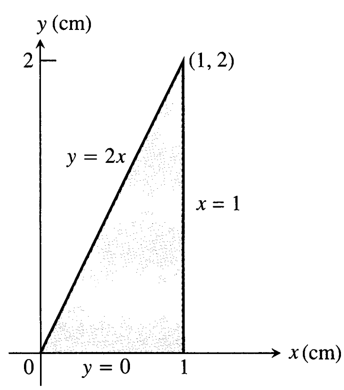
Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy-plane

Moment about the x-axis: $\quad M_x = \int \tilde{y}\,dm$

Moment about the y-axis: $\quad M_y = \int \tilde{x}\,dm \qquad$ **(6)**

Mass: $\quad M = \int dm$

Center of mass: $\quad \bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$



5.56 The plate in Example 3.

To evaluate these integrals, we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinates axes. We then express the strip's mass dm and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of x or y. Finally, we integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm between limits of integration determined by the plate's location in the plane.

EXAMPLE 3 The triangular plate shown in Fig. 5.56 has a constant density of $\delta = 3$ g/cm^2. Find (a) the plate's moment M_y about the y-axis, (b) the plate's mass M, and (c) the x-coordinate of the plate's center of mass (c.m.).

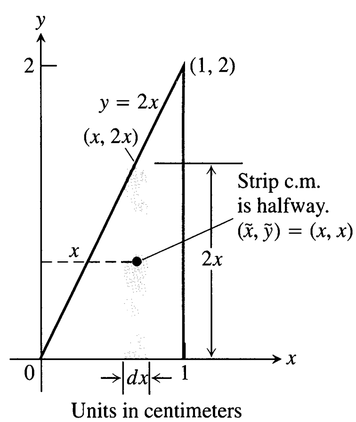
Solution

Method 1: *Vertical strips* (Fig. 5.57).

a) The moment M_y: The typical vertical strip has

center of mass (c.m.): $\quad (\tilde{x}, \tilde{y}) = (x, x)$,

length: $2x$, $\qquad$ area: $dA = 2x\,dx$,
width: dx, $\qquad$ mass: $dm = \delta\,dA = 3 \cdot 2x\,dx = 6x\,dx$,
distance of c.m. from y-axis: $\tilde{x} = x$.



5.57 Modeling the plate in Example 3 with vertical strips.

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = x \cdot 6x\,dx = 6x^2\,dx.$$

The moment of the plate about the y-axis is therefore

$$M_y = \int \tilde{x}\,dm = \int_0^1 6x^2\,dx = 2x^3\Big]_0^1 = 2\text{ g}\cdot\text{cm}.$$

b) The plate's mass:

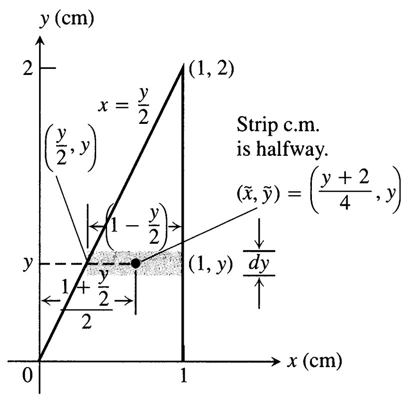
$$M = \int dm = \int_0^1 6x\,dx = 3x^2\Big]_0^1 = 3\text{ g}.$$

c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\text{ g}\cdot\text{cm}}{3\text{ g}} = \frac{2}{3}\text{ cm}.$$

By a similar computation we could find M_x and $\bar{y} = M_x/M$.

Method 2: *Horizontal strips* (Fig. 5.58).



5.58 Modeling the plate in Example 3 with horizontal strips.

a) The moment M_y: The y-coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x-coordinate is the x-coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x-value) and 1 (the strip's right-hand x-value):

$$\tilde{x} = \frac{(y/2)+1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y+2}{4}.$$

We also have

length: $1 - \frac{y}{2} = \frac{2-y}{2}$,

width: dy,

area: $dA = \frac{2-y}{2}\,dy$,

mass: $dm = \delta\,dA = 3 \cdot \frac{2-y}{2}\,dy$,

distance of c.m. to y-axis: $\tilde{x} = \frac{y+2}{4}$.

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2}\,dy = \frac{3}{8}(4-y^2)\,dy.$$

The moment of the plate about the y-axis is

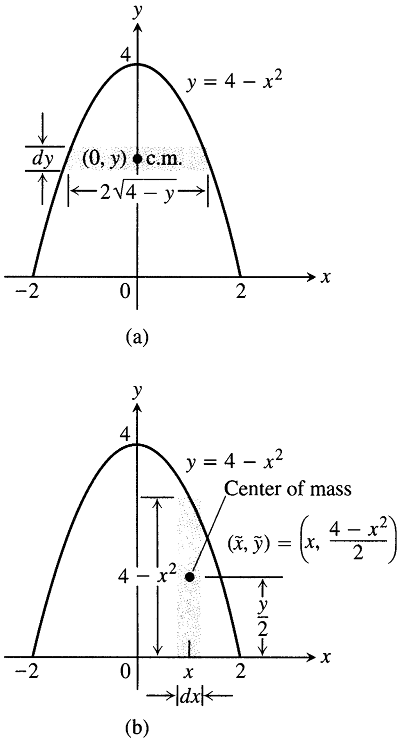
$$M_y = \int \tilde{x}\,dm = \int_0^2 \frac{3}{8}(4-y^2)\,dy = \frac{3}{8}\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{3}{8}\left(\frac{16}{3}\right) = 2\text{ g}\cdot\text{cm}.$$

b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2-y)\,dy = \frac{3}{2}\left[2y - \frac{y^2}{2}\right]_0^2 = \frac{3}{2}(4-2) = 3\text{ g}.$$

How to Find a Plate's Center of Mass

1. Picture the plate in the xy-plane.
2. Sketch a strip of mass parallel to one of the coordinate axes and find its dimensions.
3. Find the strip's mass dm and center of mass $(\tilde{x}, \tilde{y})$.
4. Integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm to find M_x, M_y, and M.
5. Divide the moments by the mass to calculate $\bar{x}$ and $\bar{y}$.



5.59 Modeling the plate in Example 4 with (a) horizontal strips leads to an inconvenient integration, so we model with (b) vertical strips instead.

c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\ \text{g} \cdot \text{cm}}{3\ \text{g}} = \frac{2}{3}\ \text{cm}.$$

By a similar computation, we could find M_x and $\bar{y}$. ❑

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

EXAMPLE 4 Find the center of mass of a thin plate of constant density δ covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Fig. 5.59).

Solution Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\bar{x} = 0$. It remains to find $\bar{y} = M_x/M$.

A trial calculation with horizontal strips (Fig. 5.59a) leads to an inconvenient integration

$$M_x = \int_0^4 2\delta\, y\sqrt{4 - y}\, dy.$$

We therefore model the distribution of mass with vertical strips instead (Fig. 5.59b). The typical vertical strip has

center of mass (c.m): $(\tilde{x}, \tilde{y}) = \left(x, \dfrac{4 - x^2}{2}\right)$,

length: $4 - x^2$,

width: dx,

area: $dA = (4 - x^2)\,dx$,

mass: $dm = \delta\, dA = \delta(4 - x^2)\,dx$,

distance from c.m to x-axis: $\tilde{y} = \dfrac{4 - x^2}{2}$.

The moment of the strip about the x-axis is

$$\tilde{y}\, dm = \frac{4 - x^2}{2} \cdot \delta(4 - x^2)\,dx = \frac{\delta}{2}(4 - x^2)^2\,dx.$$

The moment of the plate about the x-axis is

$$\begin{aligned} M_x &= \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx \\ &= \frac{\delta}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx = \frac{256}{15}\delta. \end{aligned} \tag{7}$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\,dx = \frac{32}{3}\,\delta. \tag{8}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{(256/15)\,\delta}{(32/3)\,\delta} = \frac{8}{5}.$$

The plate's center of mass is the point

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right).$$

❑

EXAMPLE 5 *Variable density*

Find the center of mass of the plate in Example 4 if the density at the point (x, y) is $\delta = 2x^2$, twice the square of the distance from the point to the y-axis.

Solution The mass distribution is still symmetric about the y-axis, so $\bar{x} = 0$. With $\delta = 2x^2$, Eqs. (7) and (8) become

$$M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx = \int_{-2}^{2} x^2(4 - x^2)^2\,dx$$

$$= \int_{-2}^{2} (16x^2 - 8x^4 + x^6)\,dx = \frac{2048}{105}, \tag{7'}$$

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\,dx = \int_{-2}^{2} 2x^2(4 - x^2)\,dx$$

$$= \int_{-2}^{2} (8x^2 - 2x^4)\,dx = \frac{256}{15}. \tag{8'}$$

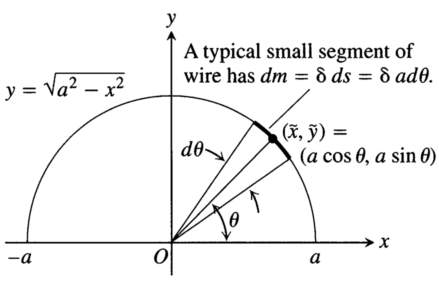
Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

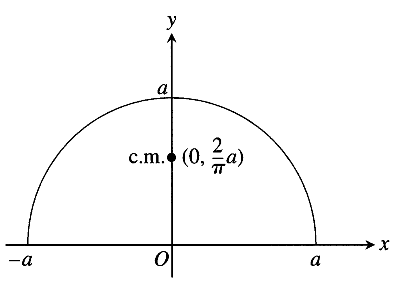
The plate's new center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

❑



(a)



(b)

5.60 The semicircular wire in Example 6. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

EXAMPLE 6 Find the center of mass of a wire of constant density δ shaped like a semicircle of radius a.

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Fig. 5.60). The distribution of mass is symmetric about the y-axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short segments. The typical segment (Fig. 5.60a) has

length: $ds = a\,d\theta$,

mass: $dm = \delta\,ds = \delta\,a\,d\theta$, Mass per unit length times length

distance of c.m. to x-axis: $\tilde{y} = a \sin\theta$.

Hence,

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_0^\pi a \sin\theta \cdot \delta a\, d\theta}{\int_0^\pi \delta a\, d\theta} = \frac{\delta a^2[-\cos\theta]_0^\pi}{\delta a \pi} = \frac{2}{\pi} a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Fig. 5.60b). ❑

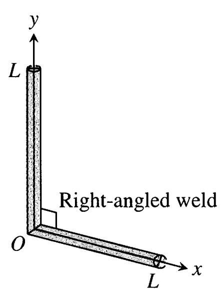
Centroids

When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. This happened in nearly every example in this section. As far as $\bar{x}$ and $\bar{y}$ were concerned, δ might as well have been 1. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases engineers may call the center of mass the **centroid** of the shape, as in "Find the centroid of a triangle or a solid cone." To do so, just set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

Exercises 5.7

Thin Rods

1. An 80-lb child and a 100-lb child are balancing on a seesaw. The 80-lb child is 5 ft from the fulcrum. How far from the fulcrum is the 100-lb child?
2. The ends of a log are placed on two scales. One scale reads 100 kg and the other 200 kg. Where is the log's center of mass?
3. The ends of two thin steel rods of equal length are welded together to make a right-angled frame. Locate the frame's center of mass. (*Hint:* Where is the center of mass of each rod?)



4. You weld the ends of two steel rods into a right-angled frame. One rod is twice the length of the other. Where is the frame's center of mass? (*Hint:* Where is the center of mass of each rod?)

Exercises 5–12 give density functions of thin rods lying along various intervals of the x-axis. Use Eqs. (3a–c) to find each rod's moment about the origin, mass, and center of mass.

5. $\delta(x) = 4, \quad 0 \le x \le 2$
6. $\delta(x) = 4, \quad 1 \le x \le 3$
7. $\delta(x) = 1 + (x/3), \quad 0 \le x \le 3$
8. $\delta(x) = 2 - (x/4), \quad 0 \le x \le 4$
9. $\delta(x) = 1 + (1/\sqrt{x}), \quad 1 \le x \le 4$
10. $\delta(x) = 3(x^{-3/2} + x^{-5/2}), \quad 0.25 \le x \le 1$
11. $\delta(x) = \begin{cases} 2 - x, & 0 \le x < 1 \\ x, & 1 \le x \le 2 \end{cases}$
12. $\delta(x) = \begin{cases} x + 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

Thin Plates with Constant Density

In Exercises 13–24, find the center of mass of a thin plate of constant density δ covering the given region.

13. The region bounded by the parabola $y = x^2$ and the line $y = 4$
14. The region bounded by the parabola $y = 25 - x^2$ and the x-axis
15. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
16. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$
17. The region bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$
18. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$
19. The region bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$

20. The region between the x-axis and the curve $y = \sec^2 x$, $-\pi/4 \le x \le \pi/4$

21. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$

22. a) The region cut from the first quadrant by the circle $x^2 + y^2 = 9$
 b) The region bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$

 Compare your answer with the answer in (a).

23. The "triangular" region in the first quadrant between the circle $x^2 + y^2 = 9$ and the lines $x = 3$ and $y = 3$. (*Hint:* Use geometry to find the area.)

24. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a\to\infty} \bar{x}$.

Thin Plates with Varying Density

25. Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.

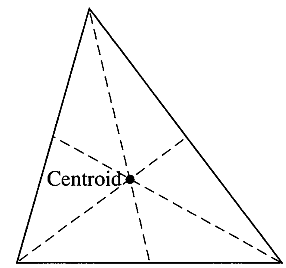
26. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.

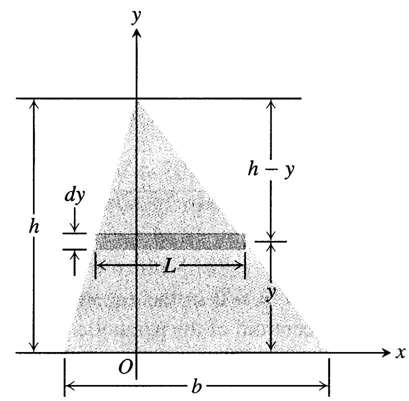
27. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.
 a) Find the volume of the solid.
 b) Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.
 c) Sketch the plate and show the center of mass in your sketch.

28. The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.
 a) Find the volume of the solid.
 b) Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.
 c) Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

29. *The centroid of a triangle lies at the intersection of the triangle's medians (Fig. 5.61a).* You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.
 1. Stand one side of the triangle on the x-axis as in Fig. 5.61(b). Express dm in terms of L and dy.
 2. Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm.





5.61 The triangle in Exercise 29. (a) The centroid. (b) The dimensions and variables to use in locating the center of mass.

 3. Show that $\bar{y} = h/3$.
 4. Extend the argument to the other sides.

Use the result in Exercise 29 to find the centroids of the triangles whose vertices appear in Exercises 30–34. (*Hint:* Draw each triangle first.)

30. $(-1, 0), (1, 0), (0, 3)$

31. $(0, 0), (1, 0), (0, 1)$

32. $(0, 0), (a, 0), (0, a)$

33. $(0, 0), (a, 0), (0, b)$

34. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

35. Find the moment about the x-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

36. Find the moment about the x-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

37. Suppose the density of the wire in Example 6 is $\delta = k \sin\theta$ (k constant). Find the center of mass.

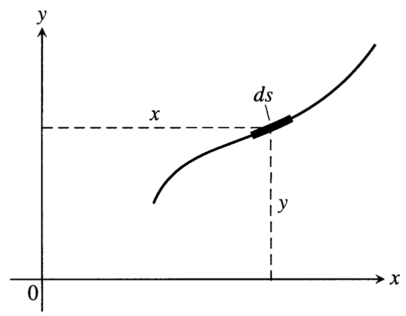
38. Suppose the density of the wire in Example 6 is $\delta = 1 + k\,|\cos\theta|$ (k constant). Find the center of mass.

Engineering Formulas

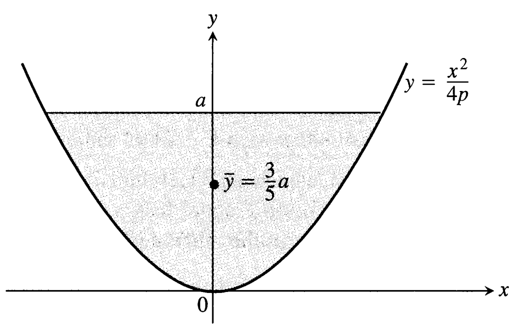
Verify the statements and formulas in Exercises 39–42.

39. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x \, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y \, ds}{\text{length}}.$$

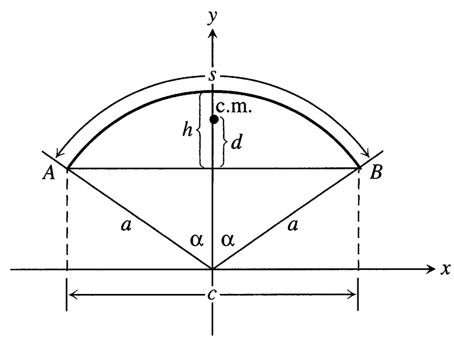


40. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



41. For wires and thin rods of constant density shaped like circular arcs centered at the origin and symmetric about the y-axis, the y-coordinate of the center of mass is

$$\bar{y} = \frac{a \sin \alpha}{\alpha} = \frac{ac}{s}.$$



42. (*Continuation of Exercise 41*)

a) Show that when α is small, the distance d from the centroid to chord AB is about $2h/3$ (in the notation of the figure here) by taking the following steps.

1. Show that

$$\frac{d}{h} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}. \tag{9}$$

2. GRAPHER Graph

$$f(\alpha) = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$$

and use TRACE to show that $\lim_{\alpha \to 0^+} f(\alpha) \approx 2/3$. (You will be able to confirm the suggested equality in Section 6.6, Exercise 74.)

b) CALCULATOR The error (difference between d and $2h/3$) is small even for angles greater than 45°. See for yourself by evaluating the right-hand side of Eq. (9) for $\alpha = 0.2$, 0.4, 0.6, 0.8, and 1.0 rad.

5.8 Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

Work Done by a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we calculate the

Joules

The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

It takes a force of about 1 N to lift an apple from a table. If you lift it 1 m you have done about 1 J of work on the apple. If you then eat the apple you will have consumed about 80 food calories, the heat equivalent of nearly 335,000 joules. If this energy were directly useful for mechanical work, it would enable you to lift 335,000 more apples up 1 m.

work W done by the force on the body with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

Right away we can see a considerable difference between what we are used to calling work and what this formula says work is. If you push a car down the street, you will be doing work on the car, both by your own reckoning and by Eq. (1). But if you push against the car and the car does not move, Eq. (1) says you will do no work on the car, even if you push for an hour.

From Eq. (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International,* or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter (N · m). This combination appears so often it has a special name, the **joule.** In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

EXAMPLE 1 If you jack up the side of a 2000-lb car 1.25 ft to change a tire (you have to apply a constant vertical force of about 1000 lb) you will perform $1000 \times 1.25 = 1250$ ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ❑

Work Done by a Variable Force

If the force you apply varies along the way, as it will if you are lifting a leaking bucket or compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts along a line that we can model with the x-axis and that its magnitude F is a continuous function of the position. We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, F, being continuous, will not vary much from x_{k-1} to x_k. The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k, the same as it would be if F were constant and we could apply Eq. (1). The total work done from a to b is therefore approximated by the Riemann sum

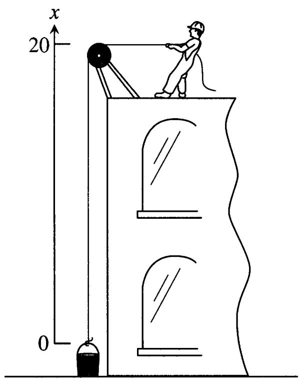
$$\sum_{k=1}^{n} F(c_k)\Delta x_k. \tag{2}$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b.

Definition

The **work** done by a variable force $F(x)$ directed along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{3}$$



5.62 The leaky bucket in Example 3.

The units of the integral are joules if F is in newtons and x is in meters, and foot-pounds if F is in pounds and x in feet.

EXAMPLE 2 The work done by a force of $F(x) = 1/x^2$ N along the x-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J.}$$

❑

EXAMPLE 3 A leaky 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Fig. 5.62). The rope weighs 0.08 lb/ft. The bucket starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent

a) lifting the water alone;
b) lifting the water and bucket together;
c) lifting the water, bucket, and rope?

Solution

a) *The water alone.* The force required to lift the water is equal to the water's weight, which varies steadily from 16 to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs

$$F(x) = \underbrace{16}_{\text{original weight of water}}\underbrace{\left(\frac{20-x}{20}\right)}_{\text{proportion left at elevation } x} = 16\left(1 - \frac{x}{20}\right) = 16 - \frac{4x}{5} \text{ lb.}$$

The work done is

$$W = \int_a^b F(x)\,dx \qquad \text{Use Eq. (3) for variable forces.}$$

$$= \int_0^{20} \left(16 - \frac{4x}{5}\right) dx = \left[16x - \frac{2x^2}{5}\right]_0^{20} = 320 - 160 = 160 \text{ ft} \cdot \text{lb.}$$

b) *The water and bucket together.* According to Eq. (1), it takes $5 \times 20 = 100$ ft · lb to lift a 5-lb weight 20 ft. Therefore

$$160 + 100 = 260 \text{ ft} \cdot \text{lb}$$

of work were spent lifting the water and bucket together.

c) *The water, bucket, and rope.* Now the total weight at level x is

$$F(x) = \underbrace{\left(16 - \frac{4x}{5}\right)}_{\text{variable weight of water}} + \underbrace{5}_{\text{constant weight of bucket}} + \underbrace{(\overset{\text{lb/ft}}{0.08})(\overset{\text{ft}}{20 - x})}_{\text{weight of rope paid out at elevation } x}.$$

The work lifting the rope is

$$\text{Work on rope} = \int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx$$

$$= \Big[1.6x - 0.04x^2\Big]_0^{20} = 32 - 16 = 16 \text{ ft} \cdot \text{lb}.$$

The total work for the water, bucket, and rope combined is

$$160 + 100 + 16 = 276 \text{ ft} \cdot \text{lb}.$$

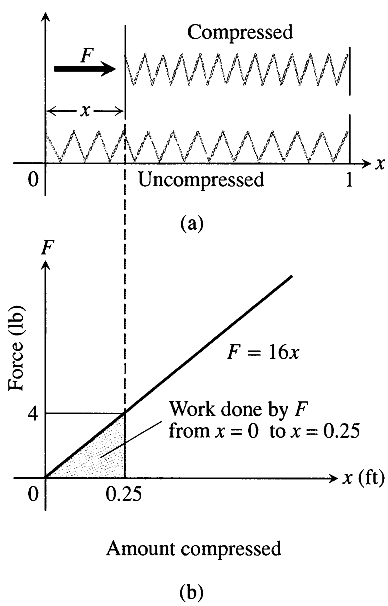
❑

Hooke's Law for Springs: $F = kx$

Hooke's law says that the force it takes to stretch or compress a spring x length units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{4}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or spring constant) of the spring. Hooke's law (Eq. 4) gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.



5.63 The force F needed to hold a spring under compression increases linearly as the spring is compressed.

EXAMPLE 4 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Fig. 5.63). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \qquad \text{to} \qquad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft} \cdot \text{lb}.$$ Eq. (3) with $a = 0$, $b = 0.25$, $F(x) = 16x$

❑

EXAMPLE 5 A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

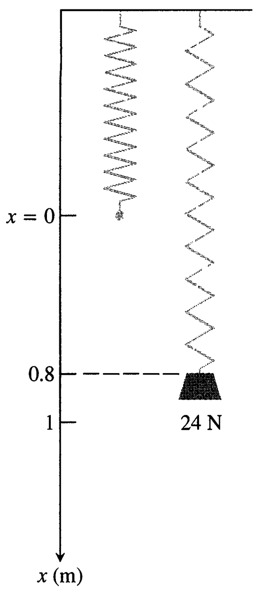
a) Find the force constant k.
b) How much work will it take to stretch the spring 2 m beyond its natural length?
c) How far will a 45-N force stretch the spring?

Solution

a) *The force constant.* We find the force constant from Eq. (4). A force of 24 N stretches the spring 0.8 m, so

$$24 = k(0.8)$$ Eq. (4) with $F = 24$, $x = 0.8$

$$k = 24/0.8 = 30 \text{ N/m}.$$



5.64 A 24-N weight stretches this spring 0.8 m beyond its unstressed length.

b) *The work to stretch the spring* 2 m. We imagine the unstressed spring hanging along the x-axis with its free end at $x = 0$ (Fig. 5.64). The force required to stretch the spring x m beyond its natural length is the force required to pull the free end of the spring x units from the origin. Hooke's law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60 \text{ J}.$$

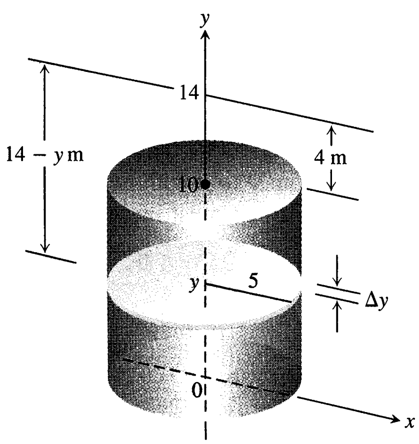
c) *How far will a* 45-N *force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \qquad \text{or} \qquad x = 1.5 \text{ m}.$$

A 45-N force will stretch the spring 1.5 m. No calculus is required to find this. ❑

Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next examples show what to do.



5.65 To find the work it takes to pump the water from a tank, think of lifting the water one thin slab at a time.

EXAMPLE 6 How much work does it take to pump the water from a full upright circular cylindrical tank of radius 5 m and height 10 m to a level of 4 m above the top of the tank?

Solution We draw the tank (Fig. 5.65), add coordinate axes, and imagine the water divided into thin horizontal slabs by planes perpendicular to the y-axis at the points of a partition P of the interval $[0, 10]$.

The typical slab between the planes at y and $y + \Delta y$ has a volume of

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(5)^2\Delta y = 25\pi\,\Delta y \text{ m}^3.$$

The force F required to lift the slab is equal to its weight,

$$\begin{aligned} F &= 9800\Delta V \\ &= 9800(25\pi\,\Delta y) = 245{,}000\pi\,\Delta y \text{ N}. \end{aligned}$$

Water weighs 9800 N/m^3.

The distance through which F must act is about $(14 - y)$ m, so the work done lifting the slab is about

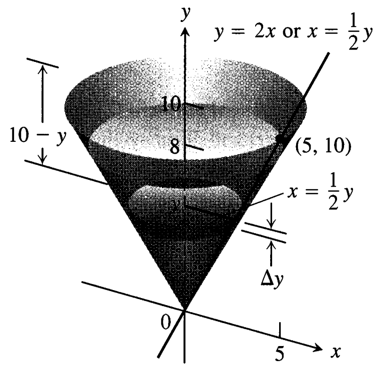
$$\Delta W = \text{force} \times \text{distance} = 245{,}000\pi(14 - y)\Delta y \text{ J}.$$

The work it takes to lift all the water is approximately

$$W \approx \sum_0^{10} \Delta W = \sum_0^{10} 245{,}000\pi(14 - y)\Delta y \text{ J}.$$

How to Find Work Done During Pumping

1. Draw a figure with a coordinate system.
2. Find the weight F of a thin horizontal slab of liquid.
3. Find the work ΔW it takes to lift the slab to its destination.
4. Integrate the work expression from the base to the surface of the liquid.



5.66 The olive oil in Example 7.

This is a Riemann sum for the function $245{,}000\pi(14 - y)$ over the interval $0 \le y \le 10$. The work of pumping the tank dry is the limit of these sums as $\| P \| \to 0$:

$$W = \int_0^{10} 245{,}000\pi(14 - y)\,dy = 245{,}000\pi \int_0^{10} (14 - y)\,dy$$

$$= 245{,}000\pi \left[14y - \frac{y^2}{2} \right]_0^{10} = 245{,}000\pi[90]$$

$$\approx 69{,}272{,}118 \approx 69.3 \times 10^6 \text{ J.}$$

A 1-horsepower output motor rated at 746 J/sec could empty the tank in a little less than 26 h. ❑

EXAMPLE 7 The conical tank in Fig. 5.66 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval [0, 8].

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi \left(\frac{1}{2}y \right)^2 \Delta y = \frac{\pi}{4} y^2 \Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4} y^2 \Delta y \text{ lb.}$$

Weight = weight per unit volume × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2 \Delta y \text{ ft} \cdot \text{lb.}$$

The work done lifting all the slabs from $y = 0$ to $y = 8$ to the rim is approximately

$$W \approx \sum_0^8 \frac{57\pi}{4}(10 - y)y^2 \Delta y \text{ ft} \cdot \text{lb.}$$

This is a Riemann sum for the function $(57\pi/4)(10 - y)y^2$ on the interval from $y = 0$ to $y = 8$. The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero.

$$W = \int_0^8 \frac{57\pi}{4}(10 - y)y^2\,dy$$

$$= \frac{57\pi}{4} \int_0^8 (10y^2 - y^3)\,dy$$

$$= \frac{57\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 30{,}561 \text{ ft} \cdot \text{lb.}$$

❑

Exercises 5.8

Work Done by a Variable Force

1. The workers in Example 3 changed to a larger bucket that held 5 gal (40 lb) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water? (Do not include the rope and bucket.)

2. The bucket in Example 3 is hauled up twice as fast so that there is still 1 gal (8 lb) of water left when the bucket reaches the top. How much work is done lifting the water this time? (Do not include the rope and bucket.)

3. A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

4. A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

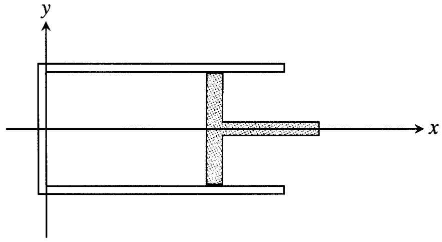
5. An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

6. When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

7. Suppose that the gas in a circular cylinder of cross-section area A is being compressed by a piston. If p is the pressure of the gas in pounds per square inch and V is the volume in cubic inches, show that the work done in compressing the gas from state (p_1, V_1) to state (p_2, V_2) is given by the equation

$$\text{Work} = \int_{(p_1,V_1)}^{(p_2,V_2)} p\,dV.$$

(*Hint:* In the coordinates suggested in the figure here, $dV = A\,dx$. The force against the piston is pA.)



8. *(Continuation of Exercise 7.)* Use the integral in Exercise 7 to find the work done in compressing the gas from $V_1 = 243$ in^3 to $V_2 = 32$ in^3 if $p_1 = 50$ lb/in^3 and p and V obey the gas law $pV^{1.4} = \text{constant}$ (for adiabatic processes).

Springs

9. It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

10. A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in. (a) Find the force constant. (b) How much work is done in stretching the spring from 10 in. to 12 in.? (c) How far beyond its natural length will a 1600-lb force stretch the spring?

11. A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming Hooke's law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

12. If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

13. *Subway car springs.* It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.
   a) What is the assembly's force constant?
   b) How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in · lb.

   (Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

14. A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming the scale behaves like a spring that obeys Hooke's law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

Pumping Liquids from Containers

The Weight of Water
Because of variations in the earth's gravitational field, the weight of a cubic foot of water at sea level can vary from about 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot that weighs about 62.4 lb in Melbourne and New York City will weigh 62.5 lb in Juneau and Stockholm. While 62.4 is a typical figure and a common textbook value, there is considerable variation.

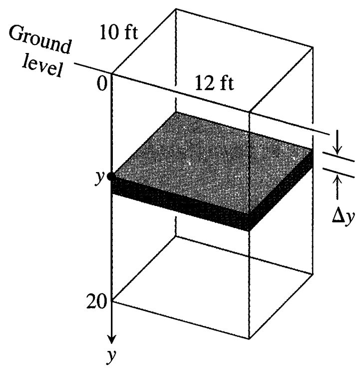
15. The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.

a) How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

b) If the water is pumped to ground level with a (5/11)-hp motor (work output 250 ft • lb/sec), how long will it take to empty the full tank (to the nearest minute)?

c) Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.

d) What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?



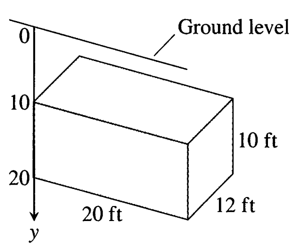
16. The rectangular cistern (storage tank for rainwater) shown here has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

a) How much work will it take to empty the cistern?

b) How long will it take a (1/2)-hp pump, rated at 275 ft • lb/sec, to pump the tank dry?

c) How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)

d) What are the answers to parts (a)–(c) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?

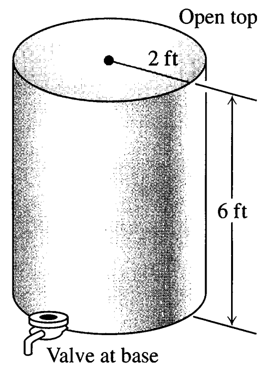


17. How much work would it take to pump the water from the tank in Example 6 to the level of the top of the tank (instead of 4 m higher)?

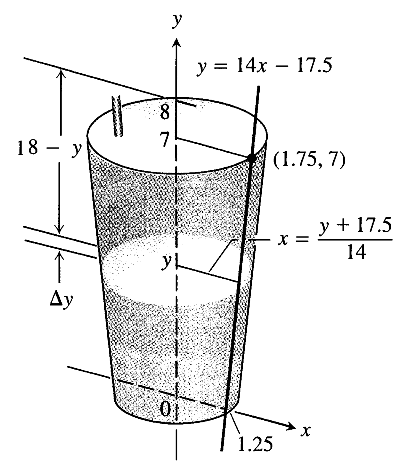
18. Suppose that, instead of being full, the tank in Example 6 is only half full. How much work does it take to pump the remaining water to a level 4 m above the top of the tank?

19. A vertical right circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank?

20. The cylindrical tank shown here can be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reasons for your answer.



21. CALCULATOR The truncated conical container shown here is full of strawberry milkshake that weighs 4/9 oz/in^3. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.



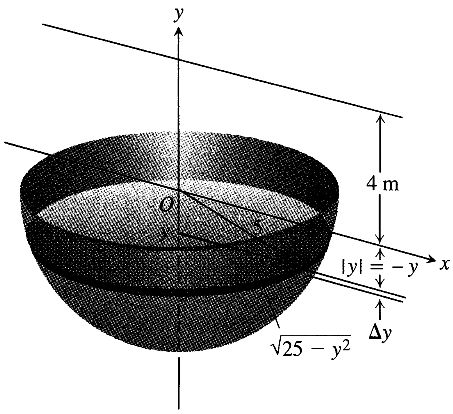
Dimensions in inches

22. a) Suppose the conical container in Example 7 contains milk (weighing 64.5 lb/ft^3) instead of olive oil. How much work will it take to pump the contents to the rim?

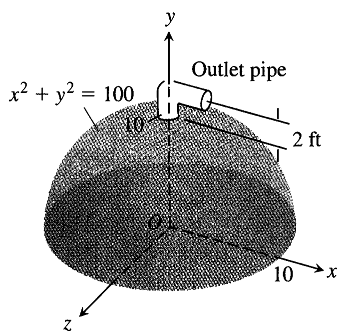
b) How much work will it take to pump the oil in Example 7 to a level 3 ft above the cone's rim?

23. To design the interior surface of a huge stainless steel tank, you revolve the curve $y = x^2, 0 \le x \le 4$, about the y-axis. The container, with dimensions in meters, is to be filled with seawater, which weighs 10,000 N/m^3. How much work will it take to empty the tank by pumping the water to the tank's top?

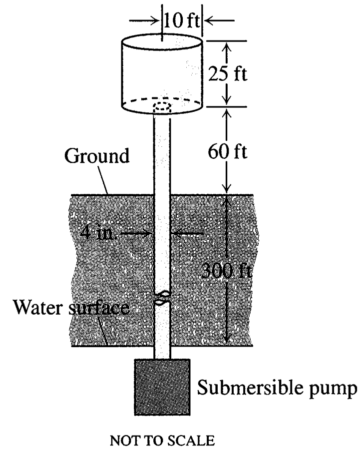
24. We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m^3.



25. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for 1/2¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?



26. Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300-ft well through a vertical 4-in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft aboveground. The pump is a 3-hp pump, rated at 1650 ft • lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume water weighs 62.4 lb/ft^3.



Other Applications

27. *Putting a satellite in orbit.* The strength of the earth's gravitational field varies with the distance r from the earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is the earth's mass, $G = 6.6720 \times 10^{-11}$ N • m^{2}kg^{-2} is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from the earth's surface to a circular orbit 35,780 km above the earth's center is therefore given by the integral

$$\text{Work} = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \text{ joules.}$$

Evaluate the integral. The lower limit of integration is the earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

28. *Forcing electrons together.* Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons.}$$

a) Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

b) Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)?

Work and Kinetic Energy

29. If a variable force of magnitude $F(x)$ moves a body of mass

m along the x-axis from x_1 to x_2, the body's velocity v can be written as dx/dt (where t represents time). Use Newton's Second Law of Motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2,$$

where v_1 and v_2 are the body's velocities at x_1 and x_2. In physics the expression $(1/2)mv^2$ is called the *kinetic energy* of the body moving with velocity v. Therefore, *the work done by the force equals the change in the body's kinetic energy,* and we can find the work by calculating this change.

Weight vs. Mass
Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{Weight} = \text{mass} \times \text{acceleration}.$$

Thus,

$$\text{Newtons} = \text{kilograms} \times \text{m/sec}^2,$$

$$\text{Pounds} = \text{slugs} \times \text{ft/sec}^2.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity.

In Exercises 30–36, use the result of Exercise 29.

30. *Tennis.* A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

31. *Baseball.* How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz = 0.3125 lb.

32. *Golf.* A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done getting the ball into the air?

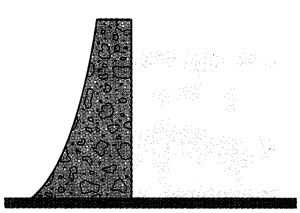
33. *Tennis.* During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed?

34. *Football.* A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to this speed?

35. *Softball.* How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?

36. *A ball bearing.* A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 3 inches and released. About how high does the ball bearing go?

5.9 Fluid Pressures and Forces



5.67 To withstand the increasing pressure, dams are built thicker as they go down.

We make dams thicker at the bottom than at the top (Fig. 5.67) because the pressure against them increases with depth. It is a remarkable fact that the pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of water in pounds per cubic foot.

The formula, pressure $= 62.4h$, makes sense when you think of the units involved:

$$\frac{\text{lb}}{\text{ft}^2} = \frac{\text{lb}}{\text{ft}^3} \times \text{ft}.$$

As you can see, this equation depends only on units and not on the fluid involved. The pressure h feet below the surface of any fluid is the fluid's weight-density times h.

Weight-density

A fluid's weight-density is its weight per unit volume. Typical values (lb/ft^3) are

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Water	62.4

The Pressure-Depth Equation

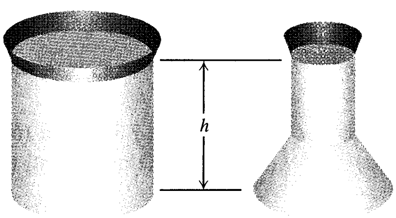
In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{1}$$

In this section we use the equation $p = wh$ to derive a formula for the total force exerted by a fluid against all or part of a vertical or horizontal containing wall.

The Constant-Depth Formula for Fluid Force

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Fig. 5.68.) If F, p, and A are the total force, pressure, and area, then



5.68 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \qquad \qquad p = wh \text{ from Eq. (1)} \end{aligned}$$

Fluid Force on a Constant-Depth Surface

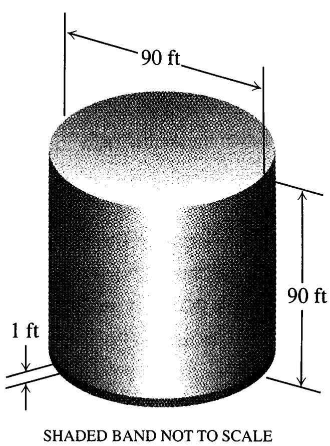
$$F = pA = whA \tag{2}$$

EXAMPLE 1 *The Great Molasses Flood*

At 1:00 P.M. on January 15, 1919, an unusually warm day, a 90-ft-high, 90-ft-diameter cylindrical metal tank in which the Puritan Distilling Company was storing molasses at the corner of Foster and Commercial streets in Boston's North End exploded. The molasses flooded into the streets, 30 ft deep, trapping pedestrians and horses, knocking down buildings, and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs (on trolley cars and people's shoes). It took weeks to clean up.

Given that the molasses weighed 100 lb/ft^3, what was the total force exerted by the molasses against the bottom of the tank at the time it blew? Assuming the tank was full, we can find out from Eq. (2):

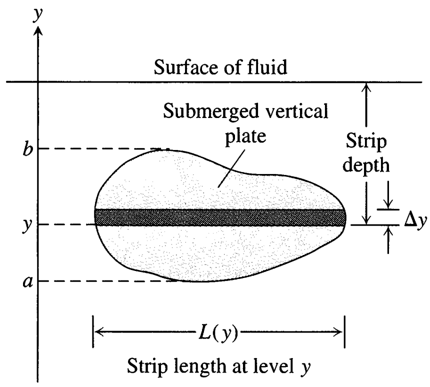
$$\text{Total force} = whA = (100)(90)(\pi(45)^2) \approx 57{,}255{,}526 \text{ lb.} \qquad \square$$



5.69 Schematic drawing of the molasses tank in Example 1. How much force did the lowest foot of the vertical wall have to withstand when the tank was full? It takes an integral to find out. Notice that the proportions of the tank were ideal.

How about the force against the walls of the tank? For example, what was the total force against the bottom foot-wide band of tank wall (Fig. 5.69)? The area of the band was

$$A = 2\pi rh = 2\pi(45)(1) = 90\pi \text{ ft}^2.$$



5.70 The force exerted by a fluid against one side of a thin horizontal strip is about $\Delta F =$ pressure $\times$ area $= w \times$ (strip depth) $\times$ $L(y)\,\Delta y$. The plate here is flat, but it might have been curved instead, like the vertical wall of a cylindrical tank. Whatever the case, the strip length is measured along the surface of the plate.

The tank was 90 ft deep, so the pressure near the bottom was approximately

$$p = wh = (100)(90) = 9000 \text{ lb/ft}^2.$$

Therefore the total force against the band was approximately

$$F = whA = (9000)(90\pi) \approx 2,544,690 \text{ lb}.$$

But this is not exactly right. The top of the band was 89 ft below the surface, not 90, and the pressure there was less. To find out exactly what the force on the band was, we need to take into account the variation of the pressure across the band.

The Variable-Depth Formula

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w. To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy-plane (Fig. 5.70). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y-axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y.

The pressure varies across the strip from top to bottom, just as it did in the molasses tank. But if the strip is narrow enough, the pressure will remain close to its bottom-edge value of $w \times$ (strip depth). The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned}\Delta F &= (\text{pressure along bottom edge}) \times (\text{area})\\ &= w \times (\text{strip depth}) \times L(y)\Delta y.\end{aligned}$$

The force against the entire plate will be about

$$\sum_a^b \Delta F = \sum_a^b (w \times (\text{strip depth}) \times L(y)\Delta y). \tag{3}$$

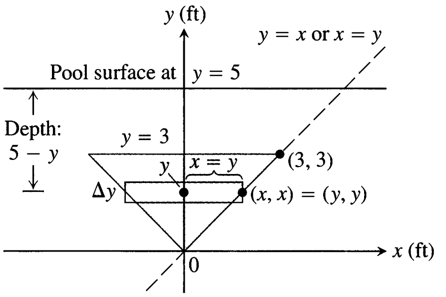
The sum in (3) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. We define the force against the plate to be the limit of these sums.

Definition
The Integral for Fluid Force

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy. \tag{4}$$

EXAMPLE 2 A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.



5.71 To find the force on one side of the submerged plate in Example 2, we can use a coordinate system like the one here.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y-axis upward along the plate's axis of symmetry (Fig. 5.71). (We will look at other coordinate systems in Exercises 3 and 4.) The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper right vertex at (3, 3). The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

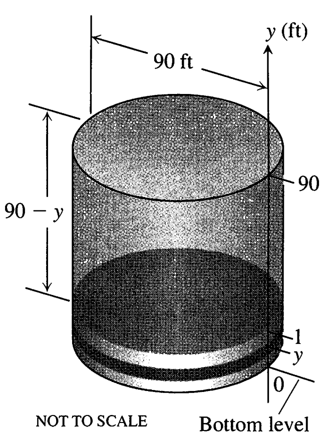
$$\begin{aligned} F &= \int_a^b w \times \begin{pmatrix} \text{strip} \\ \text{depth} \end{pmatrix} \times L(y)\,dy \qquad \text{Eq. (4)} \\ &= \int_0^3 62.4(5 - y)\,2y\,dy \\ &= 124.8 \int_0^3 (5y - y^2)\,dy \\ &= 124.8 \left[\frac{5}{2}y^2 - \frac{y^3}{3} \right]_0^3 = 1684.8 \text{ lb.} \end{aligned}$$

❑

How to Find Fluid Force

Whatever coordinate system you use, you can find the fluid force against one side of a submerged vertical plate or wall by taking these steps.

1. *Find expressions* for the length and depth of a typical thin horizontal strip.
2. *Multiply their product by the fluid's weight-density w and integrate* over the interval of depths occupied by the plate or wall.



5.72 The molasses tank with the coordinate origin at the bottom (Example 3).

EXAMPLE 3 We can now calculate exactly the force exerted by the molasses against the bottom 1-ft band of the Puritan Distilling Company's storage tank when the tank was full.

The tank was a right circular cylindrical tank 90 ft high and 90 ft in diameter. Using a coordinate system with the origin at the bottom of the tank and the y-axis pointing up (Fig. 5.72), we find that the typical horizontal strip at level y has

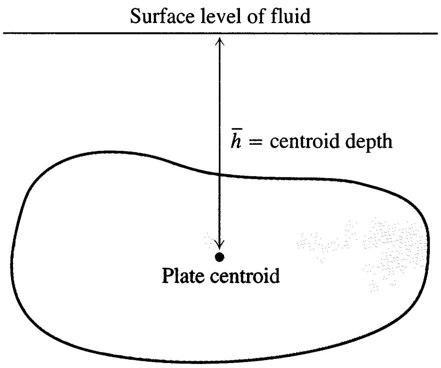
Strip depth: $\quad 90 - y,$

Strip length: $\quad \pi \times \text{tank diameter} = 90\pi.$

The force against the band is therefore

$$\begin{aligned} \text{Force} &= \int_0^1 w(\text{depth})(\text{length})\,dy = \int_0^1 100(90 - y)(90\pi)\,dy \qquad \text{For molasses, } w = 100 \\ &= 9000\pi \int_0^1 (90 - y)\,dy \approx 2{,}530{,}553 \text{ lb.} \end{aligned}$$

❑



5.73 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

As expected, the force is slightly less than the constant-depth estimate following Example 1.

Fluid Forces and Centroids

If we know the location of the centroid of a submerged flat vertical plate (Fig. 5.73), we can take a shortcut to find the force against one side of the plate. From Eq. (4),

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y)\,dy \\ &= w\int_a^b (\text{strip depth}) \times L(y)\,dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \tag{5}$$

EXAMPLE 4 Use Eq. (5) to find the force in Example 2.

Solution The centroid of the triangle (Fig. 5.71) lies on the y-axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$. The triangle's area is

$$\begin{aligned} A &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(6)(3) = 9. \end{aligned}$$

Hence,

$$\begin{aligned} F &= w\bar{h}A = (62.4)(3)(9) \\ &= 1684.8 \text{ lb.} \end{aligned}$$

❑

Equation (5) says that the fluid force on one side of a submerged flat vertical plate is the same as it would be if the plate's entire area lay $\bar{h}$ units beneath the surface. For many shapes, the location of the centroid can be found in a table, and Eq. (5) gives a practical way to find F. Of course, the centroid's location was found by someone who performed an integration equivalent to evaluating the integral in Eq. (4). We recommend for now that you practice your mathematical modeling by drawing pictures and thinking things through the way we did when we developed Eq. (4). Then check your results, when you conveniently can, with Eq. (5).

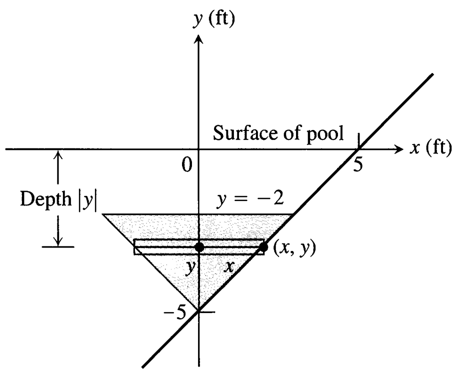
Exercises 5.9

The weight-densities of the fluids in the following exercises can be found in the table on page 428.

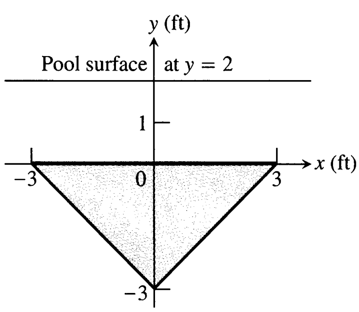
1. What was the total fluid force against the cylindrical inside wall of the molasses tank in Example 1 when the tank was full? half full?

2. What was the total fluid force against the bottom 1-ft band of the inside wall of the molasses tank in Example 1 when the tank was half full?

3. Calculate the fluid force on one side of the plate in Example 2 using the coordinate system shown here.



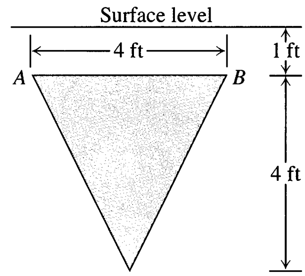
4. Calculate the fluid force on one side of the plate in Example 2 using the coordinate system shown here.



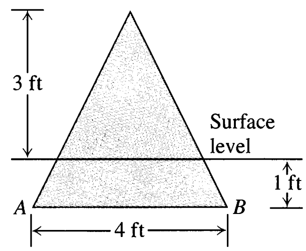
5. The plate in Example 2 is lowered another 2 ft into the water. What is the fluid force on one side of the plate now?

6. The plate in Example 2 is raised to put its top edge at the surface of the pool. What is the fluid force on one side of the plate now?

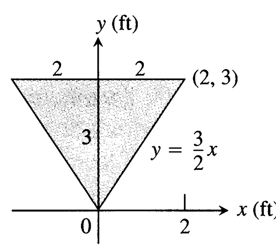
7. The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.
   a) Find the fluid force against one face of the plate.
   b) What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



8. The plate in Exercise 7 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



9. The vertical ends of a watering trough are isosceles triangles like the one shown here (dimensions in feet).
   a) Find the fluid force against the ends when the trough is full.
   b) CALCULATOR How many inches do you have to lower the water level in the trough to cut the fluid force on the ends in half? (Answer to the nearest half inch.)
   c) Does it matter how long the trough is? Give reasons for your answer.



10. The vertical ends of a watering trough are squares 3 ft on a side.
    a) Find the fluid force against the ends when the trough is full.
    b) CALCULATOR How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?
    c) Does it matter how long the trough is? Give reasons for your answer.

11. The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 lb/ft^3. (In case you were wondering, the glass is 3/4 in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

12. A horizontal rectangular freshwater fish tank with base 2×4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.

a) Find the fluid force against each side and end of the tank.

b) If the tank is sealed and stood on end (without spilling), so that one of the square ends is the base, what does that do to the fluid forces on the rectangular sides?

13. CALCULATOR A rectangular milk carton measures 3.75×3.75 in. at the base and is 7.75 in. tall. Find the force of the milk on one side when the carton is full.

14. CALCULATOR A standard olive oil can measures 5.75 by 3.5 in. at the base and is 10 in. tall. Find the fluid force against the base and each side when the can is full.

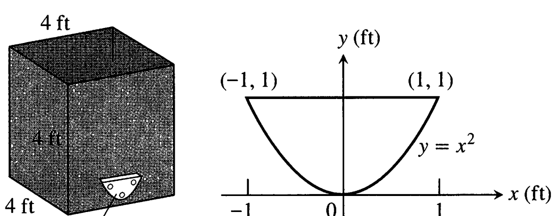
15. A semicircular plate 2 ft in diameter sticks straight down into fresh water with the diameter along the surface. Find the force exerted by the water on one side of the plate.

16. A tank truck hauls milk in a 6-ft-diameter horizontal right circular cylindrical tank. How much force does the milk exert on each end of the tank when the tank is half full?

17. The cubical metal tank shown here has a parabolic gate, held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.

a) What is the fluid force on the gate when the liquid is 2 ft deep?

b) What is the maximum height to which the container can be filled without exceeding its design limitation?



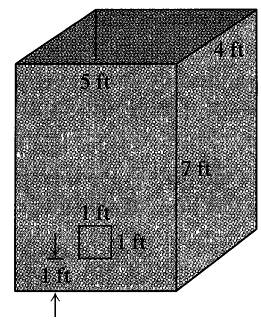
Parabolic gate

Enlarged view of parabolic gate

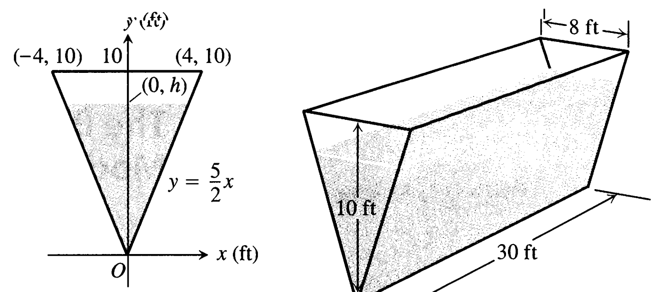
18. The rectangular tank shown here has a $1 \text{ ft} \times 1 \text{ ft}$ square window 1 ft above the base. The window is designed to withstand a fluid force of 312 lb without cracking.

a) What fluid force will the window have to withstand if the tank is filled with water to a depth of 3 ft?

b) To what level can the tank be filled with water without exceeding the window's design limitation?



19. CALCULATOR The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot.



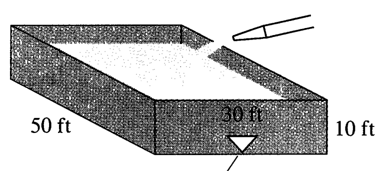
End view of trough

Dimensional view of trough

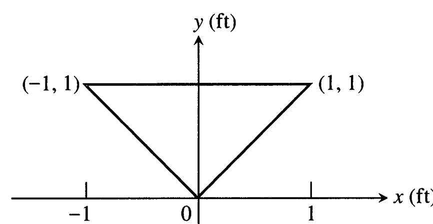
20. Water is running into the rectangular swimming pool shown here at the rate of 1000 ft^3/h.

a) Find the fluid force against the triangular drain plate after 9 h of filling.

b) The drain plate is designed to withstand a fluid force of 520 lb. How high can you fill the pool without exceeding this limitation?



Triangular drain plate

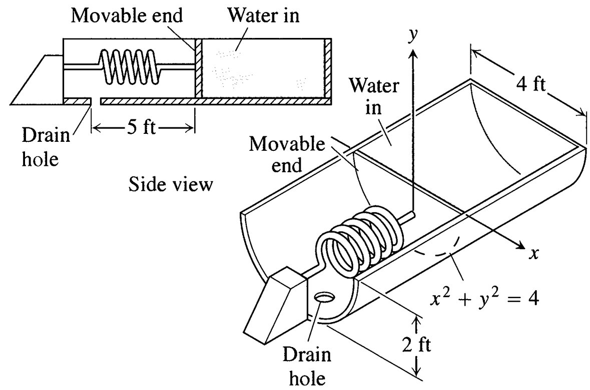


Enlarged view of drain plate

21. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

22. *(Continuation of Exercise 21.)* Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 21) times the area of the plate.

23. Water pours into the tank here at the rate of 4 ft^3/min. The tank's cross sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100$ lb/ft. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of 5 ft^3/min. Will the movable end reach the hole before the tank overflows?



5.10 The Basic Pattern and Other Modeling Applications

There is a pattern to what we did in the preceding sections. In each section we wanted to measure something that was modeled or described by one or more continuous functions. In Section 5.1 it was the area between the graphs of two continuous functions. In Section 5.2 it was the volume of a solid. In Section 5.8 it was the work done by a force whose magnitude was a continuous function, and so on. In each case we responded by partitioning the interval on which the function or functions were defined and approximating what we wanted to measure with Riemann sums over the interval. We used the integral defined by the limit of the Riemann sums to define and calculate what we wanted to measure. Table 5.2 shows the pattern.

Literally thousands of things in biology, chemistry, economics, engineering, finance, geology, medicine, and other fields (the list would fill pages) are modeled and calculated by exactly this process.

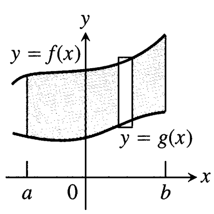
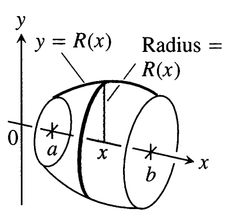
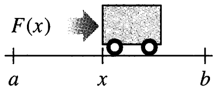
This section reviews the process and looks at a few more of the integrals it leads to.

Displacement vs. Distance Traveled

If a body with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by integrating its velocity function $v(t)$ from $t = a$ to $t = b$, as we did in Chapter 4. If the body changes direction one or more times during the trip, we need to integrate the body's *speed* $|v(t)|$ to find the total distance traveled. Integrating the velocity will give only the body's **displacement,** $s(b) - s(a)$, the difference between its initial and final positions.

To see why, partition the time interval $a \le t \le b$ into subintervals in the usual way and let Δt_k denote the length of the kth interval. If Δt_k is small enough, the body's velocity $v(t)$ will not change much from t_{k-1} to t_k and the right-hand

Table 5.2 The phases of developing an integral to calculate something

Phase 1	**Phase 2**	**Phase 3**
We describe or model something we want to measure in terms of one or more continuous functions defined on a closed interval $[a, b]$.	We partition $[a, b]$ into subintervals of length Δx_k and choose a point c_k in each subinterval. We approximate what we want to measure with a finite sum. We identify the sum as a Riemann sum of a continuous function over $[a, b]$.	The approximations improve as the norm of the partition goes to zero. The Riemann sums approach a limiting integral. We use the integral to define and calculate what we originally wanted to measure.
The area between the curves $y = f(x)$, $y = g(x)$ on $[a, b]$ when $f(x) \geq g(x)$ 	$\sum [f(c_k) - g(c_k)]\,\Delta x_k$	$A = \lim_{\Vert P\Vert\to 0} \sum [f(c_k) - g(c_k)]\,\Delta x_k$ $= \int_a^b [f(x) - g(x)]\,dx$
The volume of the solid defined by revolving the curve $y = R(x)$, $a \leq x \leq b$, about the x-axis. 	$\sum \pi[R(c_k)]^2\,\Delta x_k$	$V = \lim_{\Vert P\Vert\to 0} \sum \pi[R(c_k)]^2\,\Delta x_k$ $= \int_a^b \pi[R(x)]^2\,dx$
The work done by a continuous variable force of magnitude $F(x)$ directed along the x-axis from a to b 	$\sum F(c_k)\,\Delta x_k$	$W = \lim_{\Vert P\Vert\to 0} \sum F(c_k)\,\Delta x_k$ $= \int_a^b F(x)\,dx$

endpoint value $v(t_k)$ will give a good approximation of the velocity throughout the interval. Accordingly, the change in the body's position coordinate during the kth time interval will be about

$$v(t_k)\Delta t_k.$$

The change will be positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled during the kth interval will be about

$$|v(t_k)|\Delta t_k.$$

The total trip distance will be approximately

$$\sum_{k=1}^{n} |v(t_k)|\Delta t_k. \tag{1}$$

The sum in Eq. (1) is a Riemann sum for the speed $|v(t)|$ on the interval $[a, b]$. We expect the approximations to improve as the norm of the partition of $[a, b]$ goes to zero. It therefore looks as if we should be able to calculate the total distance traveled by the body by integrating the body's speed from a to b. In practice, this turns out to be the right thing to do. The mathematical model predicts the distance correctly every time.

$$\boxed{\text{Distance traveled} = \int_a^b |v(t)|\, dt}$$

If we wish to predict how far up or down the line from its initial position a body will end up when a trip is over, we integrate v instead of its absolute value.

To see why, let $s(t)$ be the body's position at time t and let F be an antiderivative of v. Then

$$s(t) = F(t) + C$$

for some constant C. The displacement caused by the trip from $t = a$ to $t = b$ is

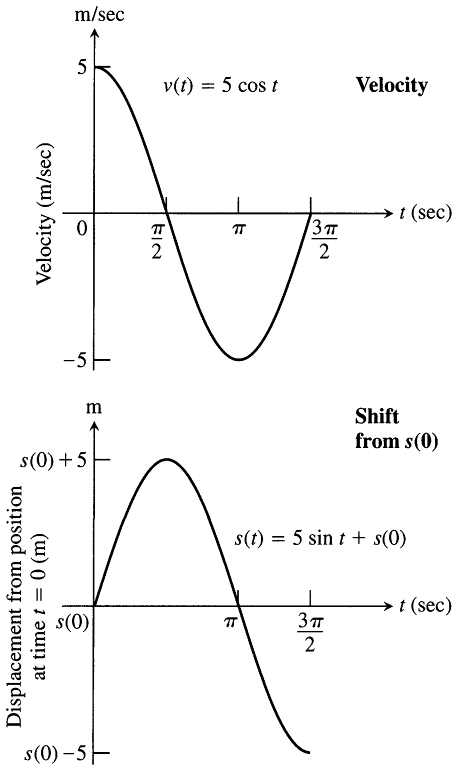
$$\begin{aligned} s(b) - s(a) &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a) = \int_a^b v(t)\, dt. \end{aligned}$$

$$\boxed{\text{Displacement} = \int_a^b v(t)\, dt}$$

EXAMPLE 1 The velocity of a body moving along a line from $t = 0$ to $t = 3\pi/2$ sec was

$$v(t) = 5\cos t \text{ m/sec}.$$

Find the total distance traveled and the body's displacement.



5.74 The velocity and displacement of the body in Example 1.

Solution

$$\text{Distance traveled} = \int_0^{3\pi/2} |5\cos t|\, dt \qquad \text{Distance is the integral of speed.}$$

$$= \int_0^{\pi/2} 5\cos t\, dt + \int_{\pi/2}^{3\pi/2} (-5\cos t)\, dt$$

$$= 5\sin t\Big]_0^{\pi/2} - 5\sin t\Big]_{\pi/2}^{3\pi/2}$$

$$= 5(1-0) - 5(-1-1) = 5 + 10 = 15 \text{ m}$$

$$\text{Displacement} = \int_0^{3\pi/2} 5\cos t\, dt \qquad \text{Displacement is the integral of velocity.}$$

$$= 5\sin t\Big]_0^{3\pi/2} = 5(-1) - 5(0) = -5 \text{ m}$$

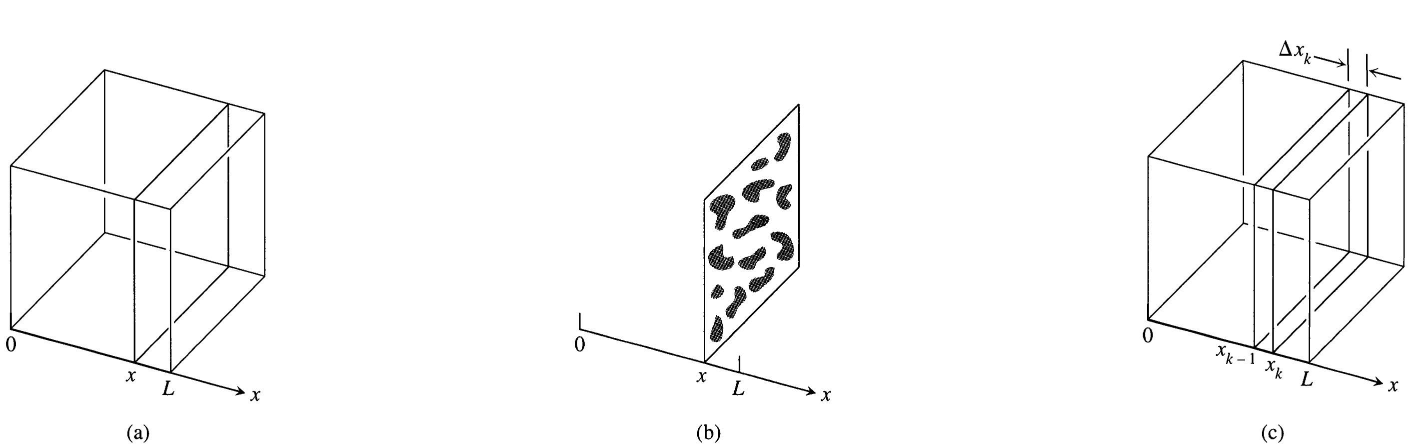
During the trip, the body traveled 5 m forward and 10 m backward for a total distance of 15 m. This displaced the body 5 m to the left (Fig. 5.74). ❑

Delesse's Rule

As you may know, the sugar in an apple starts turning into starch as soon as the apple is picked, and the longer the apple sits around, the starchier it becomes. You can tell fresh apples from stale by both flavor and consistency.

To find out how much starch is in a given apple, we can look at a thin slice under a microscope. The cross sections of the starch granules will show up clearly, and it is easy to estimate the proportion of the viewing area they occupy. This two-dimensional proportion will be the same as the three-dimensional proportion of uncut starch granules in the apple itself. The apparently magical equality of these proportions was first discovered by a French geologist, Achille Ernest Delesse, in the 1840s. Its explanation lies in the notion of average value.

Suppose we want to find the proportion of some granular material in a solid and that the sample we have chosen to analyze is a cube whose edges have length L. We picture the cube with an x-axis along one edge and imagine slicing the cube with planes perpendicular to points of the interval $[0, L]$ (Fig. 5.75). Call the proportion of the area of the slice at x occupied by the granular material of interest (starch, in our apple example) $r(x)$ and assume r is a continuous function of x.



5.75 The steps leading to Delesse's rule: (a) a slice through a sample cube; (b) the granular material in the slice; (c) the slab between consecutive slices determined by a partition of $[0, L]$.

Now partition the interval $[0, L]$ into subintervals in the usual way. Imagine the cube sliced into thin slices by planes at the subdivision points. The length Δx_k of the kth subinterval is the distance between the planes at x_{k-1} and x_k. If the planes are close enough together, the sections cut from the grains by the planes will resemble cylinders with bases in the plane at x_k. The proportion of granular material between the planes will be about the same as the proportion of cylinder base area in the plane at x_k, which in turn will be about $r(x_k)$. Thus the amount of granular material in the slab between the two planes will be about

$$\text{(Proportion)} \times \text{(slab volume)} = r(x_k)L^2\,\Delta x_k.$$

The amount of granular material in the entire sample cube will be about

$$\sum_{k=1}^{n} r(x_k)L^2\,\Delta x_k.$$

This sum is a Riemann sum for the function $r(x)L^2$ over the interval $[0, L]$. We expect the approximations by sums like these to improve as the norm of the subdivision of $[0, L]$ goes to zero and therefore expect the integral

$$\int_0^L r(x)L^2\,dx$$

to give the amount of granular material in the sample cube.

We can obtain the proportion of granular material in the sample by dividing this amount by the cube's volume, L^3. If we have chosen our sample well, this will also be the proportion of granular material in the solid from which the sample was taken. Putting it all together, we get

$$\begin{aligned}
\begin{array}{r}\text{Proportion of granular}\\ \text{material in solid}\end{array} &= \begin{array}{l}\text{Proportion of granular}\\ \text{material in the sample cube}\end{array}\\
&= \frac{\int_0^L r(x)L^2\,dx}{L^3} = \frac{L^2\int_0^L r(x)\,dx}{L^3} = \frac{1}{L}\int_0^L r(x)\,dx\\
&= \text{average value of } r(x) \text{ over } [0, L]\\
&= \begin{array}{l}\text{proportion of area occupied by granular}\\ \text{material in a typical cross section.}\end{array}
\end{aligned}$$

This is Delesse's rule. Once we have found $\bar{r}$, the average of $r(x)$ over $[0, L]$, we have found the proportions of granular material in the solid.

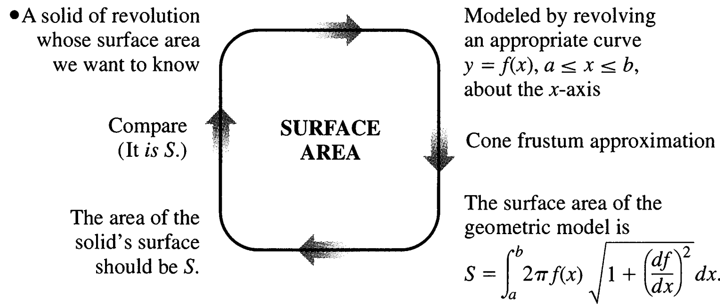
In practice, $\bar{r}$ is found by averaging over a number of cross sections. There are several things to watch out for in the process. In addition to the possibility that the granules cluster in ways that make representative samples difficult to find, there is the possibility that we might not recognize a granule's trace for what it is. Some cross sections of normal red blood cells look like disks and ovals, while others look surprisingly like dumbbells. We do not want to dismiss the dumbbells as experimental error the way one research group did a few years ago.

Delesse's rule

Achille Ernest Delesse was a mid-nineteenth-century mining engineer interested in determining the composition of rocks. To find out how much of a particular mineral a rock contained, he cut it through, polished an exposed face, and covered the face with transparent waxed paper, trimmed to size. He then traced on the paper the exposed portions of the mineral that interested him. After weighing the paper, he cut out the mineral traces and weighed them. The ratio of the weights gave not only the proportion of the surface occupied by the mineral but, more important, the proportion of the entire rock occupied by the mineral. This rule is still used by petroleum geologists today. A two-dimensional analogue of it is used to determine the porosities of the ceramic filters that extract organic molecules in chemistry laboratories and screen out microbes in water purifiers.

Useless Integrals — Bad Models

Some of the integrals we get from forming Riemann sums do what we want, but others do not. It all depends on how we choose to model the problems we want to solve. Some choices are good; others are not. Here is an example.



5.76 The modeling cycle for surface area.

We use the surface area formula

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dx \tag{2}$$

because it has predictive value and always gives results consistent with information from other sources. In other words, the model we used to derive the formula (Fig. 5.76) was a good one.

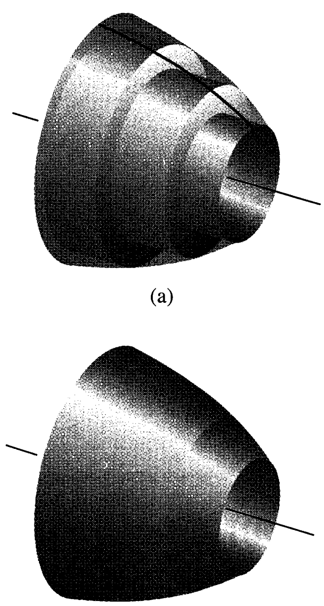
Why not find the surface area by approximating with cylindrical bands instead of conical bands, as suggested in Fig. 5.77? The Riemann sums we get this way converge just as nicely as the ones based on conical bands, and the resulting integral is simpler. Instead of Eq. (2), we get

$$S = \int_a^b 2\pi f(x)\,dx. \tag{3}$$

After all, we might argue, we used cylinders to derive good volume formulas, so why not use them again to derive surface area formulas?

The answer is that the formula in Eq. (3) has no predictive value and almost never gives results consistent with other calculations. The comparison step in the modeling process fails for this formula.

There is a moral here: Just because we end up with a nice-looking integral does not mean it will do what we want. Constructing an integral is not enough—we have to test it too (Exercises 15 and 16).



(a)

(b)

5.77 Why not use (a) cylindrical bands instead of (b) conical bands to approximate surface area?

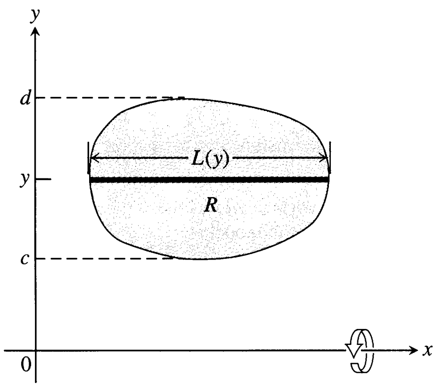
The Theorems of Pappus

In the third century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

Theorem 1
Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{4}$$



5.78 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

Proof We draw the axis of revolution as the x-axis with the region R in the first quadrant (Fig. 5.78). We let $L(y)$ denote the length of the cross section of R perpendicular to the y-axis at y. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy = 2\pi \int_c^d y\,L(y)\,dy. \tag{5}$$

The y-coordinate of R's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\,dA}{A} = \frac{\int_c^d y\,L(y)\,dy}{A},$$

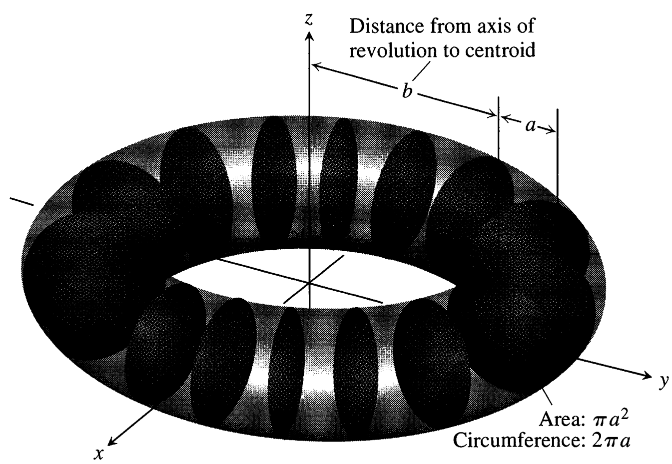
so that

$$\int_c^d y\,L(y)\,dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Eq. (5) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ❑

EXAMPLE 2 The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Fig. 5.79) is

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2.$$ ❑

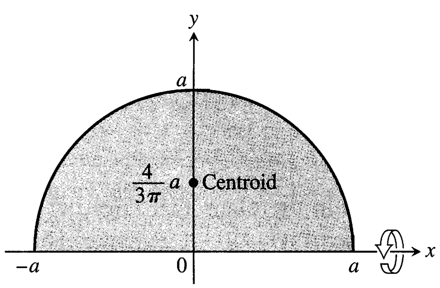


5.79 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 2).

EXAMPLE 3 Locate the centroid of a semicircular region.

Solution We model the region as the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Fig. 5.80) and the x-axis and imagine revolving the region about the x-axis to generate a solid sphere. By symmetry, the x-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Eq. (4), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a.$$ ❑

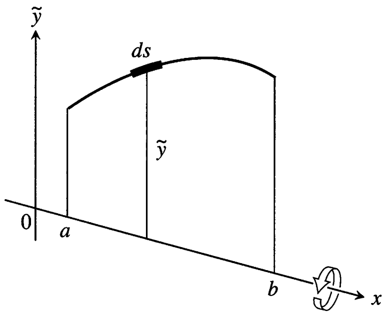


5.80 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 3).

Theorem 2
Pappus's Theorem for Surface Areas
If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{6}$$

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a smooth function of x.



5.81 Figure for Pappus's area theorem.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Fig. 5.81). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = 2\pi \int_{x=a}^{x=b} y\, ds. \tag{7}$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y}\, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y\, ds}{L}. \qquad L = \int ds \text{ is the arc's length and } \tilde{y} = y.$$

Hence

$$\int_{x=a}^{x=b} y\, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Eq. (7) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$.

EXAMPLE 4 The surface area of the torus in Example 2 is

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$

□

Exercises 5.10

Distance and Displacement

In Exercises 1–8, the function $v(t)$ is the velocity in meters per second of a body moving along a coordinate line. (a) Graph v to see where it is positive and negative. Then find (b) the total distance traveled by the body during the given time interval and (c) the body's displacement.

1. $v(t) = 5\cos t, \quad 0 \le t \le 2\pi$

2. $v(t) = \sin \pi t, \quad 0 \le t \le 2$

3. $v(t) = 6\sin 3t, \quad 0 \le t \le \pi/2$

4. $v(t) = 4\cos 2t, \quad 0 \le t \le \pi$

5. $v(t) = 49 - 9.8t, \quad 0 \le t \le 10$

6. $v(t) = 8 - 1.6t, \quad 0 \le t \le 10$

7. $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2), \quad 0 \le t \le 2$

8. $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2), \quad 0 \le t \le 3$

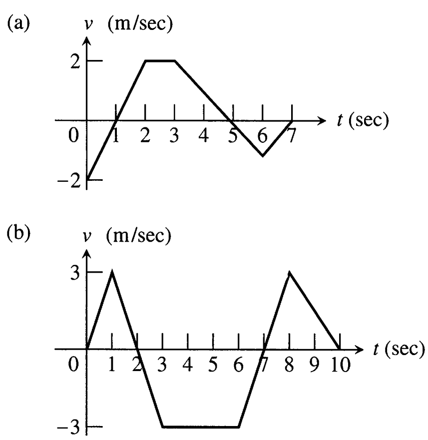
9. The function $s = (1/3)t^3 - 3t^2 + 8t$ gives the position of a body moving on the horizontal s-axis at time $t \ge 0$ (s in meters, t in seconds).

a) Show that the body is moving to the right at time $t = 0$.
b) When does the body move to the left?
c) What is the body's position at time $t = 3$?
d) When $t = 3$, what is the total distance the body has traveled?
e) GRAPHER Graph s as a function of t and comment on the relationship of the graph to the body's motion.

10. The function $s = -t^3 + 6t^2 - 9t$ gives the position of a body moving on the horizontal s-axis at time $t \ge 0$ (s in meters, t in seconds).

a) Show that the body is moving to the left at $t = 0$.
b) When does the body move to the right?
c) Does the body ever move to the right of the origin? Give reasons for your answer.
d) What is the body's position at time $t = 3$?
e) What is the total distance the particle has traveled by the time $t = 3$?
f) GRAPHER Graph s as a function of t and comment on the relationship of the graph to the body's motion.

11. Here are the velocity graphs of two bodies moving on a coordinate line. Find the total distance traveled and the body's displacement for the given time interval.

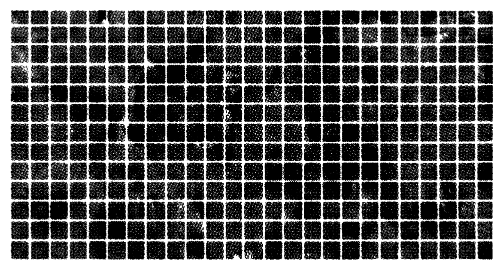


12. CALCULATOR The table at the top of the next column shows the velocity of a model train engine moving back and forth on a track for 10 sec. Use Simpson's rule to find the resulting displacement and total distance traveled.

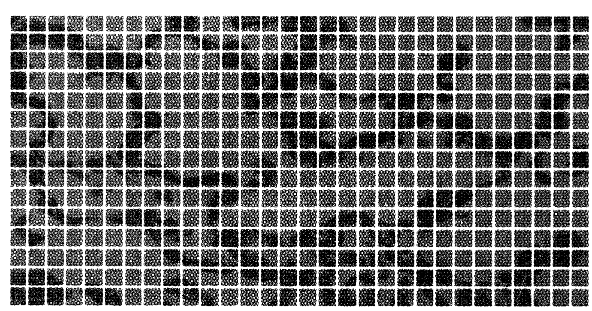
Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	−11
1	12	7	− 6
2	22	8	2
3	10	9	6
4	− 5	10	0
5	−13		

Delesse's Rule

13. The photograph here shows a grid superimposed on the polished face of a piece of granite. Use the grid and Delesse's rule to estimate the proportion of shrimp-colored granular material in the rock.

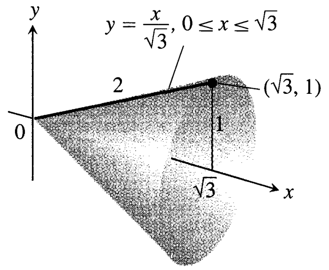


14. The photograph here shows a grid superimposed on a microscopic view of a stained section of human lung tissue. The clear spaces between the cells are cross sections of the lung's air sacks (called *alveoli,* accent on the second syllable). Use the grid and Delesse's rule to estimate the proportion of air space in the lung.



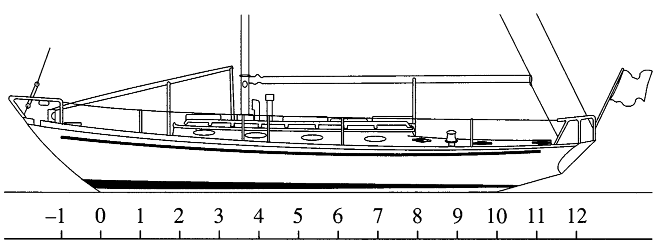
Modeling Surface Area

15. *Modeling surface area.* The lateral surface area of the cone swept out by revolving the line segment $y = x/\sqrt{3}, 0 \le x \le \sqrt{3}$, about the x-axis should be (1/2)(base circumference)(slant height) $= (1/2)(2\pi)(2) = 2\pi$. What do you get if you use Eq. (3) with $f(x) = x/\sqrt{3}$?



16. *Modeling surface area.* The only surface for which Eq. (3) gives the area we want is a cylinder. Show that Eq. (3) gives $S = 2\pi rh$ for the cylinder swept out by revolving the line segment $y = r, 0 \le x \le h$, about the x-axis.

17. *A sailboat's displacement.* To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross section area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream,* shown here. The common subinterval length (distance between consecutive stations) is $h = 2.54$ ft (about 2′ 6 1/2″, chosen for the convenience of the builder).



a) Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

Station	Submerged area (ft^2)
0	0
1	1.07
2	3.84
3	7.82
4	12.20
5	15.18
6	16.14
7	14.00
8	9.21
9	3.24
10	0

b) The figures in the table are for seawater, which weighs 64 lb/ft^3. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft, and long tons [1 long ton = 2240 lb] for larger vessels.)

(Data from *Skene's Elements of Yacht Design,* Francis S. Kinney, Dodd, Mead & Company, Inc., 1962)

18. *Prismatic coefficients (Continuation of Exercise 17).* A boat's prismatic coefficient is the ratio of the displacement volume to the volume of a prism whose height equals the boat's waterline length and whose base equals the area of the boat's largest submerged cross section. The best sailboats have prismatic coefficients between 0.51 and 0.54. Find *Pipedream*'s prismatic coefficient, given a waterline length of 25.4 ft and a largest submerged cross section area of 16.14 ft^2 (at Station 6).

The Theorems of Pappus

19. The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.

20. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$. (As you saw in Exercise 31 of Section 5.7, the centroid of a triangle lies at the intersection of the medians, one-third of the way from the midpoint of each side toward the opposite vertex.)

21. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y-axis.

22. Use the theorems of Pappus to find the lateral surface area and the volume of a right circular cone.

23. Use the second theorem of Pappus and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

24. As found in Exercise 23, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

25. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$ and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice the remarkable fact that the location is independent of a.

26. As found in Example 3, the centroid of the region enclosed by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

27. The region of Exercise 26 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

28. As found in Exercise 23, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

29. Find the moment about the x-axis of the semicircular region in Example 3. If you use results already known, you will not need to integrate.

Chapter 5 Questions to Guide Your Review

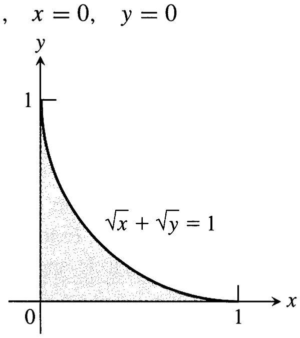
1. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.
2. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
3. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
4. Describe the method of cylindrical shells. Give an example.
5. How do you define and calculate the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
6. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x), a \le x \le b$, about the x-axis? Give an example.
7. What is a center of mass?
8. How do you locate the center of mass of a straight, narrow rod or strip of material? Give an example. If the density of the material is constant, you can tell right away where the center of mass is. Where is it?
9. How do you locate the center of mass of a thin flat plate of material? Give an example.
10. How do you define and calculate the work done by a variable force directed along a portion of the x-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
11. How do you calculate the force exerted by a liquid against a portion of a vertical wall? Give an example.
12. Suppose you know the velocity function $v(t)$ of a body that will be moving back and forth along a coordinate line from time $t = a$ to time $t = b$. How can you predict how much the motion will shift the body's position? How can you predict the total distance the body will travel?
13. What does Delesse's rule say? Give an example.
14. What do Pappus's two theorems say? Give examples of how they are used to calculate surface areas and volumes and to locate centroids.
15. There is a basic pattern to the way we constructed integrals in this chapter. What is it? Give examples.

Chapter 5 Practice Exercises

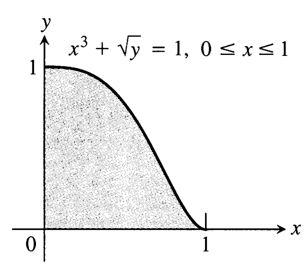
Areas

Find the areas of the regions enclosed by the curves and lines in Exercises 1–12.

1. $y = x, \quad y = 1/x^2, \quad x = 2$
2. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$
3. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



4. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



5. $x = 2y^2, \quad x = 0, \quad y = 3$
6. $x = 4 - y^2, \quad x = 0$
7. $y^2 = 4x, \quad y = 4x - 2$
8. $y^2 = 4x + 4, \quad y = 4x - 16$
9. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

10. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \leq x \leq \pi/2$

11. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \leq x \leq \pi$

12. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \leq x \leq \pi/3$

13. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

14. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

15. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

16. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

17. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

18. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 3\pi/2$.

Volumes

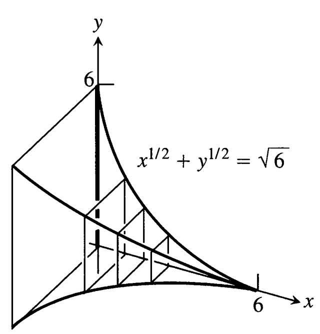
Find the volumes of the solids in Exercises 19–24.

19. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

20. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve.

21. The solid lies between planes perpendicular to the x-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

22. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



23. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

24. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy-plane. Each cross section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

25. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 3$.

26. Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 2$; (d) the line $y = 4$.

27. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 5$.

28. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 4$.

29. Find the volume of the solid generated by revolving the "triangular" region bounded by the x-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x-axis.

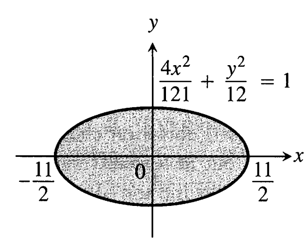
30. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

31. Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about (a) the x-axis; (b) the line $y = -1$; (c) the line $x = 2$; (d) the line $y = 2$.

32. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bow tie.)

33. A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of radius 2 ft. Find the volume of material removed from the sphere.

34. CALCULATOR The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



Lengths of Curves

Find the lengths of the curves in Exercises 35–38.

35. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \leq x \leq 4$

36. $x = y^{2/3}, \quad 1 \leq y \leq 8$

37. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

38. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

Areas of Surfaces of Revolution

In Exercises 39–42, find the areas of the surfaces generated by revolving the curves about the given axes.

39. $y = \sqrt{2x+1}, \quad 0 \le x \le 3, \quad x\text{-axis}$

40. $y = x^3/3, \quad 0 \le x \le 1, \quad x\text{-axis}$

41. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2, \quad y\text{-axis}$

42. $x = \sqrt{y}, \quad 2 \le y \le 6, \quad y\text{-axis}$

Centroids and Centers of Mass

43. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

44. Find the centroid of a thin, flat plate covering the region enclosed by the x-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

45. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$, and the line $y = 4$.

46. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

47. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

48. a) Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x-axis from $x = 1$ to $x = 9$.
    b) Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Work

49. A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately; then add.)

50. You drove an 800-gal tank truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 lb/U.S. gal.

51. If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? an additional foot?

52. A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far?

53. A reservoir shaped like a right circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

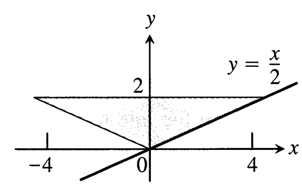
54. *(Continuation of Exercise 53.)* The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

55. A right circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft • lb/sec (1/2-hp), how long will it take to empty the tank?

56. A storage tank is a right circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

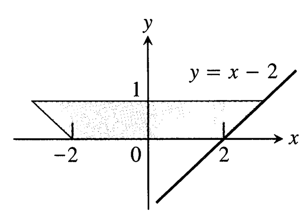
Fluid Force

57. The vertical triangular plate shown here is the end plate of a trough full of water ($w = 62.4$). What is the fluid force against the plate?



UNITS IN FEET

58. The vertical trapezoidal plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?



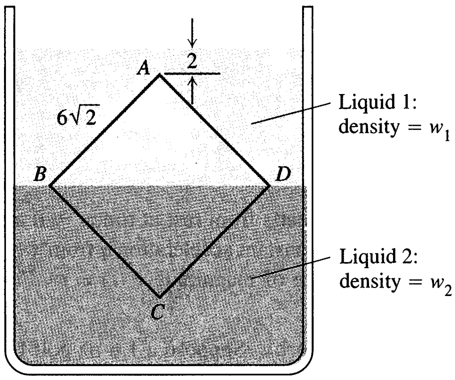
UNITS IN FEET

59. A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate ($w = 62.4$).

60. CALCULATOR You plan to store mercury ($w = 849$ lb/ft^3) in a vertical right circular cylindrical tank of radius 1 ft whose interior side wall can withstand a total fluid force of 40,000 lb. About

how many cubic feet of mercury can you store in the tank at any one time?

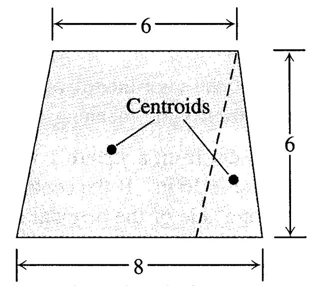
61. The container profiled in Fig. 5.82 is filled with two nonmixing liquids of weight density w_1 and w_2. Find the fluid force on one side of the vertical square plate *ABCD*. The points *B* and *D* lie in the boundary layer and the square is $6\sqrt{2}$ ft on a side.



5.82 Profile of the container in Exercise 61.

62. The isosceles trapezoidal plate shown here is submerged vertically in water ($w = 62.4$) with its upper edge 4 ft below the surface. Find the fluid force on one side of the plate in two different ways:

a) By evaluating an integral.
b) By dividing the plate into a parallelogram and an isosceles triangle, locating their centroids, and using the equation $F = w\bar{h}A$ from Section 5.9.



Distance and Displacement

In Exercises 63–66, the function $v = f(t)$ is the velocity (m/sec) of a body moving along a coordinate line. Find (a) the total distance the body travels during the given time interval and (b) the body's displacement.

63. $v = t^2 - 8t + 12, \quad 0 \le t \le 6$

64. $v = t^3 - 3t^2 + 2t, \quad 0 \le t \le 2$

65. $v = 5\cos t, \quad 0 \le t \le 3\pi/2$

66. $v = -\pi \sin \pi t, \quad 0 \le t \le 3/2$

CHAPTER 5 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Volume and Length

1. A solid is generated by revolving about the x-axis the region bounded by the graph of the continuous function $y = f(x)$, the x-axis, and the fixed line $x = a$ and the variable line $x = b$, $b > a$. Its volume, for all b, is $b^2 - ab$. Find $f(x)$.

2. A solid is generated by revolving about the x-axis the region bounded by the graph of the continuous function $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.

3. Suppose that the increasing function $f(x)$ is smooth for $x \ge 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C. What are the allowable values for C?

4. a) Show that for $0 < \alpha \le \pi/2$,

$$\int_0^\alpha \sqrt{1 + \cos^2\theta}\, d\theta > \sqrt{\alpha^2 + \sin^2\alpha}.$$

b) Generalize the result in (a).

Moments and Centers of Mass

5. Find the centroid of the region bounded below by the x-axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \to \infty$?

6. CALCULATOR If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be three feet or so behind the pole's center of mass to provide an adequate "tongue" weight. NYNEX's class 1 40-ft wooden poles have a 27-in. circumference at the top and a 43.5-in. circumference at the base. About how far from the top is the center of mass?

7. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy-plane, and let M_y be the plate's moment about the y-axis. Show that the plate's moment about the line $x = b$ is

a) $M_y - b\,\delta\, A$ if the plate lies to the right of the line, and
b) $b\,\delta\, A - M_y$ if the plate lies to the left of the line.

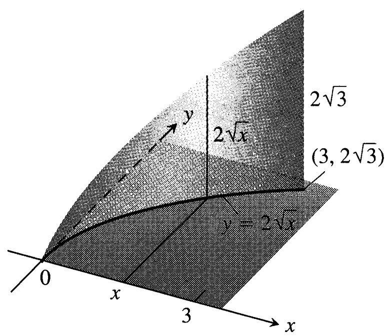
8. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, $a =$ positive constant, if the density at (x, y) is directly proportional to (a) x, (b) $|y|$.

9. a) Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b, $0 < a < b$, and their centers are at the origin.
   b) Find the limits of the coordinates of the centroid as a approaches b and discuss the meaning of the result.

10. A triangular corner is cut from a square 1 ft on a side. The area of the triangle removed is 36 in^2. If the centroid of the remaining region is 7 in. from one side of the original square, how far is it from the remaining sides?

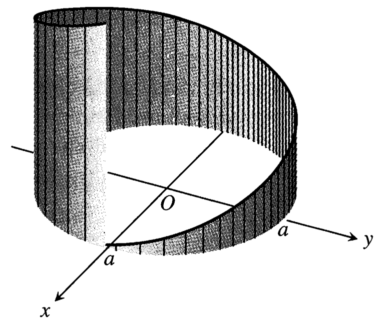
Surface Area

11. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy-plane (Fig. 5.83). Find the area of the surface formed by these perpendiculars from (0, 0) to $(3, 2\sqrt{3})$.



5.83 The surface in Exercise 11.

12. At points on a circle of radius a, line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks, where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.



Work

13. A particle of mass m starts from rest at time $t = 0$ and is moved along the x-axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.

14. *Work and kinetic energy.* Suppose a 1.6-oz golf ball is placed on a vertical spring with force constant $k = 2$ lb/in. The spring is compressed 6 in. and released. About how high does the ball go (measured from the spring's rest position)?

Fluid Force

15. A triangular plate ABC is submerged in water with its plane vertical. The side AB, 4 ft long, is 6 ft below the surface of the water, while the vertex C is 2 ft below the surface. Find the force exerted by the water on one side of the plate.

16. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

17. The *center of pressure* on one side of a plane region submerged in a fluid is defined to be the point at which the total force exerted by the fluid can be applied without changing its total moment about any axis in the plane. Find the depth to the center of pressure (a) on a vertical rectangle of height h and width b if its upper edge is in the surface of the fluid; (b) on a vertical triangle of height h and base b if the vertex opposite b is a ft and the base b is $(a + h)$ ft below the surface of the fluid.

CHAPTER 6

Transcendental Functions

OVERVIEW Many of the functions in mathematics and science are inverses of one another. The functions $\ln x$ and e^x are probably the best-known function–inverse pair, but others are nearly as important. The trigonometric functions, when suitably restricted, have important inverses, and there are other useful pairs of logarithmic and exponential functions. Less widely known are the hyperbolic functions and their inverses, functions that arise in the study of hanging cables, heat flow, and the friction encountered by objects falling through the air. We describe all of these functions in this chapter and look at the kinds of problems they solve.

6.1 Inverse Functions and Their Derivatives

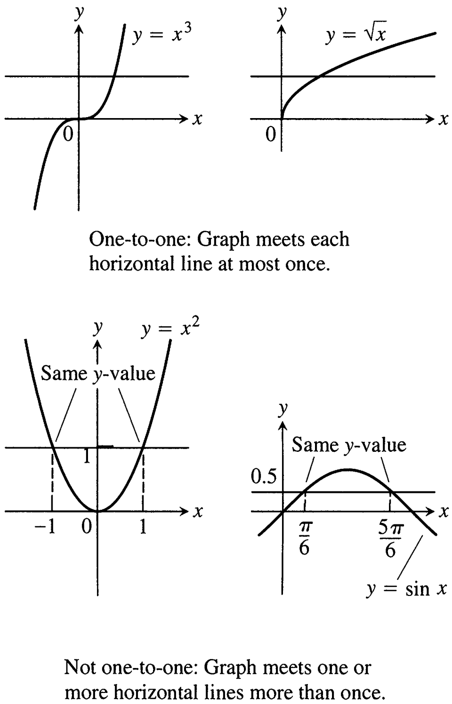
In this section, we define what it means for functions to be inverses of one another and look at what this says about the formulas, graphs, and derivatives of function–inverse pairs.

One-to-One Functions

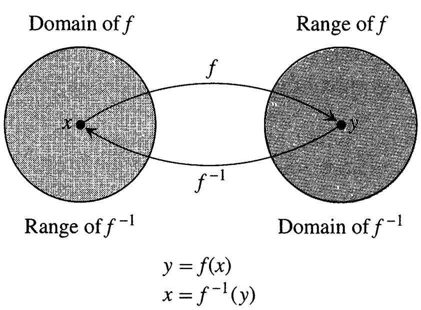
A function is a rule that assigns a value from its range to each point in its domain. Some functions assign the same value to more than one point. The squares of -1 and 1 are both 1; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions never assume a given value more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct points is called one-to-one.

Definition
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

EXAMPLE 1 $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$. ❑



6.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one, but $y = x^2$ and $y = \sin x$ are not.



6.2 The inverse of a function f sends each output back to the input from which it came.

EXAMPLE 2 $g(x) = \sin x$ *is not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because sines of angles in the first quadrant are distinct. ❑

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once it assumes the same y-value more than once, and is therefore not one-to-one (Fig. 6.1).

> **The Horizontal Line Test**
>
> A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send the outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is called the **inverse** of f. The symbol for the inverse of f is $\boldsymbol{f^{-1}}$, read "f inverse" (Fig. 6.2). The -1 in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

As Fig. 6.2 suggests, the result of composing f and f^{-1} in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverse of one another; otherwise they are not. If f cubes every number in its domain, g had better take cube roots or it isn't the inverse of f.

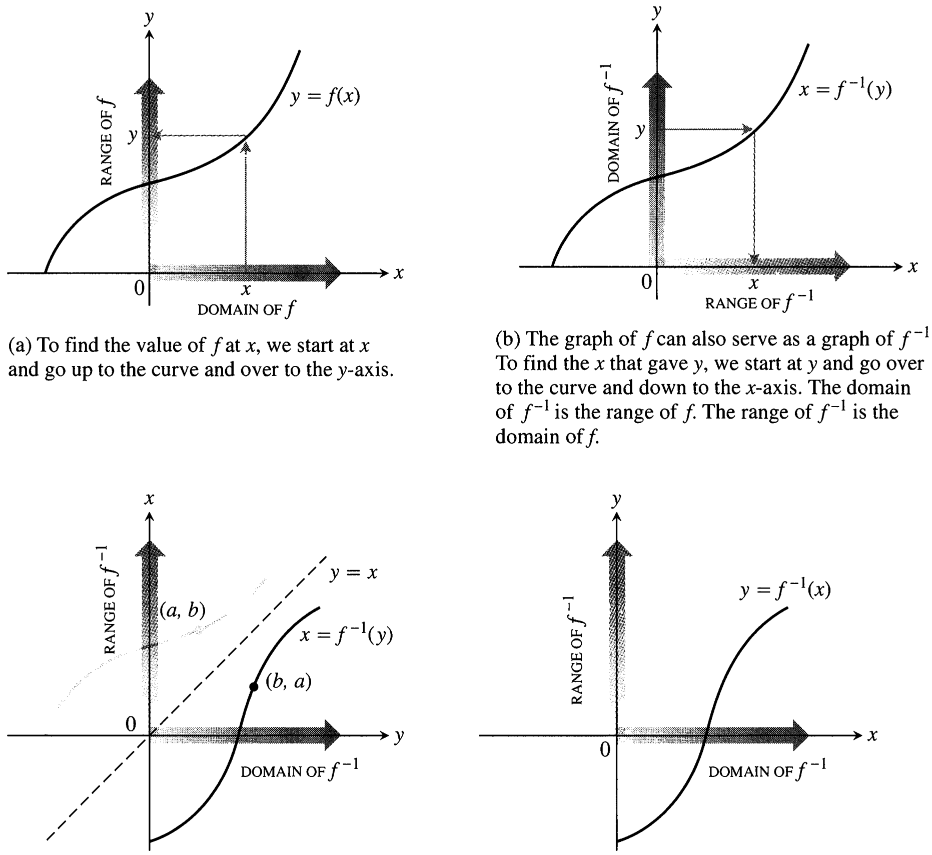
> Functions f and g are an inverse pair if and only if
>
> $$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$
>
> In this case, $g = f^{-1}$ and $f = g^{-1}$.

A function has an inverse if and only if it is one-to-one. This means, for example, that increasing functions have inverses and decreasing functions have inverses (Exercise 39). Functions with positive derivatives have inverses because they increase throughout their domains (Corollary 3 of the Mean Value Theorem, Section 3.2). Similarly, because they decrease throughout their domains, functions with negative derivatives have inverses.

Finding Inverses

How is the graph of the inverse of a function related to the graph of the function? If the function is increasing, say, its graph rises from left to right, like the graph in Fig. 6.3(a). To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Fig. 6.3b).

The graph of f is the graph of f^{-1} with the input–output pairs reversed. To display the graph in the usual way, we have to reverse the pairs by reflecting the



(a) To find the value of f at x, we start at x and go up to the curve and over to the y-axis.

(b) The graph of f can also serve as a graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the usual way, we reflect it in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a graph of f^{-1} as a function of x.

6.3 The graph of $f^{-1}(x)$.

graph in the 45° line $y = x$ (Fig. 6.3c) and interchanging the letters x and y (Fig. 6.3d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis. The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.

The pictures in Fig. 6.3 tell us how to express f^{-1} as a function of x, which is stated at the left.

How to Express f^{-1} as a Function of x

Step 1: Solve the equation $y = f(x)$ for x in terms of y.

Step 2: Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

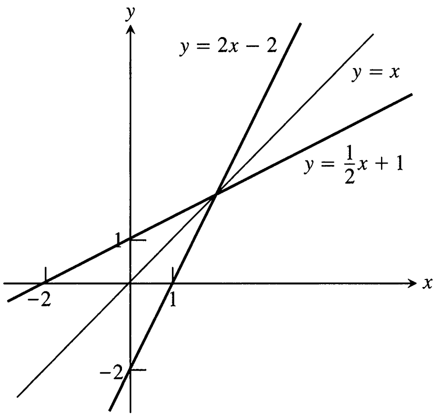
Solution

Step 1: *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

Step 2: *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$.



6.4 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

See Fig. 6.4. ❑

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution

Step 1: *Solve for x in terms of y:*

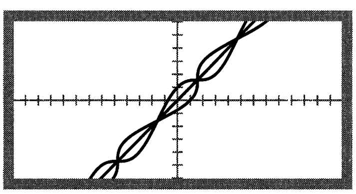
$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

Step 2: *Interchange x and y:* $y = \sqrt{x}$.

The inverse of the function $y = x^2$, $x \geq 0$, is the function $y = \sqrt{x}$. See Fig. 6.5.

Notice that, unlike the restricted function $y = x^2$, $x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse. ❑



$\begin{cases} x_1(t) = t \\ y_1(t) = t + \cos t \end{cases}$ $\begin{cases} x_2(t) = t + \cos t \\ y_2(t) = t \end{cases}$

$\begin{cases} x_3(t) = t \\ y_3(t) = t \end{cases}$

Technology *Using Parametric Equations to Graph Inverses* (See the Technology Notes in Section 2.3 for a discussion of parametric mode.) It is easy to graph the inverse of the function $y = f(x)$, using the parametric form

$$x(t) = f(t), \qquad y(t) = t.$$

You can graph the function and its inverse together, using

$$x_1(t) = t, \qquad y_1(t) = f(t) \qquad \text{(the function)}$$

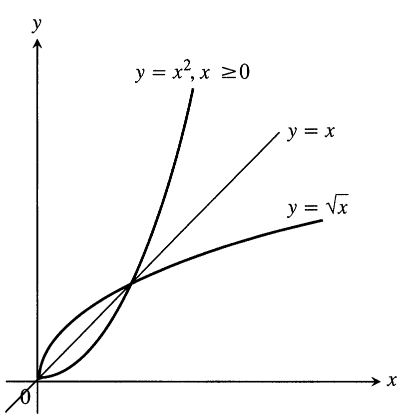
$$x_2(t) = f(t), \qquad y_2(t) = t \qquad \text{(its inverse)}$$

Even better, graph the function, its inverse, *and* the identity function $y = x$, expressed parametrically as

$$x_3(t) = t, \qquad y_3(t) = t \qquad \text{(the identity function)}$$

The graphing is particularly effective if done simultaneously.

Try it on the functions $y = x^5/(x^2 + 1)$ and $y = x + \cos x$. You will see the symmetry best if you use a square window (one in which the x- and y-axes are identically scaled).



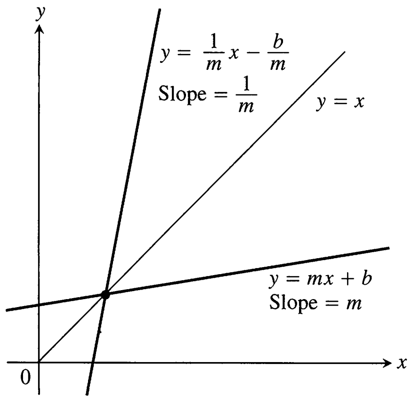
6.5 The functions $y = \sqrt{x}$ and $y = x^2$, $x \geq 0$, are inverses of one another.

Derivatives of Inverses of Differentiable Functions

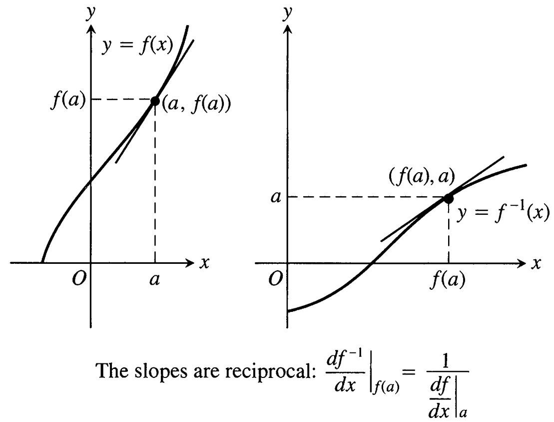
If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 3, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$



6.6 The slopes of nonvertical lines reflected across the line $y = x$ are reciprocals of one another.



6.7 The graphs of inverse functions have reciprocal slopes at corresponding points.

The derivatives are reciprocals of one another. The graph of f is the line $y = (1/2)x + 1$, and the graph of f^{-1} is the line $y = 2x - 2$ (Fig. 6.4). Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$ (Fig. 6.6), the reflected line has slope $1/m$ (Exercise 36).

The reciprocal relation between the slopes of graphs of inverses holds for other functions as well. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the corresponding point $(f(a), a)$ is $1/f'(a)$ (Fig. 6.7). Thus, the derivative of f^{-1} at $f(a)$ equals the reciprocal of the derivative of f at a. As you might imagine, we have to impose some mathematical conditions on f to be sure this conclusion holds. The usual conditions, from advanced calculus, are stated in Theorem 1.

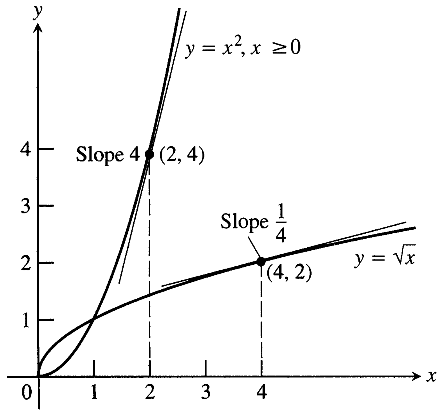
Theorem 1
The Derivative Rule for Inverses

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f^{-1} is differentiable at every point of the interval $f(I)$. The value of df^{-1}/dx at any particular point $f(a)$ is the reciprocal of the value of df/dx at a:

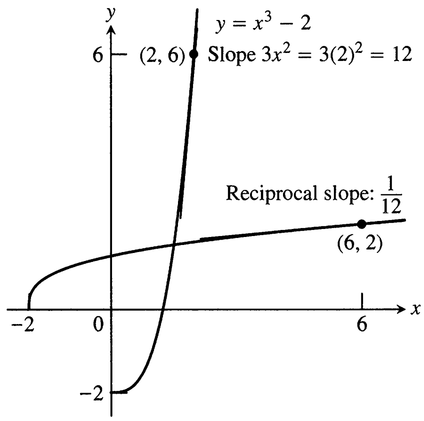
$$\left(\frac{df^{-1}}{dx}\right)_{x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=a}}. \tag{1}$$

In short,

$$(f^{-1})' = \frac{1}{f'}. \tag{2}$$



6.8 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4).



6.9 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$.

EXAMPLE 5 For $f(x) = x^2$, $x \geq 0$, and its inverse $f^{-1}(x) = \sqrt{x}$ (Fig. 6.8), we have

$$\frac{df}{dx} = \frac{d}{dx}(x^2) = 2x \qquad \text{and} \qquad \frac{df^{-1}}{dx} = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0.$$

The point (4, 2) is the mirror image of the point (2, 4) across the line $y = x$.

$$\text{At the point } (2, 4): \qquad \frac{df}{dx} = 2x = 2(2) = 4.$$

$$\text{At the point } (4, 2): \qquad \frac{df^{-1}}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4} = \frac{1}{df/dx}.$$

❑

Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

EXAMPLE 6 Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{12} \qquad \text{Eq. (1)}$$

See Fig. 6.9. ❑

Another Way to Look at Theorem 1

If $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy = f'(a)\,dx.$$

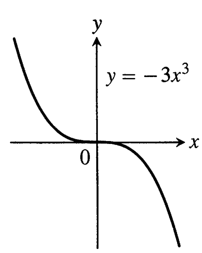
This means that y changes about $f'(a)$ times as fast as x and that x changes about $1/f'(a)$ times as fast as y.

Exercises 6.1

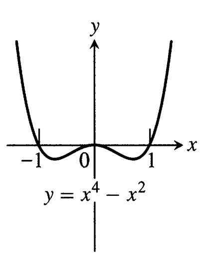
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

1.



2.



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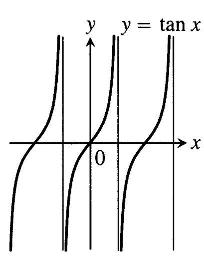

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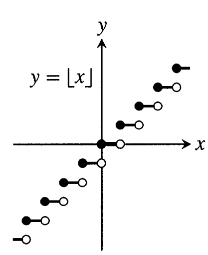

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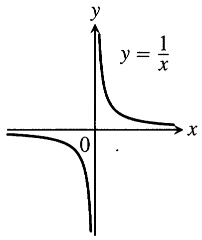

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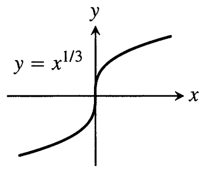
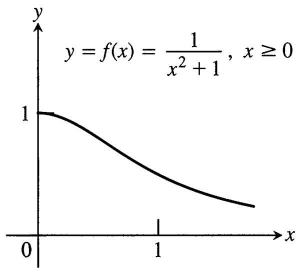


Graphing Inverse Functions

Each of Exercises 7–10 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

7.



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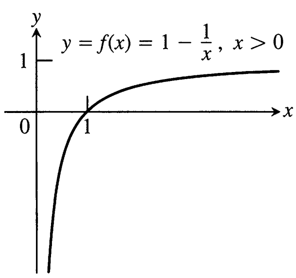

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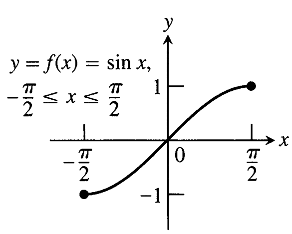
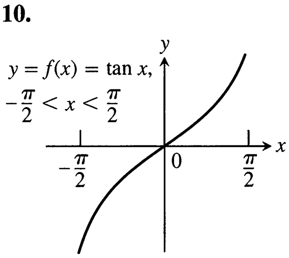


10.



11. a) Graph the function $f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1$. What symmetry does the graph have?

b) Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

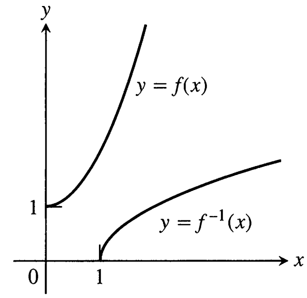
12. a) Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b) Show that f is its own inverse.

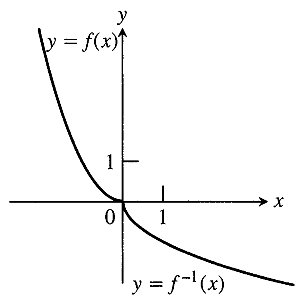
Formulas for Inverse Functions

Each of Exercises 13–18 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

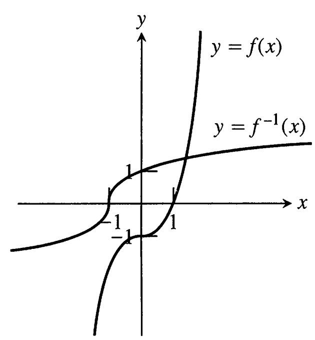
13. $f(x) = x^2 + 1, \quad x \geq 0$



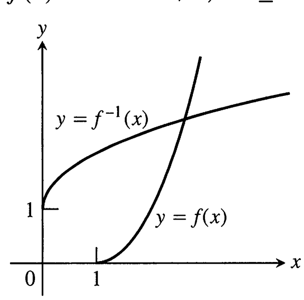
14. $f(x) = x^2, \quad x \leq 0$



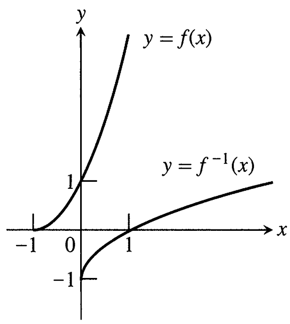
15. $f(x) = x^3 - 1$



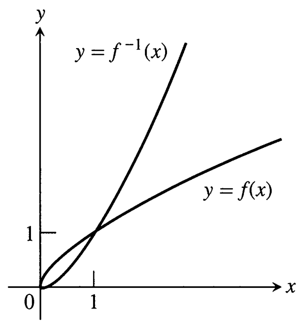
16. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



17. $f(x) = (x+1)^2, \quad x \geq -1$



18. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

19. $f(x) = x^5$

20. $f(x) = x^4, \quad x \geq 0$

21. $f(x) = x^3 + 1$

22. $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$

24. $f(x) = 1/x^3, \quad x \neq 0$

Derivatives of Inverse Functions

In Exercises 25–28:

a) Find $f^{-1}(x)$.
b) Graph f and f^{-1} together.
c) Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$

26. $f(x) = (1/5)x + 7, \quad a = -1$

27. $f(x) = 5 - 4x, \quad a = 1/2$

28. $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

29. a) Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
b) Graph f and g over an x-interval large enough to show the graphs intersecting at (1, 1) and (−1, −1). Be sure the picture shows the required symmetry in the line $y = x$.
c) Find the slopes of the tangents to the graphs of f and g at (1, 1) and (−1, −1) (four tangents in all).
d) What lines are tangent to the curves at the origin?

30. a) Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
b) Graph h and k over an x-interval large enough to show the graphs intersecting at (2, 2) and (−2, −2). Be sure the picture shows the required symmetry about the line $y = x$.
c) Find the slopes of the tangents to the graphs at h and k at (2, 2) and (−2, −2).
d) What lines are tangent to the curves at the origin?

31. Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

32. Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point (2, 4) and has a slope of 1/3 there. Find the value of df^{-1}/dx at $x = 4$.

34. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

35. a) Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.
b) What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

36. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

37. a) Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.
b) Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?
c) What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

38. a) Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?
b) Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?
c) What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Increasing and Decreasing Functions

39. *Increasing functions and decreasing functions.* As in Section 3.2, a function $f(x)$ increases on an interval I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1).$$

Similarly, a function decreases on I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1).$$

Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 39 to show that the functions in Exercises 40–44 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 1.

40. $f(x) = (1/3)x + (5/6)$

41. $f(x) = 27x^3$

42. $f(x) = 1 - 8x^3$

43. $f(x) = (1-x)^3$

44. $f(x) = x^{5/3.}$

Theory and Applications

45. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Give reasons for your answer.

46. If $f(x)$ is one-to-one and $f(x)$ is never 0, can anything be said about $h(x) = 1/f(x)$? Give reasons for your answer.

47. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

48. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

49. Suppose $f(x)$ is positive, continuous, and increasing over the interval $[a, b]$. By interpreting the graph of f show that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

50. Determine conditions on the constants a, b, c, and d so that the rational function

$$f(x) = \frac{ax+b}{cx+d}$$

has an inverse.

51. *Still another way to view Theorem 1.* If we write $g(x)$ for $f^{-1}(x)$, Eq. (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 1 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

52. *Equivalence of the washer and shell methods for finding volume.* Let f be differentiable on the interval $a \le x \le b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}. Revolve about the y-axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi\Big((f^{-1}(y))^2 - a^2\Big)dy = \int_a^b 2\pi x(f(b) - f(x))\,dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi\Big((f^{-1}(y))^2 - a^2\Big)\,dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x))\,dx.$$

Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.2, Exercise 56, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (Source: "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly*, Vol. 98, No. 2, February 1991, pp. 154–156.)

CAS Explorations and Projects

In Exercises 53–60, you will explore some functions and their inverses together with their derivatives and linear approximating functions at specified points. Perform the following steps using your CAS:

a) Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.
b) Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.
c) Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.
d) Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 1 to find the slope of this tangent line.
e) Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

53. $y = \sqrt{3x-2}, \quad \frac{2}{3} \le x \le 4, \quad x_0 = 3$

54. $y = \dfrac{3x+2}{2x-11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

55. $y = \dfrac{4x}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

56. $y = \dfrac{x^3}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

57. $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \dfrac{27}{10}$

58. $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \dfrac{3}{2}$

59. $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

60. $y = \sin x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \quad x_0 = 1$

In Exercises 61 and 62, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

61. $y^{1/3} - 1 = (x+2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

62. $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

6.2

Natural Logarithms

The most important function–inverse pair in mathematics and science is the pair consisting of the natural logarithm function $\ln x$ and the exponential function e^x. The key to understanding e^x is $\ln x$, so we introduce $\ln x$ first. The importance of logarithms came at first from the improvement they brought to arithmetic. The revolutionary properties of logarithms made possible the calculations of the great seventeenth-century advances in offshore navigation and celestial mechanics. Nowadays we do complicated arithmetic with calculators, but the properties of logarithms remain as important as ever.

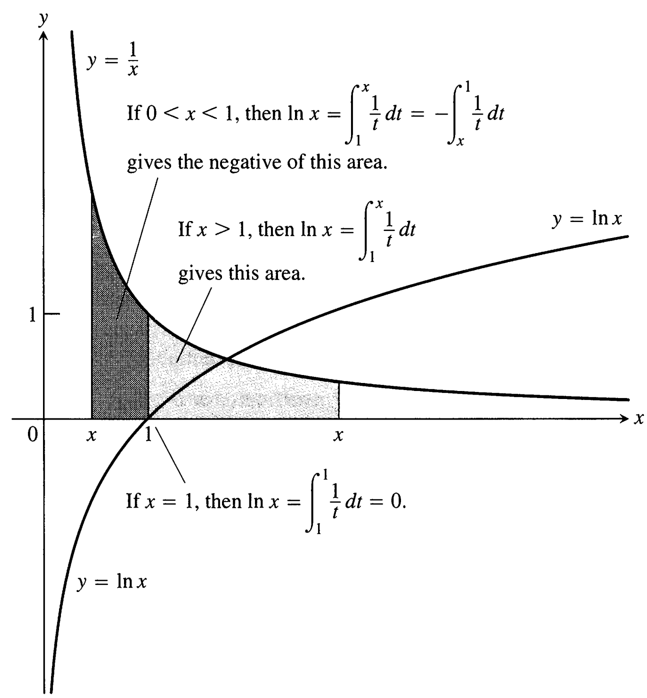
The Natural Logarithm Function

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

Definition
The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Fig. 6.10). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from



6.10 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

x to 1. The function is not defined for $x \leq 0$. We also have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0. \qquad \text{Upper and lower limits equal}$$

Notice that we show the graph of $y = 1/x$ in Fig. 6.10 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (in Section 4.6),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, therefore,

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}.}$$

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)}$$

EXAMPLE 1

$$\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$$ ❑

Notice the remarkable occurrence in Example 1. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any number a:

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \qquad (2)$$

EXAMPLE 2 Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$ ❑

In the late 1500s, a Scottish baron, John Napier, invented a device called the *logarithm* that simplified arithmetic by replacing multiplication by addition. The equation that accomplished this was

$$\ln ax = \ln a + \ln x.$$

To multiply two positive numbers a and x, you looked up their logarithms in a table, added the logarithms, found the sum in the body of the table, and read the table backward to find the product ax.

Having the table was the key, of course, and Napier spent the last 20 years of his life working on a table he never finished (while the astronomer Tycho Brahe waited in vain for the information he needed to speed his calculations). The table was completed after Napier's death (and Brahe's) by Napier's friend Henry Briggs in London. Base 10 logarithms subsequently became known as Briggs's logarithms (what else?) and some books on navigation still refer to them this way.

Napier also invented an artillery piece that could hit a cow a mile away. Horrified by the weapon's accuracy, he stopped production and suppressed the cannon's design.

Properties of Logarithms

The properties that made logarithms the single most important improvement in arithmetic before the advent of modern computers are listed in Table 6.1. The properties made it possible to replace multiplication of positive numbers by addition, and division of positive numbers by subtraction. They also made it possible to replace exponentiation by multiplication. For the moment, we add the restriction that the exponent n in Rule 4 be a rational number. You will see why when we prove the rule.

EXAMPLE 3

a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ Product

b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ Quotient

c) $\ln\frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3\ln 2$ Power ❑

EXAMPLE 4

a) $\ln 4 + \ln\sin x = \ln(4\sin x)$ Product

b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

c) $\ln\sec x = \ln\frac{1}{\cos x} = -\ln\cos x$ Reciprocal

d) $\ln\sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$ Power ❑

***Proof that* $\ln ax = \ln a + \ln x$** The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Eq. 2). According to Corollary 1 of the Mean Value Theorem, then, the functions must differ by a

Table 6.1 Properties of natural logarithms

For any numbers $a > 0$ and $x > 0$,

1.	Product Rule:	$\ln ax = \ln a + \ln x$	
2.	Quotient Rule:	$\ln\frac{a}{x} = \ln a - \ln x$	
3.	Reciprocal Rule:	$\ln\frac{1}{x} = -\ln x$	Rule 2 with $a = 1$
4.	Power Rule:	$\ln x^n = n\ln x$	

constant, which means that

$$\ln ax = \ln x + C \tag{3}$$

for some C. With this much accomplished, it remains only to show that C equals $\ln a$.

Equation (3) holds for all positive values of x, so it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(a \cdot 1) &= \ln 1 + C \\ \ln a &= 0 + C && \ln 1 = 0 \\ C &= \ln a. && \text{Rearranged} \end{aligned}$$

Substituting $C = \ln a$ in Eq. (3) gives the equation we wanted to prove:

$$\ln ax = \ln a + \ln x. \tag{4}$$

❑

***Proof that* ln (a/x) = ln a − ln x** We get this from Eq. (4) in two stages. Equation (4) with a replaced by $1/x$ gives

$$\begin{aligned} \ln\frac{1}{x} + \ln x &= \ln\left(\frac{1}{x} \cdot x\right) \\ &= \ln 1 = 0, \end{aligned}$$

so that

$$\ln\frac{1}{x} = -\ln x.$$

Equation (4) with x replaced by $1/x$ then gives

$$\begin{aligned} \ln\frac{a}{x} &= \ln\left(a \cdot \frac{1}{x}\right) = \ln a + \ln\frac{1}{x} \\ &= \ln a - \ln x. \end{aligned}$$

❑

***Proof that* ln x^n = n ln x (assuming n rational)** We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned} \frac{d}{dx}\ln x^n &= \frac{1}{x^n}\frac{d}{dx}(x^n) && \text{Eq. (1) with } u = x^n \\ &= \frac{1}{x^n} n x^{n-1} && \text{Here is where we need } n \text{ to be rational, at least for now. We have proved the Power Rule only for rational exponents.} \\ &= n \cdot \frac{1}{x} = \frac{d}{dx}(n \ln x). \end{aligned}$$

Since $\ln x^n$ and $n \ln x$ have the same derivative,

$$\ln x^n = n \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done. ❑

As for using the rule $\ln x^n = n \ln x$ for irrational values of n, go right ahead and do so. It does hold for all n, and there is no need to pretend otherwise. From the point of view of mathematical development, however, we want you to be aware that the rule is far from proved.

The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

We can estimate $\ln 2$ by numerical integration to be about 0.69. We therefore know that

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x\to\infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

The domain of $\ln x$ is the set of positive real numbers; the range is the entire real line.

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the rules in Table 6.1 to simplify the formulas before differentiating. The process, called **logarithmic differentiation,** is illustrated in the next example.

EXAMPLE 5 Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1.$

Solution We take the natural logarithm of both sides and simplify the result with the rules in Table 6.1:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) && \text{Quotient Rule} \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Product Rule} \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Power Rule}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Eq. (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right). \qquad \square$$

How to Differentiate $y = f(x) > 0$ by Logarithmic Differentiation

1.	$\ln y = \ln f(x)$	Take logs of both sides.
2.	$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}(\ln f(x))$	Differentiate both sides . . .
3.	$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln f(x))$	. . . using Eq. (1) on the left.
4.	$\dfrac{dy}{dx} = y\dfrac{d}{dx}(\ln f(x))$	Solve for dy/dx.
5.	$\dfrac{dy}{dx} = f(x)\dfrac{d}{dx}(\ln f(x))$	Substitute $y = f(x)$.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{5}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\begin{aligned} \int \frac{1}{u}\, du &= \int \frac{1}{(-u)}\, d(-u) \\ &= \ln(-u) + C. \qquad \text{Eq. (5) with } u \text{ replaced by } -u \end{aligned} \tag{6}$$

We can combine Eqs. (5) and (6) into a single formula by noticing that in each case the expression on the right is $\ln|u| + C$. In Eq. (5), $\ln u = \ln|u|$ because $u > 0$; in Eq. (6), $\ln(-u) = \ln|u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\,du$ is $\ln|u| + C$.

> If u is a nonzero differentiable function,
>
> $$\int \frac{1}{u}\, du = \ln|u| + C. \tag{7}$$

We know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Equation (7) explains what to do when n equals -1.

Equation (7) says that integrals of a certain *form* lead to logarithms. That is,

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 6

$$\int_0^2 \frac{2x}{x^2-5}\, dx = \int_{-5}^{-1} \frac{du}{u} = \ln |u| \Big]_{-5}^{-1} \qquad \begin{array}{l} u = x^2 - 5, \quad du = 2x\, dx, \\ u(0) = -5, \quad u(2) = -1 \end{array}$$

$$= \ln |-1| - \ln |-5| = \ln 1 - \ln 5 = -\ln 5$$ ❑

EXAMPLE 7

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta} d\theta = \int_1^5 \frac{2}{u}\, du \qquad \begin{array}{l} u = 3 + 2\sin\theta, \quad du = 2\cos\theta\, d\theta, \\ u(-\pi/2) = 1, \quad u(\pi/2) = 5 \end{array}$$

$$= 2 \ln |u| \Big]_1^5$$

$$= 2 \ln |5| - 2 \ln |1| = 2 \ln 5$$ ❑

The Integrals of tan x and cot x

Equation (7) tells us at last how to integrate the tangent and cotangent functions. For the tangent,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u} \qquad \begin{array}{l} u = \cos x, \\ du = -\sin x\, dx \end{array}$$

$$= -\int \frac{du}{u} = -\ln |u| + C \qquad \text{Eq. (7)}$$

$$= -\ln |\cos x| + C = \ln \frac{1}{|\cos x|} + C \qquad \text{Reciprocal Rule}$$

$$= \ln |\sec x| + C.$$

For the cotangent,

$$\int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x} = \int \frac{du}{u} \qquad \begin{array}{l} u = \sin x, \\ du = \cos x\, dx \end{array}$$

$$= \ln |u| + C = \ln |\sin x| + C = -\ln |\csc x| + C.$$

$$\boxed{\begin{array}{l} \displaystyle\int \tan u\, du = -\ln |\cos u| + C = \ln |\sec u| + C \\ \displaystyle\int \cot u\, du = \ln |\sin u| + C = -\ln |\csc x| + C \end{array}}$$

EXAMPLE 8

$$\int_0^{\pi/6} \tan 2x\,dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\,du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

Exercises 6.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.

a) $\ln 0.75$ **b)** $\ln (4/9)$ **c)** $\ln (1/2)$

d) $\ln \sqrt[3]{9}$ **e)** $\ln 3\sqrt{2}$ **f)** $\ln \sqrt{13.5}$

2. Express the following logarithms in terms of ln 5 and ln 7.

a) $\ln (1/125)$ **b)** $\ln 9.8$ **c)** $\ln 7\sqrt{7}$

d) $\ln 1225$ **e)** $\ln 0.056$

f) $(\ln 35 + \ln (1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a) $\ln \sin\theta - \ln\left(\dfrac{\sin\theta}{5}\right)$

b) $\ln (3x^2 - 9x) + \ln\left(\dfrac{1}{3x}\right)$

c) $\dfrac{1}{2}\ln (4t^4) - \ln 2$

4. a) $\ln\sec\theta + \ln\cos\theta$

b) $\ln (8x + 4) - 2\ln 2$

c) $3\ln \sqrt[3]{t^2 - 1} - \ln (t + 1)$

Derivatives of Logarithms

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$

6. $y = \ln kx$, k constant

7. $y = \ln (t^2)$

8. $y = \ln (t^{3/2})$

9. $y = \ln \dfrac{3}{x}$

10. $y = \ln \dfrac{10}{x}$

11. $y = \ln (\theta + 1)$

12. $y = \ln (2\theta + 2)$

13. $y = \ln x^3$

14. $y = (\ln x)^3$

15. $y = t(\ln t)^2$

16. $y = t\sqrt{\ln t}$

17. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$

18. $y = \dfrac{x^3}{3}\ln x - \dfrac{x^3}{9}$

19. $y = \dfrac{\ln t}{t}$

20. $y = \dfrac{1 + \ln t}{t}$

21. $y = \dfrac{\ln x}{1 + \ln x}$

22. $y = \dfrac{x\ln x}{1 + \ln x}$

23. $y = \ln (\ln x)$

24. $y = \ln (\ln (\ln x))$

25. $y = \theta(\sin (\ln\theta) + \cos (\ln\theta))$

26. $y = \ln (\sec\theta + \tan\theta)$

27. $y = \ln \dfrac{1}{x\sqrt{x + 1}}$

28. $y = \dfrac{1}{2}\ln \dfrac{1 + x}{1 - x}$

29. $y = \dfrac{1 + \ln t}{1 - \ln t}$

30. $y = \sqrt{\ln \sqrt{t}}$

31. $y = \ln (\sec(\ln\theta))$

32. $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

33. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$

34. $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

35. $y = \displaystyle\int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt$

36. $y = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

Logarithmic Differentiation

In Exercises 37–50, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

37. $y = \sqrt{x(x + 1)}$

38. $y = \sqrt{(x^2 + 1)(x - 1)^2}$

39. $y = \sqrt{\dfrac{t}{t + 1}}$

40. $y = \sqrt{\dfrac{1}{t(t + 1)}}$

41. $y = \sqrt{\theta + 3}\sin\theta$

42. $y = (\tan\theta)\sqrt{2\theta + 1}$

43. $y = t(t + 1)(t + 2)$

44. $y = \dfrac{1}{t(t + 1)(t + 2)}$

45. $y = \dfrac{\theta + 5}{\theta\cos\theta}$

46. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

47. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$

48. $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$

49. $y = \sqrt[3]{\dfrac{x(x - 2)}{x^2 + 1}}$

50. $y = \sqrt[3]{\dfrac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}}$

Integration

Evaluate the integrals in Exercises 51–68.

51. $\int_{-3}^{-2} \frac{dx}{x}$

52. $\int_{-1}^{0} \frac{3\,dx}{3x-2}$

53. $\int \frac{2y\,dy}{y^2-25}$

54. $\int \frac{8r\,dr}{4r^2-5}$

55. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$

56. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

57. $\int_1^2 \frac{2\ln x}{x}\,dx$

58. $\int_2^4 \frac{dx}{x\ln x}$

59. $\int_2^4 \frac{dx}{x(\ln x)^2}$

60. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

61. $\int \frac{3\sec^2 t}{6+3\tan t}\,dt$

62. $\int \frac{\sec y\tan y}{2+\sec y}\,dy$

63. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

64. $\int_{\pi/4}^{\pi/2} \cot t\,dt$

65. $\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$

66. $\int_0^{\pi/12} 6\tan 3x\,dx$

67. $\int \frac{dx}{2\sqrt{x}+2x}$

68. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a) $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,
b) $\cos(\ln x)$ on $[1/2, 2]$.

70. a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.
b) Using part (a), show that $\ln x < x$ if $x > 1$.

71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

72. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

73. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

74. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

75. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

76. In Section 5.4, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3+9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 5.4, Exercise 6, for a graph.)

77. Find the lengths of the following curves.

a) $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$
b) $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

78. Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

79. CALCULATOR

a) Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to 2 decimal places.
b) Sketch the region and show the centroid in your sketch.

80. a) Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.
b) Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

Solve the initial value problems in Exercises 81 and 82.

81. $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad y(1) = 3$

82. $\frac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

83. *The linearization of* $\ln(1+x)$ *at* $x = 0$. Instead of approximating x near $x = 1$, we approximate $\ln(1+x)$ near $x = 0$. We get a simpler formula this way.

a) Derive the linearization $\ln(1+x) \approx x$ at $x = 0$.
b) CALCULATOR Estimate to 5 decimal places the error involved in replacing $\ln(1+x)$ by x on the interval [0, 0.1].
c) GRAPHER Graph $\ln(1+x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1+x)$ seem best? least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

84. *Estimating values of* $\ln x$ *with Simpson's rule.* Although linearizations are good for replacing the logarithmic function over short intervals, Simpson's rule is better for estimating *particular* values of $\ln x$.

As a case in point, the values of $\ln(1.2)$ and $\ln(0.8)$ to 5 places are

$$\ln(1.2) = 0.18232, \quad \ln(0.8) = -0.22314.$$

Estimate $\ln(1.2)$ and $\ln(0.8)$ first with the formula $\ln(1+x) \approx x$ and then use Simpson's rule with $n = 2$. (Impressive, isn't it?)

85. Find

$$\lim_{x\to\infty} \frac{\ln(x^2)}{\ln x}.$$

Generalize this result.

86. *The derivative of* $\ln kx$. Could $y = \ln 2x$ and $y = \ln 3x$ possibly have the same derivative at each point? (Differentiate them to find out.) What about $y = \ln kx$, for other positive values of the constant k? Give reasons for your answer.

Grapher Explorations

87. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

88. Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22$, $-2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

89. **a)** Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2, 4, 8, 20$, and 50 together for $0 \le x \le 23$.

b) Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

90. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question (a) by graphing, (b) by using calculus.

6.3 The Exponential Function

Whenever we have a quantity y whose rate of change over time is proportional to the amount of y present, we have a function that satisfies the differential equation

$$\frac{dy}{dt} = ky.$$

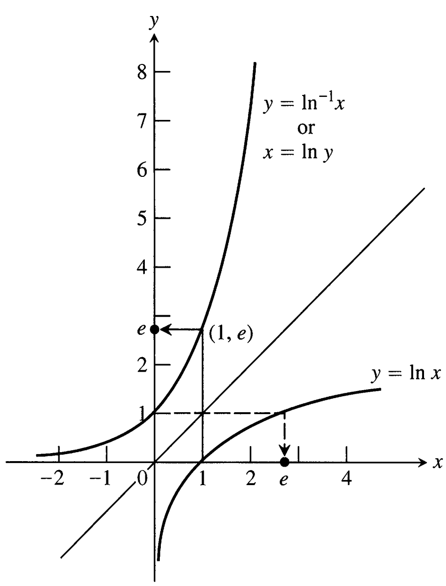
If, in addition, $y = y_0$ when $t = 0$, the function is the exponential function $y = y_0 e^{kt}$. This section defines the exponential function (it is the inverse of $\ln x$) and explores the properties that account for the amazing frequency with which the function appears in mathematics and its applications. We will look at some of these applications in Section 6.5.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The number $\ln^{-1} 1$ is denoted by the letter e (Fig. 6.11).



6.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x$. The number e is $\ln^{-1} 1$.

Definition

$$e = \ln^{-1} 1$$

Although e is not a rational number, we will see in Chapter 8 that it is possible to find its value with a computer to as many places as we want with the formula

$$e = \lim_{n\to\infty} \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!}\right).$$

To 15 places,

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45.$$

The Function $y = e^x$

We can raise the number e to a rational power x in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^x is positive too. This means that e^x has a logarithm. When we take the logarithm we find that

$$\ln e^x = x \ln e = x \cdot 1 = x. \tag{1}$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} x) = x$, Eq. (1) tells us that

$$e^x = \ln^{-1} x \qquad \text{for } x \text{ rational.} \tag{2}$$

Equation (2) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we can use it to assign a value to e^x at every point where e^x had no previous value.

Typical Values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Definition

For every real number x, $e^x = \ln^{-1} x$.

Equations Involving $\ln x$ and e^x

Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{3}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{4}$$

You might want to do parts of the next example on your calculator.

EXAMPLE 1

a) $\ln e^2 = 2$

b) $\ln e^{-1} = -1$

c) $\ln \sqrt{e} = \dfrac{1}{2}$

d) $\ln e^{\sin x} = \sin x$

e) $e^{\ln 2} = 2$

f) $e^{\ln(x^2+1)} = x^2 + 1$

g) $e^{3 \ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

h) $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ❑

Useful Operating Rules

1. To remove logarithms from an equation, exponentiate both sides.
2. To remove exponentials, take the logarithm of both sides.

EXAMPLE 2 Find y if $\ln y = 3t + 5$.

Solution Exponentiate both sides:

$$e^{\ln y} = e^{3t+5}$$
$$y = e^{3t+5}. \qquad \text{Eq. (3)}$$

❑

EXAMPLE 3 Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$\begin{aligned} e^{2k} &= 10 \\ \ln e^{2k} &= \ln 10 \\ 2k &= \ln 10 && \text{Eq. (4)} \\ k &= \frac{1}{2}\ln 10. \end{aligned}$$

❑

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra (Table 6.2).

Table 6.2 Laws of exponents for e^x

For all numbers x, x_1, and x_2,

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \qquad \text{and} \qquad y_2 = e^{x_2}. \tag{5}$$

Then

$$\begin{aligned} x_1 = \ln y_1 \quad \text{and} \quad x_2 &= \ln y_2 && \text{Take logs of both sides of Eqs. (5).} \\ x_1 + x_2 &= \ln y_1 + \ln y_2 \\ &= \ln y_1 y_2 && \text{Product Rule} \\ e^{x_1+x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.} \\ &= y_1 y_2 && e^{\ln u} = u \\ &= e^{x_1} e^{x_2}. \end{aligned}$$

❑

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4

a) $e^{x+\ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4

❑

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. Starting with $y = e^x$, we have, in order,

$$y = e^x$$

$$\ln y = x \qquad \text{Logarithms of both sides}$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \qquad \text{Derivatives of both sides with respect to } x$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = e^x. \qquad y \text{ replaced by } e^x$$

The startling conclusion we draw from this sequence of equations is that e^x is its own derivative.

As we will see in Section 6.5, the only functions that behave this way are constant multiples of e^x.

$$\frac{d}{dx}e^x = e^x \tag{6}$$

Transcendental numbers and transcendental functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic:** -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental,** a term coined by Euler to describe numbers, like e and π, that appeared to "transcend the power of algebraic methods." But it was not until a hundred years after Euler's death (1873) that Charles Hermite proved the transcendence of e in the sense that we describe. A few years later (1882), C. L. F. Lindemann proved the transcendence of π.

Today we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Polynomials and rational functions with rational coefficients are algebraic, as are all sums, products, quotients, rational powers, and rational roots of algebraic functions.

Functions that are not algebraic are called transcendental. The six basic trigonometric functions are transcendental, as are the inverses of the trigonometric functions and the exponential and logarithmic functions that are the main subject of the present chapter.

EXAMPLE 5

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x$$
$$= 5e^x$$

❑

The Chain Rule extends Eq. (6) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \tag{7}$$

EXAMPLE 6

a) $\dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (7) with $u = -x$

b) $\dfrac{d}{dx}e^{\sin x} = e^{\sin x}\dfrac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (7) with $u = \sin x$

❑

The integral equivalent of Eq. (7) is

$$\boxed{\int e^u \, du = e^u + C.}$$

EXAMPLE 7

$$\begin{aligned}
\int_0^{\ln 2} e^{3x} dx &= \int_0^{\ln 8} e^u \cdot \frac{1}{3} du && u = 3x,\ \frac{1}{3} du = dx,\ u(0) = 0, \\
& && u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8 \\
&= \frac{1}{3} \int_0^{\ln 8} e^u du \\
&= \frac{1}{3} e^u \Big]_0^{\ln 8} \\
&= \frac{1}{3}[8 - 1] = \frac{7}{3}
\end{aligned}$$

❑

EXAMPLE 8

$$\begin{aligned}
\int_0^{\pi/2} e^{\sin x} \cos x \, dx &= e^{\sin x} \Big]_0^{\pi/2} && \text{Antiderivative from Example 6} \\
&= e^1 - e^0 = e - 1
\end{aligned}$$

❑

EXAMPLE 9 *Solving an initial value problem*

Solve the initial value problem

$$e^y \frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^y = x^2 + C.$$

We use the initial condition to determine C:

$$\begin{aligned}
C &= e^0 - (2)^2 \\
&= 1 - 4 = -3.
\end{aligned}$$

This completes the formula for e^y:

$$e^y = x^2 - 3. \tag{8}$$

To find y, we take logarithms of both sides:

$$\begin{aligned}
\ln e^y &= \ln (x^2 - 3) \\
y &= \ln (x^2 - 3).
\end{aligned} \tag{9}$$

Notice that the solution is valid for $x > \sqrt{3}$.

It is always a good idea to check a solution in the original equation. From Eqs. (8) and (9), we have

$$
\begin{aligned}
e^y \frac{dy}{dx} &= e^y \frac{d}{dx} \ln (x^2 - 3) && \text{Eq. (9)} \\
&= e^y \frac{2x}{x^2 - 3} \\
&= (x^2 - 3) \frac{2x}{x^2 - 3} && \text{Eq. (8)} \\
&= 2x.
\end{aligned}
$$

The solution checks. ❑

Exercises 6.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercises 1–4.

1. a) $e^{\ln 7.2}$ **b)** $e^{-\ln x^2}$ **c)** $e^{\ln x - \ln y}$

2. a) $e^{\ln (x^2+y^2)}$ **b)** $e^{-\ln 0.3}$ **c)** $e^{\ln \pi x - \ln 2}$

3. a) $2 \ln \sqrt{e}$ **b)** $\ln (\ln e^e)$ **c)** $\ln (e^{-x^2-y^2})$

4. a) $\ln (e^{\sec \theta})$ **b)** $\ln (e^{(e^x)})$ **c)** $\ln (e^{2 \ln x})$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 5–10, solve for y in terms of t or x, as appropriate.

5. $\ln y = 2t + 4$

6. $\ln y = -t + 5$

7. $\ln (y - 40) = 5t$

8. $\ln (1 - 2y) = t$

9. $\ln (y - 1) - \ln 2 = x + \ln x$

10. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 11 and 12, solve for k.

11. a) $e^{2k} = 4$ **b)** $100e^{10k} = 200$ **c)** $e^{k/1000} = a$

12. a) $e^{5k} = \frac{1}{4}$ **b)** $80e^k = 1$ **c)** $e^{(\ln 0.8)k} = 0.8$

In Exercises 13–16, solve for t.

13. a) $e^{-0.3t} = 27$ **b)** $e^{kt} = \frac{1}{2}$ **c)** $e^{(\ln 0.2)t} = 0.4$

14. a) $e^{-0.01t} = 1000$ **b)** $e^{kt} = \frac{1}{10}$ **c)** $e^{(\ln 2)t} = \frac{1}{2}$

15. $e^{\sqrt{t}} = x^2$

16. $e^{(x^2)} e^{(2x+1)} = e^t$

Derivatives

In Exercises 17–36, find the derivative of y with respect to x, t, or θ, as appropriate.

17. $y = e^{-5x}$

18. $y = e^{2x/3}$

19. $y = e^{5-7x}$

20. $y = e^{(4\sqrt{x}+x^2)}$

21. $y = xe^x - e^x$

22. $y = (1 + 2x)e^{-2x}$

23. $y = (x^2 - 2x + 2)e^x$

24. $y = (9x^2 - 6x + 2)e^{3x}$

25. $y = e^\theta (\sin \theta + \cos \theta)$

26. $y = \ln (3\theta e^{-\theta})$

27. $y = \cos \left(e^{-\theta^2}\right)$

28. $y = \theta^3 e^{-2\theta} \cos 5\theta$

29. $y = \ln (3te^{-t})$

30. $y = \ln (2e^{-t} \sin t)$

31. $y = \ln \left(\frac{e^\theta}{1 + e^\theta}\right)$

32. $y = \ln \left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

33. $y = e^{(\cos t + \ln t)}$

34. $y = e^{\sin t}(\ln t^2 + 1)$

35. $y = \int_0^{\ln x} \sin e^t \, dt$

36. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

In Exercises 37–40, find dy/dx.

37. $\ln y = e^y \sin x$

38. $\ln xy = e^{x+y}$

39. $e^{2x} = \sin (x + 3y)$

40. $\tan y = e^x + \ln x$

Integrals

Evaluate the integrals in Exercises 41–62.

41. $\int (e^{3x} + 5e^{-x}) \, dx$

42. $\int (2e^x - 3e^{-2x}) \, dx$

43. $\int_{\ln 2}^{\ln 3} e^x \, dx$

44. $\int_{-\ln 2}^{0} e^{-x} \, dx$

45. $\int 8e^{(x+1)}\,dx$

46. $\int 2e^{(2x-1)}\,dx$

47. $\int_{\ln 4}^{\ln 9} e^{x/2}\,dx$

48. $\int_0^{\ln 16} e^{x/4}\,dx$

49. $\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr$

50. $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

51. $\int 2t\,e^{-t^2}\,dt$

52. $\int t^3 e^{(t^4)}\,dt$

53. $\int \frac{e^{1/x}}{x^2}\,dx$

54. $\int \frac{e^{-1/x^2}}{x^3}\,dx$

55. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$

56. $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$

57. $\int e^{\sec \pi t}\sec \pi t\tan \pi t\,dt$

58. $\int e^{\csc(\pi+t)}\csc(\pi+t)\cot(\pi+t)\,dt$

59. $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v\cos e^v\,dv$

60. $\int_0^{\sqrt{\ln \pi}} 2x\,e^{x^2}\cos(e^{x^2})\,dx$

61. $\int \frac{e^r}{1+e^r}\,dr$

62. $\int \frac{dx}{1+e^x}$

Initial Value Problems

Solve the initial value problems in Exercises 63–66.

63. $\frac{dy}{dt} = e^t\sin(e^t-2), \quad y(\ln 2) = 0$

64. $\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t}), \quad y(\ln 4) = 2/\pi$

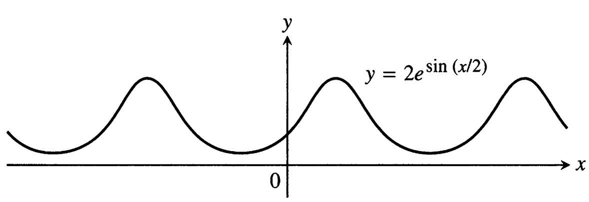
65. $\frac{d^2y}{dx^2} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$

66. $\frac{d^2y}{dt^2} = 1 - e^{2t}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

68. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



69. Find the absolute maximum value of $f(x) = x^2\ln(1/x)$ and say where it is assumed.

70. **GRAPHER** Graph $f(x) = (x-3)^2e^x$ and its first derivative together. Comment on the behavior of f in relation to the signs and values of f'. Identify significant points on the graphs with calculus, as necessary.

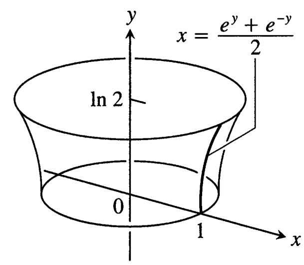
71. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

72. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

73. Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\,dx.$$

74. Find the area of the surface generated by revolving the curve $x = (e^y + e^{-y})/2$, $0 \le y \le \ln 2$, about the y-axis.



75. a) Show that $\int \ln x\,dx = x\ln x - x + C$.
b) Find the average value of $\ln x$ over $[1, e]$.

76. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

77. *The linearization of e^x at $x = 0$*
a) Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.
b) **CALCULATOR** Estimate to 5 decimal places the magnitude of the error involved in replacing e^x by $1 + x$ on the interval $[0, 0.2]$.
c) **GRAPHER** Graph e^x and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x? underestimate e^x?

78. *Laws of Exponents.*
a) Starting with the equation $e^{x_1}e^{x_2} = e^{x_1+x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1-x_2}$ for any numbers x_1 and x_2.
b) Show that $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

79. *A decimal representation of e.* Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

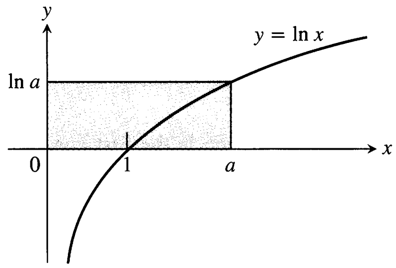
80. *The inverse relation between e^x and $\ln x$.* Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln(e^x).$$

81. Show that for any number $a > 1$

$$\int_1^a \ln x\,dx + \int_0^{\ln a} e^y dy = a \ln a.$$

(See accompanying figure.)



82. *The geometric, logarithmic, and arithmetic mean inequality*

a) Show that the graph of e^x is concave up over every interval of x-values.

b) Show, by reference to the accompanying figure, that if $0 < a < b$ then

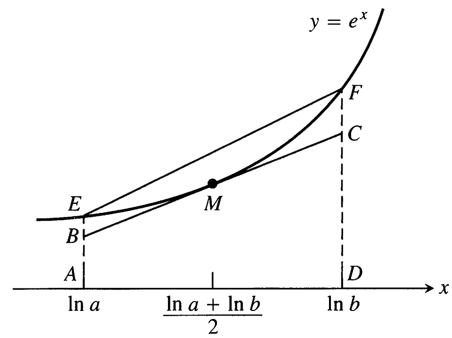
$$e^{(\ln a+\ln b)/2}\cdot(\ln b-\ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a}+e^{\ln b}}{2}\cdot(\ln b-\ln a).$$

c) Use the inequality in (b) to conclude that

$$\sqrt{ab} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

(For more about this inequality, see "The Geometric, Logarithmic, and Arithmetic Mean Inequality" by Frank Burk, *American Mathematical Monthly,* Vol. 94, No. 6, June–July 1987, pp. 527–528.)



NOT TO SCALE

6.4 a^x and $\log_a x$

While we have not yet devised a way to raise positive numbers to any but rational powers, we have an exception in the number e. The definition $e^x = \ln^{-1} x$ defines e^x for every real value of x, irrational as well as rational. In this section, we show how this enables us to raise any other positive number to an arbitrary power and thus to define an exponential function $y = a^x$ for any positive number a. We also prove the Power Rule for differentiation in its final form (good for all exponents) and define functions like x^x and $(\sin x)^{\tan x}$ that involve raising the values of one function to powers given by another.

Just as e^x is but one of many exponential functions, $\ln x$ is one of many logarithmic functions, the others being the inverses of the function a^x. These logarithmic functions have important applications in science and engineering.

The Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

Definition

For any numbers $a > 0$ and x,

$$a^x = e^{x \ln a}. \tag{1}$$

Table 6.3 Laws of exponents

For $a > 0$, and any x and y:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy} = (a^y)^x$

EXAMPLE 1

a) $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2}$

b) $2^{\pi} = e^{\pi \ln 2}$ ❑

The function a^x obeys the usual laws of exponents (Table 6.3). We omit the proofs.

The Power Rule (Final Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n \ln x}$. Therefore, the n in the equation $\ln x^n = n \ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\begin{aligned} \ln x^n = \ln (e^{n \ln x}) &= n \ln x \cdot \ln e && \ln e^u = u, \text{ any } u \\ &= n \ln x. \end{aligned}$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n \ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} && \text{Definition of } x^n,\ x > 0 \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u \\ &= x^n \cdot \frac{n}{x} && \text{The definition again} \\ &= n x^{n-1}. && \text{Table 6.3, Law 3} \end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's final form.

Power Rule (Final Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}.$$

EXAMPLE 2

a) $\dfrac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1} \qquad (x > 0)$

b) $\dfrac{d}{dx} (\sin x)^{\pi} = \pi (\sin x)^{\pi - 1} \cos x \qquad (\sin x > 0)$ ❑

The Derivative of a^x

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \text{Chain Rule}$$
$$= a^x \ln a.$$

If $a > 0$, then

$$\frac{d}{dx}a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
> $$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \tag{2}$$

Equation (2) shows why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and Eq. (2) simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

EXAMPLE 3

a) $\dfrac{d}{dx}3^x = 3^x \ln 3$

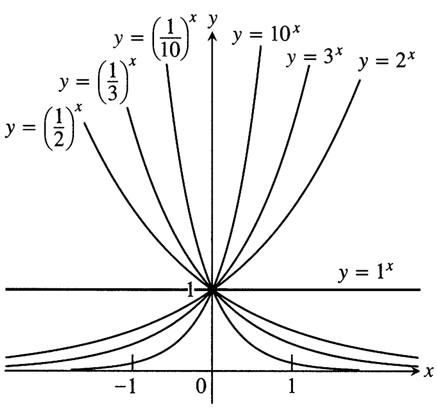
b) $\dfrac{d}{dx}3^{-x} = 3^{-x} \ln 3 \dfrac{d}{dx}(-x) = -3^{-x} \ln 3$

c) $\dfrac{d}{dx}3^{\sin x} = 3^{\sin x} \ln 3 \dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$ ❑

From Eq. (2), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Fig. 6.12).



6.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 4 Find dy/dx if $y = x^x$, $x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad \text{Eq. (1) with } a = x$$

Then differentiate as usual:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\ &= x^x (1 + \ln x). \end{aligned}$$

❑

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Eq. (2) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx} dx = \int \frac{1}{\ln a} \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} \int \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} a^u + C.$$

Writing the first integral in differential form gives

$$\boxed{\int a^u du = \frac{a^u}{\ln a} + C. \qquad (3)}$$

EXAMPLE 5

a) $\displaystyle\int 2^x\, dx = \frac{2^x}{\ln 2} + C$ Eq. (3) with $a = 2$, $u = x$

b) $\displaystyle\int 2^{\sin x} \cos x\, dx$

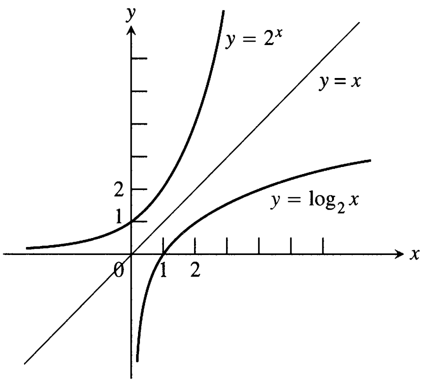
$$= \int 2^u du = \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{\sin x}}{\ln 2} + C \qquad u = \sin x \text{ in Eq. (3)}$$

❑

Logarithms with Base a

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by **$\log_a x$**.



6.13 The graph of 2^x and its inverse, $\log_2 x$.

Definition

For any positive number $a \neq 1$,

$$\log_a x = \text{ inverse of } a^x.$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$ (Fig. 6.13).

Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{4}$$

$$\log_a (a^x) = x \qquad (\text{all } x) \tag{5}$$

EXAMPLE 6

a) $\log_2 (2^5) = 5$

b) $\log_{10} (10^{-7}) = -7$

c) $2^{\log_2 (3)} = 3$

d) $10^{\log_{10} (4)} = 4$ ❑

The Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \tag{6}$$

We can derive Eq. (6) from Eq. (4):

$a^{\log_a (x)} = x$	Eq. (4)
$\ln a^{\log_a (x)} = \ln x$	Take the natural logarithm of both sides.
$\log_a (x) \cdot \ln a = \ln x$	The Power Rule in Table 6.1
$\log_a x = \dfrac{\ln x}{\ln a}$	Solve for $\log_a x$.

EXAMPLE 7

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$ ❑

Table 6.4
Properties of base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
$\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
$\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
$\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
$\log_a x^y = y \log_a x$

The arithmetic properties of $\log_a x$ are the same as the ones for $\ln x$ (Table 6.4). These rules can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\ln xy = \ln x + \ln y \qquad \text{Rule 1 for natural logarithms} \ldots$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \qquad \ldots \text{divided by } \ln a \ldots$$

$$\log_a xy = \log_a x + \log_a y. \qquad \ldots \text{gives Rule 1 for base } a \text{ logarithms.}$$

The Derivative of $\log_a u$

To find the derivative of a base a logarithm, we first convert it to a natural logarithm. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}} \qquad (7)$$

EXAMPLE 8

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)} \qquad \square$$

Integrals Involving $\log_a x$

To evaluate integrals involving base a logarithms, we convert them to natural logarithms.

EXAMPLE 9

$$\begin{aligned}
\int \frac{\log_2 x}{x}dx &= \frac{1}{\ln 2}\int \frac{\ln x}{x}dx && \log_2 x = \frac{\ln x}{\ln 2} \\
&= \frac{1}{\ln 2}\int u\,du && u = \ln x, \quad du = \frac{1}{x}dx \\
&= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C && \square
\end{aligned}$$

*Base 10 Logarithms

Base 10 logarithms, often called **common logarithms,** appear in many scientific formulas. For example, earthquake intensity is often reported on the logarithmic **Richter scale.** Here the formula is

$$\text{Magnitude } R = \log_{10}\left(\frac{a}{T}\right) + B,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

EXAMPLE 10 For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ sec, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 6.8 = 7.8.$$

An earthquake of this magnitude does great damage near its epicenter. ❑

Most foods are acidic (pH < 7).

Food	pH Value
Bananas	4.5–4.7
Grapefruit	3.0–3.3
Oranges	3.0–4.0
Limes	1.8–2.0
Milk	6.3–6.6
Soft drinks	2.0–4.0
Spinach	5.1–5.7

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$:

$$\text{pH} = \log_{10}\frac{1}{[H_3O^+]} = -\log_{10}[H_3O^+].$$

The hydronium ion concentration is measured in moles per liter. Vinegar has a pH of 3, distilled water a pH of 7, seawater a pH of 8.15, and household ammonia a pH of 12. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal (1 N) solution of sodium hydroxide.

Another example of the use of common logarithms is the **decibel** or db ("dee bee") **scale** for measuring loudness. If I is the **intensity** of sound in watts per square meter, the decibel level of the sound is

$$\textbf{Sound level} = 10\ \log_{10}(I \times 10^{12})\ \text{db}. \tag{8}$$

If you ever wondered why doubling the power of your audio amplifier increases the sound level by only a few decibels, Eq. (8) provides the answer. As the following example shows, doubling I adds only about 3 db.

Typical sound levels

Threshold of hearing	0 db
Rustle of leaves	10 db
Average whisper	20 db
Quiet automobile	50 db
Ordinary conversation	65 db
Pneumatic drill 10 feet away	90 db
Threshold of pain	120 db

EXAMPLE 11 Doubling I in Eq. (8) adds about 3 db. Writing log for $\log_{10}$ (a common practice), we have

$$\begin{aligned}
\text{Sound level with } I \text{ doubled} &= 10\log(2I \times 10^{12}) && \text{Eq. (8) with } 2I \text{ for } I\\
&= 10\log(2 \cdot I \times 10^{12})\\
&= 10\log 2 + 10\ \log(I \times 10^{12})\\
&= \text{original sound level} + 10\log 2\\
&\approx \text{original sound level} + 3. && \log_{10} 2 \approx 0.30
\end{aligned}$$

❑

Exercises 6.4

Algebraic Calculations

Simplify the expressions in Exercises 1–4.

1. a) $5^{\log_5 7}$ b) $8^{\log_8 \sqrt{2}}$ c) $1.3^{\log_{1.3} 75}$ d) $\log_4 16$ e) $\log_3 \sqrt{3}$ f) $\log_4\left(\frac{1}{4}\right)$

2. a) $2^{\log_2 3}$ b) $10^{\log_{10}(1/2)}$ c) $\pi^{\log_\pi 7}$

d) $\log_{11} 121$ **e)** $\log_{121} 11$ **f)** $\log_3 \left(\frac{1}{9}\right)$

3. a) $2^{\log_4 x}$ **b)** $9^{\log_3 x}$ **c)** $\log_2(e^{(\ln 2)(\sin x)})$

4. a) $25^{\log_5(3x^2)}$ **b)** $\log_e(e^x)$ **c)** $\log_4(2^{e^x \sin x})$

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. a) $\dfrac{\log_2 x}{\log_3 x}$ **b)** $\dfrac{\log_2 x}{\log_8 x}$ **c)** $\dfrac{\log_x a}{\log_{x^2} a}$

6. a) $\dfrac{\log_9 x}{\log_3 x}$ **b)** $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **c)** $\dfrac{\log_a b}{\log_b a}$

Solve the equations in Exercises 7–10 for x.

7. $3^{\log_3(7)} + 2^{\log_2(5)} = 5^{\log_5(x)}$

8. $8^{\log_8(3)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$

9. $3^{\log_3(x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10}(2)}$

10. $\ln e + 4^{-2\log_4(x)} = \dfrac{1}{x}\log_{10}(100)$

Derivatives

In Exercises 11–38, find the derivative of y with respect to the given independent variable.

11. $y = 2^x$

12. $y = 3^{-x}$

13. $y = 5^{\sqrt{s}}$

14. $y = 2^{(s^2)}$

15. $y = x^{\pi}$

16. $y = t^{1-e}$

17. $y = (\cos\theta)^{\sqrt{2}}$

18. $y = (\ln\theta)^{\pi}$

19. $y = 7^{\sec\theta}\ln 7$

20. $y = 3^{\tan\theta}\ln 3$

21. $y = 2^{\sin 3t}$

22. $y = 5^{-\cos 2t}$

23. $y = \log_2 5\theta$

24. $y = \log_3(1 + \theta\ln 3)$

25. $y = \log_4 x + \log_4 x^2$

26. $y = \log_{25} e^x - \log_5 \sqrt{x}$

27. $y = \log_2 r \cdot \log_4 r$

28. $y = \log_3 r \cdot \log_9 r$

29. $y = \log_3\left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$

30. $y = \log_5\sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$

31. $y = \theta\sin(\log_7\theta)$

32. $y = \log_7\left(\dfrac{\sin\theta\cos\theta}{e^{\theta}\,2^{\theta}}\right)$

33. $y = \log_5 e^x$

34. $y = \log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$

35. $y = 3^{\log_2 t}$

36. $y = 3\log_8(\log_2 t)$

37. $y = \log_2(8t^{\ln 2})$

38. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

39. $y = (x+1)^x$

40. $y = x^{(x+1)}$

41. $y = (\sqrt{t})^t$

42. $y = t^{\sqrt{t}}$

43. $y = (\sin x)^x$

44. $y = x^{\sin x}$

45. $y = x^{\ln x}$

46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercises 47–56

47. $\displaystyle\int 5^x\,dx$

48. $\displaystyle\int (1.3)^x\,dx$

49. $\displaystyle\int_0^1 2^{-\theta}\,d\theta$

50. $\displaystyle\int_{-2}^0 5^{-\theta}\,d\theta$

51. $\displaystyle\int_1^{\sqrt{2}} x2^{(x^2)}\,dx$

52. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

53. $\displaystyle\int_0^{\pi/2} 7^{\cos t}\sin t\,dt$

54. $\displaystyle\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

55. $\displaystyle\int_2^4 x^{2x}(1+\ln x)\,dx$

56. $\displaystyle\int_1^2 \frac{2^{\ln x}}{x}\,dx$

Evaluate the integrals in Exercises 57–60.

57. $\displaystyle\int 3x^{\sqrt{3}}dx$

58. $\displaystyle\int x^{\sqrt{2}-1}dx$

59. $\displaystyle\int_0^3(\sqrt{2}+1)x^{\sqrt{2}}dx$

60. $\displaystyle\int_1^e x^{(\ln 2)-1}dx$

Evaluate the integrals in Exercises 61–70.

61. $\displaystyle\int\frac{\log_{10}x}{x}\,dx$

62. $\displaystyle\int_1^4\frac{\log_2 x}{x}\,dx$

63. $\displaystyle\int_1^4\frac{\ln 2\log_2 x}{x}\,dx$

64. $\displaystyle\int_1^e\frac{2\ln 10\log_{10}x}{x}\,dx$

65. $\displaystyle\int_0^2\frac{\log_2(x+2)}{x+2}\,dx$

66. $\displaystyle\int_{1/10}^{10}\frac{\log_{10}(10x)}{x}\,dx$

67. $\displaystyle\int_0^9\frac{2\log_{10}(x+1)}{x+1}\,dx$

68. $\displaystyle\int_2^3\frac{2\log_2(x-1)}{x-1}\,dx$

69. $\displaystyle\int\frac{dx}{x\log_{10}x}$

70. $\displaystyle\int\frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 71–74.

71. $\displaystyle\int_1^{\ln x}\frac{1}{t}dt,\quad x>1$

72. $\displaystyle\int_1^{e^x}\frac{1}{t}\,dt$

73. $\displaystyle\int_1^{1/x}\frac{1}{t}\,dt,\quad x>0$

74. $\displaystyle\frac{1}{\ln a}\int_1^x\frac{1}{t}\,dt,\quad x>0$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1+x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.

76. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the x-axis.

77. *Blood pH.* The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for $[H_3O^+]$.

78. *Brain fluid pH.* The cerebrospinal fluid in the brain has a hydronium ion concentration of about $[H_3O^+] = 4.8 \times 10^{-8}$ moles per liter. What is the pH?

79. *Audio amplifiers.* By what factor k do you have to multiply the intensity of I of the sound from your audio amplifier to add 10 db to the sound level?

80. *Audio amplifiers.* You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration $[H_3O^+]$ (moles/L) and the hydroxyl ion concentration $[OH^-]$ (moles/L) is about 10^{-14}.

a) What value of $[H_3O^+]$ minimizes the sum of the concentrations, $S = [H_3O^+] + [OH^-]$? (*Hint:* Change notation. Let $x = [H_3O^+]$.)

b) What is the pH of a solution in which S has this minimum value?

c) What ratio of $[H_3O^+]$ to $[OH^-]$ minimizes S?

82. Could $\log_a b$ possibly equal $1/\log_b a$? Give reasons for your answer.

Grapher Explorations

83. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

84. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. *The linearization of 2^x*

a) Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to 2 decimal places.

b) Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

86. *The linearization of $\log_3 x$*

a) Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to 2 decimal places.

b) Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

Calculations with Other Bases

87. CALCULATOR Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the equation $\log_a x = (\ln x)/(\ln a)$.

To find $\log_2 x$, find $\ln x$ and divide by $\ln 2$: $\quad \log_2 5 = \dfrac{\ln 5}{\ln 2} \approx 2.3219.$

To find $\ln x$ given $\log_2 x$, multiply by $\ln 2$: $\quad \ln 5 = \log_2 5 \cdot \ln 2 \approx 1.6094.$

Find the following logarithms to 5-decimal places.

a) $\log_3 8$ **b)** $\log_7 0.5$
c) $\log_{20} 17$ **d)** $\log_{0.5} 7$
e) $\ln x$, given that $\log_{10} x = 2.3$
f) $\ln x$, given that $\log_2 x = 1.4$
g) $\ln x$, given that $\log_2 x = -1.5$
h) $\ln x$, given that $\log_{10} x = -0.7$

88. *Conversion factors*

a) Show that the equation for converting base 10 logarithms to base 2 logarithms is

$$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$

b) Show that the equation for converting base a logarithms to base b logarithms is

$$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

6.5

Growth and Decay

In this section, we derive the law of exponential change and describe some of the applications that account for the importance of logarithmic and exponential functions.

The Law of Exponential Change

To set the stage once again, suppose we are interested in a quantity y (velocity, temperature, electric current, whatever) that increases or decreases at a rate that at any given time t is proportional to the amount present. If we also know the amount present at time $t = 0$, call it y_0, we can find y as a function of t by solving the following initial value problem:

Differential equation:	$\dfrac{dy}{dt} = ky$	(1)
Initial condition:	$y = y_0$ when $t = 0$.	

If y is positive and increasing, then k is positive, and we use Eq. (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Eq. (1) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Eq. (1). To find the nonzero solutions, we divide Eq. (1) by y:

$$\frac{1}{y} \cdot \frac{dy}{dt} = k$$

$$\ln|y| = kt + C \qquad \text{Integrate with respect to } t;\ \int (1/u)\,du = \ln|u| + C.$$

$$|y| = e^{kt+C} \qquad \text{Exponentiate.}$$

$$|y| = e^C \cdot e^{kt} \qquad e^{a+b} = e^a \cdot e^b$$

$$y = \pm e^C e^{kt} \qquad \text{If } |y| = r, \text{ then } y = \pm r.$$

$$y = Ae^{kt}. \qquad A \text{ is a more convenient name for } \pm e^C.$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^C$, we can include the solution $y = 0$ in the formula.

We find the right value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$.

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

The derivation of Eq. (2) shows that the only functions that are their own derivatives are constant multiples of the exponential function.

Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, foxes, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, as soon as the number of individuals becomes large enough, it can safely be described with a continuous or even differentiable function.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. If, further, we neglect departures, arrivals, and deaths, the growth rate dy/dt will be the same as the birth rate ky. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here.

EXAMPLE 1 One model for the way diseases spread assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The more infected people there are, the faster the disease will spread. The fewer there are, the slower it will spread.

Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find:

1. the value of y_0,
2. the value of k,
3. the value of t that makes $y = 1000$.

Step 1: *The value of* y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000\, e^{kt}. \tag{3}$$

Step 2: *The value of* k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10{,}000\, e^{k(1)} && \text{Eq. (3) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 \\ k &= \ln 0.8. \end{aligned}$$

At any given time t,

$$y = 10{,}000\, e^{(\ln 0.8)t}. \tag{4}$$

Step 3: *The value of t that makes* $y = 1000$. We set y equal to 1000 in Eq. (4) and solve for t:

$$\begin{aligned} 1000 &= 10{,}000\, e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years.} \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. ❑

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, it turns out that the amount of money you will have at the end of t years is

$$A_t = A_0\left(1+\frac{r}{k}\right)^{kt}. \tag{5}$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, say by the hour or by the minute. But there is still a limit to how much you will earn that way, and the limit is

$$\begin{aligned}\lim_{k\to\infty} A_t &= \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}\\ &= A_0\,e^{rt}.\end{aligned}$$

Evaluating

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$$

involves what is called the indeterminate form 1^∞. We will see how to evaluate limits of this type in Section 6.6.

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0\,e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is called the **continuous interest rate.**

EXAMPLE 2 Suppose you deposit \$621 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later?

Solution We use Eq. (6) with $A_0 = 621$, $r = 0.06$, and $t = 8$:

$$A(8) = 621\,e^{(0.06)(8)} = 621\,e^{0.48} = 1003.58 \qquad \text{Nearest cent}$$

Had the bank paid interest quarterly ($k = 4$ in Eq. (5)), the amount in your account would have been \$1000.01. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of \$3.57. A bank might decide it would be worth this additional amount to be able to advertise, "We compound interest every second, night and day—better yet, we compound the interest continuously." ❑

Radioactivity

When an atom emits some of its mass as radiation, the remainder of the atom re-forms to make an atom of some new element. This process of radiation and change is called **radioactive decay,** and an element whose atoms go spontaneously through this process is called **radioactive.** Thus, radioactive carbon-14 decays into nitrogen; radium, through a number of intervening radioactive steps, decays into lead.

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$. The decay of radium in the earth's crust is the source of the radon we sometimes find in our basements.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0\,e^{-kt}, \qquad k > 0.$$

It is conventional to use $-k$ ($k > 0$) here instead of k ($k < 0$) to emphasize that y is decreasing.

EXAMPLE 3 *Half-life*

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is a remarkable fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$
\begin{aligned}
y_0 e^{-kt} &= \frac{1}{2} y_0 \\
e^{-kt} &= \frac{1}{2} \\
-kt &= \ln \frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms} \\
t &= \frac{\ln 2}{k}
\end{aligned}
$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not enter in. ❑

$$
\boxed{\text{Half-life} = \frac{\ln 2}{k} \qquad (7)}
$$

EXAMPLE 4 *Polonium-210*

The effective radioactive lifetime of polonium-210 is so short we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

Find the element's half-life.

Solution

$$
\begin{aligned}
\text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (7)} \\
&= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\
&\approx 139 \text{ days}
\end{aligned}
$$

❑

EXAMPLE 5 *Carbon-14*

People who do carbon-14 dating use a figure of 5700 years for its half-life (more about carbon-14 dating in the exercises). Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Carbon-14 dating

The decay of radioactive elements can sometimes be used to date events from the Earth's past. The ages of rocks more than 2 billion years old have been measured by the extent of the radioactive decay of uranium (half-life 4.5 billion years!). In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays. It is possible to estimate the ages of fairly old organic remains by comparing the proportion of carbon-14 they contain with the proportion assumed to have been in the organism's environment at the time it lived. Archaeologists have dated shells (which contain $CaCO_3$), seeds, and wooden artifacts this way. The estimate of 15,500 years for the age of the cave paintings at Lascaux, France, is based on carbon-14 dating. After generations of controversy, the Shroud of Turin, long believed by many to be the burial cloth of Christ, was shown by carbon-14 dating in 1988 to have been made after A.D. 1200.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find:

1. the value of k,
2. the value of t when $y_0 e^{-kt} = 0.9 y_0$, or $e^{-kt} = 0.9$ — 90% of the radioactive nuclei still present

Step 1: *The value of k.* We use the half-life equation:

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \qquad (\text{about } 1.2 \times 10^{-4})$$

Step 2: *The value of t that makes* $e^{-kt} = 0.9$.

$$\begin{aligned} e^{-kt} &= 0.9 \\ e^{-(\ln 2/5700)t} &= 0.9 \\ -\frac{\ln 2}{5700}t &= \ln 0.9 && \text{Logs of both sides} \\ t &= -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years.} \end{aligned}$$

The sample is about 866 years old. ❑

Heat Transfer: Newton's Law of Cooling

Soup left in a tin cup cools to the temperature of the surrounding air. A hot silver ingot immersed in water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's law of cooling,* although it applies to warming as well, and there is an equation for it.

If T is the temperature of the object at time t, and T_S is the surrounding temperature, then

$$\frac{dT}{dt} = -k(T - T_S). \tag{8}$$

If we substitute y for $(T - T_S)$, then

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(T - T_S) = \frac{dT}{dt} - \frac{d}{dt}(T_S) \\ &= \frac{dT}{dt} - 0 && T_S \text{ is a constant.} \\ &= \frac{dT}{dt} \end{aligned}$$

In terms of y, Eq. (8) therefore reads

$$\frac{dy}{dt} = -ky,$$

and we know that the solution to this differential equation is

$$y = y_0 e^{-kt}.$$

Thus, **Newton's law of cooling** is

$$T - T_S = (T_0 - T_S)e^{-kt}, \tag{9}$$

where T_0 is the value of T at time zero.

EXAMPLE 6 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 minutes, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 minutes that have already elapsed.

According to Eq. (9), the egg's temperature t minutes after it is put in the sink is

$$T = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $T = 38$ when $t = 5$:

$$\begin{aligned}
38 &= 18 + 80e^{-5k} \\
e^{-5k} &= \frac{1}{4} \\
-5k &= \ln\frac{1}{4} = -\ln 4 \\
k &= \frac{1}{5}\ln 4 = 0.2\ \ln 4 \qquad \text{(about 0.28).}
\end{aligned}$$

The egg's temperature at time t is $T = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $T = 20$:

$$\begin{aligned}
20 &= 18 + 80e^{-(0.2\ln 4)t} \\
80e^{-(0.2\ln 4)t} &= 2 \\
e^{-(0.2\ln 4)t} &= \frac{1}{40} \\
-(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\
t &= \frac{\ln 40}{0.2\ \ln 4} \approx 13 \text{ min.}
\end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ❑

Exercises 6.5

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

1. *Human evolution continues.* The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

   a) If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0\,e^{kt}$. Then use this value of k to answer the following questions.

   b) In about how many years will human teeth be 90% of their present size?

   c) What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

   (Source: *LSA Magazine,* Spring 1989, Vol. 12, No. 2, p. 19, Ann Arbor, MI.)

2. *Atmospheric pressure.* The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.
   a) Solve the initial value problem

   Differential equation: $dp/dh = kp$ (k a constant)

   Initial condition: $p = p_0$ when $h = 0$

   to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.
   b) What is the atmospheric pressure at $h = 50$ km?
   c) At what altitude does the pressure equal 900 millibars?

3. *First order chemical reactions.* In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,
$$\frac{dy}{dt} = -0.6y$$
when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

4. *The inversion of sugar.* The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

5. *Working underwater.* The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation
$$\frac{dL}{dx} = -kL.$$
As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

6. *Voltage in a discharging capacitor.* Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,
$$\frac{dV}{dt} = -\frac{1}{40}V.$$
Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

7. *Cholera bacteria.* Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half hour. How many bacteria will the colony contain at the end of 24 h? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

8. *Growth of bacteria.* A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000. How many bacteria were present initially?

9. *The incidence of a disease* (*Continuation of Example* 1). Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.
   a) How long will it take to reduce the number of cases to 1000?
   b) How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

10. *The U.S. population.* The Museum of Science in Boston displays a running total of the U.S. population. On May 11, 1993, the total was increasing at the rate of 1 person every 14 sec. The displayed population figure for 3:45 P.M. that day was 257,313,431.
   a) Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).
   b) At this rate, what will the U.S. population be at 3:45 P.M. Boston time on May 11, 2001?

11. *Oil depletion.* Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

12. *Continuous price discounting.* To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.
   a) Find $p(x)$ by solving the following initial value problem:

   Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$

   Initial condition: $p(100) = 20.09.$
   b) Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.
   c) The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.
   d) GRAPHER Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

13. *Continuously compounded interest.* You have just placed A_0 dollars in a bank account that pays 4% interest, compounded continuously.
   a) How much money will you have in the account in 5 years?
   b) How long will it take your money to double? to triple?

14. *John Napier's question.* John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question What happens if you invest an amount of money at 100% interest, compounded continuously?

a) What does happen?
b) How long does it take to triple your money?
c) How much can you earn in a year?

Give reasons for your answers.

15. *Benjamin Franklin's will.* The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> . . . with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. . . . If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. . . . The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. . . . At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the reception of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost exactly 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

What rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

16. (*Continuation of Exercise 15.*) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

17. *Radon-222.* The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

18. *Polonium-210.* The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

19. *The mean life of a radioactive nucleus.* Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

20. *Californium-252.* What costs $27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

a) What is the value of k in the decay equation for this isotope?
b) What is the isotope's mean life? (See Exercise 19.)
c) How long will it take 95% of a sample's radioactive nuclei to disintegrate?

21. *Cooling soup.* Suppose that a cup of soup cooled from 90°C to 60°C after 10 minutes in a room whose temperature was 20°C. Use Newton's law of cooling to answer the following questions.

a) How much longer would it take the soup to cool to 35°C?
b) Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

22. *A beam of unknown temperature.* An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°. After 10 minutes, the beam warmed to 35°F and after another 10 minutes it was 50°F. Use Newton's law of cooling to estimate the beam's initial temperature.

23. *Surrounding medium of unknown temperature.* A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's law of cooling to estimate how cold the refrigerator was.

24. *Silver cooling in air.* The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

a) 15 minutes from now?

b) two hours from now?

c) When will the silver be 10°C above room temperature?

25. *The age of Crater Lake.* The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

26. *The sensitivity of carbon-14 dating to measurement.* To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

a) A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

b) Repeat (a) assuming 18% instead of 17%.

c) Repeat (a) assuming 16% instead of 17%.

27. *Art forgery.* A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

6.6 L'Hôpital's Rule

In the late seventeenth century, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. The rule is known today as **l'Hôpital's rule,** after Guillaume François Antoine de l'Hôpital (1661–1704), Marquis de St. Mesme, a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Quotients

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a} f(x)/g(x)$ cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression known as an **indeterminate form.** Our experience so far has been that limits that lead to indeterminate forms may or may not be hard to find. It took a lot of work to find $\lim_{x\to 0} (\sin x)/x$ in Section 2.4. But we have had remarkable success with the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

from which we calculate derivatives and which always produces $0/0$. L'Hôpital's rule enables us to draw on our success with derivatives to evaluate limits that lead to indeterminate forms.

Theorem 2

L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}. \tag{1}$$

Proof Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{f(x)-0}{g(x)-0}$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)}. \qquad \square$$

Caution

To apply l'Hôpital's rule to *f/g*, divide the derivative of *f* by the derivative of *g*. Do not fall into the trap of taking the derivative of *f/g*. The quotient to use is *f'/g'*, not *(f/g)'*.

EXAMPLE 1

a) $\displaystyle \lim_{x\to 0} \frac{3x - \sin x}{x} = \frac{3-\cos x}{1}\bigg|_{x=0} = 2$

b) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{\dfrac{1}{2\sqrt{1+x}}}{1}\Bigg|_{x=0} = \frac{1}{2}$

c) $\displaystyle \lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1-\cos x}{3x^2}\bigg|_{x=0} = ?$ Still $\frac{0}{0}$ $\square$

What can we do about the limit in Example 1(c)? A stronger form of l'Hôpital's rule says that whenever the rule gives $0/0$ we can apply it again, repeating the process until we get a different result. With this stronger rule we get

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} \qquad \text{Still } \tfrac{0}{0}\text{; apply the rule again.}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \text{Still } \tfrac{0}{0}\text{; apply the rule again.}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}. \qquad \text{A different result. Stop.}$$

A misnamed rule and the first differential calculus text

In 1694 John Bernoulli agreed to accept a retainer of 300 pounds per year from his former student l'Hôpital to solve problems for him and keep him up to date on calculus. One of the problems was the so-called 0/0 problem, which Bernoulli solved as agreed. When l'Hôpital published his notes on calculus in book form in 1696, the 0/0 rule appeared as a theorem. L'Hôpital acknowledged his debt to Bernoulli and, to avoid claiming authorship of the book's entire contents, had the book published anonymously. Bernoulli nevertheless accused l'Hôpital of plagiarism, an accusation inadvertently supported after l'Hôpital's death in 1704 by the publisher's promotion of the book as l'Hôpital's. By 1721, Bernoulli, a man so jealous he once threw his son Daniel out of the house for accepting a mathematics prize from the French Academy of Sciences, claimed to have been the author of the entire work. As puzzling and fickle as ever, history accepted Bernoulli's claim (until recently), but still named the rule after l'Hôpital.

Theorem 3

L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on an open interval I containing a. Suppose also that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \qquad (2)$$

if the limit on the right exists (or is ∞ or $-\infty$).

You will find a proof of the finite-limit case of Theorem 3 in Appendix 5.

EXAMPLE 2

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - (x/2)}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found}$$

❑

When you apply l'Hôpital's rule, look for a change from $0/0$ to something else. This is where the limit is revealed.

EXAMPLE 3

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

If we continue to differentiate in an attempt to apply l'Hôpital's rule once more, we get

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. ❑

EXAMPLE 4

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Not } \frac{0}{0}\text{; answer is found.}$$

❑

L'Hôpital's rule also applies to quotients that lead to the indeterminate form ∞/∞. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists. The a here may itself be either finite or infinite.

EXAMPLE 5

a) $$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

❑

Indeterminate Products and Differences

We can sometimes handle the indeterminate forms $0 \cdot \infty$ and $\infty - \infty$ by using algebra to get $0/0$ or ∞/∞ instead. Here again, we do not mean to suggest that there is a number $0 \cdot \infty$ or $\infty - \infty$ any more than we mean to suggest that there is a number $0/0$ or ∞/∞. These forms are not numbers but descriptions of function behavior.

EXAMPLE 6

$$\lim_{x\to 0^+} x \cot x \qquad 0 \cdot \infty;\ \text{rewrite } x \cot x.$$

$$= \lim_{x\to 0^+} x \cdot \frac{1}{\tan x} \qquad \cot x = \frac{1}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{x}{\tan x} \qquad \text{Now } \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{1}{\sec^2 x} = \frac{1}{1} = 1$$

❑

EXAMPLE 7 Find $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions.

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}, \qquad \text{Common denominator is } x \sin x.$$

and then apply l'Hôpital's rule to the result:

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

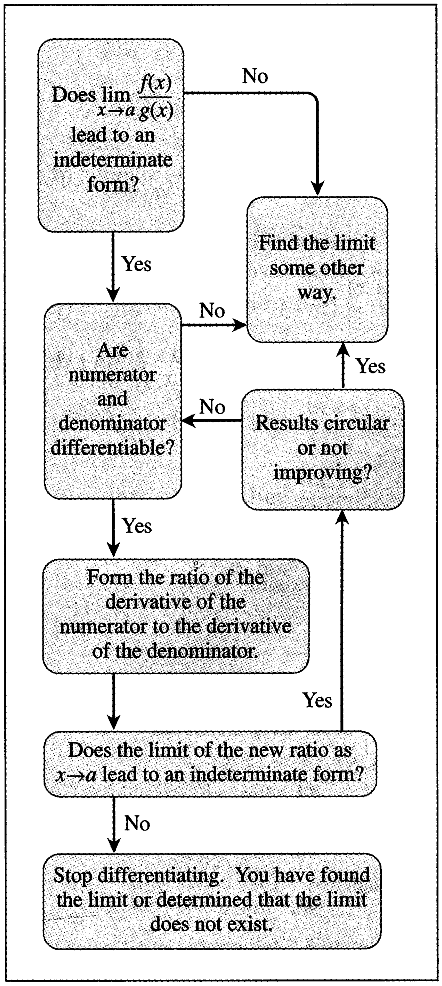
$$= \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

❑

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by taking logarithms first. We use l'Hôpital's rule to find the limit of the logarithm and then exponentiate to find the original function behavior.



Flowchart 6.1 L'Hôpital's rule

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

EXAMPLE 8 Show that $\lim_{x\to 0^+} (1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\begin{aligned}\ln f(x) &= \ln (1+x)^{1/x}\\ &= \frac{1}{x}\ln(1+x),\end{aligned}$$

l'Hôpital's rule now applies to give

$$\begin{aligned}\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}\\ &= \frac{1}{1} = 1.\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e. \qquad \square$$

EXAMPLE 9 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\begin{aligned}\ln f(x) &= \ln x^{1/x}\\ &= \frac{\ln x}{x},\end{aligned}$$

l'Hôpital's rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1. \qquad \square$$

Exercises 6.6

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 1–42.

1. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

2. $\lim_{x\to -5} \dfrac{x^2-25}{x+5}$

3. $\lim_{t\to -3} \dfrac{t^3-4t+15}{t^2-t-12}$

4. $\lim_{t\to 1} \dfrac{t^3-1}{4t^3-t-3}$

5. $\lim_{x\to \infty} \dfrac{5x^2-3x}{7x^2+1}$

6. $\lim_{x\to \infty} \dfrac{x-8x^2}{12x^2+5x}$

7. $\lim_{t\to 0} \dfrac{\sin t^2}{t}$

8. $\lim_{t\to 0} \dfrac{\sin 5t}{t}$

9. $\lim_{x\to 0} \dfrac{8x^2}{\cos x-1}$

10. $\lim_{x\to 0} \dfrac{\sin x-x}{x^3}$

11. $\lim_{\theta\to \pi/2} \dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$

12. $\lim_{\theta\to -\pi/3} \dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

13. $\lim_{\theta\to \pi/2} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

14. $\lim_{x\to 1} \dfrac{x-1}{\ln x-\sin \pi x}$

15. $\lim_{x\to 0} \dfrac{x^2}{\ln(\sec x)}$

16. $\lim_{x\to \pi/2} \dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$

17. $\lim_{t\to 0} \dfrac{t(1-\cos t)}{t-\sin t}$

18. $\lim_{t\to 0} \dfrac{t\sin t}{1-\cos t}$

19. $\lim_{x\to (\pi/2)^-} \left(x-\dfrac{\pi}{2}\right)\sec x$

20. $\lim_{x\to (\pi/2)^-} \left(\dfrac{\pi}{2}-x\right)\tan x$

21. $\lim_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$

22. $\lim_{\theta\to 0} \dfrac{(1/2)^\theta-1}{\theta}$

23. $\lim_{x\to 0} \dfrac{x2^x}{2^x-1}$

24. $\lim_{x\to 0} \dfrac{3^x-1}{2^x-1}$

25. $\lim_{x\to \infty} \dfrac{\ln(x+1)}{\log_2 x}$

26. $\lim_{x\to \infty} \dfrac{\log_2 x}{\log_3(x+3)}$

27. $\lim_{x\to 0^+} \dfrac{\ln(x^2+2x)}{\ln x}$

28. $\lim_{x\to 0^+} \dfrac{\ln(e^x-1)}{\ln x}$

29. $\lim_{y\to 0} \dfrac{\sqrt{5y+25}-5}{y}$

30. $\lim_{y\to 0} \dfrac{\sqrt{ay+a^2}-a}{y}, \quad a>0$

31. $\lim_{x\to \infty} (\ln 2x-\ln(x+1))$

32. $\lim_{x\to 0^+} (\ln x-\ln\sin x)$

33. $\lim_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

34. $\lim_{x\to 0^+} \left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$

35. $\lim_{x\to 1^+} \left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$

36. $\lim_{x\to 0^+} (\csc x-\cot x+\cos x)$

37. $\lim_{x\to \infty} \displaystyle\int_x^{2x} \frac{1}{t}\,dt$

38. $\lim_{x\to \infty} \dfrac{1}{x\ln x}\displaystyle\int_1^x \ln t\,dt$

39. $\lim_{\theta\to 0} \dfrac{\cos\theta-1}{e^\theta-\theta-1}$

40. $\lim_{h\to 0} \dfrac{e^h-(1+h)}{h^2}$

41. $\lim_{t\to \infty} \dfrac{e^t+t^2}{e^t-t}$

42. $\lim_{x\to \infty} x^2e^{-x}$

Limits Involving Bases and Exponents

Find the limits in Exercises 43–52.

43. $\lim_{x\to 1^+} x^{1/(1-x)}$

44. $\lim_{x\to 1^+} x^{1/(x-1)}$

45. $\lim_{x\to \infty} (\ln x)^{1/x}$

46. $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$

47. $\lim_{x\to 0^+} x^{-1/\ln x}$

48. $\lim_{x\to \infty} x^{1/\ln x}$

49. $\lim_{x\to \infty} (1+2x)^{1/(2\ln x)}$

50. $\lim_{x\to 0} (e^x+x)^{1/x}$

51. $\lim_{x\to 0^+} x^x$

52. $\lim_{x\to 0^+} \left(1+\dfrac{1}{x}\right)^x$

Theory and Applications

L'Hôpital's rule does not help with the limits in Exercises 53–56. Try it—you just keep on cycling. Find the limits some other way.

53. $\lim_{x\to \infty} \dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

54. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$

55. $\lim_{x\to (\pi/2)^-} \dfrac{\sec x}{\tan x}$

56. $\lim_{x\to 0^+} \dfrac{\cot x}{\csc x}$

57. Which one is correct, and which one is wrong? Give reasons for your answers.

 a) $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \lim_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$

 b) $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \dfrac{0}{6} = 0$

58. Which one is correct, and which one is wrong? Give reasons for your answers.

a) $\lim_{x\to 0} \dfrac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \dfrac{2x-2}{2x-\cos x}$

$= \lim_{x\to 0} \dfrac{2}{2+\sin x} = \dfrac{2}{2+0} = 1$

b) $\lim_{x\to 0} \dfrac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \dfrac{2x-2}{2x-\cos x} = \dfrac{-2}{0-1} = 2$

59. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a) $\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = 0$

b) $\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = -\infty$

c) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)} = \dfrac{-\infty}{\infty} = -1$

d) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)}$

$= \lim_{x\to 0^+} \dfrac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$

60. Let

$$f(x) = \begin{cases} x+2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$g(x) = \begin{cases} x+1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but that} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

Doesn't this contradict l'Hôpital's rule? Give reasons for your answers.

61. Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

62. Find a value of c that makes the function

$$g(\theta) = \begin{cases} \dfrac{(\tan\theta)^2}{\sin(4\theta^2/\pi)}, & \theta \neq 0 \\ c, & \theta = 0 \end{cases}$$

continuous from the right at $\theta = 0$. Explain why your value of c works.

63. *The continuous compound interest formula.* In deriving the formula $A(t) = A_0 e^{rt}$ in Section 6.5, we claimed that

$$\lim_{k\to\infty} A_0 \left(1+\frac{r}{k}\right)^{kt} = A_0 e^{rt}.$$

This equation will hold if

$$\lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{kt} = e^{rt},$$

and this, in turn, will hold if

$$\lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{k} = e^{r}.$$

As you can see, the limit leads to the indeterminate form 1^∞. Verify the limit using l'Hôpital's rule.

64. Given that $x > 0$, find the maximum value, if any, of

a) $x^{1/x}$

b) x^{1/x^2}

c) x^{1/x^n} (n a positive integer)

d) Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

Grapher Explorations

65. *Determining the value of e.*

a) Use l'Hôpital's rule to show that

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e.$$

b) CALCULATOR See how close you can come to

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45$$

by evaluating $f(x) = (1+(1/x))^x$ for $x = 10$, 10^2, 10^3, ... and so on. You can expect the approximations to approach e at first, but on some calculators they will move away again as round-off errors take their toll.

c) If you have a grapher, you may prefer to do part (b) by graphing $f(x) = (1+(1/x))^x$ for large values of x, using TRACE to display the coordinates along the graph. Again, you may expect to find decreasing accuracy as x increases and, beyond $x = 10^{10}$ or so, erratic behavior.

66. This exercise explores the difference between the limit

$$\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^{x}$$

and the limit

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e,$$

studied in Exercise 65.

a) Graph

$$f(x) = \left(1+\frac{1}{x^2}\right)^{x} \quad \text{and} \quad g(x) = \left(1+\frac{1}{x}\right)^{x}$$

together for $x \geq 0$. How does the behavior of f compare with that of g? Estimate the value of $\lim_{x\to\infty} f(x)$.

b) Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's rule.

67. a) Estimate the value of

$$\lim_{x\to\infty} (x - \sqrt{x^2+x})$$

by graphing $f(x) = x - \sqrt{x^2+x}$ over a suitably large interval of x-values.

b) Now confirm your estimate by finding the limit with l'Hôpital's rule. As the first step, multiply $f(x)$ by the fraction $(x + \sqrt{x^2 + x})/(x + \sqrt{x^2 + x})$ and simplify the new numerator.

68. Estimate the value of

$$\lim_{x\to 2} \frac{x^2 - 4}{\sqrt{x^2+5} - 3}$$

by graphing. Then confirm your estimate with l'Hôpital's rule.

69. Estimate the value of

$$\lim_{x\to 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's rule.

70. a) Estimate the value of

$$\lim_{x\to 1} \frac{(x-1)^2}{x \ln x - x - \cos \pi x}$$

by graphing $f(x) = (x-1)^2/(x \ln x - x - \cos \pi x)$ near $x = 1$. Then confirm your estimate with l'Hôpital's rule.

b) Graph f for $0 < x \le 11$.

71. *The continuous extension of* $(\sin x)^x$ *to* $[0, \pi]$

a) Graph $f(x) = (\sin x)^x$ on the interval $0 \le x \le \pi$. What value would you assign to f to make it continuous at $x = 0$?

b) Verify your conclusion in (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's rule.

c) Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is max f taken on?

d) Sharpen your estimate in (c) by graphing f' in the same window to see where its graph crosses the x-axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

e) Sharpen your estimate of the location of max f further still by solving the equation $f' = 0$ numerically.

f) CALCULATOR Estimate max f by evaluating f at the locations you found in (c), (d), and (e). What is your best value for max f?

72. *The function* $(\sin x)^{\tan x}$. (*Continuation of Exercise* 71.)

a) Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \le x \le 7$. How do you account for the gaps in the graph? How wide are the gaps?

b) Now graph f on the interval $0 \le x \le \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x = \pi/2$? (*Hint:* Use l'Hôpital's rule to find lim f as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

c) Continuing with the graphs in (b), find max f and min f as accurately as you can and estimate the values of x at which they are taken on.

73. *The place of* ln x *among the powers of* x. The natural logarithm

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

fills the gap in the set of formulas

$$\int t^{k-1} dt = \frac{t^k}{k} + C, \quad k \ne 0, \tag{3}$$

but the formulas themselves do not reveal how well the logarithm fits in. We can see the nice fit graphically if we select from Eq. (3) the specific antiderivatives

$$\int_1^x t^{k-1} dt = \frac{x^k - 1}{k}, \quad x > 0,$$

and compare their graphs with the graph of ln x.

a) Graph the functions $f(x) = (x^k - 1)/k$ together with ln x on the interval $0 \le x \le 50$ for $k = \pm 1, \pm 0.5, \pm 0.1$, and ± 0.05.

b) Show that

$$\lim_{k\to 0} \frac{x^k - 1}{k} = \ln x.$$

(Based on "The Place of ln x Among the Powers of x" by Henry C. Finlayson, *American Mathematical Monthly,* Vol. 94, No. 5, May 1987, p. 450.)

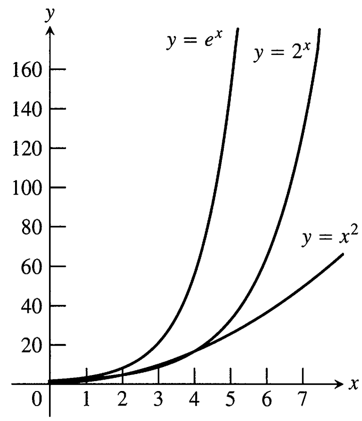
74. *Confirmation of the limit in Section 5.7, Exercise 42.* Estimate the value of

$$\lim_{\alpha\to 0^+} \frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \alpha\cos\alpha}$$

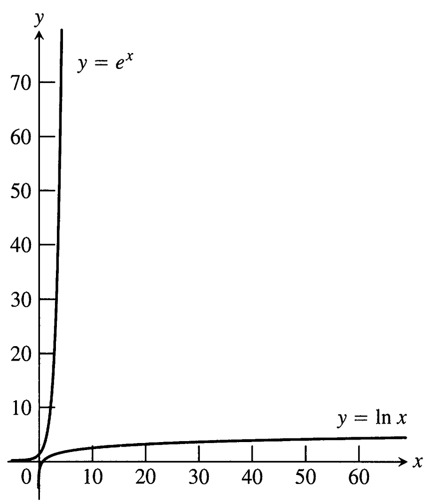
as closely as you can by graphing. Then confirm your estimate with l'Hôpital's rule.

6.7 Relative Rates of Growth

This section shows how to compare the rates at which functions of x grow as x becomes large and introduces the so-called little-oh and big-oh notation sometimes used to describe the results of these comparisons. *We restrict our attention to functions whose values eventually become and remain positive as* $x \to \infty$.



6.14 The graphs of e^x, 2^x, and x^2.



6.15 Scale drawings of the graphs of e^x and $\ln x$.

Relatives Rates of Growth

You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than the polynomials and rational functions we graphed in Chapter 3. These exponentials certainly grow more rapidly than x itself, and you can see 2^x outgrowing x^2 as x increases in Fig. 6.14. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x, even $x^{1,000,000}$ (Exercise 19).

To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high (it is about to go through the ceiling if it hasn't done so already). At $x = 10$ cm, the graph is $e^{10} \approx 22{,}026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm from the origin, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri:

$$\begin{aligned} e^{43} &\approx 4.73 \times 10^{18} \text{ cm} \\ &= 4.73 \times 10^{13} \text{ km} \\ &\approx 1.58 \times 10^{8} \text{ light-seconds} \\ &\approx 5.0 \text{ light-years} \end{aligned}$$

In a vacuum, light travels at 300,000 km/sec.

The distance to Proxima Centauri is about 4.22 light-years. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x (Exercise 21). With axes scaled in centimeters, you have to go nearly 5 light-years out on the x-axis to find a point where the graph of $y = \ln x$ is even $y = 43$ cm high. See Fig. 6.15.

These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than a function $g(x)$ as $x \to \infty$.

Definition
Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0. \qquad L \text{ finite and not zero}$$

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because

$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2,$$

which is a finite, nonzero limit. The reason for this apparent disregard of common sense is that we want "f grows faster than g" to mean that for large x-values g is negligible when compared with f.

EXAMPLE 1 e^x grows faster than x^2 as $x \to \infty$ because

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{\infty/\infty} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\infty/\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Using l'Hôpital's rule twice

❑

EXAMPLE 2

a) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

b) As part (a) suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

❑

EXAMPLE 3 x^2 grows faster than $\ln x$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty.$$

l'Hôpital's rule

❑

EXAMPLE 4 $\ln x$ grows slower than x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1}$$

l'Hôpital's rule

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

❑

EXAMPLE 5 In contrast to exponential functions, logarithmic functions with different bases a and b always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ❑

If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$. The reason is that

$$\lim_{x\to\infty} \frac{f}{g} = L_1 \quad \text{and} \quad \lim_{x\to\infty} \frac{g}{h} = L_2$$

together imply

$$\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 6 Show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function x:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2-\frac{1}{\sqrt{x}}\right)^2 = 4. \qquad \square$$

Order and Oh-Notation

Here we introduce the "little-oh" and "big-oh" notation invented by number theorists a hundred years ago and now commonplace in mathematical analysis and computer science.

Definition

A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

Notice that saying $f = o(g)$ as $x \to \infty$ is another way to say that f grows slower than g as $x \to \infty$.

EXAMPLE 7

$$\ln x = o(x) \text{ as } x \to \infty \quad \text{because} \quad \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

$$x^2 = o(x^3+1) \text{ as } x \to \infty \quad \text{because} \quad \lim_{x\to\infty} \frac{x^2}{x^3+1} = 0 \qquad \square$$

Definition

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 8

$$x + \sin x = O(x) \text{ as } x \to \infty \quad \text{because} \quad \frac{x + \sin x}{x} \le 2 \text{ for } x \text{ sufficiently large.}$$

❑

EXAMPLE 9

$$e^x + x^2 = O(e^x) \text{ as } x \to \infty \quad \text{because} \quad \frac{e^x + x^2}{e^x} \to 1 \text{ as } x \to \infty,$$

$$x = O(e^x) \text{ as } x \to \infty \quad \text{because} \quad \frac{x}{e^x} \to 0 \text{ as } x \to \infty.$$

❑

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$ (Exercise 11).

Sequential vs. Binary Search

Computer scientists sometimes measure the efficiency of an algorithm by counting the number of steps a computer must take to make the algorithm do something. There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. These differences are often described in big-oh notation. Here is an example.

Webster's Third New International Dictionary lists about 26,000 words that begin with the letter a. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This method, called sequential search, makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take 26,000 steps.

Another way to find the word or to learn it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that contains it and forget about the half that does not. (You know which half contains it because you know the list is ordered alphabetically.) This method eliminates roughly 13,000 words in a single step. If you do not find the word on the second try, then jump to the middle of the half that contains it. Continue this way until you have either found the word or divided the list in half so many times there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$(26{,}000/2^{15}) < 1.$$

That certainly beats a possible 26,000 steps.

For a list of length n, a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list. A binary search, as the second algorithm is called, takes on the order of $\log_2 n$ steps. The reason is that if $2^{m-1} < n \le 2^m$, then $m - 1 < \log_2 n \le m$, and the number of bisections required to narrow the list to one word will be at most $m = \lceil \log_2 n \rceil$, the integer ceiling for $\log_2 n$.

Big-oh notation provides a compact way to say all this. The number of steps in a sequential search of an ordered list is $O(n)$; the number of steps in a binary search is $O(\log_2 n)$. In our example, there is a big difference between the two (26,000 vs. 15), and the difference can only increase with n because n grows faster than $\log_2 n$ as $n \to \infty$.

> To find an item in a list of length n:
>
> A sequential search takes $O(n)$ steps.
>
> A binary search takes $O(\log_2 n)$ steps.

Exercises 6.7

Comparisons with the Exponential e^x

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a) $x+3$ b) $x^3+\sin^2 x$
   c) $\sqrt{x}$ d) 4^x
   e) $(3/2)^x$ f) $e^{x/2}$
   g) $e^x/2$ h) $\log_{10} x$

2. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a) $10x^4+30x+1$ b) $x\ln x - x$
   c) $\sqrt{1+x^4}$ d) $(5/2)^x$
   e) e^{-x} f) xe^x
   g) $e^{\cos x}$ h) e^{x-1}

Comparisons with the Power x^2

3. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a) x^2+4x b) x^5-x^2
   c) $\sqrt{x^4+x^3}$ d) $(x+3)^2$
   e) $x\ln x$ f) 2^x
   g) x^3e^{-x} h) $8x^2$

4. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a) $x^2+\sqrt{x}$ b) $10x^2$
   c) x^2e^{-x} d) $\log_{10}(x^2)$
   e) x^3-x^2 f) $(1/10)^x$
   g) $(1.1)^x$ h) x^2+100x

Comparisons with the Logarithm $\ln x$

5. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

   a) $\log_3 x$ b) $\ln 2x$
   c) $\ln\sqrt{x}$ d) $\sqrt{x}$
   e) x f) $5\ln x$
   g) $1/x$ h) e^x

6. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

   a) $\log_2(x^2)$ b) $\log_{10} 10x$
   c) $1/\sqrt{x}$ d) $1/x^2$
   e) $x-2\ln x$ f) e^{-x}
   g) $\ln(\ln x)$ h) $\ln(2x+5)$

Ordering Functions by Growth Rates

7. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

   a) e^x b) x^x c) $(\ln x)^x$ d) $e^{x/2}$

8. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

   a) 2^x b) x^2 c) $(\ln 2)^x$ d) e^x

Big-oh and Little-oh; Order

9. True, or false? As $x \to \infty$,

   a) $x=o(x)$ b) $x=o(x+5)$
   c) $x=O(x+5)$ d) $x=O(2x)$
   e) $e^x=o(e^{2x})$ f) $x+\ln x=O(x)$
   g) $\ln x=o(\ln 2x)$ h) $\sqrt{x^2+5}=O(x)$

10. True, or false? As $x \to \infty$,

    a) $\dfrac{1}{x+3}=O\left(\dfrac{1}{x}\right)$ b) $\dfrac{1}{x}+\dfrac{1}{x^2}=O\left(\dfrac{1}{x}\right)$
    c) $\dfrac{1}{x}-\dfrac{1}{x^2}=o\left(\dfrac{1}{x}\right)$ d) $2+\cos x=O(2)$
    e) $e^x+x=O(e^x)$ f) $x\ln x=o(x^2)$
    g) $\ln(\ln x)=O(\ln x)$ h) $\ln(x)=o(\ln(x^2+1))$

11. Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$, then $f=O(g)$ and $g=O(f)$.

12. When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

13. When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \to \infty$? Give reasons for your anwer.

14. *Simpson's rule and the trapedzoidal rule.* The definitions in the present section can be made more general by lifting the restriction that $x \to \infty$ and considering limits as $x \to a$ for any

real number a. Show that the error E_S in the Simpson's rule approximation of a definite integral is $O(h^4)$ as $h \to 0$ while the error E_T in the trapezoidal rule approximation is $O(h^2)$. This gives another way to explain the relative accuracies of the two approximation methods.

Other Comparisons

15. What do the conclusions we drew in Section 3.5 about the limits of rational functions tell us about the relative growth of polynomials as $x \to \infty$?

16. GRAPHER

a) Investigate

$$\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln(x+999)}{\ln x}.$$

Then use l'Hôpital's rule to explain what you find.

b) Show that the value of

$$\lim_{x\to\infty} \frac{\ln(x+a)}{\ln x}$$

is the same no matter what value you assign to the constant a. What does this say about the relative rates at which the functions $f(x) = \ln(x+a)$ and $g(x) = \ln x$ grow?

17. Show that $\sqrt{10x+1}$ and $\sqrt{x+1}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \to \infty$.

18. Show that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as x^2 as $x \to \infty$.

19. Show that e^x grows faster as $x \to \infty$ than x^n for any positive integer n, even $x^{1,000,000}$. (*Hint:* What is the nth derivative of x^n?)

20. *The function e^x outgrows any polynomial.* Show that e^x grows faster as $x \to \infty$ than any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

21. a) Show that $\ln x$ grows slower as $x \to \infty$ than $x^{1/n}$ for any positive integer n, even $x^{1/1,000,000}$.

b) CALCULATOR Although the values of $x^{1/1,000,000}$ eventually overtake the values of $\ln x$, you have to go way out on the x-axis before this happens. Find a value of x greater than 1 for which $x^{1/1,000,000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1,000,000}$ is equivalent to the equation $\ln(\ln x) = (\ln x)/1,000,000$.

c) CALCULATOR Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of x at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln(\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

d) GRAPHER (*Continuation of part c.*) The value of x at which $\ln x = 10 \ln(\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. *The function $\ln x$ grows slower than any polynomial.* Show that $\ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

Algorithms and Searches

23. a) Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2.$$

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

b) GRAPHER Graph the functions in part (a) together to get a sense of how rapidly each one grows.

24. Repeat Exercise 23 for the functions

$$n, \quad \sqrt{n}\log_2 n, \quad (\log_2 n)^2.$$

25. CALCULATOR Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? a binary search?

26. CALCULATOR You are looking for an item in an ordered list 450,000 items long (the length of *Webster's Third New International Dictionary*). How many steps might it take to find the item with a sequential search? a binary search?

6.8 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how the functions are defined, graphed, and evaluated.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat), but we can restrict their domains to intervals on which they are one-to-one.

Domain Restrictions That Make the Trigonometric Functions One-to-One

Function	Domain	Range
$\sin x$	$[-\pi/2\ ,\ \pi/2]$	$[-1, 1]$
$\cos x$	$[0, \pi]$	$[-1, 1]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \text{arc } \sin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \text{arc } \cos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \text{arc } \tan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \text{arc } \cot x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \text{arc } \sec x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \text{arc } \csc x$$

These equations are read "y equals the arc sine of x" or "y equals arc sin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The domains of the inverses are chosen to satisfy the following relationships.

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{1}$$

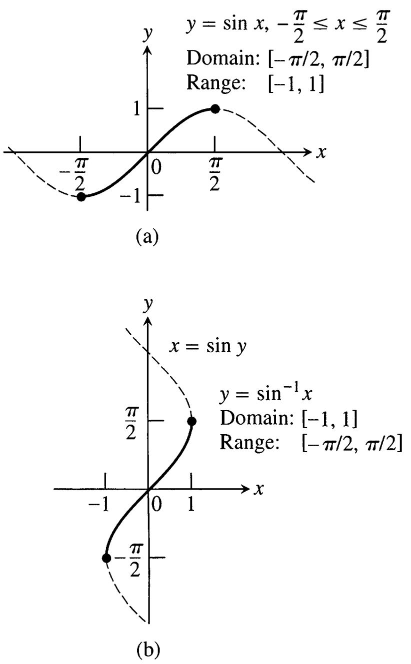
$$\csc^{-1} x = \sin^{-1}(1/x) \tag{2}$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x \tag{3}$$

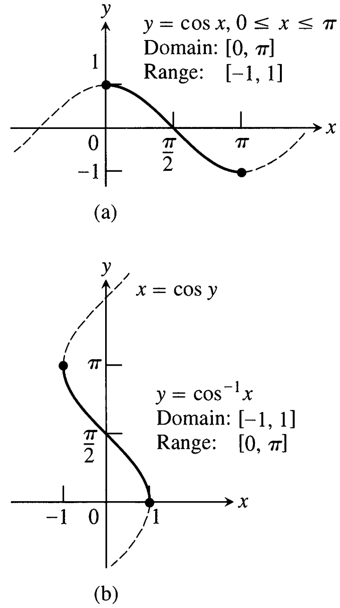
We can use these relationships to find values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ on calculators that give only $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$. As in some of the examples that follow, we can also find a few of the more common values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ using reference right triangles.

The Arc Sine and Arc Cosine

The arc sine of x is an angle whose sine is x. The arc cosine is an angle whose cosine is x.



6.16 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.



6.17 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Definition

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

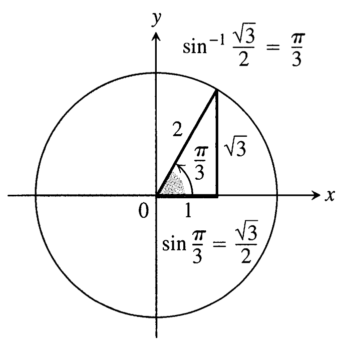
The graph of $y = \sin^{-1} x$ (Fig. 6.16) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arc sine is therefore an odd function:

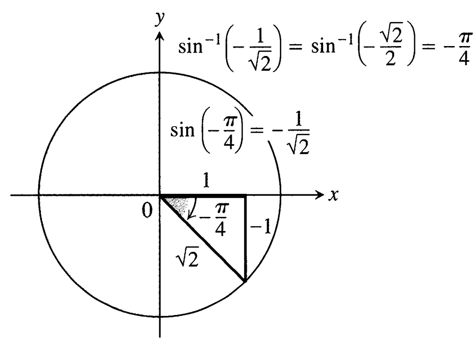
$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{4}$$

The graph of $y = \cos^{-1} x$ (Fig. 6.17) has no such symmetry.

EXAMPLE 1 *Common values of* $\sin^{-1} x$

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

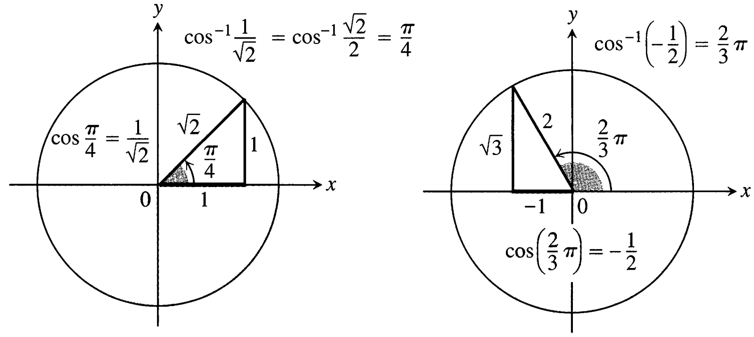




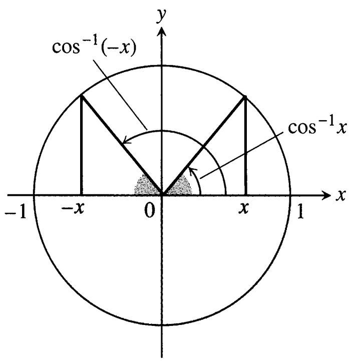
The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ❑

EXAMPLE 2 *Common values of* $\cos^{-1} x$

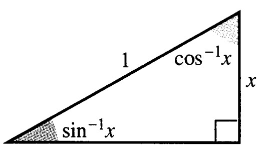
x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ❑



6.18 $\cos^{-1} x + \cos^{-1}(-x) = \pi$



6.19 In this figure,
$\sin^{-1} x + \cos^{-1} x = \pi/2$.

Identities Involving Arc Sine and Arc Cosine

As we can see from Fig. 6.18, the arc cosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{5}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{6}$$

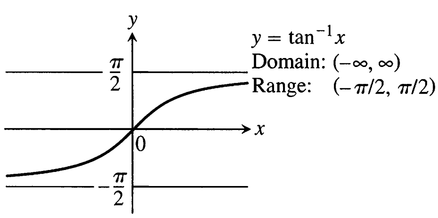
And we can see from the triangle in Fig. 6.19 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{7}$$

Equation (7) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Fig. 6.19. It is, however, a consequence of Eqs. (4) and (6) (Exercise 55).

Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The arc tangent of x is an angle whose tangent is x. The arc cotangent of x is an angle whose cotangent is x.



6.20 The graph of $y = \tan^{-1} x$.

Definition

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

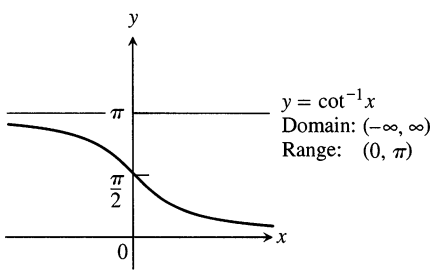
$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

We use open intervals to avoid values where the tangent and cotangent are undefined.

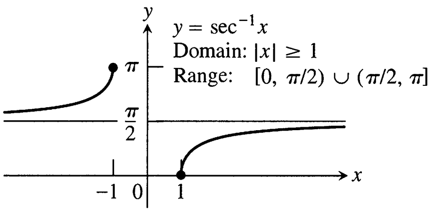
The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Fig. 6.20). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x; \tag{8}$$

the arc tangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Fig. 6.21).



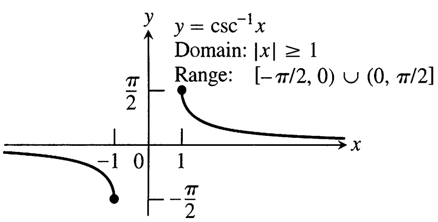
6.21 The graph of $y = \cot^{-1} x$.



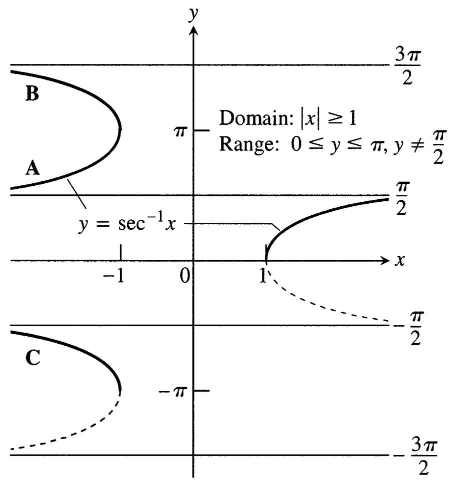
6.22 The graph of $y = \sec^{-1} x$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figs. 6.22 and 6.23.

Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Fig. 6.24). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$.



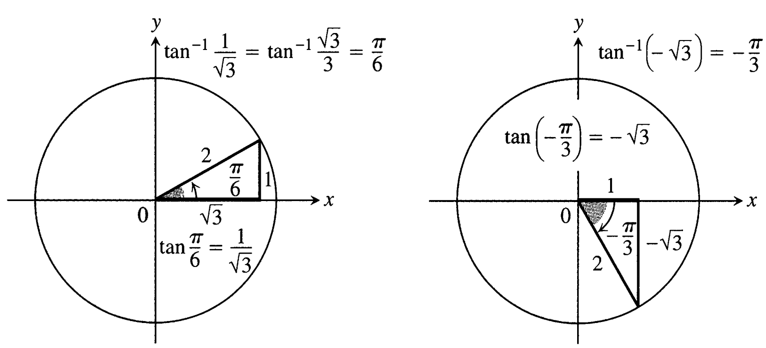
6.23 The graph of $y = \csc^{-1} x$.



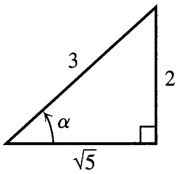
6.24 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, Eq. (1) holds, but the formula for the derivative of the arc secant is complicated by absolute value bars. Choices **B** and **C** lead to a simpler derivative formula, but Eq. (1) no longer holds. Most calculators use Eq. (1), so we chose **A**.

EXAMPLE 3 *Common values of* $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $[-\pi/2, \pi/2]$. ❑



6.25 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

EXAMPLE 4 Find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$ if

$$\alpha = \sin^{-1}\frac{2}{3}. \tag{12}$$

Solution Equation (12) says that $\sin\alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Fig. 6.25). The length of the remaining side is

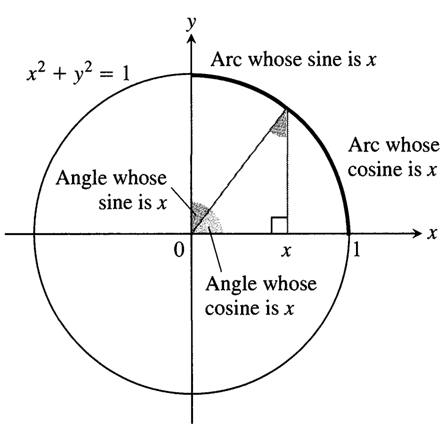
$$\sqrt{(3)^2 - (2)^2} = \sqrt{9-4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{\sqrt{5}}, \quad \sec\alpha = \frac{3}{\sqrt{5}}, \quad \csc\alpha = \frac{3}{2}, \quad \cot\alpha = \frac{\sqrt{5}}{2}. \qquad \square$$

The "arc" in arc sine and arc cosine

In case you are wondering about the "arc," look at the accompanying figure. It gives a geometric interpretation of $y = \sin^{-1}x$ and $y = \cos^{-1}x$ for angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x." When angles were measured by intercepted arc lengths, as they once were, this was a natural way to speak. Today it can sound a bit strange, but the language has stayed with us. The arc cosine has a similar interpretation.

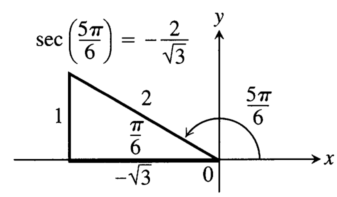


EXAMPLE 5 Find $\cot\left(\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right)$.

Solution We work from inside out, using reference triangles to exhibit ratios and angles.

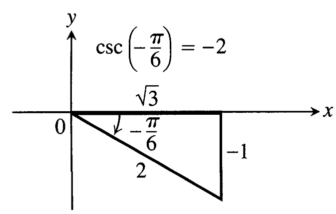
Step 1: Negative values of the secant come from second-quadrant angles:

$$\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sec^{-1}\left(\frac{2}{-\sqrt{3}}\right) = \frac{5\pi}{6}.$$



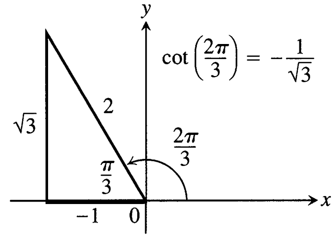
Step 2: Negative values of the cosecant come from fourth-quadrant angles:

$$\csc^{-1}(-2) = \csc^{-1}\left(\frac{2}{-1}\right) = -\frac{\pi}{6}.$$



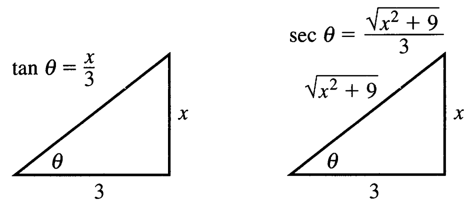
Step 3:

$$\begin{aligned}\cot\left(\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right) &= \cot\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) \\ &= \cot\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$



$\square$

EXAMPLE 6 Find $\sec\left(\tan^{-1}\dfrac{x}{3}\right)$.



Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

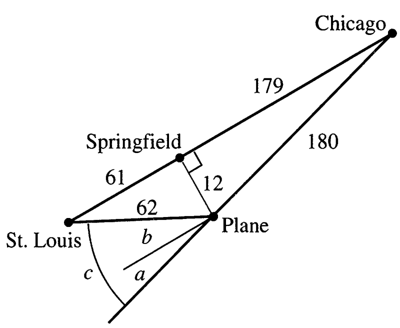
$$\sqrt{x^2+3^2} = \sqrt{x^2+9}.$$

Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

❑



6.26 Diagram for drift correction (Example 7), with distances rounded to the nearest mile (drawing not to scale).

EXAMPLE 7 *Drift correction*

During an airplane flight from Chicago to St. Louis the navigator determines that the plane is 12 mi off course, as shown in Fig. 6.26. Find the angle a for a course parallel to the original, correct course, the angle b, and the correction angle $c = a + b$.

Solution

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8^\circ$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2^\circ$$

$$c = a + b \approx 15^\circ.$$

❑

Exercises 6.8

Common Values of Inverse Trignonometric Functions

Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

1. a) $\tan^{-1} 1$ **b)** $\tan^{-1}(-\sqrt{3})$ **c)** $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

2. a) $\tan^{-1}(-1)$ **b)** $\tan^{-1}\sqrt{3}$ **c)** $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

3. a) $\sin^{-1}\left(\dfrac{-1}{2}\right)$ **b)** $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ **c)** $\sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$

4. a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

5. a) $\cos^{-1}\left(\frac{1}{2}\right)$ b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

6. a) $\cos^{-1}\left(\frac{-1}{2}\right)$ b) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ c) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

7. a) $\sec^{-1}(-\sqrt{2})$ b) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c) $\sec^{-1}(-2)$

8. a) $\sec^{-1}\sqrt{2}$ b) $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c) $\sec^{-1} 2$

9. a) $\csc^{-1}\sqrt{2}$ b) $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c) $\csc^{-1} 2$

10. a) $\csc^{-1}(-\sqrt{2})$ b) $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c) $\csc^{-1}(-2)$

11. a) $\cot^{-1}(-1)$ b) $\cot^{-1}\sqrt{3}$ c) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

12. a) $\cot^{-1} 1$ b) $\cot^{-1}(-\sqrt{3})$ c) $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

15. Given that $\alpha = \sec^{-1}(-\sqrt{5})$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercises 17–28.

17. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

18. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

20. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

21. $\csc(\sec^{-1} 2) + \cos(\tan^{-1}(-\sqrt{3}))$

22. $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2))$

23. $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

24. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sec^{-1} 2\right)$

25. $\sec(\tan^{-1} 1 + \csc^{-1} 1)$

26. $\sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1))$

27. $\sec^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$ (The answer is *not* $-\pi/6$.)

28. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$ (The answer is *not* $-\pi/4$.)

Finding Trigonometric Expressions

Evaluate the expressions in Exercises 29–40.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$

30. $\sec(\tan^{-1} 2x)$

31. $\tan(\sec^{-1} 3y)$

32. $\tan\left(\sec^{-1}\frac{y}{5}\right)$

33. $\cos(\sin^{-1} x)$

34. $\tan(\cos^{-1} x)$

35. $\sin(\tan^{-1}\sqrt{x^2 - 2x})$, $x \geq 2$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$

37. $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

38. $\cos\left(\sin^{-1}\frac{y}{5}\right)$

39. $\sin\left(\sec^{-1}\frac{x}{4}\right)$

40. $\sin\left(\sec^{-1}\frac{\sqrt{x^2+4}}{x}\right)$

Limits

Find the limits in Exercises 41–48. (If in doubt, look at the function's graph.)

41. $\lim_{x \to 1^-} \sin^{-1} x$

42. $\lim_{x \to -1^+} \cos^{-1} x$

43. $\lim_{x \to \infty} \tan^{-1} x$

44. $\lim_{x \to -\infty} \tan^{-1} x$

45. $\lim_{x \to \infty} \sec^{-1} x$

46. $\lim_{x \to -\infty} \sec^{-1} x$

47. $\lim_{x \to \infty} \csc^{-1} x$

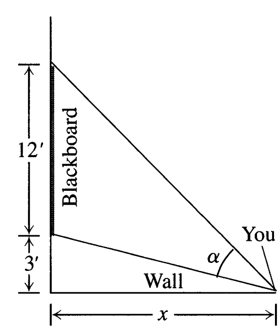
48. $\lim_{x \to -\infty} \csc^{-1} x$

Applications and Theory

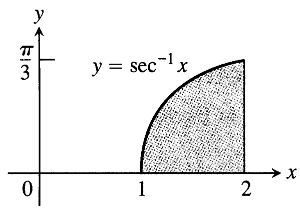
49. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

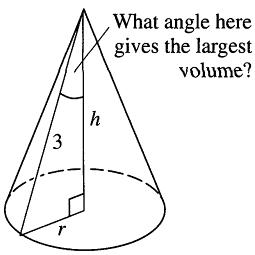
if you are x ft from the front wall.



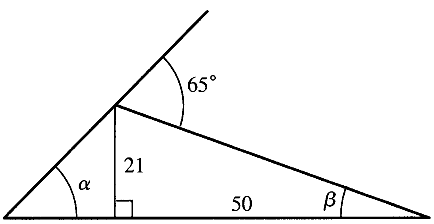
50. The region between the curve $y = \sec^{-1} x$ and the x-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y-axis to generate a solid. Find the volume of the solid.



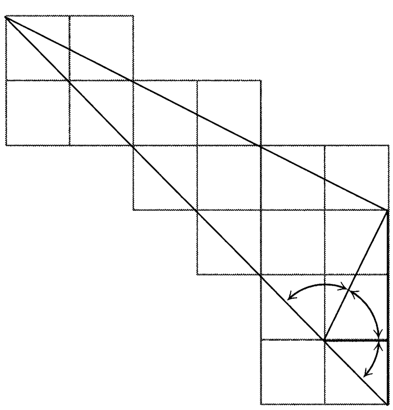
51. The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?



52. Find the angle α.

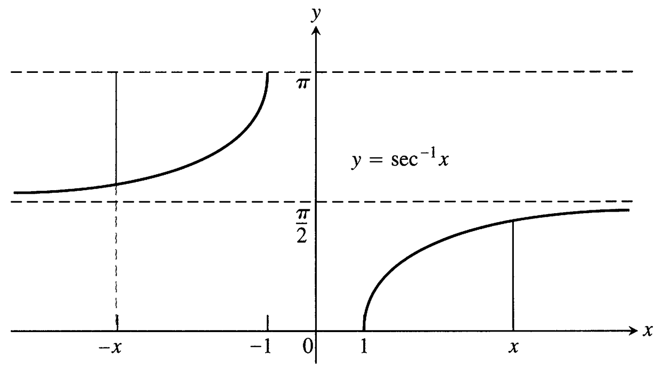


53. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



54. *Two derivations of the identity* $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

a) (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b) (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (6)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (1)}$$

55. *The identity* $\sin^{-1} x + \cos^{-1} x = \pi/2$. Figure 6.19 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1, 0$, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (4) and (6) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

56. Show that the sum $\tan^{-1} x + \tan^{-1}(1/x)$ is constant.

Which of the expressions in Exercises 57–60 are defined, and which are not? Give reasons for your answers.

57. **a)** $\tan^{-1} 2$ **b)** $\cos^{-1} 2$

58. **a)** $\csc^{-1} \dfrac{1}{2}$ **b)** $\csc^{-1} 2$

59. **a)** $\sec^{-1} 0$ **b)** $\sin^{-1} \sqrt{2}$

60. **a)** $\cot^{-1} \left(-\dfrac{1}{2}\right)$ **b)** $\cos^{-1}(-5)$

Calculator Explorations

61. Find the values of

a) $\sec^{-1} 1.5$ **b)** $\csc^{-1}(-1.5)$ **c)** $\cot^{-1} 2$

62. Find the values of

a) $\sec^{-1}(-3)$ **b)** $\csc^{-1} 1.7$ **c)** $\cot^{-1}(-2)$

Grapher Explorations

In Exercises 63–65, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

63. **a)** $y = \tan^{-1}(\tan x)$ **b)** $y = \tan(\tan^{-1} x)$

64. **a)** $y = \sin^{-1}(\sin x)$ **b)** $y = \sin(\sin^{-1} x)$

65. **a)** $y = \cos^{-1}(\cos x)$ **b)** $y = \cos(\cos^{-1} x)$

66. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

67. *Newton's serpentine.* Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

68. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

6.9 Derivatives of Inverse Trigonometric Functions; Integrals

Inverse trigonometric functions provide antiderivatives for a variety of functions that arise in mathematics, engineering, and physics. In this section we find the derivatives of the inverse trigonometric functions (Table 6.5) and discuss related integrals.

Table 6.5 Derivatives of the inverse trigonometric functions

1.	$\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad \lvert u\rvert < 1$
2.	$\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad \lvert u\rvert < 1$
3.	$\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$
4.	$\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$
5.	$\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{\lvert u\rvert\sqrt{u^2-1}}, \quad \lvert u\rvert > 1$
6.	$\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u\rvert\sqrt{u^2-1}}, \quad \lvert u\rvert > 1$

EXAMPLE 1

a) $$\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

b) $$\frac{d}{dx}\tan^{-1}\sqrt{x+1} = \frac{1}{1+(\sqrt{x+1})^2} \cdot \frac{d}{dx}(\sqrt{x+1})$$
$$= \frac{1}{x+2} \cdot \frac{1}{2\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}(x+2)}$$

c) $$\frac{d}{dx}\sec^{-1}(-3x) = \frac{1}{\lvert -3x\rvert\sqrt{(-3x)^2-1}} \cdot \frac{d}{dx}(-3x)$$
$$= \frac{-3}{\lvert 3x\rvert\sqrt{9x^2-1}} = \frac{-1}{\lvert x\rvert\sqrt{9x^2-1}}.$$ ❑

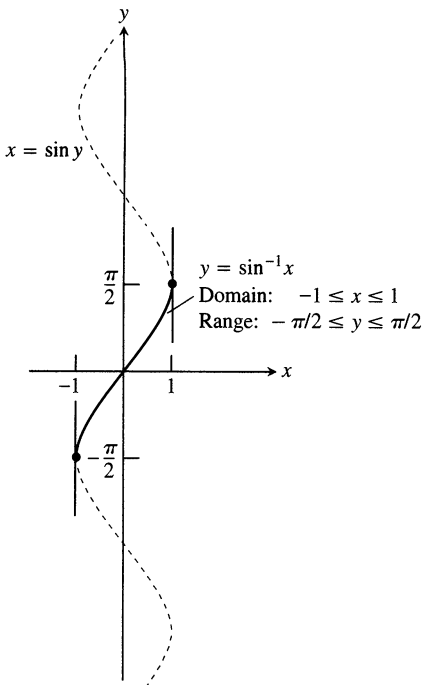
EXAMPLE 2

$$\int_0^1 \frac{e^{\tan^{-1} x}}{1+x^2}dx = \int_0^{\pi/4} e^u du \qquad u = \tan^{-1} x, \quad du = \frac{dx}{1+x^2}, \quad u(0) = 0, \quad u(1) = \pi/4$$
$$= e^u\Big]_0^{\pi/4} = e^{\pi/4} - 1$$ ❑

We derive Formulas 1 and 5 from Table 6.5. The derivation of Formula 3 is similar. Formulas 2, 4, and 6 can be derived from Formulas 1, 3, and 5 by differentiating appropriate identities (Exercises 81–83).

The Derivative of $y \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 6.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout



6.27 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Fig. 6.27).

We find the derivative of $y = \sin^{-1} x$ as follows:

$$\begin{aligned}
\sin y &= x && y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x \\
\frac{d}{dx}(\sin y) &= 1 && \text{Derivative of both sides with respect to } x \\
\cos y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\cos y} && \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2. \\
&= \frac{1}{\sqrt{1-x^2}} && \text{Fig. 6.28}
\end{aligned}$$

The derivative of $y = \sin^{-1} x$ with respect to x is

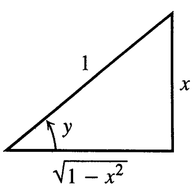
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to $y = \sin^{-1} u$ to obtain

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$



6.28 In the reference right triangle above,

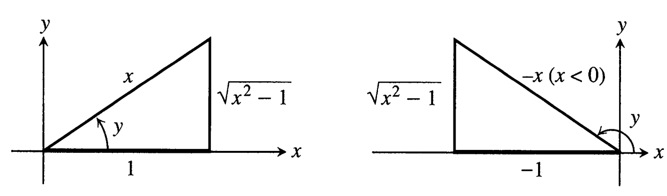
$$\sin y = \frac{x}{1} = x,$$

$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

The Derivative of $y = \sec^{-1} u$

We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, in a similar way.

$$\begin{aligned}
\sec y &= x && y = \sec^{-1} x \quad \Leftrightarrow \quad \sec y = x \\
\frac{d}{dx}(\sec y) &= 1 && \text{Derivative of both sides with respect to } x \\
\sec y \tan y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y} && \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2\ , \pi) \text{ and } \sec y \tan y \neq 0. \\
&= \pm\frac{1}{x\sqrt{x^2-1}} && \text{Fig. 6.29}
\end{aligned}$$

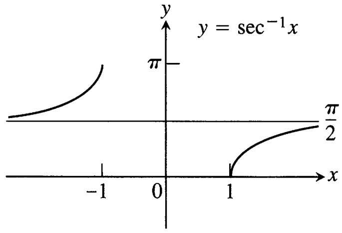


6.29 In both quadrants, $\sec y = x$. In the first quadrant,

$$\tan y = \sqrt{x^2-1}/1 = \sqrt{x^2-1}.$$

In the second quadrant,

$$\tan y = \sqrt{x^2-1}/(-1) = -\sqrt{x^2-1}.$$



6.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

What do we do about the sign? A glance at Fig. 6.30 shows that for $|x| > 1$ the slope of the graph of $y = \sec^{-1} x$ is always positive. Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases} \tag{1}$$

With absolute values, we can write Eq. (1) as a single formula:

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| > 1.$$

If u is a differentiable function of x with $|u| > 1$, we can then apply the Chain Rule to obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1.$$

Integration Formulas

The derivative formulas in Table 6.5 yield three useful integration formulas in Table 6.6.

Table 6.6 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$	(Valid for $u^2 < a^2$)	(2)
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$	(Valid for all u)	(3)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$	(Valid for $u^2 > a^2$)	(4)

The derivative formulas in Table 6.5 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 6.6 are more useful. They are readily verified by differentiating the functions on the right-hand sides.

EXAMPLE 3

a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$
$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

b) $$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

c) $$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x)\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$ ❑

EXAMPLE 4

a) $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$ Eq. (2) with $a = 3$, $u = x$

b) $$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$ $a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (2)

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$ ❑

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$.

For more about completing the square, see the end papers of this book.

Solution The expression $\sqrt{4x-x^2}$ does not match any of the formulas in Table 6.6, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2-u^2}}$$ $a = 2$, $u = x - 2$, and $du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (2)

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$ ❑

EXAMPLE 6

a) $$\int \frac{dx}{10+x^2} = \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{x}{\sqrt{10}}\right) + C$$ Eq. (3) with $a = \sqrt{10}$, $u = x$

b) $$\int \frac{dx}{7+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{a^2+u^2}$$ $a = \sqrt{7}$, $u = \sqrt{3}x$, and $du/\sqrt{3} = dx$

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (3)

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + C$$

$$= \frac{1}{\sqrt{21}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + C$$ ❑

EXAMPLE 7 Evaluate $\displaystyle\int \frac{dx}{4x^2+4x+2}$.

Solution We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then we substitute $a = 1$, $u = 2x + 1$, and $du/2 = dx$ to get

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x+1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad \begin{array}{l} a = 1, \\ u = 2x + 1, \text{ and} \\ du/2 = dx \end{array}$$

$$= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) \qquad \text{Eq. (3)}$$

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C \qquad \begin{array}{l} a = 1, \\ u = 2x + 1 \end{array}$$

❑

EXAMPLE 8 Evaluate $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 5}}$.

Solution

$$\int \frac{dx}{x\sqrt{4x^2 - 5}} = \int \frac{\frac{du}{2}}{\frac{u}{2}\sqrt{u^2 - a^2}} \qquad \begin{array}{l} u = 2x,\ x = u/2, \\ dx = du/2, \\ a = \sqrt{5} \end{array}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}} \qquad \text{The 2's cancel.}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Eq. (4)}$$

$$= \frac{1}{\sqrt{5}} \sec^{-1}\left(\frac{2|x|}{\sqrt{5}}\right) + C \qquad a = \sqrt{5}, \quad u = 2x$$

❑

EXAMPLE 9 Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Solution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}} \qquad \begin{array}{l} u = e^x, \\ du = e^x dx, \\ dx = du/e^x = du/u, \\ a = \sqrt{6} \end{array}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Eq. (4)}$$

$$= \frac{1}{\sqrt{6}} \sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

❑

Exercises 6.9

Finding Derivatives

In Exercises 1–22, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$
2. $y = \cos^{-1}(1/x)$
3. $y = \sin^{-1}\sqrt{2}\,t$
4. $y = \sin^{-1}(1 - t)$
5. $y = \sec^{-1}(2s + 1)$
6. $y = \sec^{-1} 5s$
7. $y = \csc^{-1}(x^2 + 1), \quad x > 0$
8. $y = \csc^{-1}\dfrac{x}{2}$
9. $y = \sec^{-1}\dfrac{1}{t}, \quad 0 < t < 1$
10. $y = \sin^{-1}\dfrac{3}{t^2}$
11. $y = \cot^{-1}\sqrt{t}$
12. $y = \cot^{-1}\sqrt{t - 1}$
13. $y = \ln(\tan^{-1} x)$
14. $y = \tan^{-1}(\ln x)$
15. $y = \csc^{-1}(e^t)$
16. $y = \cos^{-1}(e^{-t})$
17. $y = s\sqrt{1 - s^2} + \cos^{-1} s$
18. $y = \sqrt{s^2 - 1} - \sec^{-1} s$
19. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$
20. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$
21. $y = x\sin^{-1} x + \sqrt{1 - x^2}$
22. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

Evaluating Integrals

Evaluate the integrals in Exercises 23–46.

23. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$
24. $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$
25. $\displaystyle\int \frac{dx}{17 + x^2}$
26. $\displaystyle\int \frac{dx}{9 + 3x^2}$
27. $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$
28. $\displaystyle\int \frac{dx}{x\sqrt{5x^2 - 4}}$
29. $\displaystyle\int_0^1 \frac{4\,ds}{\sqrt{4 - s^2}}$
30. $\displaystyle\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$
31. $\displaystyle\int_0^2 \frac{dt}{8 + 2t^2}$
32. $\displaystyle\int_{-2}^2 \frac{dt}{4 + 3t^2}$
33. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$
34. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$
35. $\displaystyle\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$
36. $\displaystyle\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$
37. $\displaystyle\int \frac{dx}{2 + (x - 1)^2}$
38. $\displaystyle\int \frac{dx}{1 + (3x + 1)^2}$
39. $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$
40. $\displaystyle\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$
41. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1 + (\sin\theta)^2}$
42. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1 + (\cot x)^2}$
43. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1 + e^{2x}}$
44. $\displaystyle\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1 + \ln^2 t)}$
45. $\displaystyle\int \frac{y\,dy}{\sqrt{1 - y^4}}$
46. $\displaystyle\int \frac{\sec^2 y\,dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 47–56.

47. $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$
48. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$
49. $\displaystyle\int_{-1}^0 \frac{6\,dt}{\sqrt{3 - 2t - t^2}}$
50. $\displaystyle\int_{1/2}^1 \frac{6\,dt}{\sqrt{3 + 4t - 4t^2}}$
51. $\displaystyle\int \frac{dy}{y^2 - 2y + 5}$
52. $\displaystyle\int \frac{dy}{y^2 + 6y + 10}$
53. $\displaystyle\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$
54. $\displaystyle\int_2^4 \frac{2\,dx}{x^2 - 6x + 10}$
55. $\displaystyle\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$
56. $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 57–64.

57. $\displaystyle\int \frac{e^{\sin^{-1} x}dx}{\sqrt{1 - x^2}}$
58. $\displaystyle\int \frac{e^{\cos^{-1} x}dx}{\sqrt{1 - x^2}}$
59. $\displaystyle\int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1 - x^2}}$
60. $\displaystyle\int \frac{\sqrt{\tan^{-1} x}\,dx}{1 + x^2}$
61. $\displaystyle\int \frac{dy}{(\tan^{-1} y)(1 + y^2)}$
62. $\displaystyle\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$
63. $\displaystyle\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$
64. $\displaystyle\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

Limits

Find the limits in Exercises 65–68.

65. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} 5x}{x}$
66. $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1} x}$
67. $\displaystyle\lim_{x\to\infty} x\tan^{-1}\frac{2}{x}$
68. $\displaystyle\lim_{x\to 0} \frac{2\tan^{-1} 3x^2}{7x^2}$

Integration Formulas

Verify the integration formulas in Exercises 69–72.

69. $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

70. $\displaystyle\int x^3 \cos^{-1} 5x\,dx = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4\,dx}{\sqrt{1-25x^2}}$

71. $\displaystyle\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1} x + C$

72. $\displaystyle\int \ln\,(a^2+x^2)\,dx = x\ln\,(a^2+x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Initial Value Problems

Solve the initial value problems in Exercises 73–76.

73. $\displaystyle\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$

74. $\displaystyle\frac{dy}{dx} = \frac{1}{x^2+1} - 1, \quad y(0) = 1$

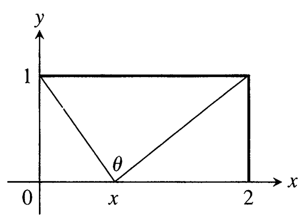
75. $\displaystyle\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1; \quad y(2) = \pi$

76. $\displaystyle\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$

Theory and Examples

77. (*Continuation of Exercise 49, Section 6.8.*) You want to position your chair along the wall to maximize your viewing angle α. How far from the front of the room should you sit?

78. What value of x maximizes the angle θ shown here? How large is θ at that point? Begin by showing that $\theta = \pi - \cot^{-1} x - \cot^{-1}(2-x)$.



79. Can the integrations in (a) and (b) both be correct? Explain.

a) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

b) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

80. Can the integrations in (a) and (b) both be correct? Explain.

a) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

b)
$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{-du}{\sqrt{1-(-u)^2}} && \begin{aligned}x &= -u,\\ dx &= -du\end{aligned}\\ &= \int \frac{-du}{\sqrt{1-u^2}}\\ &= \cos^{-1} u + C\\ &= \cos^{-1}(-x) + C && u = -x\end{aligned}$$

81. Use the identity
$$\cos^{-1} u = \frac{\pi}{2} - \sin^{-1} u$$
to derive the formula for the derivative of $\cos^{-1} u$ in Table 6.5 from the formula for the derivative of $\sin^{-1} u$.

82. Use the identity
$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$
to derive the formula for the derivative of $\cot^{-1} u$ in Table 6.5 from the formula for the derivative of $\tan^{-1} u$.

83. Use the identity
$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$
to derive the formula for the derivative of $\csc^{-1} u$ in Table 6.5 from the formula for the derivative of $\sec^{-1} u$.

84. Derive the formula
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$
for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

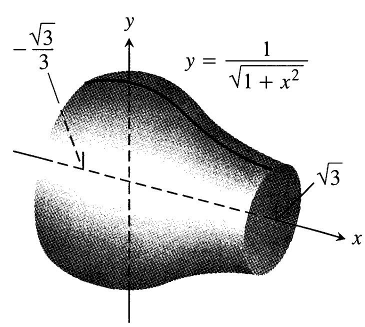
85. Use the Derivative Rule in Section 6.1, Theorem 1, to derive
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

86. Use the Derivative Rule in Section 6.1, Theorem 1, to derive
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

87. What is special about the functions
$$f(x) = \sin^{-1}\frac{x-1}{x+1}, \quad x \ge 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$
Explain.

88. What is special about the functions
$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$
Explain.

89. Find the volume of the solid of revolution shown here.



90. Find the length of the curve $y = \sqrt{1-x^2}$, $-1/2 \le x \le 1/2$.

Volumes by Slicing

Find the volumes of the solids in Exercises 91 and 92.

91. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are
   a) circles whose diameters stretch from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$;
   b) vertical squares whose base edges run from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

92. The solid lies between planes perpendicular to the x-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross sections are
   a) circles whose diameters stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.
   b) squares whose diagonals stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

Calculator and Grapher Explorations

93. CALCULATOR Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to 5 places.

94. CALCULATOR Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1+x^2}dx.$$

95. GRAPHER Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

96. GRAPHER Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

6.10 Hyperbolic Functions

Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x-e^{-x}}{2}}_{\text{odd part}}.$$

The even and odd parts of e^x, called the hyperbolic cosine and hyperbolic sine of x, respectively, are useful in their own right. They describe the motions of waves in elastic solids, the shapes of hanging electric power lines, and the temperature distributions in metal cooling fins. The center line of the Gateway Arch to the West in St. Louis is a weighted hyperbolic cosine curve.

The notation cosh x is often read "kosh x," rhyming with either "gosh x" or "gauche x," and sinh x is pronounced as if spelled "cinch x" or "shine x."

Definitions and Identities

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in Table 6.7. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. As we will see, the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named. (See Exercise 86 as well.)

Table 6.7 The six basic hyperbolic functions (See Fig. 6.31 for graphs.)

Hyperbolic cosine of x:	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic sine of x:	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic tangent:	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent:	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant:	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant:	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Table 6.8 Identities for hyperbolic functions

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Table 6.9 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Table 6.10 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$
$$\int \cosh u\, du = \sinh u + C$$
$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$
$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$
$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$
$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Identities

Hyperbolic functions satisfy the identities in Table 6.8. Except for differences in sign, these are identities we already know for trigonometric functions.

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 6.9). Again, there are similarities with trigonometric functions. The derivative formulas in Table 6.9 lead to the integral formulas in Table 6.10.

EXAMPLE 1

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2\sqrt{1+t^2}$$

❑

EXAMPLE 2

$$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u} \qquad \begin{array}{l} u = \sinh 5x, \\ du = 5\cosh 5x\, dx \end{array}$$
$$= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C$$

❑

EXAMPLE 3

$$\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx \qquad \text{Table 6.8}$$
$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1) dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

❑

Evaluating hyperbolic functions

Like many standard functions, hyperbolic functions and their inverses are easily evaluated with calculators, which have special keys or keystroke sequences for that purpose.

EXAMPLE 4

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\,dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx \\ &= [e^{2x} - 2x]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0) \\ &= 4 - 2\ln 2 - 1 \\ &\approx 1.6137\end{aligned}$$

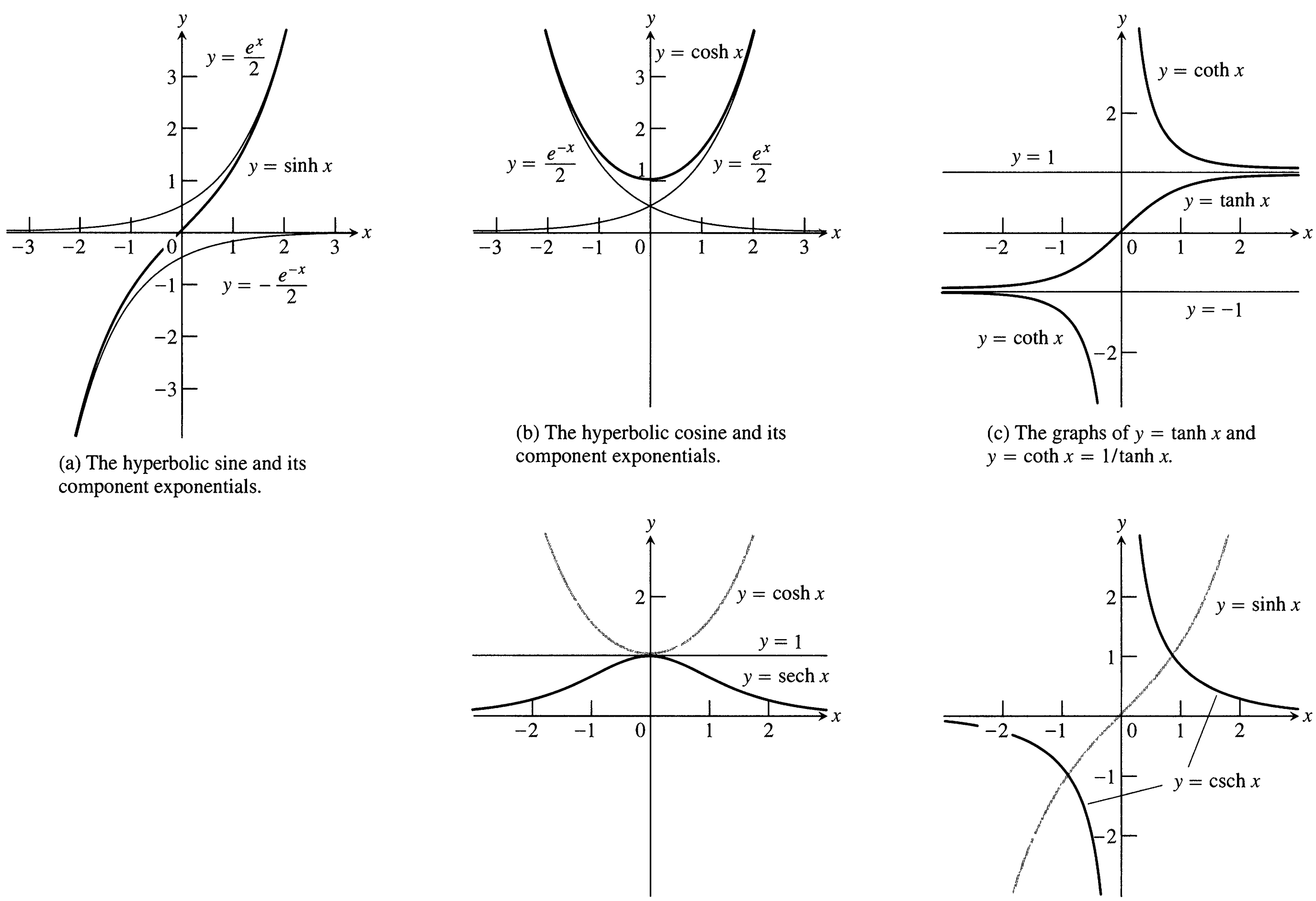
❑

The Inverse Hyperbolic Functions

We use the inverses of the six basic hyperbolic functions in integration. Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Fig. 6.32(a).



(a) The hyperbolic sine and its component exponentials.

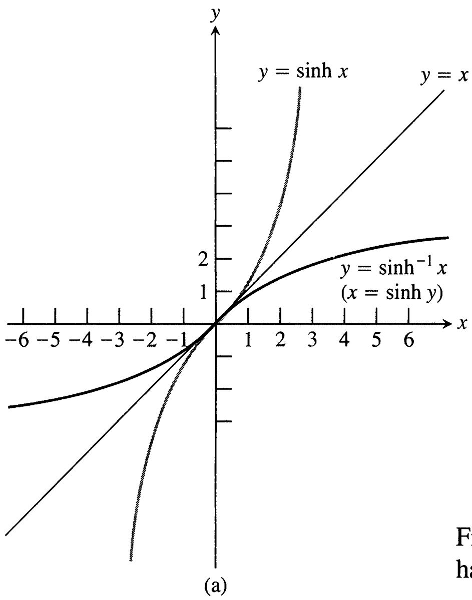
(b) The hyperbolic cosine and its component exponentials.

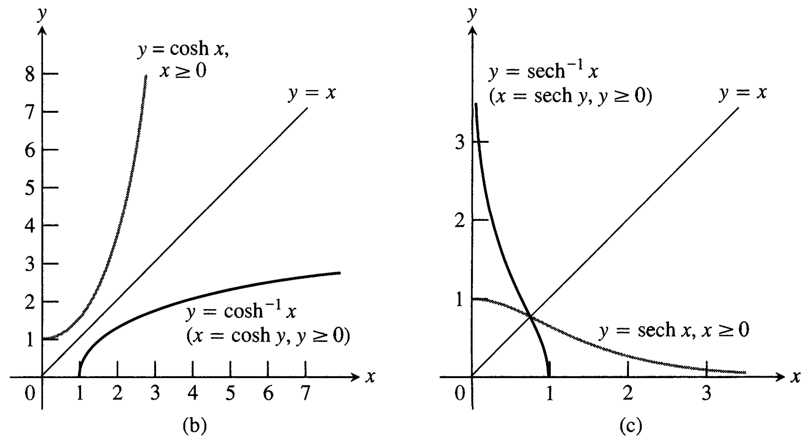
(c) The graphs of $y = \tanh x$ and $y = \coth x = 1/\tanh x$.

(d) The graphs of $y = \cosh x$ and $y = \operatorname{sech} x = 1/\cosh x$.

(e) The graphs of $y = \sinh x$ and $y = \operatorname{csch} x = 1/\sinh x$.

6.31 The graphs of the six hyperbolic functions.





6.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The function $y = \cosh x$ is not one-to-one, as we can see from the graph in Fig. 6.31. But the restricted function $y = \cosh x$, $x \geq 0$, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Fig. 6.32(b).

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Fig. 6.32(c).

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Fig. 6.33.

6.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

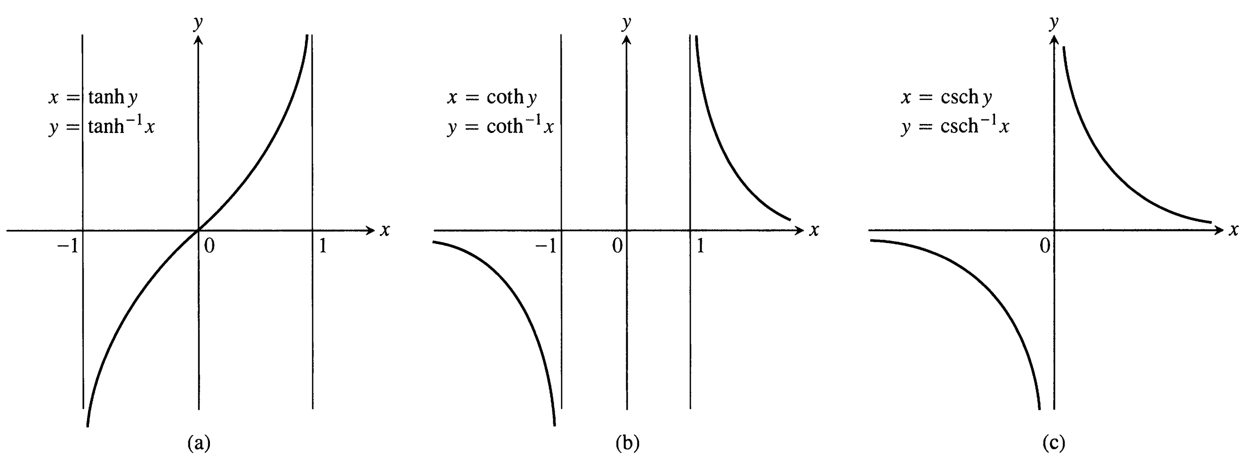


Table 6.11 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1}x = \cosh^{-1}\frac{1}{x}$$
$$\operatorname{csch}^{-1}x = \sinh^{-1}\frac{1}{x}$$
$$\coth^{-1}x = \tanh^{-1}\frac{1}{x}$$

Table 6.12 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1}u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$
$$\frac{d(\cosh^{-1}u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$
$$\frac{d(\tanh^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$
$$\frac{d(\coth^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$
$$\frac{d(\operatorname{sech}^{-1}u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$$
$$\frac{d(\operatorname{csch}^{-1}u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$$

Useful Identities

We use the identities in Table 6.11 to calculate the values of $\operatorname{sech}^{-1}x$, $\operatorname{csch}^{-1}x$, and $\coth^{-1}x$ on calculators that give only $\cosh^{-1}x$, $\sinh^{-1}x$, and $\tanh^{-1}x$.

Derivatives and Integrals

The chief use of inverse hyperbolic functions lies in integrations that reverse the derivative formulas in Table 6.12.

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1}u$ and $\coth^{-1}u$ come from the natural restrictions on the values of these functions. (See Figs. 6.33a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas. If $|u| < 1$, the integral of $1/(1-u^2)$ is $\tanh^{-1}u + C$. If $|u| > 1$, the integral is $\coth^{-1}u + C$.

EXAMPLE 5 Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1}x$ for $x > 1$:

$$\begin{aligned}
y &= \cosh^{-1}x \\
x &= \cosh y && \text{Equivalent equation} \\
1 &= \sinh y\,\frac{dy}{dx} && \text{Differentiation with respect to } x \\
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} && \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0 \\
&= \frac{1}{\sqrt{x^2-1}} && \cosh y = x
\end{aligned}$$

In short, $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2-1}}$. The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}.$$

❑

With appropriate substitutions, the derivative formulas in Table 6.12 lead to the integration formulas in Table 6.13.

EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

Solution The indefinite integral is

$$\begin{aligned}
\int \frac{2\,dx}{\sqrt{3+4x^2}} &= \int \frac{du}{\sqrt{a^2+u^2}} && u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3} \\
&= \sinh^{-1}\left(\frac{u}{a}\right) + C && \text{Formula from Table 6.13} \\
&= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.
\end{aligned}$$

Table 6.13 Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \quad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \quad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \quad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \quad u \neq 0$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

❑

Exercises 6.10

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\dfrac{3}{4}$

2. $\sinh x = \dfrac{4}{3}$

3. $\cosh x = \dfrac{17}{15}, \quad x > 0$

4. $\cosh x = \dfrac{13}{5}, \quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2\cosh(\ln x)$

6. $\sinh(2\ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

11. Use the identities

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$
$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$

to show that

a) $\sinh 2x = 2\sinh x\cosh x$;

b) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\dfrac{x}{3}$

14. $y = \dfrac{1}{2}\sinh(2x+1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$

16. $y = t^2\tanh\dfrac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech}\theta(1 - \ln\operatorname{sech}\theta)$

20. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \dfrac{1}{2}\tanh^2 v$

22. $y = \ln\sinh v - \dfrac{1}{2}\coth^2 v$

23. $y = (x^2 + 1)\operatorname{sech}(\ln x)$
(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2 - 1)\operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$

26. $y = \cosh^{-1} 2\sqrt{x+1}$

27. $y = (1-\theta)\tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

29. $y = (1-t)\coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t$

31. $y = \cos^{-1} x - x\operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\operatorname{sech}^{-1} x$

33. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

34. $y = \operatorname{csch}^{-1} 2^{\theta}$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x), \quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a) $\displaystyle\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + C$

b) $\displaystyle\int \operatorname{sech} x\,dx = \sin^{-1}(\tanh x) + C$

38. $\displaystyle\int x\operatorname{sech}^{-1} x\,dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

39. $\displaystyle\int x\coth^{-1} x\,dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

40. $\displaystyle\int \tanh^{-1} x\,dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

41. $\displaystyle\int \sinh 2x\,dx$

42. $\displaystyle\int \sinh\frac{x}{5}\,dx$

43. $\displaystyle\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx$

44. $\displaystyle\int 4\cosh(3x - \ln 2)\,dx$

45. $\displaystyle\int \tanh\frac{x}{7}\,dx$

46. $\displaystyle\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

47. $\displaystyle\int \operatorname{sech}^2\left(x - \frac{1}{2}\right)dx$

48. $\displaystyle\int \operatorname{csch}^2(5-x)\,dx$

49. $\displaystyle\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\,dt}{\sqrt{t}}$

50. $\displaystyle\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)\,dt}{t}$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

51. $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\,dx$

52. $\displaystyle\int_0^{\ln 2} \tanh 2x\,dx$

53. $\displaystyle\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\,d\theta$

54. $\displaystyle\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta$

55. $\displaystyle\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta$

56. $\displaystyle\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta$

57. $\displaystyle\int_1^2 \frac{\cosh(\ln t)}{t}dt$

58. $\displaystyle\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}dx$

59. $\displaystyle\int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right)dx$

60. $\displaystyle\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)dx$

Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms as shown in the table below.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad -\infty < x < \infty$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$$

$$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}, \quad |x| > 1$$

Use the formulas in the table here to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$

62. $\cosh^{-1}(5/3)$

63. $\tanh^{-1}(-1/2)$

64. $\coth^{-1}(5/4)$

65. $\operatorname{sech}^{-1}(3/5)$

66. $\operatorname{csch}^{-1}\left(-1/\sqrt{3}\right)$

Evaluate the integrals in Exercises 67–74 in terms of (a) inverse hyperbolic functions, (b) natural logarithms.

67. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

68. $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}}$

69. $\displaystyle\int_{5/4}^2 \frac{dx}{1-x^2}$

70. $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$

71. $\displaystyle\int_{1/5}^{3/13} \frac{dx}{\sqrt{1-16x^2}}$

72. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{4+x^2}}$

73. $\int_0^{\pi} \frac{\cos x\, dx}{\sqrt{1+\sin^2 x}}$

74. $\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Applications and Theory

75. a) Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}. \qquad (1)$$

Then show that $(f(x)+f(-x))/2$ is even and that $(f(x)-f(-x))/2$ is odd.

b) Equation (1) simplifies considerably if f itself is (i) even or (ii) odd. What are the new equations? Give reasons for your answers.

76. Derive the formula $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right), -\infty < x < \infty$. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. *Skydiving.* If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t seconds into the fall satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome.)

a) Show that

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

b) Find the body's *limiting velocity,* $\lim_{t\to\infty} v$.

c) CALCULATOR For a 160-lb skydiver ($mg = 160$), with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity?

78. *Accelerations whose magnitudes are proportional to displacement.* Suppose that the position of a body moving along a coordinate line at time t is

a) $s = a\cos kt + b\sin kt$,

b) $s = a\cosh kt + b\sinh kt$.

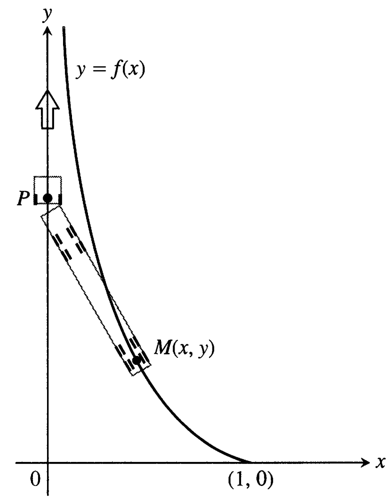
Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin while in the second case it is directed away from the origin.

79. *Tractor trailers and the tractrix.* When a tractor trailer turns into a cross street or driveway, its rear wheels follow a curve like the one shown here. (This is why the rear wheels sometimes ride up over the curb.) We can find an equation for the curve if we picture the rear wheels as a mass M at the point (1, 0) on the x-axis attached by a rod of unit length to a point P representing the cab at the origin. As the point P moves up the y-axis, it drags M along behind it. The curve traced by M, called a *tractrix* from the Latin word *tractum* for "drag," can be shown to be the graph of the function $y = f(x)$ that solves the initial value problem

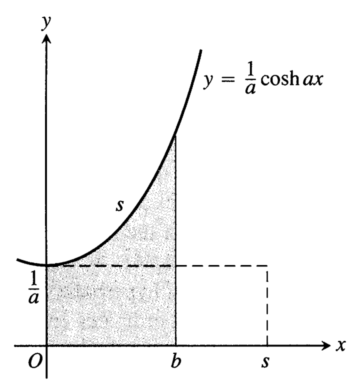
Differential equation: $\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$,

Initial condition: $y = 0$ when $x = 1$.

Solve the initial value problem to find an equation for the curve. (You need an inverse hyperbolic function.)



80. Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s, where s is the length of the curve from $x = 0$ to $x = b$.



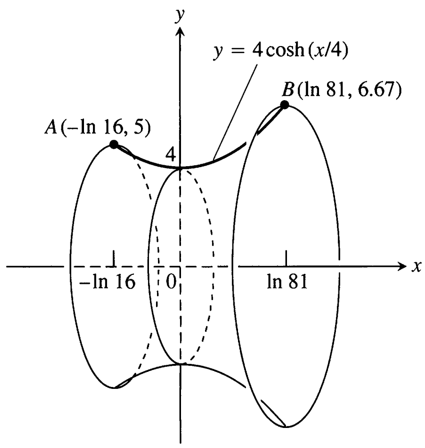
81. A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

82. The region enclosed by the curve $y = \operatorname{sech} x$, the x-axis, and the lines $x = \pm \ln \sqrt{3}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

83. **a)** Find the length of the segment of the curve $y = (1/2) \cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

b) Find the length of the segment of the curve $y = (1/a) \cosh ax$ from $x = 0$ to $x = b > 0$.

84. *A minimal surface.* Find the area of the surface swept out by revolving about the x-axis the curve $y = 4 \cosh (x/4)$, $-\ln 16 \leq x \leq \ln 81$.



It can be shown that, of all continuously differentiable curves joining points A and B in the figure, the curve $y = 4 \cosh (x/4)$ generates the surface of least area. If you made a rigid wire frame of the end-circles through A and B and dipped them in a soap-film solution, the surface spanning the circles would be the one generated by the curve.

85. **a)** Find the centroid of the curve $y = \cosh x$, $-\ln 2 \leq x \leq \ln 2$.

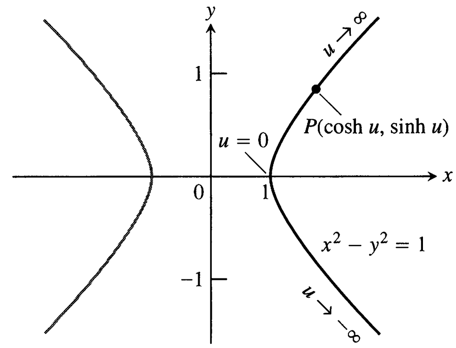
b) CALCULATOR Evaluate the coordinates to 2 decimal places. Then sketch the curve and plot the centroid to show its relation to the curve.

86. *The hyperbolic in hyperbolic functions.* In case you are wondering where the name *hyperbolic* comes from, here is the answer: Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$ (Fig. 6.34).

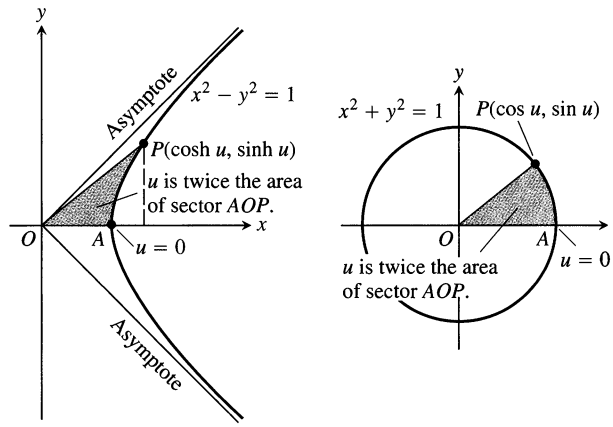
Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in Fig. 6.35. To see why this is so, carry out the following steps.

a) Show that the area $A(u)$ of sector AOP is given by the formula

$$A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx.$$



6.34 Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u (Exercise 86).



6.35 One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 86).

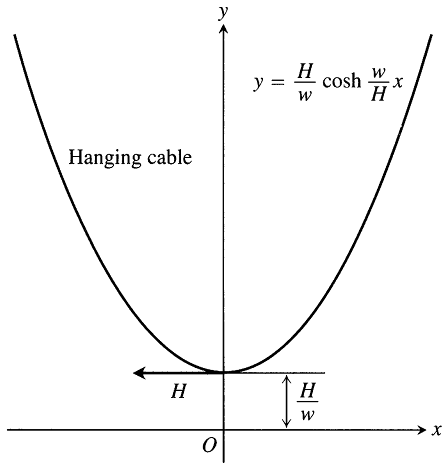
a) Differentiate both sides of the equation in (a) with respect to u to show that

$$A'(u) = \frac{1}{2}.$$

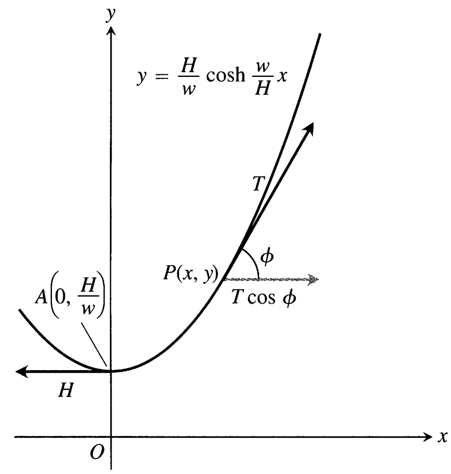
b) Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?

Hanging Cables

87. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y-axis (Fig. 6.36), then it can be shown that the cable lies



6.36 In a coordinate system chosen to match H and w in the manner shown, a hanging cable lies along the hyperbolic cosine $y = (H/w)\cosh(wx/H)$.



6.37 As discussed in Exercise 87, $T = wy$ in this coordinate system.

along the graph of the hyperbolic cosine

$$y = \frac{H}{w}\cosh\frac{w}{H}x.$$

Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena,* meaning "chain."

a) Let $P(x, y)$ denote an arbitrary point on the cable. Figure 6.37 displays the tension at P as a vector of length (magnitude) T, as well as the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\frac{w}{H}x.$$

b) Using the result from part (a) and the fact that the tension at P must equal H (the cable is not moving), show that $T = wy$. This means that the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.

88. (*Continuation of Exercise 87.*) The length of arc AP in Fig. 6.37 is $s = (1/a)\sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a}\sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

89. *The sag and horizontal tension in a cable.* The ends of a cable 32 ft long and weighing 2 lb/ft are fastened at the same level to posts 30 ft apart.

a) Model the cable with the equation

$$y = \frac{1}{a}\cosh ax, \quad -15 \le x \le 15.$$

Use information from Exercise 88 to show that a satisfies the equation

$$16a = \sinh 15a. \tag{2}$$

b) GRAPHER Solve Eq. (2) graphically by estimating the coordinates of the points where the graphs of the equations $y = 16a$ and $y = \sinh 15a$ intersect in the ay-plane.

c) EQUATION SOLVER or ROOT FINDER Solve Eq. (2) for a numerically. Compare your solution with the value you found in (b).

d) Estimate the horizontal tension in the cable at the cable's lowest point.

e) GRAPHER Graph the catenary

$$y = \frac{1}{a}\cosh ax$$

over the interval $-15 \le x \le 15$. Estimate the sag in the cable at its center.

6.11 First Order Differential Equations

In Section 6.5 we derived the law of exponential change, $y = y_0 e^{kt}$, as the solution of the initial value problem $dy/dt = ky$, $y(0) = y_0$. As we saw, this problem models population growth, radioactive decay, heat transfer, and a great many other phenomena. In the present section, we study initial value problems based on the equation

$dy/dx = f(x, y)$, in which f is a function of both the independent and dependent variables. The applications of this equation, a generalization of $dy/dt = ky$ (think of t as x), are broader still.

First Order Differential Equations

A **first order** differential equation is a relation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. A **solution** of Eq. (1) is a differentiable function $y = y(x)$ defined on an interval of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. The initial condition that $y(x_0) = y_0$ amounts to requiring the solution curve $y = y(x)$ to pass through the point (x_0, y_0).

EXAMPLE 1 The equation

$$\frac{dy}{dx} = 1 - \frac{y}{x}$$

is a first order differential equation in which $f(x, y) = 1 - (y/x)$. ❑

EXAMPLE 2 Show that the function

$$y = \frac{1}{x} + \frac{x}{2}$$

is a solution of the initial value problem

$$\frac{dy}{dx} = 1 - \frac{y}{x}, \qquad y(2) = \frac{3}{2}.$$

Solution The given function satisfies the initial condition because

$$y(2) = \left(\frac{1}{x} + \frac{x}{2}\right)_{x=2} = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}.$$

To show that it satisfies the differential equation, we show that the two sides of the equation agree when we substitute $(1/x) + (x/2)$ for y.

On the left: $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x}{2}\right) = -\frac{1}{x^2} + \frac{1}{2}$$

On the right: $$1 - \frac{y}{x} = 1 - \frac{1}{x}\left(\frac{1}{x} + \frac{x}{2}\right)$$
$$= 1 - \frac{1}{x^2} - \frac{1}{2} = -\frac{1}{x^2} + \frac{1}{2}$$

The function $y = (1/x) + (x/2)$ satisfies both the differential equation and the initial condition, which is what we needed to show. ❑

We sometimes write $y' = f(x, y)$ for $dy/dx = f(x, y)$.

Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)h(y).$$

If $h(y) \neq 0$, we can **separate the variables** by dividing both sides by h and multiplying both sides by dx, obtaining

$$\frac{1}{h(y)}\,dy = g(x)\,dx.$$

This groups the y-terms with dy on the left and the x-terms with dx on the right. We then integrate both sides, obtaining

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

The integrated equation provides the solutions we seek by expressing y either explicitly or implicitly as a function of x, up to an arbitrary constant.

EXAMPLE 3 Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)\,e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\begin{aligned}
\frac{dy}{dx} &= (1 + y^2)\,e^x \\
dy &= (1 + y^2)\,e^x\,dx && \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx. \\
\frac{dy}{1 + y^2} &= e^x\,dx && \text{Divide by } (1 + y^2). \\
\int \frac{dy}{1 + y^2} &= \int e^x\,dx && \text{Integrate both sides.} \\
\tan^{-1} y &= e^x + C && C \text{ represents the combined constants of integration.}
\end{aligned}$$

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. In this case, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\begin{aligned}
\tan\,(\tan^{-1} y) &= \tan\,(e^x + C) \\
y &= \tan\,(e^x + C).
\end{aligned}$$

❑

Linear First Order Equations

A first order differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2}$$

where P and Q are functions of x, is a **linear** first order equation. Equation (2) is the equation's **standard form**.

EXAMPLE 4 Put the following equation in standard form

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ❑

EXAMPLE 5 The equation

$$\frac{dy}{dx} = ky$$

with which we modeled bacterial growth, radioactive decay, and temperature change in Section 6.5 is a linear first order equation. Its standard form is

$$\frac{dy}{dx} - ky = 0. \qquad P(x) = -k \text{ and } Q(x) = 0$$

❑

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3}$$

by multiplying both sides by a positive function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by v works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y)$$

$$v(x) \cdot y = \int v(x)Q(x)\,dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \qquad \text{Solve for } y. \tag{4}$$

We call $v(x)$ an **integrating factor** for Eq. (3) because its presence makes the equation integrable.

Equation (4) expresses the solution of Eq. (3) in terms of the functions $v(x)$ and $Q(x)$.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Product Rule for derivatives}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated}$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0, \text{ we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{5}$$

From this, we see that any function v that satisfies Eq. (5) will enable us to solve Eq. (3) with the formula in Eq. (4). We do not need the most general possible v, only one that will work. Therefore, it will do no harm to simplify our lives by choosing the simplest possible antiderivative of P for $\int P\,dx$.

Theorem 4

The solution of the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{6}$$

is

$$y = \frac{1}{v(x)} \int v(x)\, Q(x)\, dx, \tag{7}$$

where

$$v(x) = e^{\int P(x)\,dx}. \tag{8}$$

In the formula for v, we do not need the most general antiderivative of $P(x)$. Any antiderivative will do.

EXAMPLE 6 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \quad x > 0.$$

How to Solve a Linear First Order Equation

1. Put it in standard form.
2. Find an antiderivative of $P(x)$.
3. Find $v(x) = e^{\int P(x)\,dx}$.
4. Use Eq. (7) to find y.

Solution We solve the equation in four steps.

Step 1: *Put the equation in standard form to identify P and Q.*

$$\frac{dy}{dx} - \frac{3}{x}y = x, \qquad P(x) = -\frac{3}{x}, \qquad Q(x) = x. \qquad \text{Example 4}$$

Step 2: *Find an antiderivative of P(x) (any one will do).*

$$\int P(x)\,dx = \int -\frac{3}{x}\,dx = -3\int \frac{1}{x}\,dx = -3\ln|x| = -3\ln x \qquad (x > 0)$$

Step 3: *Find the integrating factor v(x).*

$$v(x) = e^{\int P(x)\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \frac{1}{x^3} \qquad \text{Eq. (8)}$$

Step 4: *Find the solution.*

$$\begin{aligned} y &= \frac{1}{v(x)}\int v(x)Q(x)\,dx && \text{Eq. (7)} \\ &= \frac{1}{(1/x^3)}\int\left(\frac{1}{x^3}\right)(x)\,dx && \text{Values from steps 1–3} \\ &= x^3\cdot\int\frac{1}{x^2}\,dx \\ &= x^3\left(-\frac{1}{x} + C\right) && \text{Don't forget the } C \ldots \\ &= -x^2 + Cx^3 && \ldots\text{it provides part of the answer.} \end{aligned}$$

The solution is $y = -x^2 + Cx^3, \quad x > 0$. ❑

EXAMPLE 7 Solve the equation

$$xy' = x^2 + 3y, \qquad x > 0,$$

given the intial condition $y(1) = 2$.

Solution We first solve the differential equation (Example 6), obtaining

$$y = -x^2 + Cx^3, \qquad x > 0.$$

We then use the initial condition to find the right value for C:

$$\begin{aligned} y &= -x^2 + Cx^3 \\ 2 &= -(1)^2 + C(1)^3 && y = 2 \quad \text{when} \quad x = 1 \\ C &= 2 + (1)^2 = 3. \end{aligned}$$

The solution of the initial value problem is the function $y = -x^2 + 3x^3$. ❑

Resistance Proportional to Velocity

In some cases it makes sense to assume that, other forces being absent, the resistance encountered by a moving object, like a car coasting to a stop, is proportional to the object's velocity. The slower the object moves, the less its forward progress is resisted by the air through which it passes. We can describe this in mathematical terms if we picture the object as a mass m moving along a coordinate line with position s and velocity v at time t. The resisting force opposing the motion is mass $\times$ acceleration $= m(dv/dt)$, and we can write

$$m\frac{dv}{dt} = -kv \qquad (k > 0) \tag{9}$$

to say that the force decreases in proportion to velocity. If we rewrite (9) as

$$\frac{dv}{dt} + \frac{k}{m}v = 0 \qquad \text{Standard form} \tag{10}$$

and let v_0 denote the object's velocity at time $t = 0$, we can apply Theorem 4 to arrive at the solution

$$v = v_0\, e^{-(k/m)t} \tag{11}$$

(Exercise 42).

What can we learn from Eq. (11)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero. For another, we can integrate the equation to find s as a function of t.

Suppose a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Eq. (11) and solve the initial value problem

$$\frac{ds}{dt} = v_0\, e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k}e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k}e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t}).$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted } = \frac{v_0 m}{k}. \tag{12}$$

This is an ideal figure, of course. Only in mathematics can time stretch to infinity. The number $v_0\, m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least—if m is large, it will take a lot of energy to stop the body. That is why ocean liners have to be docked by tugboats. Any liner of conventional design entering a slip with enough speed to steer would smash into the pier before it could stop.

Weight vs. mass

Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{Weight} = \text{mass} \times \text{acceleration}.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity. In the metric system,

$$\text{Newtons} = \text{kilograms} \times 9.8$$

and

$$\text{Newtons}/9.8 = \text{kilograms}.$$

In the English system, where weight is measured in pounds, mass is measured in **slugs.** Thus,

$$\text{Pounds} = \text{slugs} \times 32$$

and

$$\text{Pounds}/32 = \text{slugs}.$$

A skater weighing 192 lb has a mass of

$$192/32 = 6 \text{ slugs}.$$

EXAMPLE 8 For a 192-lb ice skater, the k in Eq. (11) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Eq. (11) for t:

$$11e^{-t/18} = 1 \qquad \text{Eq. (11) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1$$

$$e^{-t/18} = 1/11$$

$$-t/18 = \ln(1/11) = -\ln 11$$

$$t = 18\ln 11 \approx 43 \text{ sec}.$$

We answer the second question with Eq. (12):

$$\text{Distance coasted} = \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}$$

$$= 198 \text{ ft}.$$

❑

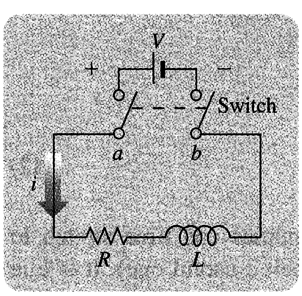
RL Circuits

The diagram in Fig. 6.38 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V, \tag{13}$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.



6.38 The *RL* circuit in Example 9.

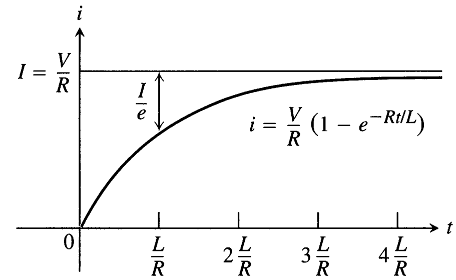
EXAMPLE 9 The switch in the RL circuit in Fig. 6.38 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (13) is a linear first order differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{14}$$

and the corresponding solution, from Theorem 4, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} \tag{15}$$



6.39 The growth of the current in the *RL* circuit in Example 9. *I* is the current's steady state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady state value in 3 time constants (Exercise 53).

(Exercise 54). Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes the current approaches the **steady state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Fig. 6.39).

Equation (15) expresses the solution of Eq. (14) as the sum of two terms: a **steady state solution** V/R and a **transient solution** $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ❑

Exercises 6.11

Verifying Solutions

In Exercises 1 and 2, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$

a) $y = e^{-x}$
b) $y = e^{-x} + e^{-(3/2)x}$
c) $y = e^{-x} + Ce^{-(3/2)x}$

2. $y' = y^2$

a) $y = -\frac{1}{x}$ **b)** $y = -\frac{1}{x+3}$ **c)** $y = -\frac{1}{x+C}$

In Exercises 3 and 4, show that the function $y = f(x)$ is a solution of the given differential equation.

3. $y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt, \quad y' + \frac{2x^3}{1+x^4}y = 1$

In Exercises 5–8, show that each function is a solution of the given initial value problem.

	Differential equation	Initial condition	Solution candidate
5.	$y' + y = \frac{2}{1+4e^{2x}}$	$y(-\ln 2) = \frac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
6.	$y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
7.	$xy' + y = -\sin x$, $x > 0$	$y\left(\frac{\pi}{2}\right) = 0$	$y = \frac{\cos x}{x}$
8.	$x^2y' = xy - y^2$, $x > 1$	$y(e) = e$	$y = \frac{x}{\ln x}$

Separable Equations

Solve the differential equations in Exercises 9–14.

9. $\frac{dy}{dx} = 2(x + y^2x)$ **10.** $(y+1)\frac{dy}{dx} = y(x-1)$

11. $2\sqrt{xy}\dfrac{dy}{dx} = 1, \quad x, y > 0$

12. $\dfrac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

13. $\dfrac{dy}{dx} = e^{x-y}$

14. $\dfrac{dy}{dx} = \dfrac{2x^2+1}{xe^y}, \quad x > 0$

Linear First Order Equations

Solve the differential equations in Exercises 15–20.

15. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

16. $e^x\dfrac{dy}{dx} + 2e^x y = 1$

17. $xy' + 3y = \dfrac{\sin x}{x^2}, \quad x > 0$

18. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

19. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

20. $(1 + x)y' + y = \sqrt{x}$

First Order Equations

Solve the differential equations in Exercises 21–34.

21. $2y' = e^{x/2} + y$

22. $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

23. $e^{2x}y' + 2e^{2x}y = 2x$

24. $xy' - y = 2x\ln x$

25. $\sec x\dfrac{dy}{dx} = e^{y+\sin x}$

26. $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y, \quad x > 0$

27. $(t-1)^3\dfrac{ds}{dt} + 4(t-1)^2 s = t + 1, \quad t > 1$

28. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

29. $(\sec^2\sqrt{x})\dfrac{dx}{dt} = \sqrt{x}$

30. $\sin t - (x\cos^2 t)\dfrac{dx}{dt} = 0, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

31. $\sin\theta\dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

32. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

33. $\cosh x\dfrac{dy}{dx} + (\sinh x)y = e^{-x}$

34. $\sinh x\dfrac{dy}{dx} + 3(\cosh x)y = \cosh x \sinh x$

Solving Initial Value Problems

Solve the initial value problems in Exercises 35–40.

Differential equation	Initial condition
35. $\dfrac{dy}{dt} + 2y = 3$	$y(0) = 1$
36. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0$	$y(2) = 1$
37. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0$	$y(\pi/2) = 1$
38. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3\sec\theta\tan\theta, \quad \theta > 0$	$y(\pi/3) = 2$
39. $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1$	$y(0) = 5$
40. $\dfrac{dy}{dx} + xy = x$	$y(0) = -6$

41. What do you get when you use Theorem 4 to solve the following initial value problem for y as a function of t?

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

42. Use Theorem 4 to solve the following initial value problem for v as a function of t.

$$\frac{dv}{dt} + \frac{k}{m}v = 0 \quad (k \text{ and } m \text{ positive constants}), \quad v(0) = v_0$$

Theory and Examples

43. Is either of the following equations correct? Give reasons for your answers.

a) $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$

b) $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

44. Is either of the following equations correct? Give reasons for your answers.

a) $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b) $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

45. *Blood sugar.* If glucose is fed intravenously at a constant rate, the change in the overall concentration $c(t)$ of glucose in the blood with respect to time may be described by the differential equation

$$\frac{dc}{dt} = \frac{G}{100V} - kc.$$

In this equation, G, V, and k are positive constants, G being the rate at which glucose is admitted, in milligrams per minute, and V the volume of blood in the body, in liters (around 5 liters for an adult). The concentration $c(t)$ is measured in milligrams per centiliter. The term $-kc$ is included because the glucose is assumed to be changing continually into other molecules at a rate proportional to its concentration.

a) Solve the equation for $c(t)$, using c_0 to denote $c(0)$.
b) Find the steady state concentration, $\lim_{t\to\infty} c(t)$.

46. *Continuous compounding.* You have \$1000 with which to open an account and plan to add \$1000 per year. All funds in the account will earn 10% interest per year, compounded continuously. If the added deposits are also credited to your account continuously, the number of dollars x in your account at time t (years) will satisfy the initial value problem

$$\frac{dx}{dt} = 1000 + 0.10x, \quad x(0) = 1000.$$

a) Solve the initial value problem for x as a function of t.
b) CALCULATOR About how many years will it take for the amount in your account to reach \$100,000?

47. *How long will it take a tank to drain?* If we drain the water from a vertical cylindrical tank by opening a valve at the base of the tank, the water will flow fast when the tank is full but slow down as the tank drains. It turns out that the rate at which the water level drops is proportional to the square root of the water's depth, y. This means that

$$\frac{dy}{dt} = -k\sqrt{y}.$$

The value of k depends on the acceleration of gravity, the shape of the hole, the fluid, and the cross-section areas of the tank and drain hole.

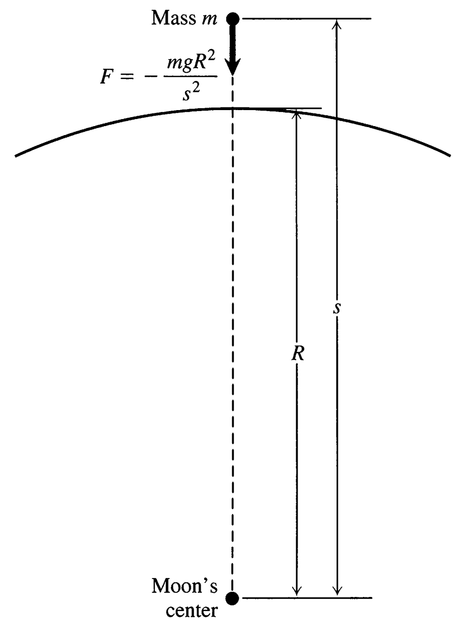
Suppose t is measured in minutes and $k = 1/10$. How long does it take the tank to drain if the water is 9 ft deep to start with?

48. *Escape velocity.* The gravitational attraction F exerted by an airless moon on a body of mass m at a distance s from the moon's center is given by the equation $F = -mg\,R^2 s^{-2}$, where g is the acceleration of gravity at the moon's surface and R is the moon's radius (Fig. 6.40). The force F is negative because it acts in the direction of decreasing s.

a) If the body is projected vertically upward from the moon's surface with an initial velocity v_0 at time $t = 0$, use Newton's second law, $F = ma$, to show that the body's velocity at position s is given by the equation

$$v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR.$$

Thus, the velocity remains positive as long as $v_0 \ge \sqrt{2gR}$. The velocity $v_0 = \sqrt{2gR}$ is the moon's **escape velocity.** A body projected upward with this velocity or a greater one will escape from the moon's gravitational pull.



6.40 Diagram for Exercise 48.

b) Show that if $v_0 = \sqrt{2gR}$, then

$$s = R\left(1 + \frac{3v_0}{2R}t\right)^{2/3}.$$

RESISTANCE PROPORTIONAL TO VELOCITY

49. For a 145-lb cyclist on a 15-lb bicycle on level ground, the k in Eq. (11) is about 1/5 slug/sec and $m = 160/32 = 5$ slugs. The cyclist starts coasting at 22 ft/sec (15 mph).

a) About how far will the cyclist coast before reaching a complete stop?
b) To the nearest second, about how long will it take the cyclist's speed to drop to 1 ft/sec?

50. For a 56,000-ton Iowa class battleship, $m = 1{,}750{,}000$ slugs and the k in Eq. (11) might be 3000 slugs/sec. Suppose the battleship loses power when it is moving at a speed of 22 ft/sec (13.2 knots).

a) About how far will the ship coast before it stops?
b) About how long will it take the ship's speed to drop to 1 ft/sec?

RL CIRCUITS

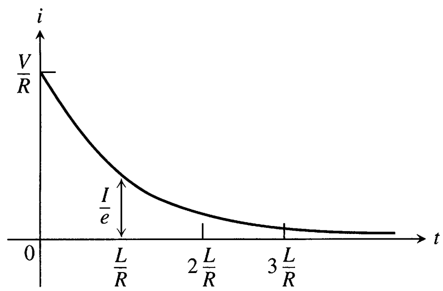
51. *Current in a closed RL circuit.* How many seconds after the switch in an RL circuit is closed will it take the current i to reach half of its steady state value? Notice that the time depends on R and L and not on how much voltage is applied.

52. *Current in an open RL circuit.* If the switch is thrown open after the current in an RL circuit has built up to its steady state value, the decaying current (graphed here) obeys the equation

$$L\frac{di}{dt} + Ri = 0, \tag{16}$$

which is Eq. (13) with $V = 0$.

a) Solve Eq. (16) to express i as a function of t.
b) How long after the switch is thrown will it take the current to fall to half its original value?
c) What is the value of the current when $t = L/R$? (The significance of this time is explained in the next exercise.)



53. *Time constants.* Engineers call the number L/R the *time constant* of the RL circuit in Fig. 6.39. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Fig. 6.39). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

a) Find the value of i in Eq. (15) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady state value $I = V/R$.
b) Approximately what percentage of the steady state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

54. (*Derivation of Eq. (15) in Example 9.*)

a) Use Theorem 4 to show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

b) Then use the initial condition $i(0) = 0$ to determine the value of C. This will complete the derivation of Eq. (15).
c) Show that $i = V/R$ is a solution of Eq. (14) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

MIXTURE PROBLEMS

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix} \tag{17}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate})\\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate}).\end{aligned} \tag{18}$$

Accordingly, Eq. (17) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{19}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Eq. (19) are

$$\frac{\text{pounds}}{\text{min}} = \frac{\text{pounds}}{\text{min}} - \frac{\text{pounds}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}}.$$

55. A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

a) At what rate (lb/min) does salt enter the tank at time t?
b) What is the volume of brine in the tank at time t?
c) At what rate (lb/min) does salt leave the tank at time t?
d) Write down and solve the initial value problem describing the mixing process.
e) Find the concentration of salt in the tank 25 min after the process starts.

56. In an oil refinery a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. Find the amount of additive in the tank 20 min after the process starts.

57. A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

58. An executive conference room of a corporation contains 4500 cubic feet of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft ^{3}/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft ^{3}/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

6.12 Euler's Numerical Method; Slope Fields

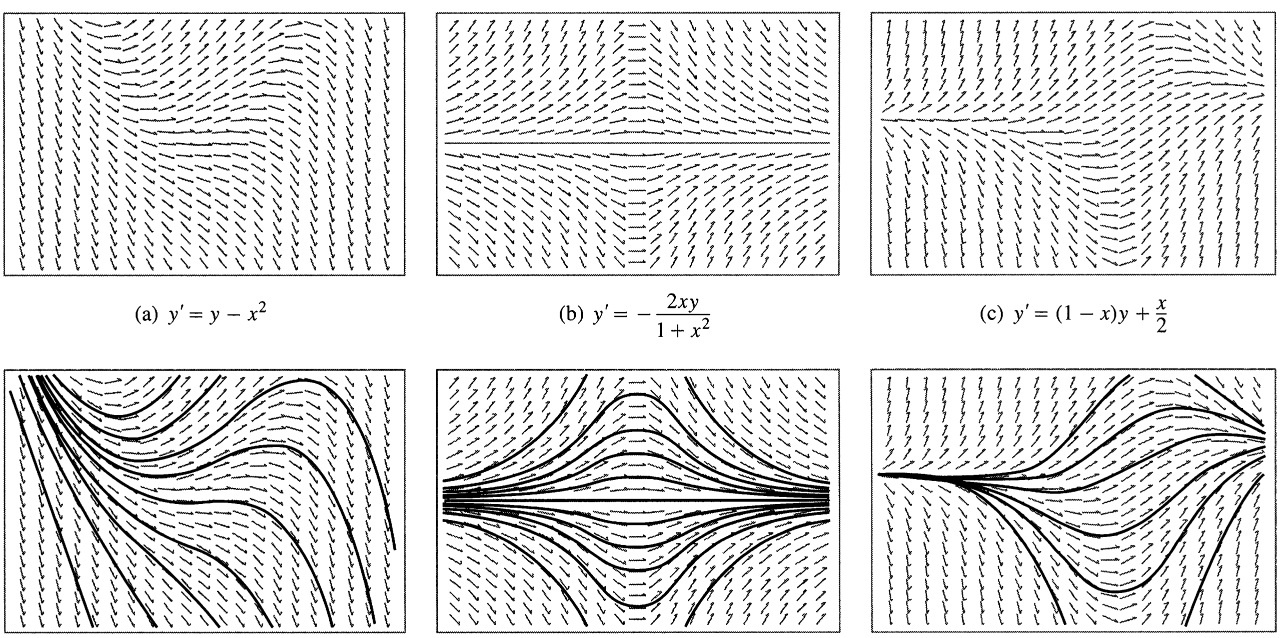
If we do not require or cannot immediately find an *exact* solution for an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, we can probably use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem and the method by which we generate the table is called a **numerical method.** Numerical methods are generally fast and accurate and are often the methods of choice when exact formulas are unnecessary, unavailable, or overly complicated. In the present section, we study one such method, called Euler's method, upon which all other numerical methods are based.

Slope Fields

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the solution curve is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. We see how the curves behave by following these tangents (Fig. 6.41).

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer. Let us see how a computer might obtain one of the solution curves.

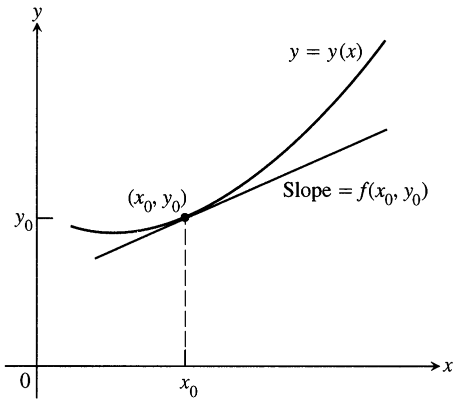
6.41 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with vectors, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.



(a) $y' = y - x^2$

(b) $y' = -\dfrac{2xy}{1+x^2}$

(c) $y' = (1-x)y + \dfrac{x}{2}$



6.42 The equation of the tangent line is $y = L(x) = y_0 + f(x_0, y_0)(x - x_0)$.

Using Linearizations

If we are given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution curve $y = y(x)$ by its linearization

$$L(x) = y(x_0) + \left.\frac{dy}{dx}\right|_{x=x_0}(x - x_0)$$

or

$$L(x) = y_0 + f(x_0, y_0)(x - x_0). \tag{1}$$

The function $L(x)$ will give a good approximation to the solution $y(x)$ in a short interval about x_0 (Fig. 6.42). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

We know the point (x_0, y_0) lies on the solution curve. Suppose we specify a new value for the independent variable to be $x_1 = x_0 + dx$. If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve.

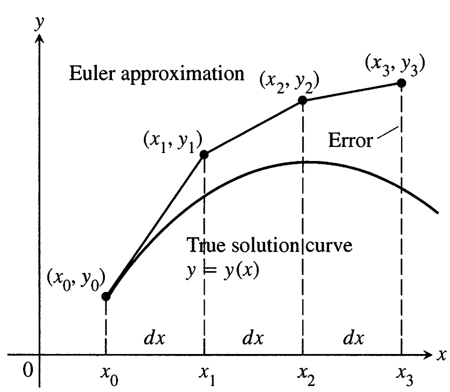
Using the point (x_1, y_1) and the slope $f(x_1, y_1)$, we take a second step. Setting $x_2 = x_1 + dx$, we calculate

$$y_2 = y_1 + f(x_1, y_1)\,dx,$$

to obtain another approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Fig. 6.43). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the next approximation

$$y_3 = y_2 + f(x_2, y_2)\,dx,$$

and so on.



6.43 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y)$, $y = y_0$ when $x = x_0$. The errors involved usually accumulate as we take more steps.

EXAMPLE 1 Find the first three approximations y_1, y_2, y_3 using the Euler approximation for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ and using $dx = 0.1$.

Solution

First:
$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\,dx \\ &= y_0 + (1 + y_0)\,dx \\ &= 1 + (1 + 1)(0.1) = 1.2 \end{aligned}$$

Second:
$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)\,dx \\ &= y_1 + (1 + y_1)\,dx \\ &= 1.2 + (1 + 1.2)(0.1) = 1.42 \end{aligned}$$

Third:
$$\begin{aligned} y_3 &= y_2 + (1 + y_2)\,dx \\ &= 1.42 + (1 + 1.42)(0.1) = 1.662 \end{aligned}$$

❑

The Euler Method

To continue our discussion, the Euler method is a numerical process for generating a table of approximate values of the function that solves the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If we use equally spaced values for the independent variable in the table and generate n of them, we first set

$$\begin{aligned} x_1 &= x_0 + dx, \\ x_2 &= x_1 + dx, \\ &\vdots \\ x_n &= x_{n-1} + dx. \end{aligned} \tag{2}$$

Then we calculate the solution approximations in turn:

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\,dx, \\ y_2 &= y_1 + f(x_1, y_1)\,dx, \\ &\vdots \\ y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\,dx. \end{aligned} \tag{3}$$

The number n of steps can be as large as we like, but errors may accumulate if n is too large.

EXAMPLE 2 Investigate the accuracy of the Euler approximation method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1$$

in Example 1 over the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking $dx = 0.1$.

Solution The exact solution to the initial value problem is $y = 2e^x - 1$ (using either method discussed in Section 6.11). Table 6.14 shows the results of the Euler approximation method using Eqs. (2) and (3) and compares them to the exact results

Table 6.14 Euler solution of $y' = 1 + y$, $y(0) = 1$, increment size $dx = 0.1$

x	y (approx)	y (exact)	Error = y (exact) $-y$ (approx)
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

rounded to 4 decimal places. By the time we reach $x = 1$ (after 10 steps), the error is about 5.6%. ❑

EXAMPLE 3 Investigate the accuracy of the Euler method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1$$

over the interval $0 \leq x \leq 1$, starting at $x_0 = 0$ and taking $dx = 0.05$.

Solution Table 6.15 shows the results and their comparisons with the exact solution. Notice that in doubling the number of steps from 10 to 20 we have reduced the error. This time when we reach $x = 1$ the error is only about 2.9%. ❑

It might be tempting to reduce the increment size even further to obtain greater accuracy. However, each additional calculation not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers in the calculations.

The analysis of error and the investigation of methods to reduce it when making numerical calculations is important, but appropriate for a more advanced course. There are numerical methods that are more accurate than Euler's method, as you will see when you study differential equations. In the exercises you will have the opportunity to explore the trade-offs involved in trying to reduce error by taking more but smaller increment steps.

Table 6.15 Euler solution of $y' = 1 + y$, $y(0) = 1$, increment size $dx = 0.05$

x	y (approx)	y (exact)	Error = y (exact) $-y$ (approx)
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0339
0.50	2.2578	2.2974	0.0396
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0854
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

Exercises 6.12

Calculating Euler Approximations

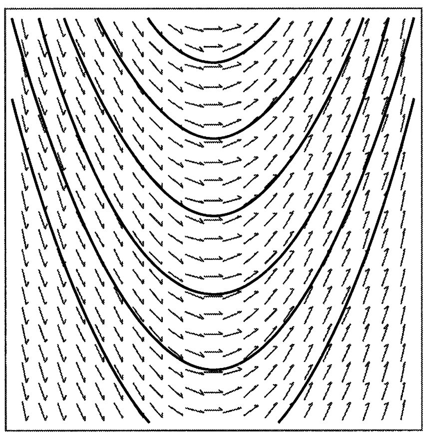
In Exercises 1–6, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to 4 decimal places.

1. $y' = 1 - \dfrac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$
2. $y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$
3. $y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$
4. $y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$
5. CALCULATOR $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1$
6. CALCULATOR $y' = y + e^x - 2, \quad y(0) = 2, \quad dx = 0.5$
7. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?
8. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?
9. CALCULATOR Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?
10. CALCULATOR Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = y - e^{2x}$ and $y(0) = 1$. What is the exact value of $y(2)$?

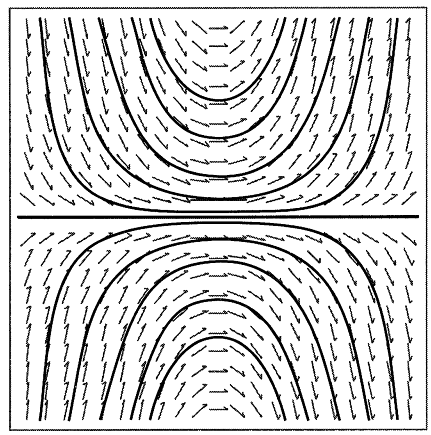
Slope Fields

In Exercises 11–14, match the differential equations with the solution curves sketched below in the slope fields (a)–(d).

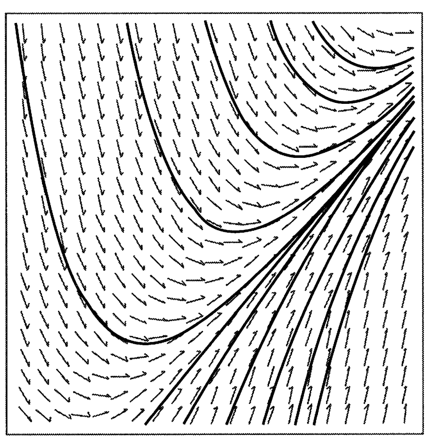
11. $y' = xy$
12. $y' = x + y$
13. $y' = x$
14. $y' = x - y$



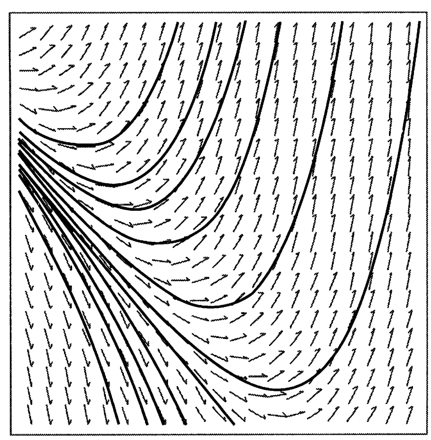
(a)



(b)



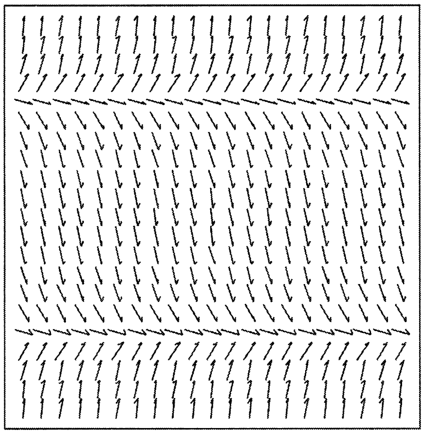
(c)



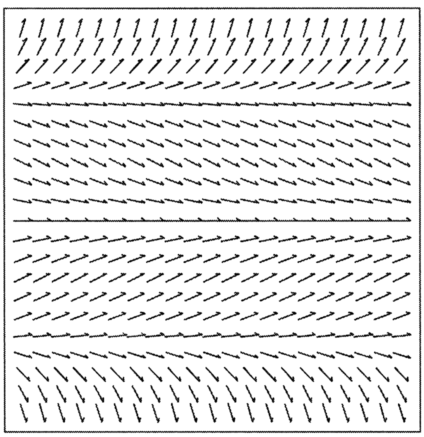
(d)

In Exercises 15 and 16, copy the slope fields and sketch in some of the solution curves.

15. $y' = (y+2)(y-2)$



16. $y' = y(y+1)(y-1)$



In Exercises 17–20, sketch part of the slope field. Using the slope field, sketch the solution curves that pass through the given points.

17. $y' = y$ with (a) $(0, 1)$, (b) $(0, 2)$, (c) $(0, -1)$

18. $y' = 2(y-4)$ with (a) $(0, 1)$, (b) $(0, 4)$, (c) $(0, 5)$

19. $y' = y(2-y)$ with (a) $(0, 1/2)$, (b) $(0, 3/2)$, (c) $(0, 2)$, (d) $(0, 3)$

20. $y' = y^2$ with (a) $(0, 1)$, (b) $(0, 2)$, (c) $(0, -1)$, (d) $(0, 0)$

CAS Explorations and Projects

Use a CAS to explore graphically each of the differential equations in Exercises 21–24. Perform the following steps to help with your explorations.

a) Plot a slope field for the differential equation in the given xy-window.

b) Find the general solution of the differential equation using your CAS DE solver.

c) Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

d) Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

e) Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the x-interval and plot the Euler approximation superimposed on the graph produced in part (d).

f) Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

g) Find the error y (exact) $-y$ (Euler) at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

21. $y' = x + y$, $y(0) = -7/10$; $-4 \le x \le 4$, $-4 \le y \le 4$; $b = 1$

22. $y' = -x/y$, $y(0) = 2$; $-3 \le x \le 3$, $-3 \le y \le 3$; $b = 2$

23. *A logistic equation.* $y' = y(2-y)$, $y(0) = 1/2$; $0 \le x \le 4$, $0 \le y \le 3$; $b = 3$

24. $y' = (\sin x)(\sin y)$, $y(0) = 2$; $-6 \le x \le 6$, $-6 \le y \le 6$; $b = 3\pi/2$

Exercises 25 and 26 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations, performing as many of the steps (a)–(g) above as possible.

25. $y' = \cos(2x - y)$, $y(0) = 2$; $0 \le x \le 5$, $0 \le y \le 5$; $y(2)$

26. *A Gompertz equation.* $y' = y(1/2 - \ln y)$, $y(0) = 1/3$; $0 \le x \le 4$, $0 \le y \le 3$; $y(3)$

27. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a) $2x$

b) $\sin 2x$

c) $3e^{x/2}$

d) $2e^{-x/2}\cos 2x$.

Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

28. a) Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

b) Separate the variables and use a CAS integrator to find the general solution in implicit form.

c) Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d) Find and graph the solution that satisfies the initial condition $y(0) = -1$.

CHAPTER 6 QUESTIONS TO GUIDE YOUR REVIEW

1. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
2. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
3. How can you sometimes express the inverse of a function of x as a function of x?
4. Under what circumstances can you be sure that the inverse of a function f is differentiable? How are the derivatives of f and f^{-1} related?
5. What is the natural logarithm function? What are its domain, range, and derivative? What arithmetic properties does it have? Comment on its graph.
6. What is logarithmic differentiation? Give an example.
7. What integrals lead to logarithms? Give examples. What are the integrals of $\tan x$ and $\cot x$?
8. How is the exponential function e^x defined? What are its domain, range, and derivative? What laws of exponents does it obey? Comment on its graph.
9. How are the functions a^x and $\log_a x$ defined? Are there any restrictions on a? How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there is really only one exponential function and one logarithmic function?
10. Describe some of the applications of base 10 logarithms.
11. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
12. How do you compare the growth rates of positive functions as $x \to \infty$?
13. What roles do the functions e^x and $\ln x$ play in growth comparisons?
14. Describe big-oh and little-oh notation. Give examples.
15. Which is more efficient—a sequential search or a binary search? Explain.
16. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.
17. How can you find values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ using a calculator's keys for $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$?
18. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?
19. What integrals lead to inverse trigonometric functions? How do substitution and completing the square broaden the application of these integrals?
20. What are the six basic hyperbolic functions? Comment on their domains, ranges, and graphs. What are some of the identities relating them?
21. What are the derivatives of the six basic hyperbolic functions? What are the corresponding integral formulas? What similarities do you see here with the six basic trigonometric functions?
22. How are the inverse hyperbolic functions defined? Comment on their domains, ranges, and graphs. How can you find values of $\operatorname{sech}^{-1}x$, $\operatorname{csch}^{-1}x$, and $\coth^{-1}x$ using a calculator's keys for $\cosh^{-1}x$, $\sinh^{-1}x$, and $\tanh^{-1}x$?
23. What integrals lead naturally to inverse hyperbolic functions?
24. What is a first order differential equation? When is a function a solution of such an equation?
25. How do you solve separable first order differential equations?
26. How do you solve linear first order differential equations?
27. What is the slope field of a differential equation $y' = f(x, y)$? What can we learn from such fields?
28. Describe Euler's method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ numerically. Give an example. Comment on the method's accuracy. Why might you want to solve an initial value problem numerically?

Chapter 6 Practice Exercises

Differentiation

In Exercises 1–24, find the derivative of y with respect to the appropriate variable.

1. $y = 10e^{-x/5}$
2. $y = \sqrt{2}\, e^{\sqrt{2}x}$
3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
4. $y = x^2 e^{-2/x}$
5. $y = \ln(\sin^2\theta)$
6. $y = \ln(\sec^2\theta)$
7. $y = \log_2(x^2/2)$
8. $y = \log_5(3x - 7)$
9. $y = 8^{-t}$
10. $y = 9^{2t}$
11. $y = 5x^{3.6}$
12. $y = \sqrt{2}\, x^{-\sqrt{2}}$
13. $y = (x + 2)^{x+2}$
14. $y = 2(\ln x)^{x/2}$
15. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$
16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$
17. $y = \ln\cos^{-1} x$
18. $y = z\cos^{-1} z - \sqrt{1 - z^2}$
19. $y = t\tan^{-1} t - \frac{1}{2}\ln t$
20. $y = (1 + t^2)\cot^{-1} 2t$
21. $y = z\sec^{-1} z - \sqrt{z^2 - 1}, \quad z > 1$
22. $y = 2\sqrt{x - 1}\sec^{-1}\sqrt{x}$
23. $y = \csc^{-1}(\sec\theta), \quad 0 < \theta < \pi/2$
24. $y = (1 + x^2)e^{\tan^{-1} x}$

Logarithmic Differentiation

In Exercises 25–30, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

25. $y = \dfrac{2(x^2 + 1)}{\sqrt{\cos 2x}}$
26. $y = \sqrt[10]{\dfrac{3x + 4}{2x - 4}}$
27. $y = \left(\dfrac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$
28. $y = \dfrac{2u2^u}{\sqrt{u^2 + 1}}$
29. $y = (\sin\theta)^{\sqrt{\theta}}$
30. $y = (\ln x)^{1/(\ln x)}$

Integration

Evaluate the integrals in Exercises 31–50.

31. $\int e^x \sin(e^x)\, dx$
32. $\int e^t \cos(3e^t - 2)\, dt$
33. $\int e^x \sec^2(e^x - 7)\, dx$
34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\, dy$
35. $\int \sec^2(x)e^{\tan x}\, dx$
36. $\int \csc^2 x\, e^{\cot x}\, dx$
37. $\int_{-1}^{1} \dfrac{dx}{3x - 4}$
38. $\int_1^e \dfrac{\sqrt{\ln x}}{x}\, dx$
39. $\int_0^{\pi} \tan\dfrac{x}{3}\, dx$
40. $\int_{1/6}^{1/4} 2\cot\pi x\, dx$
41. $\int_0^4 \dfrac{2t}{t^2 - 25}\, dt$
42. $\int_{-\pi/2}^{\pi/6} \dfrac{\cos t}{1 - \sin t}\, dt$
43. $\int \dfrac{\tan(\ln v)}{v}\, dv$
44. $\int \dfrac{dv}{v\ln v}$
45. $\int \dfrac{(\ln x)^{-3}}{x}\, dx$
46. $\int \dfrac{\ln(x - 5)}{x - 5}\, dx$
47. $\int \dfrac{1}{r}\csc^2(1 + \ln r)\, dr$
48. $\int \dfrac{\cos(1 - \ln v)}{v}\, dv$
49. $\int x3^{x^2}\, dx$
50. $\int 2^{\tan x}\sec^2 x\, dx$

Evaluate the integrals in Exercises 51–64.

51. $\int_1^7 \dfrac{3}{x}\, dx$
52. $\int_1^{32} \dfrac{1}{5x}\, dx$
53. $\int_1^4 \left(\dfrac{x}{8} + \dfrac{1}{2x}\right) dx$
54. $\int_1^8 \left(\dfrac{2}{3x} - \dfrac{8}{x^2}\right) dx$
55. $\int_{-2}^{-1} e^{-(x+1)}\, dx$
56. $\int_{-\ln 2}^{0} e^{2w}\, dw$
57. $\int_0^{\ln 5} e^r(3e^r + 1)^{-3/2}\, dr$
58. $\int_0^{\ln 9} e^{\theta}(e^{\theta} - 1)^{1/2} d\theta$
59. $\int_1^e \dfrac{1}{x}(1 + 7\ln x)^{-1/3}\, dx$
60. $\int_e^{e^2} \dfrac{1}{x\sqrt{\ln x}}\, dx$
61. $\int_1^3 \dfrac{(\ln(v + 1))^2}{v + 1}\, dv$
62. $\int_2^4 (1 + \ln t)t\ln t\, dt$
63. $\int_1^8 \dfrac{\log_4\theta}{\theta}\, d\theta$
64. $\int_1^e \dfrac{8\ln 3\log_3\theta}{\theta}\, d\theta$

Evalute the integrals in Exercises 65–78.

65. $\int_{-3/4}^{3/4} \dfrac{6\, dx}{\sqrt{9 - 4x^2}}$
66. $\int_{-1/5}^{1/5} \dfrac{6\, dx}{\sqrt{4 - 25x^2}}$

67. $\int_{-2}^{2} \frac{3\,dt}{4+3t^2}$

68. $\int_{\sqrt{3}}^{3} \frac{dt}{3+t^2}$

69. $\int \frac{dy}{y\sqrt{4y^2-1}}$

70. $\int \frac{24\,dy}{y\sqrt{y^2-16}}$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2-1}}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2-3}}$

73. $\int \frac{dx}{\sqrt{-2x-x^2}}$

74. $\int \frac{dx}{\sqrt{-x^2+4x-1}}$

75. $\int_{-2}^{-1} \frac{2\,dv}{v^2+4v+5}$

76. $\int_{-1}^{1} \frac{3\,dv}{4v^2+4v+4}$

77. $\int \frac{dt}{(t+1)\sqrt{t^2+2t-8}}$

78. $\int \frac{dt}{(3t+1)\sqrt{9t^2+6t}}$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 79–84, solve for y.

79. $3^y = 2^{y+1}$

80. $4^{-y} = 3^{y+2}$

81. $9e^{2y} = x^2$

82. $3^y = 3\ln x$

83. $\ln(y-1) = x + \ln y$

84. $\ln(10\ln y) = \ln 5x$

Evaluating Limits

Find the limits in Exercises 85–96.

85. $\lim_{x\to 0} \frac{10^x - 1}{x}$

86. $\lim_{\theta\to 0} \frac{3^\theta - 1}{\theta}$

87. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

88. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

89. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

90. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

91. $\lim_{t\to 0^+} \frac{t - \ln(1+2t)}{t^2}$

92. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

93. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

94. $\lim_{y\to 0^+} e^{-1/y} \ln y$

95. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

96. $\lim_{x\to 0^+} \left(1 + \frac{3}{x}\right)^x$

Comparing Growth Rates of Functions

97. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a) $f(x) = \log_2 x,\quad g(x) = \log_3 x$

b) $f(x) = x,\quad g(x) = x + \frac{1}{x}$

c) $f(x) = x/100,\quad g(x) = xe^{-x}$

d) $f(x) = x,\quad g(x) = \tan^{-1} x$

e) $f(x) = \csc^{-1} x,\quad g(x) = 1/x$

f) $f(x) = \sinh x,\quad g(x) = e^x$

98. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a) $f(x) = 3^{-x},\quad g(x) = 2^{-x}$

b) $f(x) = \ln 2x,\quad g(x) = \ln x^2$

c) $f(x) = 10x^3 + 2x^2,\quad g(x) = e^x$

d) $f(x) = \tan^{-1}(1/x),\quad g(x) = 1/x$

e) $f(x) = \sin^{-1}(1/x),\quad g(x) = 1/x^2$

f) $f(x) = \text{sech}\, x,\quad g(x) = e^{-x}$

99. True, or false? Give reasons for your answers.

a) $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$

b) $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$

c) $x = o(x + \ln x)$

d) $\ln(\ln x) = o(\ln x)$

e) $\tan^{-1} x = O(1)$

f) $\cosh x = O(e^x)$

100. True, or false? Give reasons for your answers.

a) $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$

b) $\frac{1}{x^4} = o\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$

c) $\ln x = o(x+1)$

d) $\ln 2x = O(\ln x)$

e) $\sec^{-1} x = O(1)$

f) $\sinh x = O(e^x)$

Theory and Applications

101. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

102. Find the inverse of the function $f(x) = 1 + (1/x)$, $x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}.$$

In Exercises 103 and 104, find the absolute maximum and minimum values of each function on the given interval.

103. $y = x\ln 2x - x,\quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

104. $y = 10x(2 - \ln x),\quad (0, e^2]$

105. Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

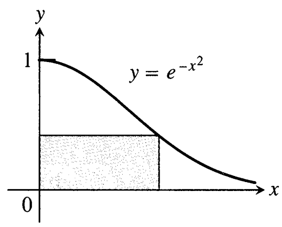
106. **a)** Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b) Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

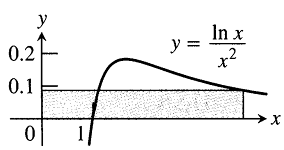
107. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x-coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/sec. At what rate is the y-coordinate changing at the point $(e^2, 2)$?

108. A girl is sliding down a slide shaped like the curve $y = 9e^{-x/3}$. Her y-coordinate is changing at the rate $dy/dt = (-1/4)\sqrt{9-y}$ ft/sec. At approximately what rate is her x-coordinate changing when she reaches the bottom of the slide at $x = 9$ ft? (Take e^3 to be 20 and round your answer to the nearest ft/sec.)

109. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



110. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



111. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

112. **a)** If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?
b) If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

113. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

114. *$\log_x(2)$ vs. $\log_2(x)$.* How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out:

a) Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b) Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

115. GRAPHER Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a) $y = (\ln x)/\sqrt{x}$ **b)** $y = e^{-x^2}$
c) $y = (1 + x)e^{-x}$

116. GRAPHER Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

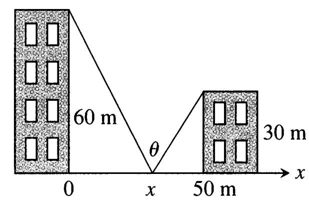
117. CALCULATOR What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

118. *Cooling a pie.* A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

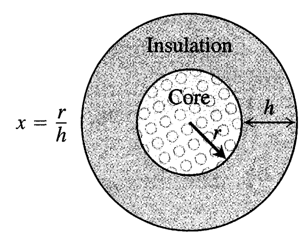
119. *Locating a solar station.* You are under contract to build a solar station at ground level on the east–west line between the two buildings shown here. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when the sun passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\frac{x}{60} - \cot^{-1}\frac{50 - x}{30}.$$

Then find the value of x that maximizes θ.



120. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



Initial Value Problems

Solve the initial value problems in Exercises 121–124.

	Differential equation	Initial condition
121.	$\dfrac{dy}{dx} = e^{-x-y-2}$	$y(0) = -2$
122.	$\dfrac{dy}{dx} = -\dfrac{y \ln y}{1 + x^2}$	$y(0) = e^2$
123.	$(x + 1)\dfrac{dy}{dx} + 2y = x, \quad x > -1$	$y(0) = 1$
124.	$x\dfrac{dy}{dx} + 2y = x^2 + 1, \quad x > 0$	$y(1) = 1$

Slope Fields and Euler's Method

In Exercises 125–128, sketch part of the equation's slope field. Then add to your sketch the solution curve that passes through the point $P(1, -1)$. Use Euler's method with $x_0 = 1$ and $dx = 0.2$ to estimate $y(2)$. Round your answers to 4 decimal places. Find the exact value of $y(2)$ for comparison.

125. $y' = x$

126. $y' = 1/x$

127. $y' = xy$

128. $y' = 1/y$

CHAPTER 6 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Limits

Find the limits in Exercises 1–6

1. $\lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$

2. $\lim_{x \to \infty} \frac{1}{x} \int_0^x \tan^{-1} t \, dt$

3. $\lim_{x \to 0^+} (\cos \sqrt{x})^{1/x}$

4. $\lim_{x \to \infty} (x + e^x)^{2/x}$

5. $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$

6. $\lim_{n \to \infty} \frac{1}{n} \left(e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n} \right)$

7. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the vertical line $x = t$, $t > 0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

a) $\lim_{t \to \infty} A(t)$ **b)** $\lim_{t \to \infty} V(t)/A(t)$ **c)** $\lim_{t \to 0^+} V(t)/A(t)$

8. *Varying a logarithm's base*

a) Find $\lim \log_a 2$ as $a \to 0^+, 1^-, 1^+$, and ∞.

b) GRAPHER Graph $y = \log_a 2$ as a function of a over the interval $0 < a \le 4$.

Determining Parameter Values

9. Find values of a and b for which

$$\lim_{x \to 0} \frac{\sin ax + bx}{x^3} = -\frac{4}{3}.$$

10. Find values of a and b for which

$$\lim_{x \to 0} \frac{a \cos x - \cos bx}{x^2} = 4.$$

Theory and Examples

11. Find the areas between the curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$ and the x-axis from $x = 1$ to $x = e$. What is the ratio of the larger area to the smaller?

12. GRAPHER Graph $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$ for $-5 \le x \le 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5, 5]$? Give reasons for your answer.

13. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

14. GRAPHER Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

15. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4} \, dt$.

16. a) Find df/dx if

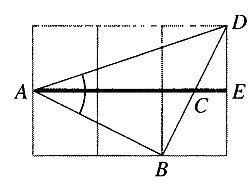
$$f(x) = \int_1^{e^x} \frac{2 \ln t}{t} \, dt.$$

b) Find $f(0)$.

c) What can you conclude about the graph of f? Give reasons for your answer.

17. The figure here shows an informal proof that

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}.$$

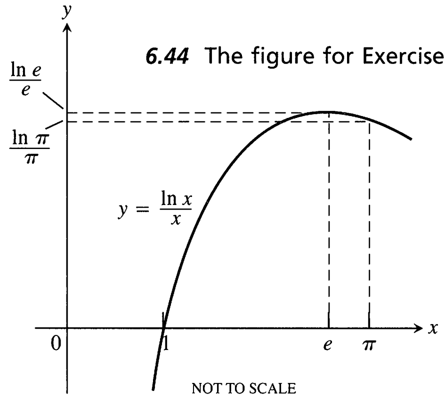


How does the argument go? (Source: "Behold! Sums of Arctan," by Edward M. Harris, *College Mathematics Journal,* Vol. 18, No. 2, March 1987, p. 141.)

18. $\pi^e < e^\pi$

a) Why does Fig. 6.44 (on the following page) "prove" that $\pi^e < e^\pi$? (Source: "Proof Without Words," by Fouad Nakhil, *Mathematics Magazine,* Vol. 60, No. 3, June 1987, p. 165.)

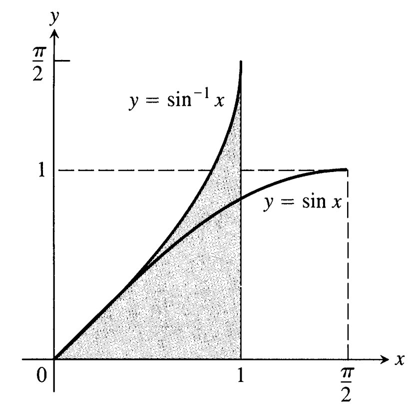
b) Figure 6.44 assumes that $f(x) = (\ln x)/x$ has an absolute maximum value at $x = e$. How do you know it does?



6.44 The figure for Exercise 18.

19. Use the accompanying figure to show that

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

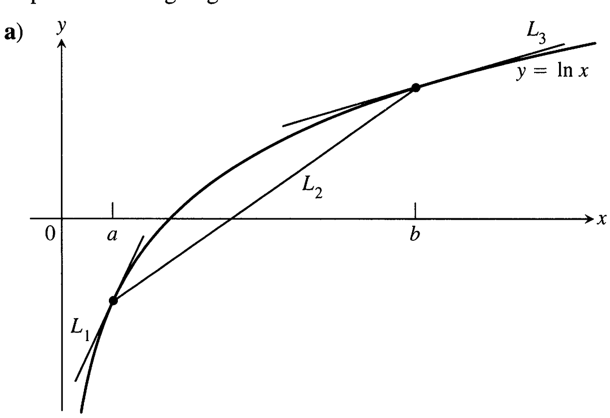


20. *Napier's inequality.* Here are two pictorial proofs that

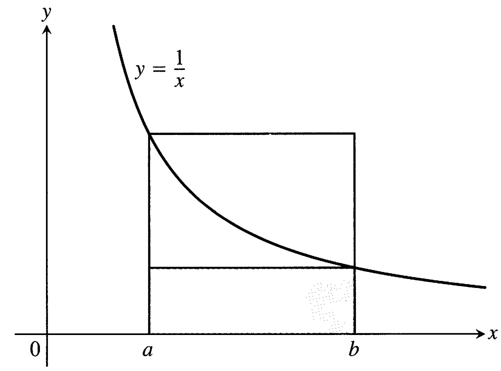
$$b > a > 0 \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.

a)



b)



(Source: Roger B. Nelson, *College Mathematics Journal,* Vol. 24, No. 2, March 1993, p. 165.)

21. *Even–odd decompositions*

a) Suppose that g is an even function of x and h is an odd function of x. Show that if $g(x) + h(x) = 0$ for all x then $g(x) = 0$ for all x and $h(x) = 0$ for all x.

b) Use the result in (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then

$$f_E(x) = (f(x) + f(-x))/2 \quad \text{and} \quad f_O(x) = (f(x) - f(-x))/2.$$

c) What is the significance of the result in (b)?

22. Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i) $g(x + y) = \dfrac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers x, y, and $x + y$ in the domain of g.

ii) $\lim_{h \to 0} g(h) = 0$

iii) $\lim_{h \to 0} \dfrac{g(h)}{h} = 1$

a) Show that $g(0) = 0$.

b) Show that $g'(x) = 1 + [g(x)]^2$.

c) Find $g(x)$ by solving the differential equation in (b).

Applications

23. Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.

24. The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid.

a) Find the volume of the solid.

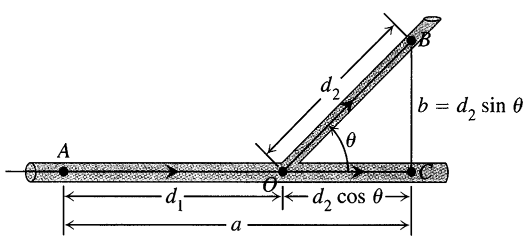
b) Find the centroid of the region.

25. *The Rule of 70.* If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

26. *Free fall in the fourteenth century.* In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

27. *The best branching angles for blood vessels and pipes.* When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section AOB shown in Fig. 6.45. In this diagram, B is a given point to be reached by the smaller pipe, A is a point in the larger pipe upstream from B, and O is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along AO is $(kd_1)/R^4$ and along OB is $(kd_2)/r^4$, where k is a constant, d_1 is the length of AO, d_2 is the length of OB, R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k\,\frac{d_1}{R^4} + k\,\frac{d_2}{r^4}.$$



6.45 Diagram for Exercise 27.

In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2 \cos\theta = a \qquad d_2 \sin\theta = b,$$

so that

$$d_2 = b\csc\theta,$$

$$d_1 = a - d_2\cos\theta = a - b\cot\theta.$$

We can express the total loss L as a function of θ:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

a) Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

b) CALCULATOR If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

The mathematical analysis described here is also used to explain the angles at which arteries branch in an animal's body. (See *Introduction to Mathematics for Life Scientists,* Second Edition, by E. Batschelet [New York: Springer-Verlag, 1976].)

28. *Group blood testing.* During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y. Also find the integer value of x that maximizes y. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q.

29. *Transport through a cell membrane.* Under some conditions the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\,\frac{A}{V}(c - y).$$

In this equation, y is the concentration of the substance inside the cell and dy/dt is the rate at which y changes over time. The letters k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

a) Solve the equation for $y(t)$, using y_0 to denote $y(0)$.

b) Find the steady state concentration, $\lim_{t\to\infty} y(t)$.

(Based on *Some Mathematical Models in Biology* by R. M. Thrall, J. A. Mortimer, K. R. Rebman, R. F. Baum, Eds., Revised Edition, December 1967, PB-202 364, pp. 101–103; distributed by N.T.I.S., U.S. Department of Commerce.)

CHAPTER 7

Techniques of Integration

OVERVIEW We have seen how integrals arise in modeling real phenomena and in measuring objects in the world around us, and we know in theory how integrals are evaluated with antiderivatives. The more sophisticated our models become, however, the more involved our integrals become. We need to know how to change these more involved integrals into forms we can work with. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer.

7.1 Basic Integration Formulas

As we saw in Section 4.1, we evaluate an indefinite integral by finding an antiderivative of the integrand and adding an arbitrary constant. Table 7.1 (on the following page) shows the basic forms of the integrals we have evaluated so far. There is a more extensive table at the back of the book; we will discuss it in Section 7.5.

Algebraic Procedures

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 *A simplifying substitution*

Evaluate $\displaystyle\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx.$

Solution

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx &= \int \frac{du}{\sqrt{u}} && \begin{array}{l} u = x^2-9x+1 \\ du = (2x-9)\,dx \end{array} \\
&= \int u^{-1/2}\,du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C && \text{Table 7.1, Formula 4, with } n = -1/2 \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

❑

Table 7.1 Basic integration formulas

1. $\int du = u + C$	11. $\int \csc u \cot u\, du = -\csc u + C$
2. $\int k\, du = ku + C \quad$ (any number k)	12. $\int \tan u\, du = -\ln\|\cos u\| + C$ $= \ln\|\sec u\| + C$
3. $\int (du + dv) = \int du + \int dv$	13. $\int \cot u\, du = \ln\|\sin u\| + C$ $= -\ln\|\csc u\| + C$
4. $\int u^n\, du = \dfrac{u^{n+1}}{n+1} + C \quad (n \neq -1)$	14. $\int e^u du = e^u + C$
5. $\int \dfrac{du}{u} = \ln\|u\| + C$	15. $\int a^u du = \dfrac{a^u}{\ln a} + C \quad (a > 0,\quad a \neq 1)$
6. $\int \sin u\, du = -\cos u + C$	16. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
7. $\int \cos u\, du = \sin u + C$	17. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
8. $\int \sec^2 u\, du = \tan u + C$	18. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left\|\dfrac{u}{a}\right\| + C$
9. $\int \csc^2 u\, du = -\cot u + C$	
10. $\int \sec u \tan u\, du = \sec u + C$	

EXAMPLE 2 *Completing the square*

Evaluate $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

Solution We complete the square to write the radicand as

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4,\quad u = (x - 4) \\ & && du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 7.1, Formula 16} \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

❑

EXAMPLE 3 *Expanding a power and using a trigonometric identity*

Evaluate $\int (\sec x + \tan x)^2 \, dx$.

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are old friends; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2 \, dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) \, dx \\ &= 2 \int \sec^2 x \, dx + 2 \int \sec x \tan x \, dx - \int 1 \, dx \\ &= 2 \tan x + 2 \sec x - x + C.\end{aligned}$$

❑

EXAMPLE 4 *Eliminating a square root*

Evaluate $\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx$.

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx && \sqrt{u^2} = |u| \\ &= \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0 \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} \\ &= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}.\end{aligned}$$

❑

EXAMPLE 5 *Reducing an improper fraction*

Evaluate $\int \frac{3x^2 - 7x}{3x + 2} \, dx$.

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2-7x}{3x+2} = x - 3 + \frac{6}{3x+2}.$$

$$\begin{array}{r} x \;\; - 3 \\ 3x+2 \,\big)\, \overline{3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -\,9x \\ \underline{-\,9x - 6} \\ +\,6 \end{array}$$

Therefore,

$$\int \frac{3x^2-7x}{3x+2}\,dx = \int \left(x - 3 + \frac{6}{3x+2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x+2| + C.$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We will see what to do about that in Section 7.3.

EXAMPLE 6 *Separating a fraction*

Evaluate $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}\,dx.$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{and} \qquad x\,dx = -\frac{1}{2}\,du.$$

$$\begin{aligned} 3\int \frac{x\,dx}{\sqrt{1-x^2}} &= 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du \\ &= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1. \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C.$$

EXAMPLE 7 *Multiplying by a form of 1*

Evaluate $\displaystyle\int \sec x\,dx.$

Solution

$$\begin{aligned} \int \sec x\,dx &= \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \end{aligned}$$

$$= \int \frac{du}{u} \qquad \begin{aligned} u &= \tan x + \sec x \\ du &= (\sec^2 x + \sec x \tan x)\, dx \end{aligned}$$

$$= \ln |u| + C = \ln |\sec x + \tan x| + C$$

❑

Table 7.2 The secant and cosecant integrals

1.	$\int \sec u \, du = \ln |\sec u + \tan u| + C$
2.	$\int \csc u \, du = -\ln |\csc u + \cot u| + C$

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\, dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x \tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,|\cos 2x|$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

Exercises 7.1

Basic Substitutions

Evaluate the integrals in Exercises 1–36 by using substitutions that reduce them to standard forms.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$

10. $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^\theta\csc(e^\theta+1)\,d\theta$

12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$

14. $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$

16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx$

18. $\displaystyle\int_{\pi/2}^{\pi} \sin(y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$

20. $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$

22. $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$

26. $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\displaystyle\int_0^1 \frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

34. $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate the integrals in Exercises 37–42 by completing the square and using substitutions to reduce them to standard forms.

37. $\displaystyle\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\displaystyle\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate the integrals in Exercises 43–46 by using trigonometric identities and substitutions to reduce them to standard forms.

43. $\displaystyle\int (\sec x+\cot x)^2 dx$

44. $\displaystyle\int (\csc x-\tan x)^2 dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\displaystyle\int \frac{x}{x+1}dx$

48. $\displaystyle\int \frac{x^2}{x^2+1}dx$

49. $\displaystyle\int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1}dx$

50. $\displaystyle\int_{-1}^3 \frac{4x^2-7}{2x+3}dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}dt$

52. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}dx$

54. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}dx$

55. $\displaystyle\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}dx$

56. $\displaystyle\int_0^{1/2} \frac{2-8x}{1+4x^2}dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\int \frac{1}{1+\sin x}dx$ **58.** $\int \frac{1}{1+\cos x}dx$

59. $\int \frac{1}{\sec\theta+\tan\theta}d\theta$ **60.** $\int \frac{1}{\csc\theta+\cot\theta}d\theta$

61. $\int \frac{1}{1-\sec x}dx$ **62.** $\int \frac{1}{1-\csc x}dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$ **64.** $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\,dt$ **66.** $\int_{-\pi}^{0} \sqrt{1+\cos t}\,dt$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\,d\theta$ **68.** $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\,d\theta$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 y}\,dy$ **70.** $\int_{-\pi/4}^{0} \sqrt{\sec^2 y-1}\,dy$

Assorted Integrations

Evaluate the integrals in Exercises 71–82 using any technique you think is appropriate.

71. $\int_{\pi/4}^{3\pi/4} (\csc x-\cot x)^2\,dx$ **72.** $\int_0^{\pi/4} (\sec x+4\cos x)^2\,dx$

73. $\int \cos\theta\,\csc(\sin\theta)\,d\theta$ **74.** $\int \left(1+\frac{1}{x}\right)\cot(x+\ln x)\,dx$

75. $\int (\csc x-\sec x)(\sin x+\cos x)\,dx$

76. $\int (\csc x+\sec x)(\tan x+\cot x)\,dx$

77. $\int \frac{6\,dy}{\sqrt{y}(1+y)}$ **78.** $\int \frac{dx}{x\sqrt{4x^2-1}}$

79. $\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$ **80.** $\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}}$

81. $\int \sec^2 t\tan(\tan t)\,dt$ **82.** $\int \frac{\tan\theta\,d\theta}{2\sec\theta+1}$

Trigonometric Powers

83. a) Evaluate $\int \cos^3\theta\,d\theta$. (*Hint:* $\cos^2\theta = 1-\sin^2\theta$.)

b) Evaluate $\int \cos^5\theta\,d\theta$.

c) Without actually evaluating the integral, explain how you would evaluate $\int \cos^9\theta\,d\theta$.

84. a) Evaluate $\int \sin^3\theta\,d\theta$. (*Hint:* $\sin^2\theta = 1-\cos^2\theta$.)

b) Evaluate $\int \sin^5\theta\,d\theta$.

c) Evaluate $\int \sin^7\theta\,d\theta$.

d) Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13}\theta\,d\theta$.

85. a) Express $\int \tan^3\theta\,d\theta$ in terms of $\int \tan\theta\,d\theta$. Then evaluate $\int \tan^3\theta\,d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta-1$.)

b) Express $\int \tan^5\theta\,d\theta$ in terms of $\int \tan^3\theta\,d\theta$.

c) Express $\int \tan^7\theta\,d\theta$ in terms of $\int \tan^5\theta\,d\theta$.

d) Express $\int \tan^{2k+1}\theta\,d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1}\theta\,d\theta$.

86. a) Express $\int \cot^3\theta\,d\theta$ in terms of $\int \cot\theta\,d\theta$. Then evaluate $\int \cot^3\theta\,d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta-1$.)

b) Express $\int \cot^5\theta\,d\theta$ in terms of $\int \cot^3\theta\,d\theta$.

c) Express $\int \cot^7\theta\,d\theta$ in terms of $\int \cot^5\theta\,d\theta$.

d) Express $\int \cot^{2k+1}\theta\,d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1}\theta\,d\theta$.

Theory and Examples

87. Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. *The integral of* $\csc x$. Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\,dx = -\ln|\csc x+\cot x| + C.$$

96. Show that the integral

$$\int \left((x^2-1)(x+1)\right)^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a) $u = 1/(x+1)$

b) $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c) $u = \tan^{-1} x$

d) $u = \tan^{-1}\sqrt{x}$

e) $u = \tan^{-1}((x-1)/2)$

f) $u = \cos^{-1} x$

g) $u = \cosh^{-1} x$

What is the value of the integral? (From "Problems and Solutions," *College Mathematics Journal,* Vol. 21, No. 5, Nov. 1990, pp. 425–426.)

7.2

Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx \tag{1}$$

in which f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx,$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

The Formula

The formula for integration by parts comes from the Product Rule,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In its differential form, the rule becomes

$$d(uv) = u\,dv + v\,du,$$

which is then written as

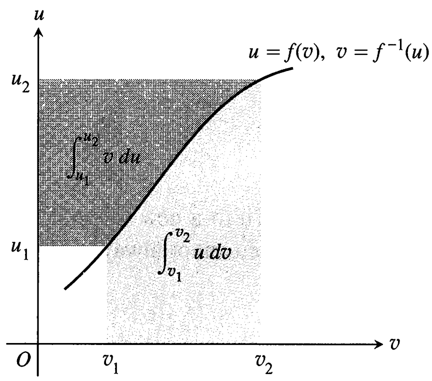
$$u\,dv = d(uv) - v\,du$$

and integrated to give the following formula.

The Integration-by-Parts Formula

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

The integration-by-parts formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. This is the reason for the importance of the formula. When faced with an integral we cannot handle, we can replace it by one with which we might have more success.



7.1 The area of the blue region, $\int_{v_1}^{v_2} u\,dv$, equals the area of the large rectangle, u_2v_2, minus the areas of the small rectangle, u_1v_1, and the gray region,

$$\int_{u_1}^{u_2} v\,du.$$

In symbols,

$$\int_{v_1}^{v_2} u\,dv = (u_2v_2 - u_1v_1) - \int_{u_1}^{u_2} v\,du.$$

When and How to Use Integration by Parts

When: If substitution doesn't work, try integration by parts.

How: Start with an integral of the form

$$\int f(x)g(x)\,dx.$$

Match this with an integral of the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and possibly $f(x)$ or $g(x)$.

Guideline for choosing u and dv: The formula

$$\int u\,dv = uv - \int v\,du$$

gives a new integral on the right side of the equation. If the new integral is more complex than the original one, try a different choice for u and dv.

The equivalent formula for definite integrals is

$$\boxed{\int_{v_1}^{v_2} u\,dv = (u_2v_2 - u_1v_1) - \int_{u_1}^{u_2} v\,du. \qquad (3)}$$

Figure 7.1 shows how the different parts of the formula may be interpreted as areas.

EXAMPLE 1 Find $\int x\cos x\,dx$.

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

❑

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 *Example 1 revisited*

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x\cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x\cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x\cos x\,dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$u = x\cos x, \qquad dv = dx,$$
$$du = (\cos x - x\sin x)\,dx, \qquad v = x,$$

and the new integral

$$\int v\,du = \int (x\cos x - x^2\sin x)\,dx.$$

This is worse than the integral we started with.

Choice 4 leads to

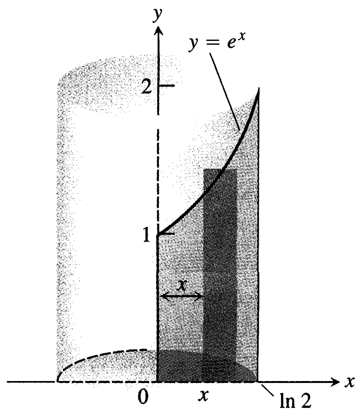
$$u = \cos x, \qquad dv = x\,dx,$$
$$du = -\sin x\,dx, \qquad v = x^2/2,$$

so the new integral is

$$\int v\,du = -\int \frac{x^2}{2}\sin x\,dx.$$

This, too, is worse.

Summary. Keep in mind that the object is to go from $\int u\,dv$ to a new integral that is simpler. Integration by parts does not always work, so we cannot always achieve the goal. ❑



7.2 The solid in Example 3.

EXAMPLE 3 Find the volume of the solid generated by revolving about the y-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$ and the line $x = \ln 2$ (Fig. 7.2).

Solution Using the method of cylindrical shells, we find

$$\begin{aligned} V &= \int_a^b 2\pi x\, f(x)\,dx && \text{The shell volume formula} \\ &= 2\pi \int_0^{\ln 2} x\,e^x\,dx. \end{aligned}$$

To evaluate the integral, we use the formula $\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= e^x\,dx \\ du &= dx, & v &= e^x. && \text{Simplest antiderivative of } e^x \end{aligned}$$

Then

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx,$$

so

$$\begin{aligned} \int_0^{\ln 2} x\,e^x dx &= x\,e^x\Big]_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx \\ &= \left[\ln 2\, e^{\ln 2} - 0\right] - \left[e^x\right]_0^{\ln 2} \\ &= 2\ln 2 - [2 - 1] \\ &= 2\ln 2 - 1. \end{aligned}$$

The solid's volume is therefore

$$\begin{aligned} V &= 2\pi \int_0^{\ln 2} x\,e^x\,dx \\ &= 2\pi(2\ln 2 - 1). \end{aligned}$$ ❑

Integration by parts can be useful even when the integrand has only a single factor. For example, we can use this method to find $\int \ln x\,dx$ (next example) or $\int \cos^{-1} x\,dx$ (Exercise 47).

EXAMPLE 4 Find $\int \ln x\,dx$.

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\ dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = \ln x \quad \text{Simplifies when differentiated} \qquad dv = dx \quad \text{Easy to integrate}$$

$$du = \frac{1}{x}dx \qquad\qquad v = x. \quad \text{Simplest antiderivative}$$

Then

$$\int \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}dx = x \ln x - \int dx = x \ln x - x + C.$$

❑

Repeated Use

Sometimes we have to use integration by parts more than once to obtain an answer.

EXAMPLE 5 Find $\displaystyle\int x^2 e^x\,dx.$

Solution We use the formula $\displaystyle\int u\,dv = uv - \int v\,du$ with

$$u = x^2, \qquad dv = e^x\,dx, \qquad v = e^x, \qquad du = 2x\,dx.$$

This gives

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

It takes a second integration by parts to find the integral on the right. As in Example 3, its value is $xe^x - e^x + C'$. Hence

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

❑

Solving for the Unknown Integral

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 6 Find $\displaystyle\int e^x \cos x\,dx.$

Solution We first use the formula $\displaystyle\int u\,dv = uv - \int v\,du$ with

$$u = e^x, \qquad dv = \cos x\,dx, \qquad v = \sin x, \qquad du = e^x dx.$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx. \tag{4}$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Combining the two expressions gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C'.$$

The choice of $u = e^x$ and $dv = \sin x\,dx$ in the second integration may have seemed arbitrary but it wasn't. In theory, we could have chosen $u = \sin x$ and $dv = e^x\,dx$. Doing so, however, would have turned Eq. (4) into

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(e^x \sin x - \int e^x \cos x\,dx\right)\\ &= \int e^x \cos x\,dx.\end{aligned}$$

The resulting identity is correct, but useless. *Moral:* Once you have decided on what to differentiate and integrate in circumstances like these, stick with them. Formulas for the integrals of $e^{ax}\cos bx$ and the closely related $e^{ax}\sin bx$ can be found in the integral table at the end of this book. ❑

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Find $\displaystyle\int x^2 e^x\,dx$ by tabular integration.

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x.

We add the products of the functions connected by the arrows, with the middle sign changed, to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C. \qquad \square$$

EXAMPLE 8 Find $\displaystyle\int x^3 \sin x \, dx$ by tabular integration.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x.$

For more about tabular integration, see the Additional Exercises at the end of this chapter.

Again we add the products of the functions connected by the arrows, with every other sign changed, to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \qquad \square$$

Exercises 7.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\displaystyle\int x \sin \frac{x}{2} \, dx$

2. $\displaystyle\int \theta \cos \pi\theta \, d\theta$

3. $\displaystyle\int t^2 \cos t \, dt$

4. $\displaystyle\int x^2 \sin x \, dx$

5. $\displaystyle\int_1^2 x \ln x \, dx$

6. $\displaystyle\int_1^e x^3 \ln x \, dx$

7. $\displaystyle\int \tan^{-1} y \, dy$

8. $\displaystyle\int \sin^{-1} y \, dy$

9. $\displaystyle\int x \sec^2 x \, dx$

10. $\displaystyle\int 4x \sec^2 2x \, dx$

11. $\displaystyle\int x^3 e^x \, dx$

12. $\displaystyle\int p^4 e^{-p} \, dp$

13. $\displaystyle\int (x^2 - 5x)e^x \, dx$

14. $\displaystyle\int (r^2 + r + 1)e^r \, dr$

15. $\displaystyle\int x^5 e^x \, dx$

16. $\displaystyle\int t^2 e^{4t} \, dt$

17. $\displaystyle\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta$

18. $\displaystyle\int_0^{\pi/2} x^3 \cos 2x \, dx$

19. $\displaystyle\int_{2/\sqrt{3}}^2 t \sec^{-1} t \, dt$

20. $\displaystyle\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx$

21. $\displaystyle\int e^\theta \sin \theta \, d\theta$

22. $\displaystyle\int e^{-y} \cos y \, dy$

23. $\displaystyle\int e^{2x} \cos 3x \, dx$

24. $\displaystyle\int e^{-2x} \sin 2x \, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\displaystyle\int e^{\sqrt{3s+9}} \, ds$

26. $\displaystyle\int_0^1 x\sqrt{1-x} \, dx$

27. $\displaystyle\int_0^{\pi/3} x \tan^2 x \, dx$

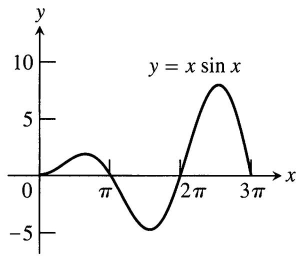
28. $\displaystyle\int \ln (x + x^2) \, dx$

29. $\displaystyle\int \sin (\ln x) \, dx$

30. $\displaystyle\int z (\ln z)^2 \, dz$

Theory and Examples

31. Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis for (a) $0 \le x \le \pi$, (b) $\pi \le x \le 2\pi$, (c) $2\pi \le x \le 3\pi$. (d) What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.

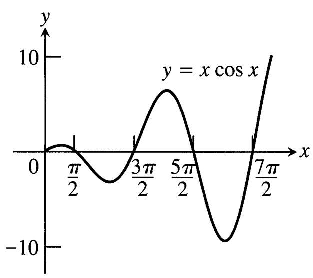


32. Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the figure below) for

a) $\pi/2 \le x \le 3\pi/2$,
b) $3\pi/2 \le x \le 5\pi/2$,
c) $5\pi/2 \le x \le 7\pi/2$.
d) What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$ (a) about the y-axis, (b) about the line $x = 1$.

35. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about (a) the y-axis, (b) the line $x = \pi/2$.

36. Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about (a) the y-axis, (b) the line $x = \pi$. (See Exercise 31 for a graph.)

37. a) Find the centroid of a thin plate of constant density covering the region in the first quadrant enclosed by the curve $y = x^2 e^x$, the x-axis, and the line $x = 1$.

b) CALCULATOR Find the coordinates of the centroid to 2 decimal places. Show the center of mass in a rough sketch of the plate.

38. a) Find the centroid of a thin plate of constant density covering the region enclosed by the curve $y = \ln x$, the x-axis, and the line $x = e$.

b) CALCULATOR Find the coordinates of the centroid to 2 decimal places. Show the centroid in a rough sketch of the plate.

39. Find the moment about the y-axis of a thin plate of density $\delta = 1 + x$ covering the region bounded by the x-axis and the curve $y = \sin x$, $0 \le x \le \pi$.

40. Although we usually drop the constant of integration in determining v as $\int dv$ in integration by parts, choosing the constant to be different from zero can occasionally be helpful. As a case in point, evaluate

$$\int x \tan^{-1} x \, dx,$$

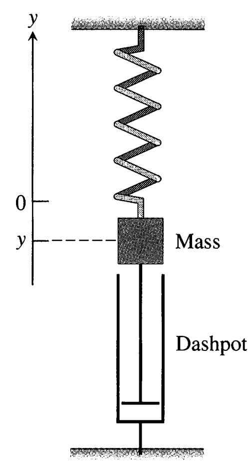
with $u = \tan^{-1} x$ and $v = (x^2/2) + C$, and find a value of C that simplifies the resulting formula.

41. A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \ge 0.$$

a) Find the average value of y over the interval $0 \le t \le 2\pi$.

b) GRAPHER Graph y over the interval $0 \le t \le 2\pi$. Copy the graph and mark the average value of y as a point on the y-axis.



42. In a mass-spring-dashpot system like the one in Exercise 41, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \quad t \ge 0.$$

a) Find the average value of y over the interval $0 \le t \le 2\pi$.

b) GRAPHER Graph y over the interval $0 \le t \le 2\pi$. Copy the graph and mark the average value of y as a point on the y-axis.

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy \qquad \begin{matrix} y = f^{-1}(x), \quad x = f(y) \\ dx = f'(y)\,dy \end{matrix}$$

$$= y f(y) - \int f(y)\,dy \qquad \begin{matrix} \text{Integration by parts with} \\ u = y,\ dv = f'(y)\,dy \end{matrix}$$

$$= x f^{-1}(x) - \int f(y)\,dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int y e^y\,dy \qquad \begin{matrix} y = \ln x, \quad x = e^y \\ dx = e^y\,dy \end{matrix}$$

$$= ye^y - e^y + C$$

$$= x \ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = x \cos^{-1} x - \int \cos y\,dy \qquad y = \cos^{-1} x$$

$$= x \cos^{-1} x - \sin y + C$$

$$= x \cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{5}$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\,dx$

44. $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$

46. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x)\right) dx. \tag{6}$$

Exercises 47 and 48 compare the results of using Eqs. (5) and (6).

47. Equations (5) and (6) give different formulas for the integral of $\cos^{-1} x$:

a) $\int \cos^{-1} x\,dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (5)

b) $\int \cos^{-1} x\,dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (6)

Can both integrations be correct? Explain.

48. Equations (5) and (6) lead to different formulas for the integral of $\tan^{-1} x$:

a) $\int \tan^{-1} x\,dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C$ Eq. (5)

b) $\int \tan^{-1} x\,dx = x \tan^{-1} x - \ln \sqrt{1 + x^2} + C$ Eq. (6)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with (a) Eq. (5) and (b) Eq. (6). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\,dx$

50. $\int \tanh^{-1} x\,dx$

7.3 Partial Fractions

A theorem from advanced algebra (mentioned later in more detail) says that every rational function, no matter how complicated, can be rewritten as a sum of simpler fractions that we can integrate with techniques we already know. For instance,

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}, \tag{1}$$

so we can integrate the rational function on the left by integrating the fractions on the right instead.

The method for rewriting rational functions this way is called the **method of partial fractions.** In this particular case, it consists of finding constants A and B

such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \tag{2}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Eq. (2) of fractions, obtaining

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

EXAMPLE 1 *Two distinct linear factors in the denominator*

Find

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx.$$

Solution From the preceding discussion,

$$\begin{aligned}\int \frac{5x-3}{(x+1)(x-3)}\,dx &= \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C.\end{aligned}$$ ❑

EXAMPLE 2 *A repeated linear factor in the denominator*

Express

$$\frac{6x+7}{(x+2)^2}$$

as a sum of partial fractions.

Solution Since the denominator has a repeated linear factor, $(x+2)^2$, we must express the fraction in the form

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}. \tag{3}$$

Clearing Eq. (3) of fractions gives

$$6x + 7 = A(x+2) + B = Ax + (2A + B).$$

Matching coefficients of like terms gives $A = 6$ and

$$7 = 2A + B = 12 + B, \qquad \text{or} \qquad B = -5.$$

Hence,

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}.$$ ❑

How to Evaluate Undetermined Coefficients

1. Clear the given equation of fractions.
2. Equate the coefficients of like terms (powers of x).
3. Solve the resulting equations for the coefficients.

EXAMPLE 3 *An improper fraction*

Express

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}$$

as a sum of partial fractions.

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction. Then we write the proper fraction as a sum of partial fractions. Long division gives

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{\smash{)}\, 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

Hence,

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3} \qquad \text{Result of the division}$$

$$= 2x + \frac{2}{x + 1} + \frac{3}{x - 3}. \qquad \text{Proper fraction expanded as in Example 1}$$

❑

EXAMPLE 4 *An irreducible quadratic factor in the denominator*

Express

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2}$$

as a sum of partial fractions.

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{4}$$

Notice the numerator over $x^2 + 1$: For quadratic factors, we use first degree numerators, not constant numerators. Clearing the equation of fractions gives

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$= (A + C)x^3 + (-2A + B - C + D)x^2$$

$$+ (A - 2B + C)x + (B - C + D).$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$
Coefficients of x^2:	$0 = -2A + B - C + D$
Coefficients of x^1:	$-2 = A - 2B + C$
Coefficients of x^0:	$4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$-4 = -2A, \quad A = 2 \qquad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \qquad \text{From the first equation}$$

$$B = 1 \qquad A = 2 \text{ and } C = -2 \text{ in third equation.}$$

$$D = 4 - B + C = 1. \qquad \text{From the fourth equation}$$

We substitute these values into Eq. (4), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

❑

EXAMPLE 5 Evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$

Solution We expand the integrand by partial fractions, as in Example 4, and integrate the terms of the expansion:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \qquad \text{Example 4}\\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx\\ &= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C.\end{aligned}$$

❑

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

1. *The degree of $f(x)$ must be less than the degree of $g(x)$.* (If it isn't, divide and work with the remainder term.)
2. *We must know the factors of $g(x)$.* (In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.)

A theorem from advanced algebra says that when these two conditions are met, we may write $f(x)/g(x)$ as the sum of partial fractions by taking these steps.

Cases discussed so far

Proper fraction		*Decomposition*
$\dfrac{\text{numerator}}{(x+p)(x+q)}$	$=$	$\dfrac{A}{(x+p)} + \dfrac{B}{(x+q)}$
$\dfrac{\text{numerator}}{(x+p)^2}$	$=$	$\dfrac{A}{(x+p)} + \dfrac{B}{(x+p)^2}$
$\dfrac{\text{numerator}}{(x^2+p)(x+q)^2}$	$=$	$\dfrac{Ax+B}{x^2+p} + \dfrac{C}{x+q} + \dfrac{D}{(x+q)^2}$

The Method of Partial Fractions (*f*(*x*)/*g*(*x*) *Proper*)

Step 1 Let $x - r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then assign the sum of m partial fractions to this factor, as follows:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

Step 2 Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$. Suppose $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then to this factor assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

Step 3 Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

Step 4 Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

⁎ The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$, and

$$g(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \tag{5}$$

Solution If we multiply both sides of Eq. (5) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$
$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} \tag{6}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\underbrace{\boxed{(x - 1)}}_{\Uparrow\ \text{Cover}}(1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.$$

Similarly, we find the value of B in Eq. (5) by covering the factor $(x-2)$ in (6) and evaluating the rest at $x=2$:

$$B = \frac{(2)^2+1}{(2-1)\,\boxed{(x-2)}\,(2-3)} = \frac{5}{(1)(-1)} = -5.$$

$$\Uparrow$$
Cover

Finally, C is found by covering the $(x-3)$ in (6) and evaluating the rest at $x=3$:

$$C = \frac{(3)^2+1}{(3-1)(3-2)\,\boxed{(x-3)}} = \frac{10}{(2)(1)} = 5.$$

$$\Uparrow$$
Cover

❑

The steps in the cover-up method are these:

Step 1: Write the quotient with $g(x)$ factored:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}. \tag{7}$$

Step 2: Cover the factors $(x-r_i)$ of $g(x)$ in (7) one at a time, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)},$$

$$A_2 = \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)},$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}.$$

Step 3: Write the partial-fraction expansion of $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

EXAMPLE 7 Evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx.$$

Solution The degree of $f(x)=x+4$ is less than the degree of $g(x)=x^3+3x^2-10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}.$$

The roots of $g(x)$ are $r_1=0$, $r_2=2$, and $r_3=-5$. We find

$$A_1 = \frac{0+4}{\boxed{x}\,(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5},$$

$$\Uparrow$$
Cover

$$A_2 = \frac{2+4}{2\,\boxed{(x-2)}\,(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7},$$

$\Uparrow$
Cover

$$A_3 = \frac{-5+4}{(-5)(-5-2)\,\boxed{(x+5)}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

$\Uparrow$
Cover

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C. \qquad \square$$

Other Ways to Determine the Constants

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 *Differentiation*

Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear of fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}. \qquad \square$$

In some problems, assigning small values to x such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 *Assigning numerical values to x*

Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution Clear of fractions to get

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned} x = 1: \qquad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \\ x = 2: \qquad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\ 5 &= -B \\ B &= -5 \\ x = 3: \qquad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\ 10 &= 2C \\ C &= 5. \end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$ ❑

Exercises 7.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$

2. $\dfrac{5x-7}{x^2-3x+2}$

3. $\dfrac{x+4}{(x+1)^2}$

4. $\dfrac{2x+2}{x^2-2x+1}$

5. $\dfrac{z+1}{z^2(z-1)}$

6. $\dfrac{z}{z^3-z^2-6z}$

7. $\dfrac{t^2+8}{t^2-5t+6}$

8. $\dfrac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$

10. $\displaystyle\int \frac{dx}{x^2+2x}$

11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$

12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$

16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

19. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

23. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$

25. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$

26. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

27. $\displaystyle\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$

28. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

30. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

31. $\displaystyle\int \frac{9x^2 - 3x + 1}{x^3 - x^2}\,dx$

32. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

33. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

34. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

36. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

38. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

39. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

40. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 41–44 for x as a function of t.

41. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

42. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

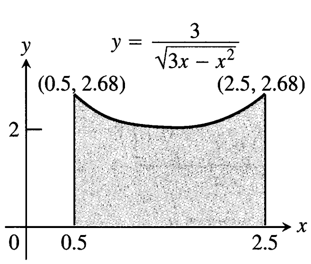
43. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

44. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4$

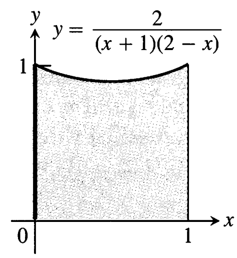
Applications and Examples

In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The x-axis

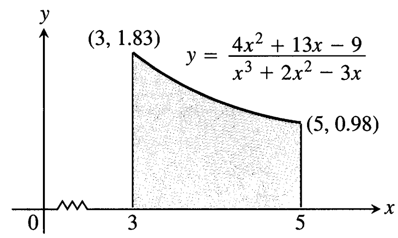


46. The y-axis



47. CALCULATOR Find, to 2 decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

48. CALCULATOR Find the x-coordinate of the centroid of this region to 2 decimal places.



49. *Social diffusion.* Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a) Find x as a function of t.

b) When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. *Second order chemical reactions.* Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t (a) if $a = b$, and (b) if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

51. *An integral connecting π to the approximation 22/7*

a) Evaluate $\displaystyle\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx.$

b) CALCULATOR How good is the approximation $\pi \approx 22/7$? Find out by expressing $(\pi - 22/7)$ as a percentage of π.

c) GRAPHER Graph the function $y = \dfrac{x^4(x-1)^4}{x^2+1}$ for $0 \le x \le 1$. Experiment with the range on the y-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second degree polynomial $P(x)$ such that $P(0) = 1$, $P'(0) = 0$, and

$$\int \frac{P(x)}{x^3(x-1)^2}\,dx$$

is a rational function.

7.4 Trigonometric Substitutions

Trigonometric substitutions enable us to replace the binomials $a^2 + x^2$, $a^2 - x^2$, and $x^2 - a^2$ by single squared terms and thereby transform a number of integrals containing square roots into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. They come from the reference right triangles in Fig. 7.3.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta. \tag{1}$$

With $x = a\sin\theta$,

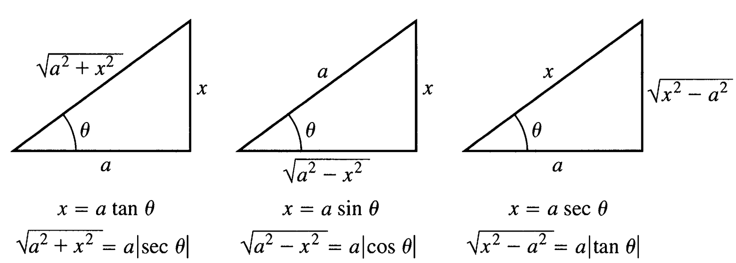
$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta. \tag{2}$$

With $x = a\sec\theta$,

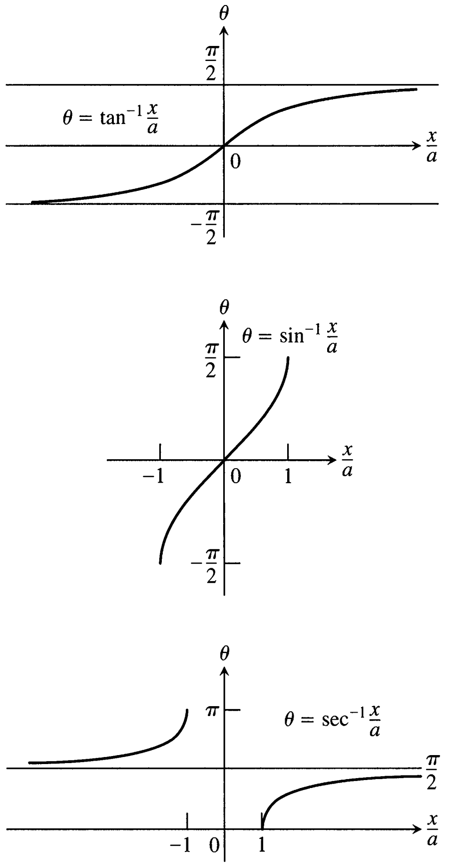
$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \tag{3}$$

Trigonometric Substitutions

1.	$x = a\tan\theta$	replaces $a^2 + x^2$	by $a^2\sec^2\theta$.
2.	$x = a\sin\theta$	replaces $a^2 - x^2$	by $a^2\cos^2\theta$.
3.	$x = a\sec\theta$	replaces $x^2 - a^2$	by $a^2\tan^2\theta$.



7.3 Reference triangles for trigonometric substitutions that change binomials into single squared terms.



7.4 The arc tangent, arc sine, and arc secant of *x/a*, graphed as functions of *x/a*.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a \sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a \sec\theta$.

As we know from Section 6.8, the functions in these substitutions have inverses only for selected values of θ (Fig. 7.4). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\[2ex] \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4 + x^2}}$.

Solution We set

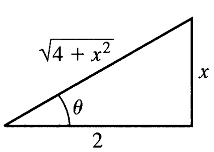
$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\ &= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C && \text{From Fig. 7.5} \\ &= \ln\left|\sqrt{4+x^2} + x\right| + C'. && \text{Taking } C' = C - \ln 2 \end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Fig. 7.5) and read the ratios from the triangle. ❑



7.5 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2\, dx}{\sqrt{9 - x^2}}$.

Solution To replace $9 - x^2$ by a single squared term, we set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$9 - x^2 = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\, d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1 - \cos 2\theta}{2}\, d\theta$$

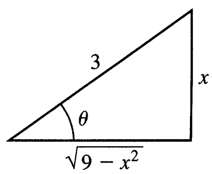
$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{Fig. 7.6}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

❑



7.6 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 3 Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

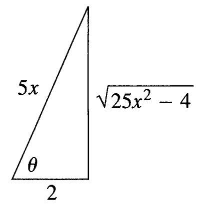
to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

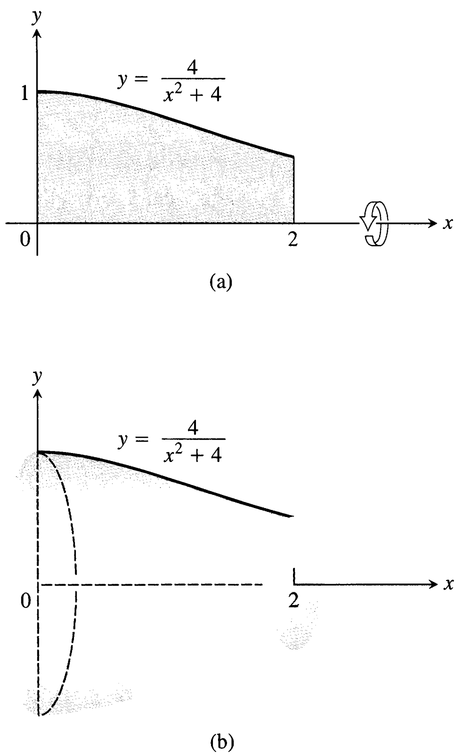
$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta,$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \tan\theta > 0 \text{ for } 0 < \theta < \pi/2$$



7.7 If $x = (2/5)\sec\theta$, $0 \le \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$ and we can read the values of the other trigonometric functions of θ from this right triangle.



7.8 The region (a) and solid (b) in Example 4.

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C \qquad \text{Fig. 7.7}$$

❑

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

EXAMPLE 4 Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2+4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Fig. 7.8) and use the disk method (Section 5.3):

$$V = \int_0^2 \pi[R(x)]^2\,dx = 16\pi\int_0^2 \frac{dx}{(x^2+4)^2}. \qquad R(x) = \frac{4}{x^2+4}$$

To evaluate the integral, we set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\,d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$

$$x^2+4 = 4\tan^2\theta + 4 = 4(\tan^2\theta+1) = 4\sec^2\theta$$

(Fig. 7.9). With these substitutions,

$$V = 16\pi\int_0^2 \frac{dx}{(x^2+4)^2}$$

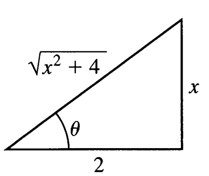
$$= 16\pi\int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{(4\sec^2\theta)^2} \qquad \begin{array}{l}\theta = 0\\ \text{when } x = 0;\\ \theta = \pi/4\\ \text{when } x = 2\end{array}$$

$$= 16\pi\int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \pi\int_0^{\pi/4} 2\cos^2\theta\,d\theta$$

$$= \pi\int_0^{\pi/4}(1+\cos 2\theta)d\theta = \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} \qquad \begin{array}{l}2\cos^2\theta =\\ 1+\cos 2\theta\end{array}$$

$$= \pi\left[\frac{\pi}{4}+\frac{1}{2}\right] \approx 4.04.$$

❑



7.9 Reference triangle for $x = 2\tan\theta$ (Example 4).

Exercises 7.4

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\int \frac{dy}{\sqrt{9+y^2}}$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\int \sqrt{25-t^2}\,dt$

8. $\int \sqrt{1-9t^2}\,dt$

9. $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\int \frac{dx}{1+x^2}$

35. $\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \quad y(2) = 0$

38. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $(x^2+4)\,\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $(x^2+1)^2\,\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1+x^2)$, and the line $x = 1$.

The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan \frac{x}{2} \tag{4}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions. Thus the substitution (4) is a powerful tool. It is cumbersome, however, and is used only when simpler methods fail.

Figure 7.10 shows how $\tan(x/2)$ expresses a rational function of $\sin x$ and $\cos x$. To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1+\tan^2(x/2)} - 1 = \frac{2}{1+z^2} - 1$$

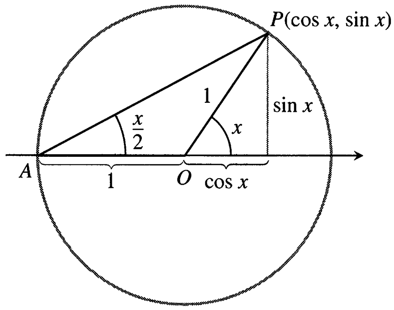
$$\cos x = \frac{1-z^2}{1+z^2}, \tag{5}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$$

$$\sin x = \frac{2z}{1+z^2}. \tag{6}$$



7.10 From this figure, we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1+z^2}. \qquad (7)$$

EXAMPLE

a)
$$\begin{aligned}\int \frac{1}{1+\cos x}\,dx &= \int \frac{1+z^2}{2}\,\frac{2\,dz}{1+z^2}\\ &= \int dz = z + C\\ &= \tan\left(\frac{x}{2}\right) + C\end{aligned}$$

b)
$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx &= \int \frac{1+z^2}{2+2z+2z^2}\,\frac{2\,dz}{1+z^2}\\ &= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+(1/2))^2+3/4}\\ &= \int \frac{du}{u^2+a^2}\\ &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}+C\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1+2\tan(x/2)}{\sqrt{3}}+C\end{aligned}$$ ❑

Use the substitutions in Eqs. (4)–(7) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1-\sin x}$

44. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t-\cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1-\cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$

7.5 Integral Tables and CAS

As you know, the basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. But where do the integrals in the tables come from? They come from applying substitutions and integration by parts. We could derive them all from scratch if we had to, but having the table saves us the trouble of repeating laborious calculations. When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, we have a ready-made solution for the problem at hand. The examples and exercises of this section show how the formulas in integral tables are derived and used. The emphasis is on use. The integration formulas at the back of this book are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 5 requires $n \neq -1$, for example, and Formula 11 requires $n \neq -2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 8 assumes $a \neq 0$, and Formula 13(a) cannot be used unless b is negative.

Many indefinite integrals can also be evaluated with a Computer Algebra System (CAS). These systems are generally faster than tables and usually do not require you to rewrite integrals in special recognizable forms first. We discuss computer algebra systems in the last third of the section.

Integration with Tables

EXAMPLE 1 Find $\displaystyle\int x(2x+5)^{-1}\,dx.$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$

❑

EXAMPLE 2 Find $\displaystyle\int \frac{dx}{x\sqrt{2x+4}}.$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2}\right| + C.$$

Formula 13(a), which requires $b < 0$, would not have been appropriate here. It *is* appropriate, however, in the next example. ❑

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x\sqrt{2x-4}}.$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1}\sqrt{\frac{2x-4}{4}} + C = \tan^{-1}\sqrt{\frac{x-2}{2}} + C. \qquad \square$$

EXAMPLE 4 Find $\displaystyle\int \frac{dx}{x^2\sqrt{2x-4}}$.

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2\cdot 4}\int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x-2}{2}} + C. \qquad \square$$

EXAMPLE 5 Find $\displaystyle\int x \sin^{-1} x \, dx$.

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2dx}{\sqrt{1-x^2}} = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned}\int x \sin^{-1} x \, dx &= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2}\right) + C' \\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C'.\end{aligned} \qquad \square$$

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \qquad (1)$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \qquad (2)$$

$$\int \sin^n x\cos^m x\,dx = -\frac{\sin^{n-1} x\cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x\cos^m x\,dx \qquad (n \neq -m). \qquad (3)$$

Formulas like these are called **reduction formulas** because they replace an integral containing some power of a function with an integral of the same form with the power reduced. By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly.

EXAMPLE 6 Find $\displaystyle\int \tan^5 x\,dx.$

Solution We apply Eq. (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Eq. (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$

❑

As their form suggests, reduction formulas are derived by integration by parts.

EXAMPLE 7 *Deriving a reduction formula*

Show that for any positive integer n,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Solution We use the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx. \qquad \square$$

Sometimes two reduction formulas come into play.

EXAMPLE 8 Find $\displaystyle\int \sin^2 x \cos^3 x\, dx.$

Solution 1 We apply Eq. (3) with $n = 2$ and $m = 3$ to get

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= -\frac{\sin x \cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x \cos^3 x\, dx \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\int \cos^3 x\, dx.\end{aligned}$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax\, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\, dx.$$

With $n = 3$ and $a = 1$, we have

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\ &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.\end{aligned}$$

The combined result is

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C\right) \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15}\sin x + C'.\end{aligned}$$

Solution 2 Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x\, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x\, dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\int \sin^2 x \cos x\, dx \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\left(\frac{\sin^3 x}{3}\right) + C \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x + C.\end{aligned}$$

As you can see, it is faster to use Formula 69, but we often cannot tell beforehand how things will work out. Do not spend a lot of time looking for the "best" formula. Just find one that will work and forge ahead.

Notice also that Formulas 68 (Solution 1) and 69 (Solution 2) lead to different-looking answers. That is often the case with trigonometric integrals and is no cause for concern. The results are equivalent, and we may use whichever one we please. ❑

Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 8) or numerical methods for their evaluation. Examples of the latter include the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1+x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1-k^2\sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives—they are, after all, continuous—but none of the antiderivatives is elementary.

None of the integrals you are asked to evaluate in the present chapter falls into this category, but you may encounter nonelementary integrals from time to time in your other work.

A General Procedure for Indefinite Integration

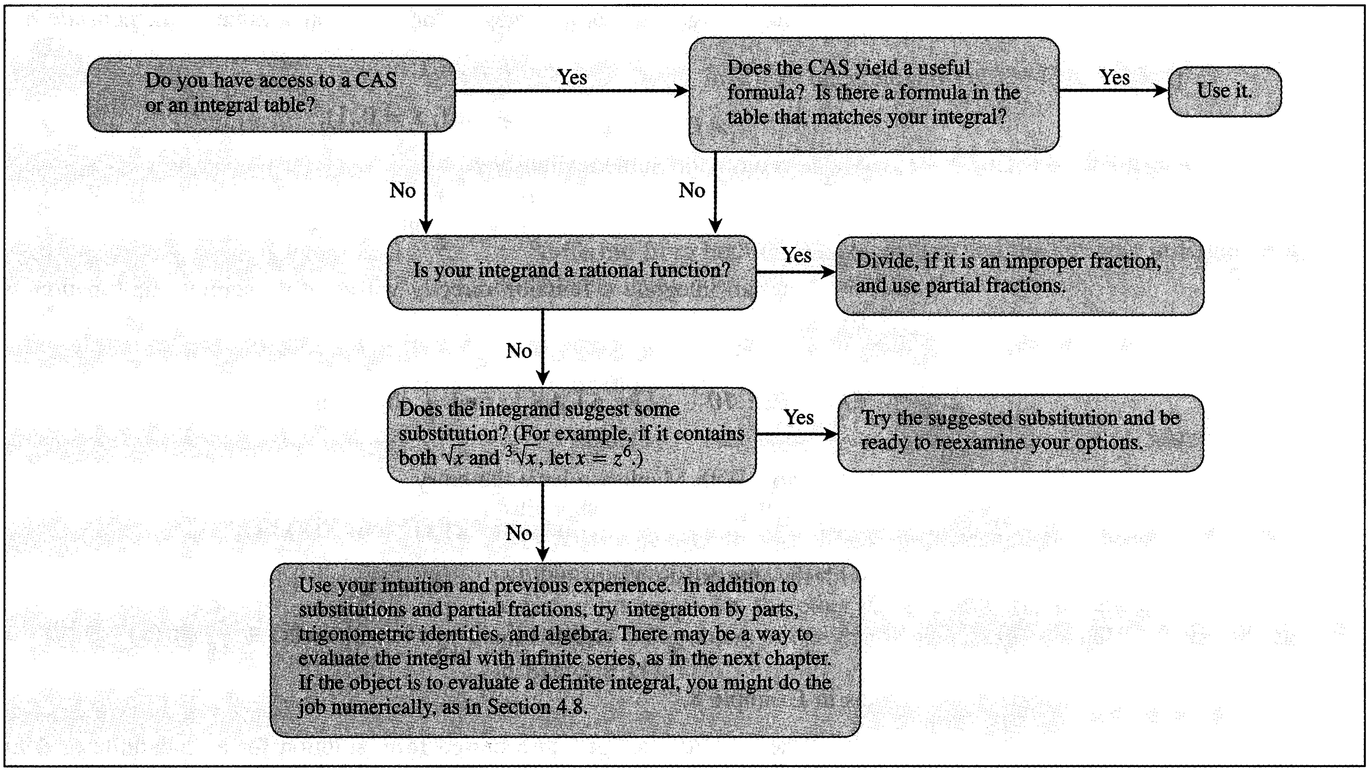
While there is no surefire way to evaluate all indefinite integrals, the procedure in Flowchart 7.1 may help.

Integration with a Computer Algebra System (CAS)

A powerful capability of Computer Algebra Systems is their facility to integrate symbolically. This is performed with the **integrate command** specified by the particular system (e.g., **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 9 Suppose you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2+x^2}.$$



Flowchart 7.1 Procedure for indefinite integration

Using Maple you first define the function:

> f := x^2 * sqrt (a^2 + x^2);

Then you use the integrate command on f, identifying the variable of integration:

> int(f, x);

Maple returns the answer

$$\frac{1}{4}x(a^2+x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2+x^2} - \frac{1}{8}a^4\ln\left(x+\sqrt{a^2+x^2}\right).$$

If you want to see if the answer can be simplified, enter

> simplify(");

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2+x^2} + \frac{1}{4}x^3\sqrt{a^2+x^2} - \frac{1}{8}a^4\ln\left(x+\sqrt{a^2+x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

> int(f, x = 0..Pi/2);

Maple (Version 3.0) will return the expression

$$\frac{1}{64}(4a^2+\pi^2)^{3/2}\pi - \frac{1}{8}a^4\ln\left(\frac{1}{2}\pi+\frac{1}{2}\sqrt{4a^2+\pi^2}\right) - \frac{1}{32}a^2\sqrt{4a^2+\pi^2}\,\pi$$
$$+\frac{1}{8}a^4\ln\left(\sqrt{a^2}\right).$$

You can also find the definite integral for a particular value of the constant a:

> a := 1;

> int(f, x = 0..1);

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} - \frac{1}{8}\ln(1+\sqrt{2})$$ ❑

You can integrate a function directly without first naming the function as in Example 9.

EXAMPLE 10 Use a CAS to find $\displaystyle\int \sin^2 x \cos^3 x \, dx$.

Solution With Maple we have the entry

> int ((sin ^2)(x) * (cos ^3)(x),x);

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x)$$

as in Example 8. ❑

When a CAS cannot find a closed form solution for an indefinite or definite integral it just returns the integral expression you asked for.

EXAMPLE 11 Use a CAS to find $\displaystyle\int (\cos^{-1} ax)^2 \, dx$.

Solution Using Maple we enter

> int ((arccos(a*x))^2, x);

and Maple returns the expression

$$\int \arccos(ax)^2 \, dx$$

indicating it does not have a closed form solution. In the next chapter you will see how series expansion may help to evaluate such an integral. ❑

Computer Algebra Systems vary in how they process integrations. We used Maple in Examples 9–11. Mathematica would have returned somewhat different results:

1. In Example 9, given

   In [1]: = Integrate [x^2 * Sqrt [a^2 + x^2], x]

   Mathematica returns

   $$\text{Out}[1] = \text{Sqrt}\,[a^2+x^2]\left(\frac{a^2x}{8}+\frac{x^3}{4}\right) - \frac{a^4\,\text{Log}\,[x+\text{Sqrt}\,[a^2+x^2]]}{8}$$

   without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

2. The Mathematica answer to the integral

$$\text{In [2]: = Integrate [Sin [x]^2 * Cos [x]^3, x]}$$

in Example 10 is

$$\text{Out [2]} = \frac{30\,\text{Sin}\,[x] - 5\,\text{Sin}\,[3x] - 3\,\text{Sin}\,[5x]}{240}$$

differing from both the Maple answer and the answers in Example 8.

3. Mathematica does give a result for the integration

$$\text{In [3]: = Integrate [ArcCos [a * x]^2, x]}$$

in Example 11:

$$\text{Out [3]} = -2x - \frac{2\,\text{Sqrt}\,[1 - a^2x^2]\ \text{ArcCos}\,[ax]}{a} + x\text{ArcCos}\,[ax]^2$$

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 111.

Exercises 7.5

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–38.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\displaystyle\int x\sqrt{2x-3}\,dx$

6. $\displaystyle\int x(7x+5)^{3/2}\,dx$

7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\displaystyle\int \sqrt{25-p^2}\,dp$

16. $\displaystyle\int q^2\sqrt{25-q^2}\,dq$

17. $\displaystyle\int \frac{r^2}{\sqrt{4-r^2}}\,dr$

18. $\displaystyle\int \frac{ds}{\sqrt{s^2-2}}$

19. $\displaystyle\int \frac{d\theta}{5+4\sin 2\theta}$

20. $\displaystyle\int \frac{d\theta}{4+5\sin 2\theta}$

21. $\displaystyle\int e^{2t}\cos 3t\,dt$

22. $\displaystyle\int e^{-3t}\sin 4t\,dt$

23. $\displaystyle\int x\cos^{-1}x\,dx$

24. $\displaystyle\int x\sin^{-1}x\,dx$

25. $\displaystyle\int \frac{ds}{(9-s^2)^2}$

26. $\displaystyle\int \frac{d\theta}{(2-\theta^2)^2}$

27. $\displaystyle\int \frac{\sqrt{4x+9}}{x^2}\,dx$

28. $\displaystyle\int \frac{\sqrt{9x-4}}{x^2}\,dx$

29. $\displaystyle\int \frac{\sqrt{3t-4}}{t}\,dt$

30. $\displaystyle\int \frac{\sqrt{3t+9}}{t}\,dt$

31. $\displaystyle\int x^2\tan^{-1}x\,dx$

32. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

33. $\displaystyle\int \sin 3x\cos 2x\,dx$

34. $\displaystyle\int \sin 2x\cos 3x\,dx$

35. $\int 8 \sin 4t \sin \frac{t}{2}\, dt$

36. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt$

37. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta$

38. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta$

Substitution and Integral Tables

In Exercises 39–52, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

39. $\int \frac{x^3 + x + 1}{(x^2+1)^2}\, dx$

40. $\int \frac{x^2 + 6x}{(x^2+3)^2}\, dx$

41. $\int \sin^{-1} \sqrt{x}\, dx$

42. $\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}}\, dx$

43. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$

44. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\, dx$

45. $\int \cot t \sqrt{1 - \sin^2 t}\, dt,\ 0 < t < \pi/2$

46. $\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$

47. $\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$

48. $\int \frac{\cos \theta\, d\theta}{\sqrt{5 + \sin^2 \theta}}$

49. $\int \frac{3\, dr}{\sqrt{9r^2 - 1}}$

50. $\int \frac{3\, dy}{\sqrt{1 + 9y^2}}$

51. $\int \cos^{-1} \sqrt{x}\, dx$

52. $\int \tan^{-1} \sqrt{y}\, dy$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 53–72.

53. $\int \sin^5 2x\, dx$

54. $\int \sin^5 \frac{\theta}{2}\, d\theta$

55. $\int 8 \cos^4 2\pi t\, dt$

56. $\int 3 \cos^5 3y\, dy$

57. $\int \sin^2 2\theta \cos^3 2\theta\, d\theta$

58. $\int 9 \sin^3 \theta \cos^{3/2} \theta\, d\theta$

59. $\int 2 \sin^2 t \sec^4 t\, dt$

60. $\int \csc^2 y \cos^5 y\, dy$

61. $\int 4 \tan^3 2x\, dx$

62. $\int \tan^4 \left(\frac{x}{2}\right) dx$

63. $\int 8 \cot^4 t\, dt$

64. $\int 4 \cot^3 2t\, dt$

65. $\int 2 \sec^3 \pi x\, dx$

66. $\int \frac{1}{2} \csc^3 \frac{x}{2}\, dx$

67. $\int 3 \sec^4 3x\, dx$

68. $\int \csc^4 \frac{\theta}{3}\, d\theta$

69. $\int \csc^5 x\, dx$

70. $\int \sec^5 x\, dx$

71. $\int 16x^3 (\ln x)^2\, dx$

72. $\int (\ln x)^3\, dx$

Powers of x Times Exponentials

Evaluate the integrals in Exercises 73–80 using table Formulas 103–106. These integrals can also be evaluated using tabular integration (Section 7.2).

73. $\int x e^{3x}\, dx$

74. $\int x e^{-2x}\, dx$

75. $\int x^3 e^{x/2}\, dx$

76. $\int x^2 e^{\pi x}\, dx$

77. $\int x^2 2^x\, dx$

78. $\int x^2 2^{-x}\, dx$

79. $\int x \pi^x\, dx$

80. $\int x 2^{\sqrt{2}x}\, dx$

Substitutions with Reduction Formulas

Evaluate the integrals in Exercises 81–86 by making a substitution (possibly trigonometric) and then applying a reduction formula.

81. $\int e^t \sec^3 (e^t - 1)\, dt$

82. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

83. $\int_0^1 2\sqrt{x^2 + 1}\, dx$

84. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

85. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r}\, dr$

86. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercises 87–92.

87. $\int \frac{1}{8} \sinh^5 3x\, dx$

88. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\, dx$

89. $\int x^2 \cosh 3x\, dx$

90. $\int x \sinh 5x\, dx$

91. $\int \operatorname{sech}^7 x \tanh x\, dx$

92. $\int \operatorname{csch}^3 2x \coth 2x\, dx$

Theory and Examples

Exercises 93–100 refer to formulas in the table of integrals at the back of the book.

93. Derive Formula 9 by using the substitution $u = ax + b$ to evaluate

$$\int \frac{x}{(ax+b)^2}\, dx.$$

94. Derive Formula 17 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{(a^2 + x^2)^2}.$$

95. Derive Formula 29 by using a trigonometric substitution to evaluate

$$\int \sqrt{a^2 - x^2}\, dx.$$

96. Derive Formula 46 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}}.$$

97. Derive Formula 80 by evaluating

$$\int x^n \sin ax\, dx$$

by integration by parts.

98. Derive Formula 110 by evaluating

$$\int x^n (\ln ax)^m\, dx$$

by integration by parts.

99. Derive Formula 99 by evaluating

$$\int x^n \sin^{-1} ax\, dx$$

by integration by parts.

100. Derive Formula 101 by evaluating

$$\int x^n \tan^{-1} ax\, dx$$

by integration by parts.

101. Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the x-axis.

102. Find the length of the curve $y = x^2$, $0 \le x \le \sqrt{3}/2$.

103. Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x+1}$ and the line $x = 3$.

104. A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x+3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the y-axis.

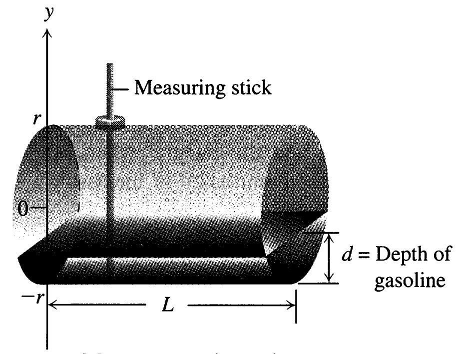
105. CALCULATOR Use the integral table and a calculator to find to 2 decimal places the area of the surface generated by revolving the curve $y = x^2$, $-1 \le x \le 1$, about the x-axis.

106. The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L, mounted horizontally, as shown here. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a) Show, in the notation of the figure here, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

b) Evaluate the integral.



Measurements in centimeters

107. What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

108. What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

CAS Explorations and Projects

In Exercises 109 and 110, use a CAS to perform the integrations.

109. Evaluate the integrals

a) $\int x \ln x\, dx$ **b)** $\int x^2 \ln x\, dx$

c) $\int x^3 \ln x\, dx$.

d) What pattern do you see? Predict the formula for $\int x^4 \ln x\, dx$ and then see if you are correct by evaluating it with a CAS.

e) What is the formula for $\int x^n \ln x\, dx$, $n \ge 1$? Check your answer using a CAS.

110. Evaluate the integrals

a) $\int \frac{\ln x}{x^2}\, dx$ **b)** $\int \frac{\ln x}{x^3}\, dx$

c) $\int \frac{\ln x}{x^4}\, dx$.

d) What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\, dx$$

and then see if you are correct by evaluating it with a CAS.

e) What is the formula for

$$\int \frac{\ln x}{x^n}\, dx, n \ge 2?$$

Check your answer using a CAS.

111. a) Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

where n is an arbitrary positive integer. Does your CAS find the result?

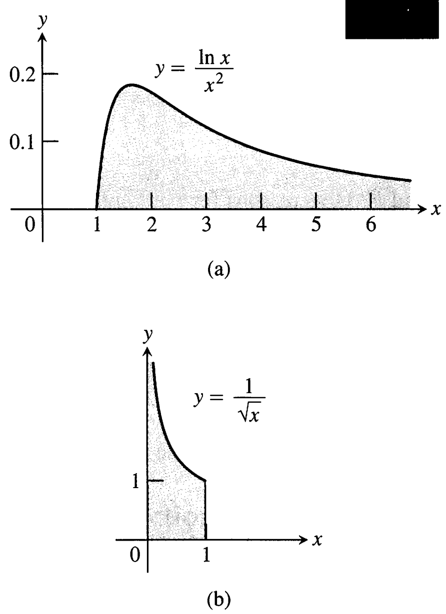
b) In succession, find the integral when $n = 1, 2, 3, 5, 7$. Comment on the complexity of the results.

c) Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

7.6 Improper Integrals



7.11 Are the areas under these infinite curves finite? See Examples 1 and 2.

Up to now, we have required our definite integrals to have two properties. First, that the domain of integration, from a to b, be finite. Second, that the range of the integrand be finite on this domain. In practice, however, we frequently encounter problems that fail to meet one or both of these conditions. As an example of an infinite domain, we might want to consider the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ (Fig. 7.11a). As an example of an infinite range, we might want to consider the area under the curve $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ (Fig. 7.11b). We treat both examples in the same reasonable way. We ask, "What is the integral when the domain is slightly less?" and examine the answer as the domain increases to the limit. We do the finite case and then see what happens as we approach infinity.

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the infinite curve (Fig. 7.12). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2} dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

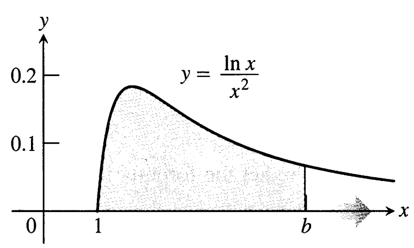
$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = -\left[\lim_{b\to\infty} \frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty} \frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

l'Hôpital's rule

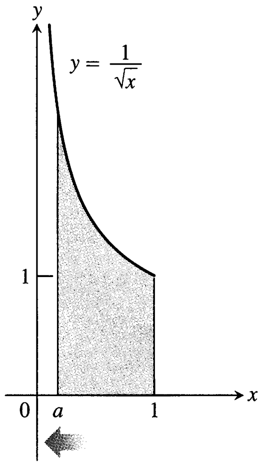


7.12 The area under this curve is $\lim_{b\to\infty} \int_1^b ((\ln x)/x^2)\, dx$ (Example 1).

In integral notation, the area under the infinite curve from 1 to ∞ is

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx = 1.$$

❑



7.13 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 (1/\sqrt{x})\, dx$$

(Example 2).

EXAMPLE 2 Is the area under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ finite? If so, what is it?

Solution We find the area under the curve from a to 1 and examine the limit as $a \to 0^+$. If the limit is finite, we take it to be the area under the infinite curve (Fig. 7.13). The area from a to 1 is

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

The limit as $a \to 0^+$ is

$$\lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2 - 0 = 2.$$

In integral notation, the area under the infinite curve from 0 to 1 is

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = 2.$$

❑

Improper Integrals

The integrals for the areas in Examples 1 and 2 are improper integrals.

Definition

Integrals with infinite limits of integration and integrals of functions that become infinite at a point within the interval of integration are **improper integrals.** When the limits involved exist, we evaluate such integrals with the following definitions:

1. If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx. \tag{1}$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx. \tag{2}$$

3. If f is continuous on $(a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx. \tag{3}$$

4. If f is continuous on $[a, b)$, then

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx. \tag{4}$$

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist the improper integral **diverges.**

Example 1 illustrates Part 1 of the definition:

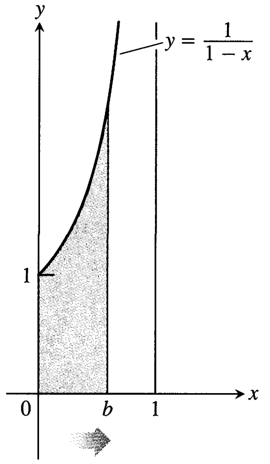
$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x^2}\, dx = 1 \qquad \text{Infinite upper limit of integration}$$

Example 2 illustrates Part 3 of the definition:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = 2$$

Integrand becomes infinite at lower limit of integration

In each case, the integral converges. The integral in the next example diverges.



7.14 If the limit exists $\int_0^1 (1/(1-x))\,dx = \lim_{b\to 1^-} \int_0^b (1/(1-x))\,dx$ (Example 3).

EXAMPLE 3 *A divergent improper integral*

Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but becomes infinite as $x \to 1^-$ (Fig. 7.14). We evaluate the integral as

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx &= \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b \\ &= \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ❑

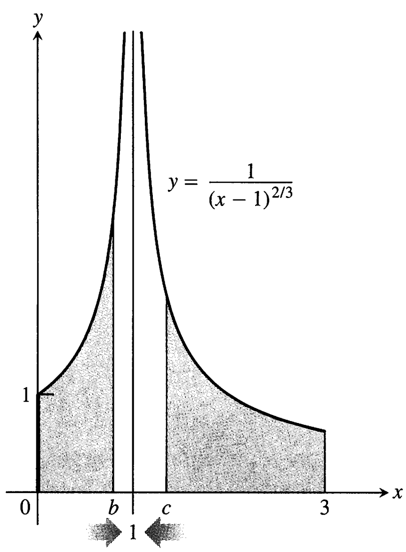
The list in the preceding definition extends in a natural way to integrals with two infinite limits of integration. We will treat these later in the section. The list also extends to integrals of functions that become infinite at an interior point d of the interval of integration. In this case, we define the integral from a to b to be the sum of the integrals from a to d and d to b.

Definition

If f becomes infinite at an interior point d of $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx. \tag{5}$$

The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges.**



7.15 Example 4 investigates the convergence of

$\int_0^3 (1/(x-1)^{2/3})\,dx.$

EXAMPLE 4 *Infinite at an interior point*

Investigate the convergence of

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

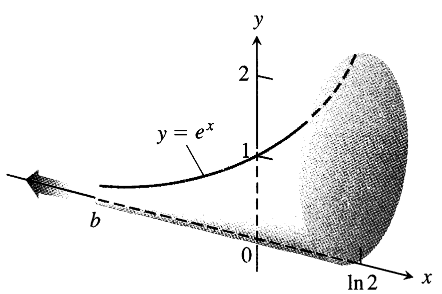
Solution The integrand $f(x) = 1/(x-1)^{2/3}$ becomes infinite at $x = 1$ but is continuous on $[0, 1)$ and $(1, 3]$ (Fig. 7.15). The convergence of the integral over $[0, 3]$ depends on the integrals from 0 to 1 and 1 to 3. On $[0, 1]$ we have

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b\to 1^-}[3(b-1)^{1/3} - 3(0-1)^{1/3}] = 3.\end{aligned}$$

On [1, 3] we have

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$
$$= \lim_{c\to 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}.$$

Both limits are finite, so the integral of f from 0 to 3 converges and its value is $3 + 3\sqrt[3]{2}$. ❑



7.16 The calculation in Example 5 shows that this infinite horn has a finite volume.

EXAMPLE 5 The cross sections of the solid horn in Fig. 7.16 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \le \ln 2$. Find the volume of the horn.

Solution The area of a typical cross section is

$$A(x) = \pi(\text{radius})^2 = \pi\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from b to ln 2. As in Section 5.2 (the method of slicing), the volume of this portion is

$$V = \int_b^{\ln 2} A(x)\,dx = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}e^{2x}\Big]_b^{\ln 2}$$
$$= \frac{\pi}{8}(e^{\ln 4} - e^{2b}) = \frac{\pi}{8}(4 - e^{2b}).$$

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4-0) = \pi/2$. The volume of the horn is $\pi/2$. ❑

EXAMPLE 6 Evaluate $\displaystyle\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx$.

Solution

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty} \int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty} \int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right) dx \qquad \text{Partial fractions}$$

$$= \lim_{b\to\infty} \Big[2\ln(x-1) - \ln(x^2+1) - \tan^{-1} x\Big]_2^b$$

$$= \lim_{b\to\infty} \left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1} x\right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b\to\infty} \left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1} b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1} 2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1} 2 \approx 1.1458$$

Notice that we combined the logarithms in the antiderivative *before* we calculated the limit as $b \to \infty$. Had we not done so, we would have encountered the indeterminate form

$$\lim_{b\to\infty} (2 \ln (b-1) - \ln (b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. But our original route was shorter. ❑

Integrals from $-\infty$ to ∞

In the mathematics underlying studies of light, electricity, and sound we encounter integrals with two infinite limits of integration. The next definition addresses the convergence of such integrals.

Definition

If f is continuous on $(-\infty, \infty)$ and if $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ **converges** and define its value to be

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx. \tag{6}$$

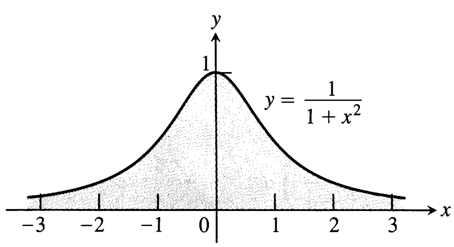
If either or both of the integrals on the right-hand side of this equation diverge, the integral of f from $-\infty$ to ∞ **diverges.**

It can be shown that the choice of a in Eq. (6) is unimportant. We can evaluate or determine the convergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

The integral of f from $-\infty$ to ∞ need not equal $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$, which may exist even if $\int_{-\infty}^{\infty} f(x)\,dx$ does not converge (Exercise 75).

EXAMPLE 7

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \qquad \text{Eq. (6) with } a = 0\\
&= \lim_{b\to-\infty} \left[\tan^{-1} x\right]_b^0 + \lim_{c\to\infty} \left[\tan^{-1} x\right]_0^c\\
&= \lim_{b\to-\infty} \left[\tan^{-1} 0 - \tan^{-1} b\right] + \lim_{c\to\infty} \left[\tan^{-1} c - \tan^{-1} 0\right]\\
&= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi.
\end{aligned}$$

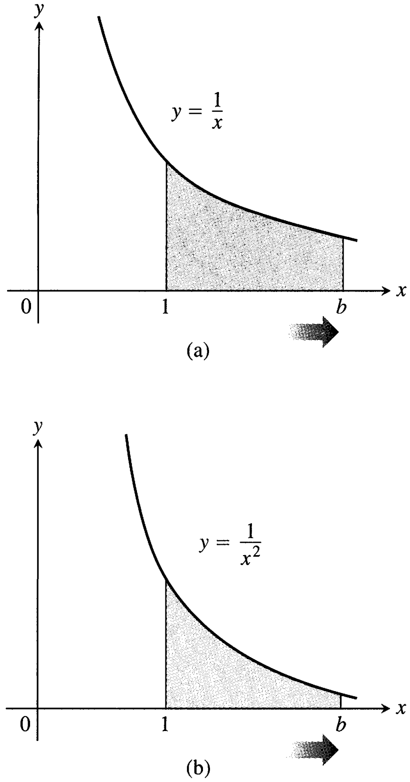


7.17 The area under this "doubly" infinite curve is finite (Example 7).

We interpret the integral as the area of the infinite region between the curve $y = 1/(1+x^2)$ and the x-axis (Fig. 7.17). ❑

The Integral $\int_1^{\infty} dx/x^p$

The convergence of the integral $\int_1^{\infty} dx/x^p$ depends on p. The next example illustrates this with $p = 1$ and $p = 2$.



7.18 One of these limits is finite; the other is not (Example 8).

EXAMPLE 8 Investigate the convergence of

$$\int_1^{\infty} \frac{dx}{x} \quad \text{and} \quad \int_1^{\infty} \frac{dx}{x^2}.$$

Solution The functions involved are continuous on $[1, \infty)$ and their graphs both approach the x-axis as $x \to \infty$ (Fig. 7.18), so it is reasonable to think that the areas under these infinite curves might be finite. In the first case,

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to\infty} (\ln b - \ln 1) = \infty,$$

so the integral diverges. In the second case,

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2} = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1,$$

so the integral converges and its value is 1. ❑

Generally, $\int_1^{\infty} dx/x^p$ converges if $p > 1$ and diverges if $p \le 1$ (Exercise 67).

Tests for Convergence and Divergence

When an improper integral cannot be evaluated directly (often the case in practice) we turn to the two-step procedure of first establishing the fact of convergence and then approximating the integral numerically. The principal tests for convergence are the direct comparison and limit comparison tests.

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^{\infty} e^{-x^2} dx$.

Solution By definition,

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b\to\infty} \int_1^{b} e^{-x^2} dx.$$

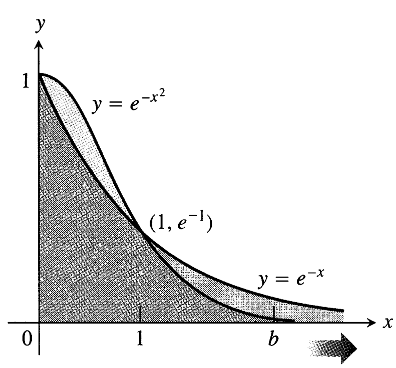
We cannot evaluate the latter integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2} dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \ge 1$ we have $e^{-x^2} \le e^{-x}$ (Fig. 7.19), so that

$$\int_1^{b} e^{-x^2} dx \le \int_1^{b} e^{-x} dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b\to\infty} \int_1^{b} e^{-x^2} dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something less than 0.37. ❑



7.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 9).

The comparison of e^{-x^2} and e^{-x} in Example 9 is a special case of the following test.

Theorem 1

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ and suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

EXAMPLE 10

a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ on $[1, \infty)$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges.

b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because $\dfrac{1}{\sqrt{x^2 - 0.1}} \geq \dfrac{1}{x}$ on $[1, \infty)$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges.

❑

Theorem 2

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \qquad (0 < L < \infty),$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

In the language of Section 6.7, Theorem 2 says that if two positive functions grow at the same rate as $x \to \infty$, then their integrals from a to ∞ behave alike: They both converge or both diverge. This does not mean that their integrals have the same value, however, as the next example shows.

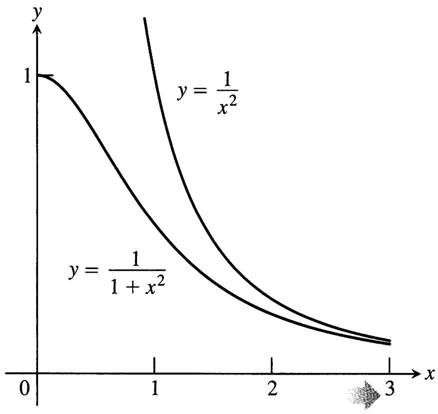
EXAMPLE 11 Compare

$$\int_1^\infty \frac{dx}{x^2} \qquad \text{and} \qquad \int_1^\infty \frac{dx}{1+x^2}$$

with the Limit Comparison Test.

Solution With $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} \\ &= \lim_{x\to\infty} \frac{1+x^2}{x^2} = \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,\end{aligned}$$



7.20 The functions in Example 11.

a positive finite limit (Fig. 7.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however.

$$\int_1^\infty \frac{dx}{x^2} = 1, \qquad \text{Example 8}$$

and

$$\begin{aligned} \int_1^\infty \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

❑

EXAMPLE 12

$$\int_1^\infty \frac{3}{e^x+5}\,dx \quad \text{converges because} \quad \int_1^\infty \frac{1}{e^x}\,dx \text{ converges}$$

and

$$\begin{aligned} \lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} &= \lim_{x\to\infty} \frac{e^x+5}{3e^x} \\ &= \lim_{x\to\infty} \left(\frac{1}{3} + \frac{5}{3e^x}\right) = \frac{1}{3} + 0 = \frac{1}{3}, \end{aligned}$$

a positive finite limit. As far as the convergence of the improper integral is concerned, $3/(e^x+5)$ behaves like $1/e^x$. ❑

Computer Algebra Systems

Computer Algebra Systems can evaluate many convergent improper integrals.

EXAMPLE 13 Evaluate the integral $\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx$ from Example 6.

Solution Using Maple, enter

```
> f: = (x + 3)/((x − 1)*(x^2 + 1));
```

Then use the integration command

```
> int(f, x=2..infinity);
```

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result use the evaluation command **evalf** and specify the number of digits, as follows:

```
> evalf(", 6);
```

The ditto symbol (") instructs the computer to evaluate the last expression on the screen, in this case $-\frac{1}{2}\pi + \ln(5) + \text{arc tan}(2)$. Maple returns 1.14579.

Using Mathematica, entering

In [1]: = Integrate $[(x+3)/((x-1)(x\hat{}2+1)),\ \{x, 2, \text{Infinity}\}]$

returns

$$\text{Out [1]} = \frac{-\text{Pi}}{2} + \text{ArcTan [2]} + \text{Log [5]}.$$

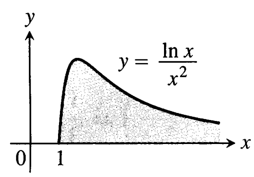
To obtain a numerical result with six digits, use the command "N[%, 6]" which also yields 1.14579. ❑

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION

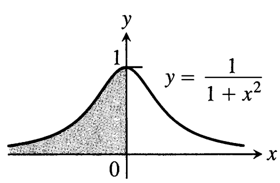
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



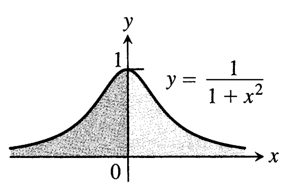
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$



3. Both limits

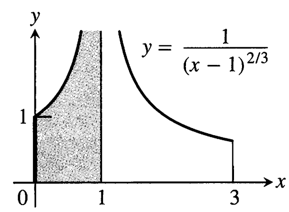
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE

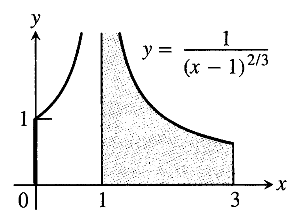
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



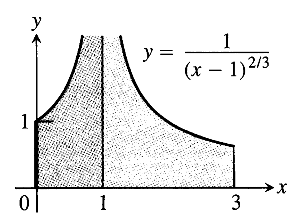
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Exercises 7.6

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\int_0^\infty \frac{dx}{x^2+1}$
2. $\int_1^\infty \frac{dx}{x^{1.001}}$
3. $\int_0^1 \frac{dx}{\sqrt{x}}$
4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$
5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$
6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$
7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
8. $\int_0^1 \frac{dr}{r^{0.999}}$
9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$
10. $\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$
11. $\int_2^\infty \frac{2}{v^2-v}\,dv$
12. $\int_2^\infty \frac{2\,dt}{t^2-1}$
13. $\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$
14. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$
15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$
16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$
17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$
18. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$
19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$
20. $\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$
21. $\int_{-\infty}^0 \theta e^\theta\,d\theta$
22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$
23. $\int_{-\infty}^0 e^{-|x|}\,dx$
24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx$
25. $\int_0^1 x\ln x\,dx$
26. $\int_0^1 (-\ln x)\,dx$
27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$
28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$
29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$
30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$
31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$
32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$
33. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$
34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan\theta\,d\theta$
36. $\int_0^{\pi/2} \cot\theta\,d\theta$
37. $\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$
38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$
39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$
40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$
41. $\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$
42. $\int_0^1 \frac{dt}{t-\sin t}$ $\left(\textit{Hint: } t \ge \sin t \text{ for } t \ge 0\right)$
43. $\int_0^2 \frac{dx}{1-x^2}$
44. $\int_0^2 \frac{dx}{1-x}$
45. $\int_{-1}^1 \ln|x|\,dx$
46. $\int_{-1}^1 -x\ln|x|\,dx$
47. $\int_1^\infty \frac{dx}{x^3+1}$
48. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$
49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$
50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$
51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$
52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$
53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$
54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$
55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$
56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

58. $\int_2^\infty \frac{1}{\ln x}\,dx$

59. $\int_1^\infty \frac{e^x}{x}\,dx$

60. $\int_{e^e}^\infty \ln(\ln x)\,dx$

61. $\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\int_1^\infty \frac{1}{e^x-2^x}\,dx$

63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

64. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. *Estimating the value of a convergent improper integral whose domain is infinite*

a) Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

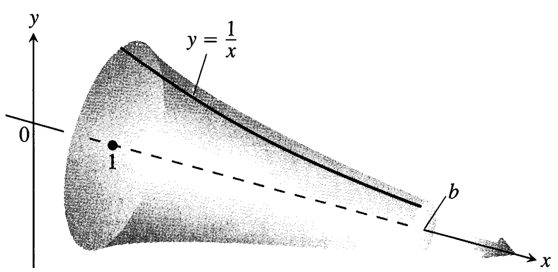
b) NUMERICAL INTEGRATOR Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

66. *The infinite paint can or Gabriel's horn.* As Example 8 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges. (a) Calculate it. (b) This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

67. a) Show that

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$$

if $p > 1$ but that the integral is infinite if $p < 1$. Example 8 shows what happens if $p = 1$.

b) Show that

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}$$

if $p < 1$ but that the integral diverges if $p \ge 1$.

68. Find the values of p for which each integral converges:

a) $\int_1^2 \frac{dx}{x(\ln x)^p}$, **b)** $\int_2^\infty \frac{dx}{x(\ln x)^p}$.

Exercises 69–72 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

69. Find the area of the region.

70. Find the centroid of the region.

71. Find the volume of the solid generated by revolving the region about the y-axis.

72. Find the volume of the solid generated by revolving the region about the x-axis.

73. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

74. The region in Exercise 73 is revolved about the x-axis to generate a solid.

a) Find the volume of the solid.

b) Show that the inner and outer surfaces of the solid have infinite area.

75. *$\int_{-\infty}^\infty f(x)\,dx$ may not equal $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$.* Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^b \frac{2x\,dx}{x^2+1} = 0.$$

76. Here is an argument that ln 3 equals $\infty - \infty$. Where does the argument go wrong? Give reasons for your answer.

$$\begin{aligned}
\ln 3 &= \ln 1 + \ln 3 = \ln 1 - \ln \frac{1}{3} \\
&= \lim_{b\to\infty} \ln\left(\frac{b-2}{b}\right) - \ln\frac{1}{3} \\
&= \lim_{b\to\infty} \left[\ln \frac{x-2}{x}\right]_3^b \\
&= \lim_{b\to\infty} \left[\ln(x-2) - \ln x\right]_3^b \\
&= \lim_{b\to\infty} \int_3^b \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \frac{1}{x-2} dx - \int_3^\infty \frac{1}{x} dx \\
&= \lim_{b\to\infty} \left[\ln(x-2)\right]_3^b - \lim_{b\to\infty} \left[\ln x\right]_3^b \\
&= \infty - \infty.
\end{aligned}$$

77. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a) $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b) $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

78. a) Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

b) Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Use direct evaluation, the comparison tests, and the results in Exercise 78, as appropriate, to determine the convergence or divergence of the integrals in Exercises 79–86. If more than one method applies, use whatever method you prefer.

79. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}}$

80. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^6+1}}$

81. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$

82. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$

83. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

84. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2}$

85. $\displaystyle\int_{-\infty}^{\infty} \frac{|\sin x| + |\cos x|}{|x|+1}\,dx$

(*Hint:* $|\sin\theta| + |\cos\theta| \ge \sin^2\theta + \cos^2\theta$.)

86. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2)}$

CAS Explorations and Projects

In Exercises 87–90, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

87. $\displaystyle\int_{0}^{e} x^p \ln x\,dx$

88. $\displaystyle\int_{e}^{\infty} x^p \ln x\,dx$

89. $\displaystyle\int_{0}^{\infty} x^p \ln x\,dx$

90. $\displaystyle\int_{-\infty}^{\infty} x^p \ln|x|\,dx$

91. The integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the **sine-integral function,** has important applications in optics.

a) Plot the integrand $(\sin t)/t$ for $t > 0$. Is the Si function everywhere increasing or decreasing? Do you think $\mathrm{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\mathrm{Si}(x)$ for $0 \le x \le 25$.

b) Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

92. The function

$$\mathrm{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the **error function,** has important applications in probability and statistics.

a) Plot the error function for $0 \le x \le 25$.

b) Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 13.3, Exercise 37.

CHAPTER 7 QUESTIONS TO GUIDE YOUR REVIEW

1. What basic integration formulas do you know?
2. What procedures do you know for matching integrals to basic formulas?
3. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
4. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
5. What is tabular integration? Give an example.
6. What is the goal of the method of partial fractions?
7. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
   a) is a product of distinct linear factors?
   b) consists of a repeated linear factor?
   c) contains an irreducible quadratic factor?

   What do you do if the degree of f is *not* less than the degree of g?
8. What substitutions are sometimes used to change quadratic binomials into single squared terms? Why might you want to make such a change?
9. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?
10. What is a reduction formula? How are reduction formulas typically derived? How are reduction formulas used? Give an example.
11. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
12. What is an improper integral? How are the values of various types of improper integrals defined? Give examples.
13. What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.

CHAPTER 7 PRACTICE EXERCISES

Integration Using Substitutions

Evaluate the integrals in Exercises 1–82. To transform each integral into a recognizable basic form, it may be necessary to use one or more of the techniques of algebraic substitution, completing the square, separating fractions, long division, or trigonometric substitution.

1. $\int x\sqrt{4x^2-9}\,dx$

2. $\int 6x\sqrt{3x^2+5}\,dx$

3. $\int x(2x+1)^{1/2}\,dx$

4. $\int x(1-x)^{-1/2}\,dx$

5. $\int \frac{x\,dx}{\sqrt{8x^2+1}}$

6. $\int \frac{x\,dx}{\sqrt{9-4x^2}}$

7. $\int \frac{y\,dy}{25+y^2}$

8. $\int \frac{y^3\,dy}{4+y^4}$

9. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}}$

10. $\int \frac{2t\,dt}{t^4+1}$

11. $\int z^{2/3}(z^{5/3}+1)^{2/3}\,dz$

12. $\int z^{-1/5}(1+z^{4/5})^{-1/2}\,dz$

13. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2}$

14. $\int \frac{\cos\theta\,d\theta}{(1+\sin\theta)^{1/2}}$

15. $\int \frac{\sin t}{3+4\cos t}\,dt$

16. $\int \frac{\cos 2t}{1+\sin 2t}\,dt$

17. $\int \sin 2x\,e^{\cos 2x}\,dx$

18. $\int \sec x\tan x\,e^{\sec x}\,dx$

19. $\int e^\theta\sin(e^\theta)\cos^2(e^\theta)\,d\theta$

20. $\int e^\theta\sec^2(e^\theta)\,d\theta$

21. $\int 2^{x-1}\,dx$

22. $\int 5^{x\sqrt{2}}\,dx$

23. $\int \frac{dv}{v\ln v}$

24. $\int \frac{dv}{v(2+\ln v)}$

25. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$

26. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

27. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$

28. $\int \frac{dx}{\sqrt{49-x^2}}$

29. $\int \frac{dt}{\sqrt{16-9t^2}}$

30. $\int \frac{dt}{\sqrt{9-4t^2}}$

31. $\int \frac{dt}{9+t^2}$

32. $\int \frac{dt}{1+25t^2}$

33. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}}$

34. $\int \frac{6\,dx}{x\sqrt{4x^2-9}}$

35. $\int \frac{dx}{\sqrt{4x-x^2}}$

36. $\int \frac{dx}{\sqrt{4x-x^2-3}}$

37. $\int \frac{dy}{y^2-4y+8}$

38. $\int \frac{dt}{t^2+4t+5}$

39. $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$

40. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}}$

41. $\int \sin^2 x\,dx$

42. $\int \cos^2 3x\,dx$

43. $\int \sin^3 \frac{\theta}{2}\,d\theta$

44. $\int \sin^3\theta\cos^2\theta\,d\theta$

45. $\int \tan^3 2t\,dt$

46. $\int 6\sec^4 t\,dt$

47. $\int \frac{dx}{2\sin x\cos x}$

48. $\int \frac{2\,dx}{\cos^2 x-\sin^2 x}$

49. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y-1}\,dy$

50. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t+1}\,dt$

51. $\int_0^{\pi} \sqrt{1-\cos^2 2x}\,dx$

52. $\int_0^{2\pi} \sqrt{1-\sin^2\frac{x}{2}}\,dx$

53. $\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos 2t}\,dt$

54. $\int_{\pi}^{2\pi} \sqrt{1+\cos 2t}\,dt$

55. $\int \frac{x^2}{x^2+4}\,dx$

56. $\int \frac{x^3}{9+x^2}\,dx$

57. $\int \frac{4x^2+3}{2x-1}\,dx$

58. $\int \frac{2x}{x-4}\,dx$

59. $\int \frac{2y-1}{y^2+4}\,dy$

60. $\int \frac{y+4}{y^2+1}\,dy$

61. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt$

62. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt$

63. $\int \frac{\tan x\,dx}{\tan x+\sec x}$

64. $\int \frac{\cot x}{\cot x+\csc x}\,dx$

65. $\int \sec(5-3x)\,dx$

66. $\int x\csc(x^2+3)\,dx$

67. $\int \cot\left(\frac{x}{4}\right)dx$

68. $\int \tan(2x-7)\,dx$

69. $\int x\sqrt{1-x}\,dx$

70. $\int 3x\sqrt{2x+1}\,dx$

71. $\int \sqrt{z^2+1}\,dz$

72. $\int (16+z^2)^{-3/2}\,dz$

73. $\int \frac{dy}{\sqrt{25+y^2}}$

74. $\int \frac{dy}{\sqrt{25+9y^2}}$

75. $\int \frac{dx}{x^2\sqrt{1-x^2}}$

76. $\int \frac{x^3\,dx}{\sqrt{1-x^2}}$

77. $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

78. $\int \sqrt{4-x^2}\,dx$

79. $\int \frac{dx}{\sqrt{x^2-9}}$

80. $\int \frac{12\,dx}{(x^2-1)^{3/2}}$

81. $\int \frac{\sqrt{w^2-1}}{w}\,dw$

82. $\int \frac{\sqrt{z^2-16}}{z}\,dz$

Integration by Parts

Evaluate the integrals in Exercises 83–90 using integration by parts.

83. $\int \ln(x+1)\,dx$

84. $\int x^2\ln x\,dx$

85. $\int \tan^{-1} 3x\,dx$

86. $\int \cos^{-1}\left(\frac{x}{2}\right)dx$

87. $\int (x+1)^2 e^x\,dx$

88. $\int x^2\sin(1-x)\,dx$

89. $\int e^x\cos 2x\,dx$

90. $\int e^{-2x}\sin 3x\,dx$

Partial Fractions

Evaluate the integrals in Exercises 91–110. It may be necessary to use a substitution first.

91. $\int \frac{x\,dx}{x^2-3x+2}$

92. $\int \frac{x\,dx}{x^2+4x+3}$

93. $\int \frac{dx}{x(x+1)^2}$

94. $\int \frac{x+1}{x^2(x-1)}\,dx$

95. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

96. $\int \frac{\cos\theta\,d\theta}{\sin^2\theta+\sin\theta-6}$

97. $\int \frac{3x^2+4x+4}{x^3+x}\,dx$

98. $\int \frac{4x\,dx}{x^3+4x}$

99. $\int \frac{v+3}{2v^3-8v}\,dv$

100. $\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)}$

101. $\int \frac{dt}{t^4+4t^2+3}$

102. $\int \frac{t\,dt}{t^4-t^2-2}$

103. $\int \frac{x^3+x^2}{x^2+x-2}\,dx$

104. $\int \frac{x^3+1}{x^3-x}\,dx$

105. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx$

106. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

107. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$

108. $\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)}$

109. $\int \frac{ds}{e^s - 1}$

110. $\int \frac{ds}{\sqrt{e^s + 1}}$

Improper Integrals

Evaluate the improper integrals in Exercises 111–120.

111. $\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$

112. $\int_0^1 \ln x \, dx$

113. $\int_{-1}^1 \frac{dy}{y^{2/3}}$

114. $\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$

115. $\int_3^\infty \frac{2\,du}{u^2 - 2u}$

116. $\int_1^\infty \frac{3v - 1}{4v^3 - v^2} dv$

117. $\int_0^\infty x^2 e^{-x} \, dx$

118. $\int_{-\infty}^0 x e^{3x} \, dx$

119. $\int_{-\infty}^\infty \frac{dx}{4x^2 + 9}$

120. $\int_{-\infty}^\infty \frac{4\,dx}{x^2 + 16}$

Convergence or Divergence

Which of the improper integrals in Exercises 121–126 converge and which diverge?

121. $\int_6^\infty \frac{d\theta}{\sqrt{\theta^2 + 1}}$

122. $\int_0^\infty e^{-u} \cos u \, du$

123. $\int_1^\infty \frac{\ln z}{z} dz$

124. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$

125. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$

126. $\int_{-\infty}^\infty \frac{dx}{x^2(1 + e^x)}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 127–130 (a) without using a trigonometric substitution, (b) using a trigonometric substitution.

127. $\int \frac{y\,dy}{\sqrt{16 - y^2}}$

128. $\int \frac{x\,dx}{\sqrt{4 + x^2}}$

129. $\int \frac{x\,dx}{4 - x^2}$

130. $\int \frac{t\,dt}{\sqrt{4t^2 - 1}}$

Quadratic Terms

Evaluate the integrals in Exercises 131–134.

131. $\int \frac{x\,dx}{9 - x^2}$

132. $\int \frac{dx}{x(9 - x^2)}$

133. $\int \frac{dx}{9 - x^2}$

134. $\int \frac{dx}{\sqrt{9 - x^2}}$

Assorted Integrations

Evaluate the integrals in Exercises 135–202. The integrals are listed in random order.

135. $\int \frac{x\,dx}{1 + \sqrt{x}}$

136. $\int \frac{x^3 + 2}{4 - x^2} dx$

137. $\int \frac{dx}{x(x^2 + 1)^2}$

138. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

139. $\int \frac{dx}{\sqrt{-2x - x^2}}$

140. $\int \frac{(t - 1)\,dt}{\sqrt{t^2 - 2t}}$

141. $\int \frac{du}{\sqrt{1 + u^2}}$

142. $\int e^t \cos e^t \, dt$

143. $\int \frac{2 - \cos x + \sin x}{\sin^2 x} dx$

144. $\int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta$

145. $\int \frac{9\,dv}{81 - v^4}$

146. $\int \frac{\cos x\,dx}{1 + \sin^2 x}$

147. $\int \theta \cos (2\theta + 1)\,d\theta$

148. $\int_2^\infty \frac{dx}{(x - 1)^2}$

149. $\int \frac{x^3\,dx}{x^2 - 2x + 1}$

150. $\int \frac{d\theta}{\sqrt{1 + \sqrt{\theta}}}$

151. $\int \frac{2 \sin \sqrt{x}\,dx}{\sqrt{x} \sec \sqrt{x}}$

152. $\int \frac{x^5\,dx}{x^4 - 16}$

153. $\int \frac{dy}{\sin y \cos y}$

154. $\int \frac{d\theta}{\theta^2 - 2\theta + 4}$

155. $\int \frac{\tan x}{\cos^2 x} dx$

156. $\int \frac{dr}{(r + 1)\sqrt{r^2 + 2r}}$

157. $\int \frac{(r + 2)\,dr}{\sqrt{-r^2 - 4r}}$

158. $\int \frac{y\,dy}{4 + y^4}$

159. $\int \frac{\sin 2\theta\,d\theta}{(1 + \cos 2\theta)^2}$

160. $\int \frac{dx}{(x^2 - 1)^2}$

161. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x}\,dx$

162. $\int (15)^{2x+1}\,dx$

163. $\int \frac{x\,dx}{\sqrt{2 - x}}$

164. $\int \frac{\sqrt{1 - v^2}}{v^2} dv$

165. $\int \frac{dy}{y^2 - 2y + 2}$

166. $\int \ln \sqrt{x - 1}\,dx$

167. $\int \theta^2 \tan (\theta^3)\,d\theta$

168. $\int \frac{x\,dx}{\sqrt{8 - 2x^2 - x^4}}$

169. $\int \frac{z + 1}{z^2(z^2 + 4)} dz$

170. $\int x^3 e^{(x^2)}\,dx$

171. $\int \frac{t\,dt}{\sqrt{9 - 4t^2}}$

172. $\int_0^{\pi/10} \sqrt{1 + \cos 5\theta}\,d\theta$

173. $\int \frac{\cot \theta\,d\theta}{1 + \sin^2 \theta}$

174. $\int \frac{\tan^{-1} x}{x^2} dx$

175. $\int \frac{\tan \sqrt{y}}{2\sqrt{y}} dy$

176. $\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

177. $\int \frac{\theta^2\,d\theta}{4 - \theta^2}$

178. $\int \frac{1 - \cos 2x}{1 + \cos 2x} dx$

179. $\int \frac{\cos (\sin^{-1} x)}{\sqrt{1 - x^2}} dx$

180. $\int \frac{\cos x\,dx}{\sin^3 x - \sin x}$

181. $\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx$

182. $\int \frac{x^2-x+2}{(x^2+2)^2}\,dx$

183. $\int \frac{e^t\,dt}{1+e^t}$

184. $\int \tan^3 t\,dt$

185. $\int_1^{\infty} \frac{\ln y}{y^3}\,dy$

186. $\int \frac{3+\sec^2 x+\sin x}{\tan x}\,dx$

187. $\int \frac{\cot v\,dv}{\ln \sin v}$

188. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$

189. $\int e^{\ln\sqrt{x}}\,dx$

190. $\int e^{\theta}\sqrt{3+4e^{\theta}}\,d\theta$

191. $\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2}$

192. $\int \frac{dv}{\sqrt{e^{2v}-1}}$

193. $\int (27)^{3\theta+1}\,d\theta$

194. $\int x^5\sin x\,dx$

195. $\int \frac{dr}{1+\sqrt{r}}$

196. $\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx$

197. $\int \frac{8\,dy}{y^3(y+2)}$

198. $\int \frac{(t+1)\,dt}{(t^2+2t)^{2/3}}$

199. $\int \frac{8\,dm}{m\sqrt{49m^2-4}}$

200. $\int \frac{dt}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}$

201. $\int_0^1 3(x-1)^2\left(\int_0^x \sqrt{1+(t-1)^4}\,dt\right)dx$

202. $\int_2^{\infty} \frac{4v^3+v-1}{v^2(v-1)(v^2+1)}\,dv$

203. Suppose for a certain function f it is known that

$$f'(x)=\frac{\cos x}{x},\quad f(\pi/2)=a,\quad \text{and}\quad f(3\pi/2)=b.$$

Use integration by parts to evaluate

$$\int_{\pi/2}^{3\pi/2} f(x)\,dx.$$

204. Find a positive number a satisfying

$$\int_0^a \frac{dx}{1+x^2}=\int_a^{\infty}\frac{dx}{1+x^2}.$$

Chapter 7 Additional Exercises–Theory, Examples, Applications

Challenging Integrals

Evaluate the integrals in Exercises 1–10.

1. $\int (\sin^{-1} x)^2\,dx$

2. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x\sin^{-1} x\,dx$

4. $\int \sin^{-1}\sqrt{y}\,dy$

5. $\int \frac{d\theta}{1-\tan^2\theta}$

6. $\int \ln(\sqrt{x}+\sqrt{1+x})\,dx$

7. $\int \frac{dt}{t-\sqrt{1-t^2}}$

8. $\int \frac{(2e^{2x}-e^x)\,dx}{\sqrt{3e^{2x}-6e^x-1}}$

9. $\int \frac{dx}{x^4+4}$

10. $\int \frac{dx}{x^6-1}$

Limits

Evaluate the limits in Exercises 11 and 12.

11. $\lim_{x\to\infty}\int_{-x}^{x}\sin t\,dt$

12. $\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$

Evaluate the limits in Exercises 13 and 14 by identifying them with definite integrals and evaluating the integrals.

13. $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{1+\frac{k}{n}}$

14. $\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}}$

Theory and Applications

15. Find the length of the curve

$$y=\int_0^x \sqrt{\cos 2t}\,dt,\quad 0\le x\le \pi/4.$$

16. Find the length of the curve $y=\ln(1-x^2)$, $0\le x\le 1/2$.

17. The region in the first quadrant that is enclosed by the x-axis and the curve $y=3x\sqrt{1-x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

18. The region in the first quadrant that is enclosed by the x-axis, the curve $y=5/(x\sqrt{5-x})$, and the lines $x=1$ and $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

19. The region in the first quadrant enclosed by the coordinate axes, the curve $y=e^x$, and the line $x=1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

20. The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

21. Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

a) the x-axis
b) the line $y = 1$.

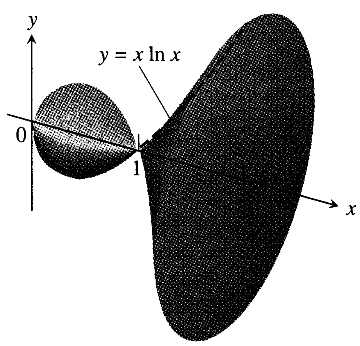
22. (*Continuation of Exercise 21.*) Find the volume of the solid generated by revolving the shaded region about (a) the y-axis, (b) the line $x = 1$.

23. The region between the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a) Show that f is continuous at $x = 0$.
b) Find the volume of the solid.



24. The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

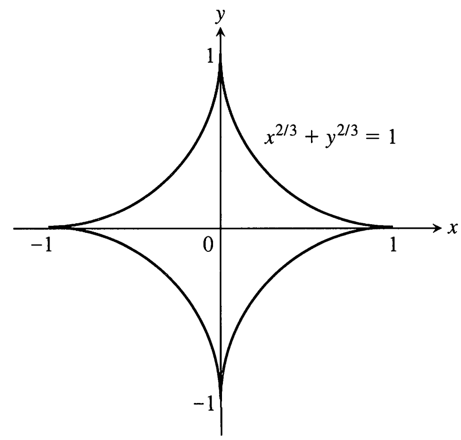
25. Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

26. Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

27. Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

28. Find the area of the surface generated by revolving the curve in Exercise 27 about the y-axis.

29. *The length of an astroid.* The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (Fig. 7.21). Find the length of this particular astroid.



7.21 The astroid in Exercises 29 and 30.

30. *The surface generated by an astroid.* Find the area of the surface generated by revolving the curve in Fig. 7.21 about the x-axis.

31. Find a curve through the origin whose length is

$$\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

32. Without evaluating either integral, explain why

$$2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}.$$

(Source: Peter A. Lindstrom, *Mathematics Magazine,* Vol. 45, No. 1, January 1972, p. 47.)

33. a) GRAPHER Graph the function $f(x) = e^{(x - e^x)}$, $-5 \le x \le 3$.

b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$ converges and find its value.

34. Find $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{n\, y^{n-1}}{1 + y}\, dy.$

35. Derive the integral formula

$$\int x\left(\sqrt{x^2 - a^2}\right)^n dx = \frac{\left(\sqrt{x^2 - a^2}\right)^{n+2}}{n + 2} + C, \quad n \ne -2.$$

36. Prove that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

(*Hint:* Observe that for $0 < x < 1$, we have $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$, with the left-hand side becoming an equality for $x = 0$ and the right-hand side becoming an equality for $x = 1$.)

37. For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

38. For each $x > 0$, let $G(x) = \int_0^\infty e^{-xt}\,dt$. Prove that $xG(x) = 1$ for each $x > 0$.

39. *Infinite area and finite volume.* What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

40. *Infinite area and finite volume.* What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval [0, 1] on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)\,g(x)\,dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x\,dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} and its derivatives		$\cos x$ and its integrals	
e^{2x}	$+$	$\cos x$	
$2e^{2x}$	$-$	$\sin x$	
$4e^{2x}$	$+$	$-\cos x$	$\longleftarrow$ *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, -1 on the right)

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x} \cos x\,dx$$

$$= +(e^{2x} \sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x)\,dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5\int e^{2x} \cos x\,dx = e^{2x} \sin x + 2e^{2x} \cos x$$

or

$$\int e^{2x} \cos x\,dx = \frac{e^{2x} \sin x + 2e^{2x} \cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 41–48.

41. $\int e^{2x} \cos 3x\,dx$

42. $\int e^{3x} \sin 4x\,dx$

43. $\int \sin 3x \sin x\,dx$

44. $\int \cos 5x \sin 4x\,dx$

45. $\int e^{ax} \sin bx\,dx$

46. $\int e^{ax} \cos bx\,dx$

47. $\int \ln(ax)\,dx$

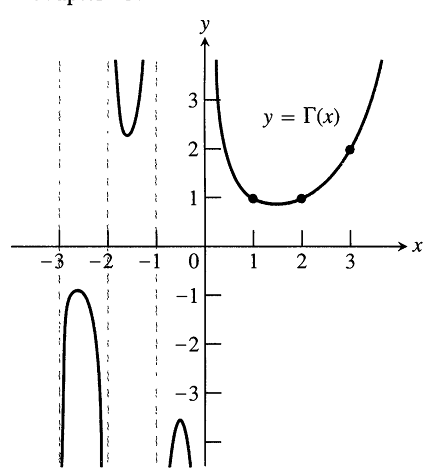
48. $\int x^2 \ln(ax)\,dx$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt, \quad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 7.22 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 31 in Chapter 13.



7.22 $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x + 1))/x$, which is the subject of Exercise 49.

49. *If n is a nonnegative integer,* $\Gamma(n+1) = n!$

a) Show that $\Gamma(1) = 1$.

b) Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives

$$\Gamma(2) = 1\Gamma(1) = 1$$
$$\Gamma(3) = 2\Gamma(2) = 2$$
$$\Gamma(4) = 3\Gamma(3) = 6$$
$$\vdots$$
$$\Gamma(n+1) = n\Gamma(n) = n! \qquad (1)$$

c) Use mathematical induction to verify Eq. (1) for every nonnegative integer n.

50. *Stirling's formula.* Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x\sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}(1+\epsilon(x)), \quad \epsilon(x)\to 0 \text{ as } x\to\infty. \qquad (2)$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}. \qquad \textbf{(Stirling's formula)} \qquad (3)$$

a) *Stirling's approximation for n!.* Use Eq. (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2n\pi}. \qquad \textbf{(Stirling's approximation)} \qquad (4)$$

As you will see if you do Exercise 68 in Section 8.2, Eq. (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \qquad (5)$$

b) CALCULATOR Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots$, as far as your calculator can go.

c) CALCULATOR A refinement of Eq. (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}e^{1/(12x)}(1+\epsilon(x)),$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}e^{1/(12x)}$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2n\pi}\,e^{1/(12n)}. \qquad (6)$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Eq. (6).

CHAPTER 8

Infinite Series

OVERVIEW In this chapter we develop a remarkable formula that enables us to express many functions as "infinite polynomials" and at the same time tells how much error we will incur if we truncate those polynomials to make them finite. In addition to providing effective polynomial approximations of differentiable functions, these infinite polynomials (called power series) have many other uses. They provide an efficient way to evaluate nonelementary integrals and they solve differential equations that give insight into heat flow, vibration, chemical diffusion, and signal transmission. What you will learn here sets the stage for the roles played by series of functions of all kinds in science and mathematics.

8.1 Limits of Sequences of Numbers

Informally, a sequence is an ordered list of things, but in this chapter the things will usually be numbers. We have seen sequences before, such as the sequence $x_0, x_1, \ldots, x_n, \ldots$ of numbers generated by Newton's method and the sequence $c_1, c_2, \ldots, c_n, \ldots$ of polygons that define Helga von Koch's snowflake. These sequences have limits, but many equally important sequences do not.

Definitions and Notation

We can list the integer multiples of 3 by assigning each multiple a position:

$$\begin{array}{lcccc} \text{Domain:} & 1 & 2 & 3 \ldots & n \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Range:} & 3 & 6 & 9 & 3n \end{array}$$

The first number is 3, the second 6, the third 9, and so on. The assignment is a function that assigns $3n$ to the nth place. And that is the basic idea for constructing sequences. There is a function that tells us where each item is to be placed.

Definition

An **infinite sequence** (or **sequence**) of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0.

Usually n_0 is 1 and the domain of the sequence is the set of positive integers. But sometimes we want to start sequences elsewhere. We take $n_0 = 0$ when we begin Newton's method. We might take $n_0 = 3$ if we were defining a sequence of n-sided polygons.

Sequences are defined the way other functions are, some typical rules being

$$a(n) = \sqrt{n}, \qquad a(n) = (-1)^{n+1}\frac{1}{n}, \qquad a(n) = \frac{n-1}{n}$$

(Example 1 and Fig. 8.1).

To indicate that the domains are sets of integers, we use a letter like n from the middle of the alphabet for the independent variable, instead of the x, y, z, and t used widely in other contexts. The formulas in the defining rules, however, like those above, are often valid for domains larger than the set of positive integers. This can be an advantage, as we will see.

The number $a(n)$ is the ***n*th term** of the sequence, or the **term with index *n*.** If $a(n) = (n-1)/n$, we have

First term	Second term	Third term		*n*th term
$a(1) = 0$	$a(2) = \frac{1}{2}$,	$a(3) = \frac{2}{3}$,	$\dots,$	$a(n) = \frac{n-1}{n}$.

When we use the subscript notation a_n for $a(n)$, the sequence is written

$$a_1 = 0, \qquad a_2 = \frac{1}{2}, \qquad a_3 = \frac{2}{3}, \qquad \dots, \qquad a_n = \frac{n-1}{n}.$$

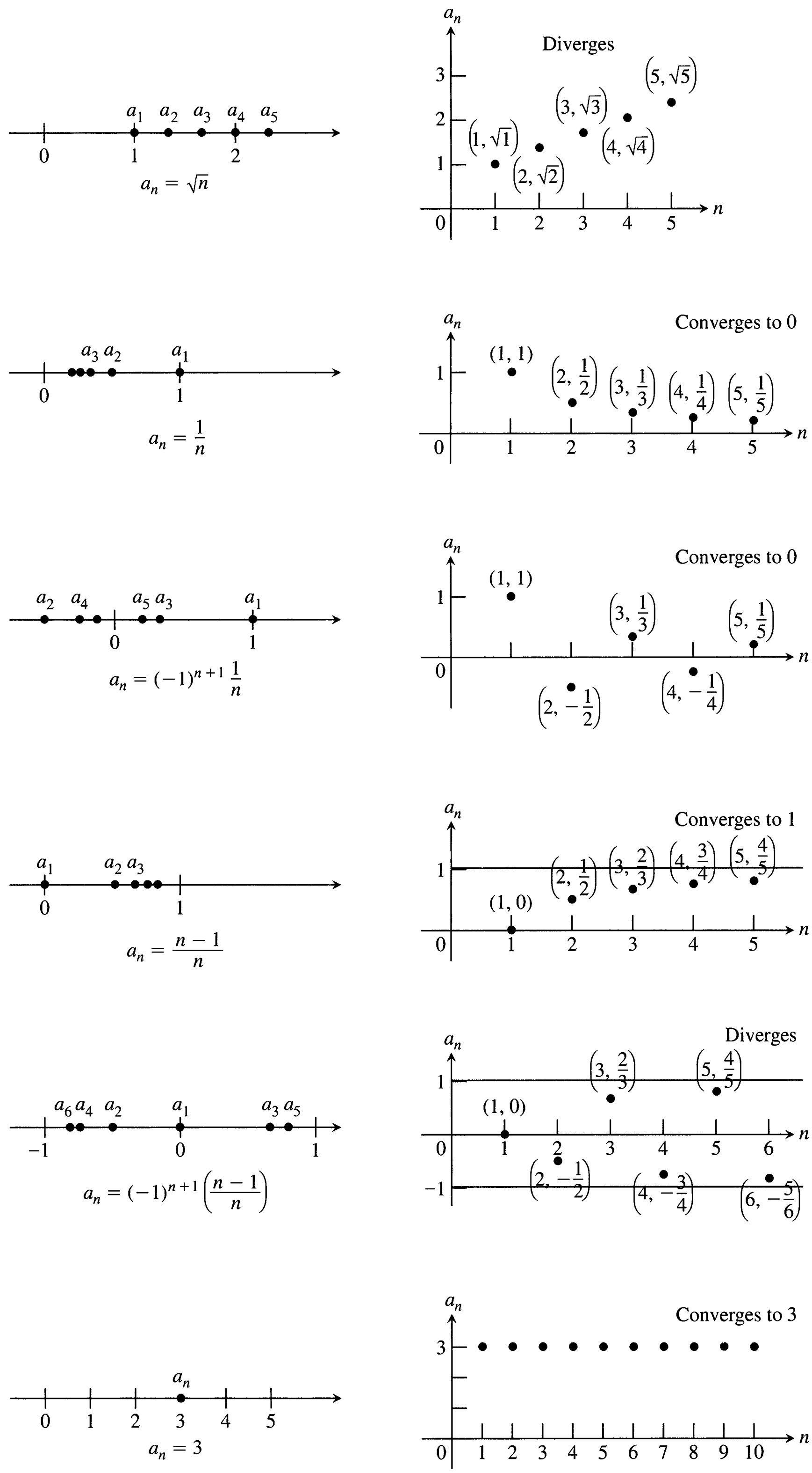
To describe sequences, we often write the first few terms as well as a formula for the nth term.

EXAMPLE 1

We write	For the sequence whose defining rule is
$1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{n}, \dots$	$a_n = \sqrt{n}$
$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$	$a_n = \frac{1}{n}$
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}\frac{1}{n}, \dots$	$a_n = (-1)^{n+1}\frac{1}{n}$
$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots$	$a_n = \frac{n-1}{n}$
$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \dots, (-1)^{n+1}\left(\frac{n-1}{n}\right), \dots$	$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$
$3, 3, 3, \dots, 3, \dots$	$a_n = 3$

❑

Notation We refer to the sequence whose nth term is a_n with the notation $\{a_n\}$ ("the sequence a sub n"). The second sequence in Example 1 is $\{1/n\}$ ("the sequence 1 over n"); the last sequence is $\{3\}$ ("the constant sequence 3").



8.1 The sequences of Example 1 are graphed here in two different ways: by plotting the numbers a_n on a horizontal axis and by plotting the points (n, a_n) in the coordinate plane.

The terms $a_n = \sqrt{n}$ eventually surpass every integer, so the sequence $\{a_n\}$ diverges, . . .

. . . but the terms $a_n = 1/n$ decrease steadily and get arbitrarily close to 0 as n increases, so the sequence $\{a_n\}$ converges to 0.

The terms $a_n = (-1)^{n+1}(1/n)$ alternate in sign but still converge to 0.

The terms $a_n = (n-1)/n$ approach 1 steadily and get arbitrarily close as n increases, so the sequence $\{a_n\}$ converges to 1.

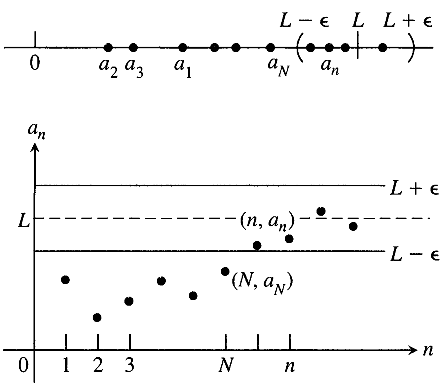
The terms $a_n = (-1)^{n+1}[(n-1)/n]$ alternate in sign. The positive terms approach 1. But the negative terms approach -1 as n increases, so the sequence $\{a_n\}$ diverges.

The terms in the sequence of constants $a_n = 3$ have the same value regardless of n, so the sequence $\{a_n\}$ converges to 3.

Convergence and Divergence

As Fig. 8.1 shows, the sequences of Example 1 do not behave the same way. The sequences $\{1/n\}$, $\{(-1)^{n+1}(1/n)\}$, and $\{(n-1)/n\}$ each seem to approach a single limiting value as n increases, and $\{3\}$ is at a limiting value from the very first. On the other hand, terms of $\{(-1)^{n+1}(n-1)/n\}$ seem to accumulate near two different values, -1 and 1, while the terms of $\{\sqrt{n}\}$ become increasingly large and do not accumulate anywhere.

To distinguish sequences that approach a unique limiting value L, as n increases, from those that do not, we say that the former sequences *converge*, according to the following definition.



8.2 $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

Definitions

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges.**

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Fig. 8.2).

EXAMPLE 2 *Testing the definition*

Show that

a) $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$ **b)** $\displaystyle\lim_{n\to\infty} k = k$ (any constant k)

Solution

a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

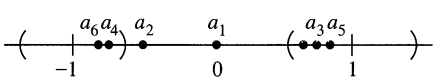
$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty}(1/n) = 0$.

b) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

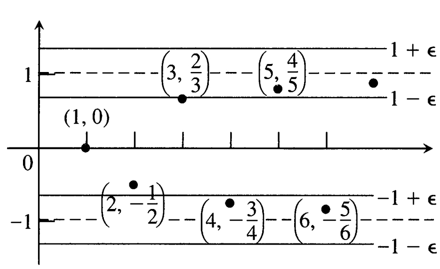
$$n > N \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k. ❑



$$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$$

Neither the ϵ-interval about 1 nor the ϵ-interval about -1 contains a complete tail of the sequence.



8.3 The sequence $\{(-1)^{n+1}[(n-1)/n]\}$ diverges.

EXAMPLE 3 Show that $\{(-1)^{n+1}[(n-1)/n]\}$ diverges.

Solution Take a positive ϵ smaller than 1 so that the bands shown in Fig. 8.3 about the lines $y = 1$ and $y = -1$ do not overlap. Any $\epsilon < 1$ will do. Convergence to 1 would require every point of the graph beyond a certain index N to lie inside the upper band, but this will never happen. As soon as a point (n, a_n) lies in the upper band, every alternate point starting with $(n+1, a_{n+1})$ will lie in the lower band. Hence the sequence cannot converge to 1. Likewise, it cannot converge to -1. On the other hand, because the terms of the sequence get alternately closer to 1 and -1, they never accumulate near any other value. Therefore, the sequence diverges. ❑

The behavior of $\{(-1)^{n+1}[(n-1)/n]\}$ is qualitatively different from that of $\{\sqrt{n}\}$, which diverges because it outgrows every real number L. To describe the behavior of $\{\sqrt{n}\}$ we write

$$\lim_{n\to\infty} (\sqrt{n}) = \infty.$$

In speaking of infinity as a limit of a sequence $\{a_n\}$, we do not mean that the difference between a_n and infinity becomes small as n increases. We mean that a_n becomes numerically large as n increases.

Recursive Definitions

Recursion formulas arise regularly in computer programs and numerical routines for solving differential equations.

So far, we have calculated each a_n directly from the value of n. But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula,** for calculating any later term from terms that precede it.

Factorial notation

The notation $n!$ ("n factorial") means the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$ of the integers from 1 to n. Notice that $(n+1)! = (n+1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define $0!$ to be 1. Factorials grow even faster than exponentials, as the following table suggests.

n	e^n (rounded)	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	4.9×10^8	2.4×10^{18}

EXAMPLE 4 *Sequences constructed recursively*

a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ define the sequence $1, 2, 3, \ldots, n, \ldots$ of positive integers. With $a_1 = 1$, we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.

b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ define the sequence 1, 2, 6, 24, $\ldots, n!, \ldots$ of factorials. With $a_1 = 1$, we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 4 \cdot a_3 = 24$, and so on.

c) The statements $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ define the sequence 1, 1, 2, 3, 5, ... of **Fibonacci numbers.** With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, and so on.

d) As we can see by applying Newton's method, the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$ define a sequence that converges to a solution of the equation $\sin x - x^2 = 0$. ❑

Subsequences

If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second.

EXAMPLE 5 *Subsequences of the sequence of positive integers*

a) The subsequence of even integers: $2, 4, 6, \ldots, 2n, \ldots$

b) The subsequence of odd integers: $1, 3, 5, \ldots, 2n-1, \ldots$

c) The subsequence of primes: $2, 3, 5, 7, 11, \ldots$ ❑

Subsequences are important for two reasons:

1. If a sequence $\{a_n\}$ converges to L, then all of its subsequences converge to L. If we know that a sequence converges, it may be quicker to find or estimate its limit by examining a particular subsequence.
2. If any subsequence of a sequence $\{a_n\}$ diverges, or if two subsequences have different limits, then $\{a_n\}$ diverges. For example, the sequence $\{(-1)^n\}$ diverges because the subsequence $-1, -1, -1, \ldots$ of odd numbered terms converges to -1 while the subsequence $1, 1, 1, \ldots$ of even numbered terms converges to 1, a different limit.

   Subsequences also provide a new way to view convergence. A **tail** of a sequence is a subsequence that consists of all terms of the sequence from some index N on. In other words, a tail is one of the sets $\{a_n \mid n \geq N\}$. Another way to say that $a_n \to L$ is to say that every ϵ-interval about L contains a tail of the sequence.

The convergence or divergence of a sequence has nothing to do with how the sequence begins. It depends only on how the tails behave.

Bounded Nondecreasing Sequences

Definition

A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a **nondecreasing sequence.**

EXAMPLE 6 *Nondecreasing sequences*

a) The sequence $1, 2, 3, \ldots, n, \ldots$ of natural numbers

b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$

c) The constant sequence $\{3\}$ ❑

There are two kinds of nondecreasing sequences—those whose terms increase beyond any finite bound and those whose terms do not.

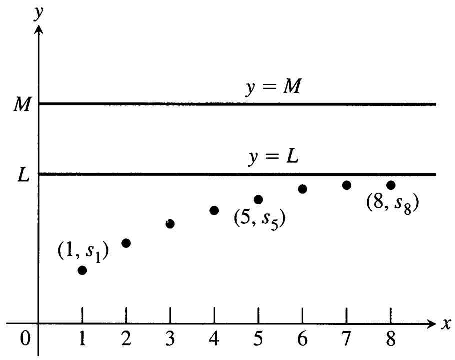
Definitions

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

EXAMPLE 7

a) The sequence $1, 2, 3, \ldots, n, \ldots$ has no upper bound.

b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by $M = 1$.
No number less than 1 is an upper bound for the sequence, so 1 is the least upper bound (Exercise 47). ❑



8.4 If the terms of a nondecreasing sequence have an upper bound *M*, they have a limit $L \le M$.

A nondecreasing sequence that is bounded from above always has a least upper bound. This fact is a consequence of the completeness property of real numbers but we will not prove it here. Instead, we will prove that if L is the least upper bound, then the sequence converges to L.

Suppose we plot the points $(1, s_1)$, $(2, s_2)$, ..., (n, s_n), ... in the xy-plane. If M is an upper bound of the sequence, all these points will lie on or below the line $y = M$ (Fig. 8.4). The line $y = L$ is the lowest such line. None of the points (n, s_n) lies above $y = L$, but some do lie above any lower line $y = L - \epsilon$, if ϵ is a positive number. The sequence converges to L because

a) $s_n \le L$ for *all* values of n and

b) given any $\epsilon > 0$, there exists at least one integer N for which $s_N > L - \epsilon$.

The fact that $\{s_n\}$ is nondecreasing tells us further that

$$s_n \ge s_N > L - \epsilon \qquad \text{for all } n \ge N.$$

Thus, *all* the numbers s_n beyond the Nth number lie within ϵ of L. This is precisely the condition for L to be the limit of the sequence s_n.

The facts for nondecreasing sequences are summarized in the following theorem. A similar result holds for nonincreasing sequences (Exercise 41).

Theorem 1
The Nondecreasing Sequence Theorem
A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Exercises 8.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \dfrac{1-n}{n^2}$

2. $a_n = \dfrac{1}{n!}$

3. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \dfrac{2^n}{2^{n+1}}$

6. $a_n = \dfrac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1} a_n/2$

10. $a_1 = -2, \quad a_{n+1} = n a_n/(n+1)$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

Finding a Sequence's Formula

In Exercises 13–22, find a formula for the nth term of the sequence.

13. The sequence $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs

14. The sequence $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs

15. The sequence $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers, with alternating signs

16. The sequence $1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \dfrac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs

17. The sequence $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1

18. The sequence $-3, -2, -1, 0, 1, \ldots$ — Integers beginning with -3

19. The sequence $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer

20. The sequence $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer

21. The sequence $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's

22. The sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated

Calculator Explorations of Limits

In Exercises 23–26, experiment with a calculator to find a value of N that will make the inequality hold for all $n > N$. Assuming that the inequality is the one from the formal definition of the limit of a sequence, what sequence is being considered in each case and what is its limit?

23. $|\sqrt[n]{0.5} - 1| < 10^{-3}$ **24.** $|\sqrt[n]{n} - 1| < 10^{-3}$

25. $(0.9)^n < 10^{-3}$ **26.** $2^n/n! < 10^{-7}$

27. *Sequences generated by Newton's method.* Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f. The recursion formula for the sequence is

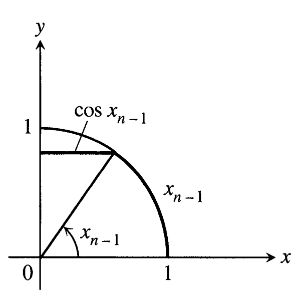
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

a) Show that the recursion formula for $f(x) = x^2 - a$, $a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

b) Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

28. (*Continuation of Exercise 27.*) Repeat part (b) of Exercise 27 with $a = 2$ in place of $a = 3$.

29. *A recursive definition of $\pi/2$.* If you start with $x_1 = 1$ and define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. (a) Try it. (b) Use the accompanying figure to explain why the convergence is so rapid.



30. According to a front-page article in the December 15, 1992, issue of *The Wall Street Journal,* Ford Motor Company now uses about $7\frac{1}{4}$ hours of labor to produce stampings for the average vehicle, down from an estimated 15 hours in 1980. The Japanese need only about $3\frac{1}{2}$ hours.

Ford's improvement since 1980 represents an average decrease of 6% per year. If that rate continues, then n years from now Ford will use about

$$S_n = 7.25(0.94)^n$$

hours of labor to produce stampings for the average vehicle. Assuming that the Japanese continue to spend $3\frac{1}{2}$ hours per vehicle, how many more years will it take Ford to catch up? Find out two ways:

a) Find the first term of the sequence $\{S_n\}$ that is less than or equal to 3.5.

b) GRAPHER Graph $f(x) = 7.25(0.94)^x$ and use TRACE to find where the graph crosses the line $y = 3.5$.

Theory and Examples

In Exercises 31–34, determine if the sequence is nondecreasing and if it is bounded from above.

31. $a_n = \dfrac{3n+1}{n+1}$ **32.** $a_n = \dfrac{(2n+3)!}{(n+1)!}$

33. $a_n = \dfrac{2^n 3^n}{n!}$ **34.** $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

Which of the sequences in Exercises 35–40 converge, and which diverge? Give reasons for your answers.

35. $a_n = 1 - \dfrac{1}{n}$ **36.** $a_n = n - \dfrac{1}{n}$

37. $a_n = \dfrac{2^n - 1}{2^n}$ **38.** $a_n = \dfrac{2^n - 1}{3^n}$

39. $a_n = ((-1)^n + 1)\left(\dfrac{n+1}{n}\right)$

40. The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n+1)\}.$$

41. *Nonincreasing sequences.* A sequence of numbers $\{a_n\}$ in which $a_n \geq a_{n+1}$ for every n is called a **nonincreasing sequence.** A sequence $\{a_n\}$ is **bounded from below** if there is a number M with $M \leq a_n$ for every n. Such a number M is called a **lower bound** for the sequence. Deduce from Theorem 1 that a nonincreasing sequence that is bounded from below converges and that a nonincreasing sequence that is not bounded from below diverges.

(*Continuation of Exercise 41.*) Using the conclusion of Exercise 41, determine which of the sequences in Exercises 42–46 converge and which diverge.

42. $a_n = \dfrac{n+1}{n}$

43. $a_n = \dfrac{1+\sqrt{2n}}{\sqrt{n}}$

44. $a_n = \dfrac{1-4^n}{2^n}$

45. $a_n = \dfrac{4^{n+1}+3^n}{4^n}$

46. $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

47. *The sequence $\{n/(n+1)\}$ has a least upper bound of 1.* Show that if M is a number less than 1, then the terms of $\{n/(n+1)\}$ eventually exceed M. That is, if $M < 1$ there is an integer N such that $n/(n+1) > M$ whenever $n > N$. Since $n/(n+1) < 1$ for every n, this proves that 1 is a least upper bound for $\{n/(n+1)\}$.

48. *Uniqueness of least upper bounds.* Show that if M_1 and M_2 are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

49. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

50. Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ϵ there corresponds an integer N such that for all m and n,

$$m > N \quad \text{and} \quad n > N \quad \Rightarrow \quad |a_m - a_n| < \epsilon.$$

51. *Uniqueness of limits.* Prove that limits of sequences are unique. That is, show that if L_1 and L_2 are numbers such that $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.

52. *Limits and subsequences.* Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

53. For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

54. Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

CAS Explorations and Projects

Use a CAS to perform the following steps for the sequences in Exercises 55–66.

a) Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit L?

b) If the sequence converges, find an integer N such that $|a_n - L| \leq 0.01$ for $n \geq N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of L?

55. $a_n = \sqrt[n]{n}$

56. $a_n = \left(1 + \dfrac{0.5}{n}\right)^n$

57. $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{5^n}$

58. $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

59. $a_n = \sin n$

60. $a_n = n \sin \dfrac{1}{n}$

61. $a_n = \dfrac{\sin n}{n}$

62. $a_n = \dfrac{\ln n}{n}$

63. $a_n = (0.9999)^n$

64. $a_n = 123456^{1/n}$

65. $a_n = \dfrac{8^n}{n!}$

66. $a_n = \dfrac{n^{41}}{19^n}$

67. *Compound interest, deposits, and withdrawals.* If you invest an amount of money A_0 at a fixed annual interest rate r compounded m times per year, and if the constant amount b is added to the account at the end of each compounding period (or taken from the account if $b < 0$), then the amount you have after $n+1$ compounding periods is

$$A_{n+1} = \left(1 + \frac{r}{m}\right) A_n + b. \tag{1}$$

a) If $A_0 = 1000$, $r = 0.02015$, $m = 12$, and $b = 50$, calculate and plot the first 100 points (n, A_n). How much money is in your account at the end of 5 years? Does $\{A_n\}$ converge? Is $\{A_n\}$ bounded?

b) Repeat part (a) with $A_0 = 5000$, $r = 0.0589$, $m = 12$, and $b = -50$.

c) If you invest 5000 dollars in a certificate of deposit (CD) that pays 4.5% annually, compounded quarterly, and you make no further investments in the CD, approximately how many years will it take before you have 20,000 dollars? What if the CD earns 6.25%?

d) It can be shown that for any $k \geq 0$, the sequence defined recursively by Eq. (1) satisfies the relation

$$A_k = \left(1 + \frac{r}{m}\right)^k \left(A_0 + \frac{mb}{r}\right) - \frac{mb}{r}. \tag{2}$$

For the values of the constants A_0, r, m, and b given in part (a), validate this assertion by comparing the values of the first 50 terms of both sequences. Then show by direct substitution that the terms in Eq. (2) satisfy the recursion formula (1).

68. *Logistic difference equation.* The recursive relation

$$a_{n+1} = r a_n (1 - a_n)$$

is called the **logistic difference equation,** and when the initial value a_0 is given the equation defines the **logistic sequence** $\{a_n\}$. Throughout this exercise we choose a_0 in the interval $0 < a_0 < 1$, say $a_0 = 0.3$.

a) Choose $r = 3/4$. Calculate and plot the points (n, a_n) for the first 100 terms in the sequence. Does it appear to converge? What do you guess is the limit? Does the limit seem to depend on your choice of a_0?

b) Choose several values of r in the interval $1 < r < 3$ and repeat the procedures in part (a). Be sure to choose some points near the endpoints of the interval. Describe the behavior of the sequences you observe in your plots.

c) Now examine the behavior of the sequence for values of r near the endpoints of the interval $3 < r < 3.45$. The transition value $r = 3$ is called a **bifurcation value** and the new behavior of the sequence in the interval is called an **attracting 2-cycle.** Explain why this reasonably describes the behavior.

d) Next explore the behavior for r values near the endpoints of each of the intervals $3.45 < r < 3.54$ and $3.54 < r < 3.55$. Plot the first 200 terms of the sequences. Describe in your own words the behavior observed in your plots for each interval. Among how many values does the sequence appear to oscillate for each interval? The values $r = 3.45$ and $r = 3.54$ (rounded to 2 decimal places) are also called bifurcation values because the behavior of the sequence changes as r crosses over those values.

e) The situation gets even more interesting. There is actually an increasing sequence of bifurcation values $3 < 3.45 < 3.54 < \cdots < c_n < c_{n+1} \cdots$ such that for $c_n < r < c_{n+1}$ the logistic sequence $\{a_n\}$ eventually oscillates steadily among 2^n values, called an **attracting 2^n-cycle.** Moreover, the bifurcation sequence $\{c_n\}$ is bounded above by 3.57 (so it converges). If you choose a value of $r < 3.57$ you will observe a 2^n-cycle of some sort. Choose $r = 3.5695$ and plot 300 points.

f) Let us see what happens when $r > 3.57$. Choose $r = 3.65$ and calculate and plot the first 300 terms of $\{a_n\}$. Observe how the terms wander around in an unpredictable, chaotic fashion. You cannot predict the value of a_{n+1} from the value of a_n.

g) For $r = 3.65$ choose two starting values of a_0 that are close together, say, $a_0 = 0.3$ and $a_0 = 0.301$. Calculate and plot the first 300 values of the sequences determined by each starting value. Compare the behaviors observed in your plots. How far out do you go before the corresponding terms of your two sequences appear to depart from each other? Repeat the exploration for $r = 3.75$. Can you see how the plots look different depending on your choice of a_0? We say that the logistic sequence is **sensitive to the initial condition a_0.**

8.2 Theorems for Calculating Limits of Sequences

The study of limits would be cumbersome if we had to answer every question about convergence by applying the definition. Fortunately, three theorems make this largely unnecessary. The first is a version of Theorem 1, Section 1.2.

Theorem 2

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty} (a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty} (a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty} (k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

EXAMPLE 1 By combining Theorem 2 with the limit results in Example 2 of the preceding section, we have

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty} \frac{1}{n} = -1 \cdot 0 = 0$$

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$$

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$$

$$\lim_{n\to\infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$$

❑

One consequence of Theorem 2 is that every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. For suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 2, we see that the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

The next theorem is the sequence version of the Sandwich Theorem in Section 1.2.

Theorem 3

The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

An immediate consequence of Theorem 3 is that, if $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \leq b_n \leq c_n$. We use this fact in the next example.

EXAMPLE 2 Since $1/n \to 0$, we know that

a) $\dfrac{\cos n}{n} \to 0$ because $\left|\dfrac{\cos n}{n}\right| = \dfrac{|\cos n|}{n} \leq \dfrac{1}{n}$;

b) $\dfrac{1}{2^n} \to 0$ because $\dfrac{1}{2^n} \leq \dfrac{1}{n}$;

c) $(-1)^n\dfrac{1}{n} \to 0$ because $\left|(-1)^n\dfrac{1}{n}\right| \leq \dfrac{1}{n}$. ❑

The application of Theorems 2 and 3 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence. We state the theorem without proof.

Theorem 4

The Continuous Function Theorem for Sequences

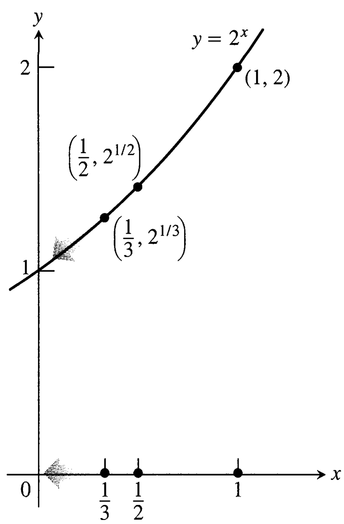
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE 3 Show that $\sqrt{(n+1)/n} \to 1$.

Solution We know that $(n+1)/n \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 4 gives $\sqrt{(n+1)/n} \to \sqrt{1} = 1$. ❑

Technology *The Sequence* $\{2^{1/n}\}$ What happens if you enter 2 in your calculator and take square roots repeatedly? The numbers form a sequence that appears to converge to 1, as suggested in the accompanying table. Try it for yourself.

n	$2^{1/n}$
2	1.4142 13562
4	1.1892 07115
8	1.0905 07733
64	1.0108 89286
256	1.0027 11275
1024	1.0006 77131
16384	1.0000 42307



8.5 As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$.

What is happening in the table above? The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 4, we see that $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1$. Since the successive square roots of 2 form a subsequence $2^{1/2}, 2^{1/4}, 2^{1/8}, \ldots$ of $\{2^{1/n}\}$, the square roots must converge to 1 also (Fig. 8.5).

Using l'Hôpital's Rule

The next theorem enables us to use l'Hôpital's rule to find the limits of some sequences.

Theorem 5

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

Proof Suppose that $\lim_{x\to\infty} f(x) = L$. Then for each positive number ϵ there is a number M such that for all x,

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

Let N be an integer greater than M and greater than or equal to n_0. Then

$$n > N \qquad \Rightarrow \qquad a_n = f(n) \qquad \text{and} \qquad |a_n - L| = |f(n) - L| < \epsilon. \qquad \square$$

EXAMPLE 4 Show that $\lim_{n\to\infty} (\ln n)/n = 0$.

Solution The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 5, $\lim_{n\to\infty} (\ln n)/n$ will equal $\lim_{x\to\infty} (\ln x)/x$ if the latter exists. A single application of l'Hôpital's rule shows that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty} (\ln n)/n = 0$. ❑

When we use l'Hôpital's rule to find the limit of a sequence, we often treat n as a continuous real variable and differentiate directly with respect to n. This saves us from having to rewrite the formula for a_n as we did in Example 4.

EXAMPLE 5 Find $\lim_{n\to\infty} (2^n/5n)$.

Solution By l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$$
$$= \infty.$$ ❑

Limits That Arise Frequently

The limits in Table 8.1 arise frequently. The first limit is from Example 4. The next two can be proved by taking logarithms and applying Theorem 4 (Exercises 71 and 72). The remaining proofs can be found in Appendix 6.

Table 8.1

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{Any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{Any } x)$

In formulas (3)–(6), x remains fixed as $n \to \infty$.

EXAMPLE 6 *Limits from Table 8.1*

1. $\frac{\ln (n^2)}{n} = \frac{2 \ln n}{n} \to 2 \cdot 0 = 0$ — Formula 1
2. $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ — Formula 2
3. $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ — Formula 3 with $x = 3$, and Formula 2
4. $\left(-\frac{1}{2}\right)^n \to 0$ — Formula 4 with $x = -\frac{1}{2}$
5. $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ — Formula 5 with $x = -2$
6. $\frac{100^n}{n!} \to 0$ — Formula 6 with $x = 100$ ❑

EXAMPLE 7 Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln \left(\frac{n+1}{n-1}\right)^n$$
$$= n \ln \left(\frac{n+1}{n-1}\right).$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) && \infty \cdot 0 \\
&= \lim_{n\to\infty} \frac{\ln\left(\dfrac{n+1}{n-1}\right)}{1/n} && \frac{0}{0} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} && \text{l'Hôpital's rule} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.
\end{aligned}$$

Since $\ln a_n \to 2$, and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2. ❑

* Picard's Method for Finding Roots

The problem of solving the equation

$$f(x) = 0 \tag{1}$$

is equivalent to that of solving the equation

$$g(x) = f(x) + x = x, \tag{2}$$

obtained by adding x to both sides of Eq. (1). By this simple change, we cast Eq. (1) into a form that may render it solvable on a computer by a powerful method called **Picard's method** (after the French mathematician Charles Émile Picard, 1856–1941).

If the domain of g contains the range of g, we can start with a point x_0 in the domain and apply g repeatedly to get

$$x_1 = g(x_0), \qquad x_2 = g(x_1), \qquad x_3 = g(x_2), \qquad \ldots. \tag{3}$$

Under simple restrictions that we will describe shortly, the sequence generated by the recursion formula $x_{n+1} = g(x_n)$ will converge to a point x for which $g(x) = x$. This point solves the equation $f(x) = 0$ because

$$f(x) = g(x) - x = x - x = 0. \tag{4}$$

A point x for which $g(x) = x$ is a **fixed point** of g. We see in Eq. (4) that the fixed points of g are precisely the roots of f.

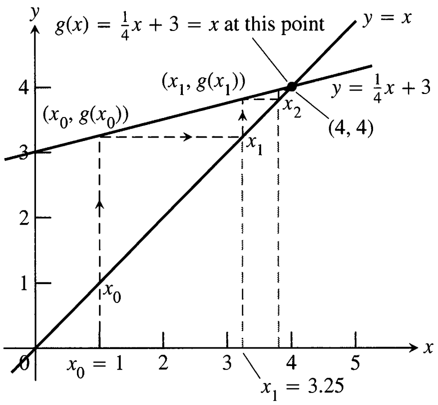
EXAMPLE 8 *Testing the method*

Solve the equation

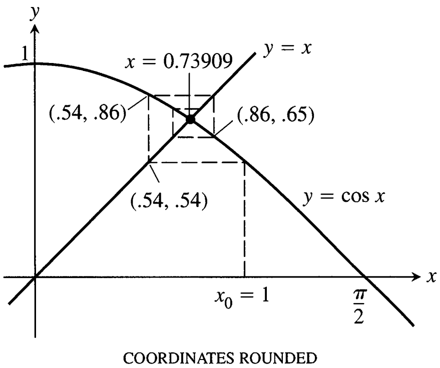
$$\frac{1}{4}x + 3 = x.$$

Solution By algebra, we know that the solution is $x = 4$. To apply Picard's method, we take

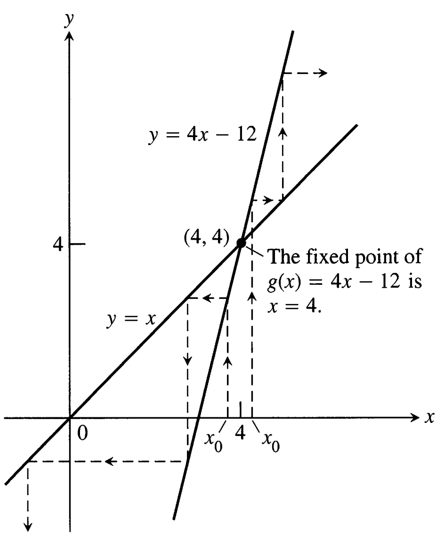
$$g(x) = \frac{1}{4}x + 3,$$



8.6 The Picard solution of the equation $g(x) = (1/4)x + 3 = x$ (Example 8).



8.7 The solution of $\cos x = x$ by Picard's method starting at $x_0 = 1$ (Example 9).



8.8 Applying the Picard method to $g(x) = 4x - 12$ will not find the fixed point unless x_0 is the fixed point 4 itself (Example 10).

choose a starting point, say $x_0 = 1$, and calculate the initial terms of the sequence $x_{n+1} = g(x_n)$. Table 8.2 lists the results. In 10 steps, the solution of the original equation is found with an error of magnitude less than 3×10^{-6}.

Figure 8.6 shows the geometry of the solution. We start with $x_0 = 1$ and calculate the first value $g(x_0)$. This becomes the second x-value x_1. The second y-value $g(x_1)$ becomes the third x-value x_2, and so on. The process is shown as a path (called the *iteration path*) that starts at $x_0 = 1$, moves up to $(x_0, g(x_0)) = (x_0, x_1)$, over to (x_1, x_1), up to $(x_1, g(x_1))$, and so on. The path converges to the point where the graph of g meets the line $y = x$. This is the point where $g(x) = x$. ❑

Table 8.2 Successive iterates of $g(x) = (1/4)x + 3$, starting with $x_0 = 1$

x_n	$x_{n+1} = g(x_n) = (1/4)x_n + 3$
$x_0 = 1$	$x_1 = g(x_0) = (1/4)(1) + 3 = 3.25$
$x_1 = 3.25$	$x_2 = g(x_1) = (1/4)(3.25) + 3 = 3.8125$
$x_2 = 3.8125$	$x_3 = g(x_2) = 3.9531\,25$
$x_3 = 3.9531\,25$	$x_4 = 3.9882\ 8125$
$\vdots$	$x_5 = 3.9970\ 70313$
	$x_6 = 3.9992\ 67578$
	$x_7 = 3.9998\ 16895$
	$x_8 = 3.9999\ 54224$
	$x_9 = 3.9999\ 88556$
	$x_{10} = 3.9999\ 97139$
	$\vdots$

EXAMPLE 9 Solve the equation $\cos x = x$.

Solution We take $g(x) = \cos x$, choose $x_0 = 1$ as a starting value, and use the recursion formula $x_{n+1} = g(x_n)$ to find

$$x_0 = 1, \qquad x_1 = \cos 1, \qquad x_2 = \cos(x_1), \ldots.$$

We can approximate the first 50 terms or so on a calculator in radian mode by entering 1 and taking the cosine repeatedly. The display stops changing when $\cos x = x$ to the number of decimal places in the display.

Try it for yourself. As you continue to take the cosine, the successive approximations lie alternately above and below the fixed point $x = 0.739085133\ldots$.

Figure 8.7 shows that the values oscillate this way because the path of the procedure spirals around the fixed point. ❑

EXAMPLE 10 Picard's method will not solve the equation

$$g(x) = 4x - 12 = x.$$

As Fig. 8.8 shows, any choice of x_0 except $x_0 = 4$, the solution itself, generates a divergent sequence that moves away from the solution. ❑

The difficulty in Example 10 can be traced to the fact that the slope of the line $y = 4x - 12$ exceeds 1, the slope of the line $y = x$. Conversely, the process worked in Example 8 because the slope of the line $y = (1/4)x + 3$ was numerically less than 1. A theorem from advanced calculus tells us that if $g'(x)$ is continuous on a closed interval I whose interior contains a solution of the equation $g(x) = x$, and if $|g'(x)| < 1$ on I, then any choice of x_0 in the interior of I will lead to the solution. (See the introduction to Exercises 83 and 84 about what to do if $|g'(x)| > 1$.)

Exercises 8.2

Finding Limits

Which of the sequences $\{a_n\}$ in Exercises 1–62 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 2 + (0.1)^n$

2. $a_n = \dfrac{n + (-1)^n}{n}$

3. $a_n = \dfrac{1 - 2n}{1 + 2n}$

4. $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$

5. $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$

6. $a_n = \dfrac{n + 3}{n^2 + 5n + 6}$

7. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$

8. $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

9. $a_n = 1 + (-1)^n$

10. $a_n = (-1)^n \left(1 - \dfrac{1}{n}\right)$

11. $a_n = \left(\dfrac{n + 1}{2n}\right)\left(1 - \dfrac{1}{n}\right)$

12. $a_n = \left(2 - \dfrac{1}{2^n}\right)\left(3 + \dfrac{1}{2^n}\right)$

13. $a_n = \dfrac{(-1)^{n+1}}{2n - 1}$

14. $a_n = \left(-\dfrac{1}{2}\right)^n$

15. $a_n = \sqrt{\dfrac{2n}{n + 1}}$

16. $a_n = \dfrac{1}{(0.9)^n}$

17. $a_n = \sin\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$

18. $a_n = n\pi \cos(n\pi)$

19. $a_n = \dfrac{\sin n}{n}$

20. $a_n = \dfrac{\sin^2 n}{2^n}$

21. $a_n = \dfrac{n}{2^n}$

22. $a_n = \dfrac{3^n}{n^3}$

23. $a_n = \dfrac{\ln(n + 1)}{\sqrt{n}}$

24. $a_n = \dfrac{\ln n}{\ln 2n}$

25. $a_n = 8^{1/n}$

26. $a_n = (0.03)^{1/n}$

27. $a_n = \left(1 + \dfrac{7}{n}\right)^n$

28. $a_n = \left(1 - \dfrac{1}{n}\right)^n$

29. $a_n = \sqrt[n]{10n}$

30. $a_n = \sqrt[n]{n^2}$

31. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$

32. $a_n = (n + 4)^{1/(n+4)}$

33. $a_n = \dfrac{\ln n}{n^{1/n}}$

34. $a_n = \ln n - \ln(n + 1)$

35. $a_n = \sqrt[n]{4^n n}$

36. $a_n = \sqrt[n]{3^{2n+1}}$

37. $a_n = \dfrac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

38. $a_n = \dfrac{(-4)^n}{n!}$

39. $a_n = \dfrac{n!}{10^{6n}}$

40. $a_n = \dfrac{n!}{2^n \cdot 3^n}$

41. $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$

42. $a_n = \ln\left(1 + \dfrac{1}{n}\right)^n$

43. $a_n = \left(\dfrac{3n + 1}{3n - 1}\right)^n$

44. $a_n = \left(\dfrac{n}{n + 1}\right)^n$

45. $a_n = \left(\dfrac{x^n}{2n + 1}\right)^{1/n}, \quad x > 0$

46. $a_n = \left(1 - \dfrac{1}{n^2}\right)^n$

47. $a_n = \dfrac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

48. $a_n = \dfrac{(10/11)^n}{(9/10)^n + (11/12)^n}$

49. $a_n = \tanh n$

50. $a_n = \sinh(\ln n)$

51. $a_n = \dfrac{n^2}{2n - 1} \sin \dfrac{1}{n}$

52. $a_n = n\left(1 - \cos \dfrac{1}{n}\right)$

53. $a_n = \tan^{-1} n$

54. $a_n = \dfrac{1}{\sqrt{n}} \tan^{-1} n$

55. $a_n = \left(\dfrac{1}{3}\right)^n + \dfrac{1}{\sqrt{2^n}}$

56. $a_n = \sqrt[n]{n^2 + n}$

57. $a_n = \dfrac{(\ln n)^{200}}{n}$

58. $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$

59. $a_n = n - \sqrt{n^2 - n}$

60. $a_n = \dfrac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$

61. $a_n = \dfrac{1}{n} \displaystyle\int_1^n \dfrac{1}{x}\,dx$

62. $a_n = \displaystyle\int_1^n \dfrac{1}{x^p}\,dx, \quad p > 1$

Theory and Examples

63. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

64. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the nth fraction $r_n = x_n/y_n$.

a) Verify that $x_1^2 - 2y_1^2 = -1$, $x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a+2b)^2 - 2(a+b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

b) The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint:* Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n.)

65. *Newton's method.* The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

a) $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$

b) $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{\tan x_n - 1}{\sec^2 x_n}$

c) $x_0 = 1, \quad x_{n+1} = x_n - 1$

66. a) Suppose that $f(x)$ is differentiable for all x in $[0, 1]$ and that $f(0) = 0$. Define the sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that $\lim_{n\to\infty} a_n = f'(0)$.

Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

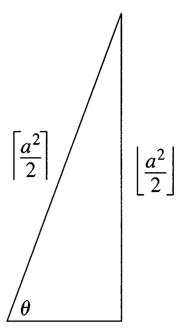
b) $a_n = n \tan^{-1} \dfrac{1}{n}$ **c)** $a_n = n(e^{1/n} - 1)$

d) $a_n = n \ln \left(1 + \dfrac{2}{n}\right)$

67. *Pythagorean triples.* A triple of positive integers a, b, and c is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let a be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.



a) Show that $a^2 + b^2 = c^2$. (*Hint:* Let $a = 2n + 1$ and express b and c in terms of n.)

b) By direct calculation, or by appealing to the figure here, find

$$\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

68. *The nth root of n!*

a) Show that $\lim_{n\to\infty} (2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 7, Additional Exercise 50a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

b) CALCULATOR Test the approximation in (a) for $n = 40, 50, 60, \ldots$, as far as your calculator will allow.

69. a) Assuming that $\lim_{n\to\infty} (1/n^c) = 0$ if c is any positive constant, show that

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$$

if c is any positive constant.

b) Prove that $\lim_{n\to\infty} (1/n^c) = 0$ if c is any positive constant. (*Hint:* If $\epsilon = 0.001$ and $c = 0.04$, how large should N be to ensure that $|1/n^c - 0| < \epsilon$ if $n > N$?)

70. *The zipper theorem.* Prove the "zipper theorem" for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L, then the sequence

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to L.

71. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

72. Prove that $\lim_{n\to\infty} x^{1/n} = 1, (x > 0)$.

73. Prove Theorem 3.

74. Prove Theorem 4.

Picard's Method

CALCULATOR Use Picard's method to solve the equations in Exercises 75–80.

75. $\sqrt{x} = x$ **76.** $x^2 = x$

77. $\cos x + x = 0$ **78.** $\cos x = x + 1$

79. $x - \sin x = 0.1$

80. $\sqrt{x} = 4 - \sqrt{1+x}$ (*Hint:* Square both sides first.)

81. Solving the equation $\sqrt{x} = x$ by Picard's method finds the solution $x = 1$ but not the solution $x = 0$. Why? (*Hint:* Graph $y = x$ and $y = \sqrt{x}$ together.)

82. Solving the equation $x^2 = x$ by Picard's method with $|x_0| \neq 1$ can find the solution $x = 0$ but not the solution $x = 1$. Why? (*Hint:* Graph $y = x^2$ and $y = x$ together.)

Slope greater than 1. Example 10 showed that we cannot apply Picard's method to find a fixed point of $g(x) = 4x - 12$. But we can apply the method to find a fixed point of $g^{-1}(x) = (1/4)x + 3$ because the derivative of g^{-1} is 1/4, whose value is less than 1 in magnitude on any interval. In Example 8, we found the fixed point of g^{-1} to be $x = 4$. Now notice that 4 is also a fixed point of g, since

$$g(4) = 4(4) - 12 = 4.$$

In finding the fixed point of g^{-1}, we found the fixed point of g.

A function and its inverse always have the same fixed points. The graphs of the functions are symmetric about the line $y = x$ and therefore intersect the line at the same points.

We now see that the application of Picard's method is quite broad. For suppose g is one-to-one, with a continuous first derivative whose magnitude is greater than 1 on a closed interval I whose interior contains a fixed point of g. Then the derivative of g^{-1}, being the reciprocal of g', has magnitude less than 1 on I. Picard's method applied to g^{-1} on I will find the fixed point of g. As cases in point, find the fixed points of the functions in Exercises 83 and 84.

83. $g(x) = 2x + 3$

84. $g(x) = 1 - 4x$

8.3 Infinite Series

In mathematics and science we often write functions as infinite polynomials, such as

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \qquad |x| < 1,$$

(we will see the importance of doing so as the chapter continues). For any allowable value of x, we evaluate the polynomial as an infinite sum of constants, a sum we call an *infinite series.* The goal of this section and the next four is to familiarize ourselves with infinite series.

Series and Partial Sums

We begin by asking how to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

The way to do so is not to try to add all the terms at once (we cannot) but rather to add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

Partial sum		**Value**
first:	$s_1 = 1$	$2 - 1$
second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$
third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

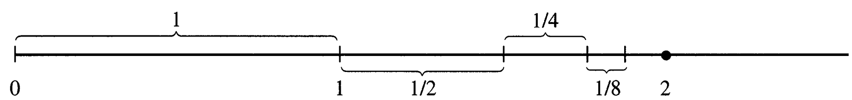
Indeed there is a pattern. The partial sums form a sequence whose nth term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence converges to 2 because $\lim_{n\to\infty} (1/2^n) = 0$. We say

$$\text{“the sum of the infinite series } 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots \text{ is 2.”}$$

Is the sum of any finite number of terms in this series equal to 2? No. Can we actually add an infinite number of terms one by one? No. But we can still define their sum by defining it to be the limit of the sequence of partial sums as $n \to \infty$, in this case 2 (Fig. 8.9). Our knowledge of sequences and limits enables us to break away from the confines of finite sums.



8.9 As the lengths 1, 1/2, 1/4, 1/8, ... are added one by one, the sum approaches 2.

Definitions

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series.** The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum.** If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges.**

When we begin to study a given series $a_1 + a_2 + \cdots + a_n + \cdots$, we might not know whether it converges or diverges. In either case, it is convenient to use sigma notation to write the series as

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \text{or} \quad \sum a_n$$

A useful shorthand when summation from 1 to ∞ is understood

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \tag{1}$$

in which a and r are fixed real numbers and $a \neq 0$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $r = 1$, the nth partial sum of the series in (1) is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \qquad \text{Multiply } s_n \text{ by } r.$$

$$s_n - rs_n = a - ar^n \qquad \text{Subtract } rs_n \text{ from } s_n. \text{ Most of the terms on the right cancel.}$$

$$s_n(1 - r) = a(1 - r^n) \qquad \text{Factor.}$$

$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1). \qquad \text{We can solve for } s_n \text{ if } r \neq 1.$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 8.2) and $s_n \to a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

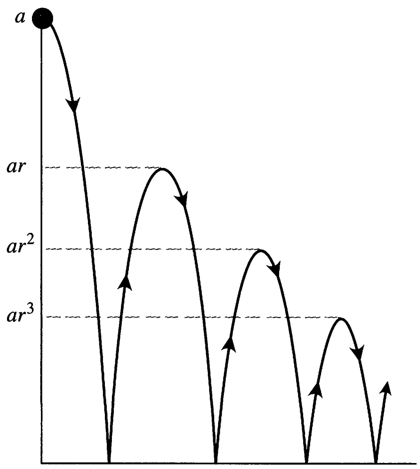
Equation (2) holds *only* if the summation begins with $n = 1$.

> If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:
>
> $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1. \tag{2}$$
>
> If $|r| \geq 1$, the series diverges.

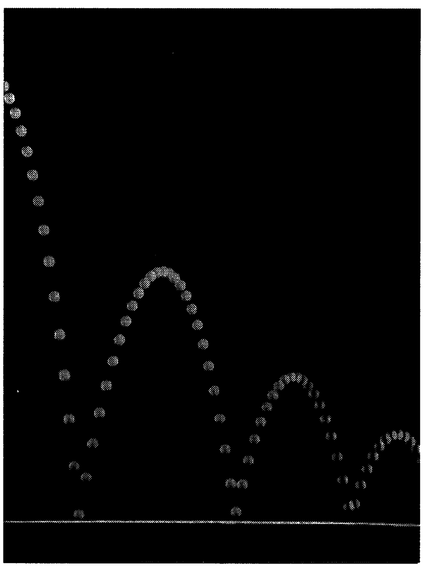
EXAMPLE 1 The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

❑



(a)



(b)

8.10 (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor r. (b) A stroboscopic photo of a bouncing ball.

EXAMPLE 2 The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 5}{4^n} = -\frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = -5/4$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{-5/4}{1+(1/4)} = -1. \qquad \square$$

EXAMPLE 3 You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down (Fig. 8.10).

Solution The total distance is

$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1-r).} = a + \frac{2ar}{1-r} = a\,\frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6\,\frac{1+(2/3)}{1-(2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30\,\text{m}. \qquad \square$$

EXAMPLE 4 *Repeating decimals*

Express the repeating decimal 5.23 23 23 ... as the ratio of two integers.

Solution

$$5.23\ 23\ 23\ldots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{1/(1-0.01)} \qquad a = 1,\ r = 1/100$$

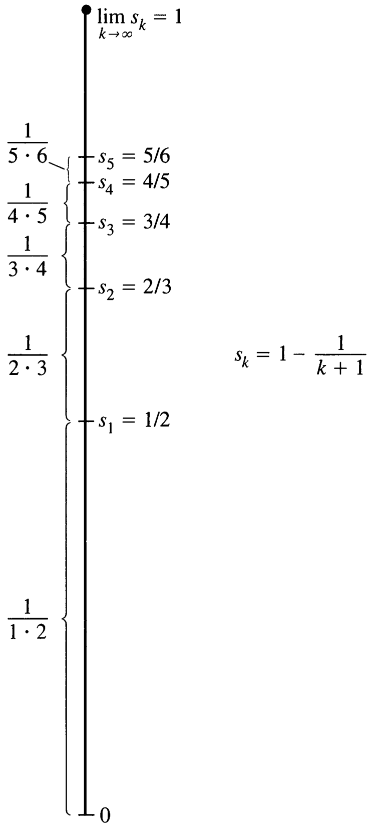
$$= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99} \qquad \square$$

Telescoping Series

Unfortunately, formulas like the one for the sum of a convergent geometric series are rare and we usually have to settle for an estimate of a series' sum (more about this later). The next example, however, is another case in which we can find the sum exactly.

EXAMPLE 5 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution We look for a pattern in the sequence of partial sums that might lead to



8.11 The partial sums of the series in Example 5.

a formula for s_k. The key, as in the integration

$$\int \frac{dx}{x(x+1)} = \int \frac{dx}{x} - \int \frac{dx}{x+1},$$

is partial fractions. The observation that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \tag{3}$$

permits us to write the partial sum

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)}$$

as

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right). \tag{4}$$

Removing parentheses and canceling the terms of opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}. \tag{5}$$

We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1 (Fig. 8.11).

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

❑

Divergent Series

Geometric series with $|r| \geq 1$ are not the only series to diverge.

EXAMPLE 6 The series

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + \cdots + n^2 + \cdots$$

diverges because the partial sums grow beyond every number L. After $n = 1$, the partial sum $s_n = 1 + 4 + 9 + \cdots + n^2$ is greater than n^2. ❑

EXAMPLE 7 The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n. ❑

The *n*th-Term Test for Divergence

Observe that $\lim_{n\to\infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let S represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the nth partial sum. When n is large, both s_n and s_{n-1} are close to S, so their difference, a_n, is close to zero. More formally,

$$a_n = s_n - s_{n-1} \quad \to \quad S - S = 0.$$

Difference Rule for sequences

Caution

Theorem 6 *does not say* that $\sum_{n=1}^{\infty} a_n$ converges if $a_n \to 0$. It is possible for a series to diverge when $a_n \to 0$.

Theorem 6

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 6 leads to a test for detecting the kind of divergence that occurred in Examples 6–8.

The *n*th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

EXAMPLE 8 In applying the *n*th-Term Test, we can see that

a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$

c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist

d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0.$ ❑

EXAMPLE 9 *$a_n \to 0$ but the series diverges*

The series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2\text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4\text{ terms}} + \cdots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n\text{ terms}} + \cdots$$

diverges even though its terms form a sequence that converges to 0. ❑

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

Theorem 7

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k \sum a_n = kA$ (Any number k).

Proof The three rules for series follow from the analogous rules for sequences in Theorem 2, Section 8.2. To prove the Sum Rule for series, let

$$A_n = a_1 + a_2 + \cdots + a_n, \quad B_n = b_1 + b_2 + \cdots + b_n.$$

Then the partial sums of $\sum(a_n + b_n)$ are

$$\begin{aligned} S_n &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) \\ &= A_n + B_n. \end{aligned}$$

Since $A_n \to A$ and $B_n \to B$, we have $S_n \to A + B$ by the Sum Rule for sequences. The proof of the Difference Rule is similar.

To prove the Constant Multiple Rule for series, observe that the partial sums of $\sum ka_n$ form the sequence

$$S_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n,$$

which converges to kA by the Constant Multiple Rule for sequences. ❑

As corollaries of Theorem 7, we have

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

We omit the proofs.

EXAMPLE 10 Find the sums of the following series.

a)
$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\ &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} && \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6 \\ &= 2 - \frac{6}{5} \\ &= \frac{4}{5} \end{aligned}$$

b) $$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = 4 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{1-(1/2)}\right) \qquad \text{Geometric series with } a = 1,\ r = 1/2$$

$$= 8$$

❑

Adding or Deleting Terms

We can always add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n. \tag{6}$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n} \tag{7}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}. \tag{8}$$

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots.$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots.$$

It works like a horizontal shift.

EXAMPLE 11 We can write the geometric series that starts with

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose. ❑

We usually give preference to indexings that lead to simple expressions.

Exercises 8.3

Finding *n*th Partial Sums

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$
2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$
3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$
4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$
5. $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$
6. $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$
8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$
9. $\sum_{n=1}^{\infty} \frac{7}{4^n}$
10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$
11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$
12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$
13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$
14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

Telescoping Series

Use partial fractions to find the sum of each series in Exercises 15–22.

15. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$
16. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$
17. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$
18. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
19. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$
20. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$
21. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$
22. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

Convergence or Divergence

Which series in Exercises 23–40 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

23. $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$
24. $\sum_{n=0}^{\infty} (\sqrt{2})^n$
25. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$
26. $\sum_{n=1}^{\infty} (-1)^{n+1} n$
27. $\sum_{n=0}^{\infty} \cos n\pi$
28. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$
29. $\sum_{n=0}^{\infty} e^{-2n}$
30. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$
31. $\sum_{n=1}^{\infty} \frac{2}{10^n}$
32. $\sum_{n=0}^{\infty} \frac{1}{x^n}, \quad |x| > 1$
33. $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$
34. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$
35. $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$
36. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$
37. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$
38. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$
39. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$
40. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

Geometric Series

In each of the geometric series in Exercises 41–44, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

41. $\sum_{n=0}^{\infty} (-1)^n x^n$
42. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
43. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$
44. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3 + \sin x}\right)^n$

In Exercises 45–50, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

45. $\sum_{n=0}^{\infty} 2^n x^n$
46. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$
47. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$
48. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$
49. $\sum_{n=0}^{\infty} \sin^n x$
50. $\sum_{n=0}^{\infty} (\ln x)^n$

Repeating Decimals

Express each of the numbers in Exercises 51–58 as the ratio of two integers.

51. $0.\overline{23} = 0.23\ 23\ 23\ \ldots$

52. $0.\overline{234} = 0.234\ 234\ 234\ \ldots$

53. $0.\overline{7} = 0.7777\ldots$

54. $0.\overline{d} = 0.dddd\ldots,$ where d is a digit

55. $0.0\overline{6} = 0.06666\ldots$

56. $1.\overline{414} = 1.414\ 414\ 414\ \ldots$

57. $1.24\overline{123} = 1.24\ 123\ 123\ 123\ \ldots$

58. $3.\overline{142857} = 3.142857\ 142857\ \ldots$

Theory and Examples

59. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with (a) $n = -2$, (b) $n = 0$, (c) $n = 5$.

60. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with (a) $n = -1$, (b) $n = 3$, (c) $n = 20$.

61. Make up an infinite series of nonzero terms whose sum is

a) 1 **b)** -3 **c)** 0.

Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

62. Make up an example of two divergent infinite series whose term-by-term sum converges.

63. Show by example that $\sum(a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no b_n equals 0.

64. Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB.

65. Show by example that $\sum(a_n/b_n)$ may converge to something other than A/B even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no b_n equals 0.

66. If $\sum a_n$ converges and $a_n > 0$ for all n, can anything be said about $\sum(1/a_n)$? Give reasons for your answer.

67. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

68. If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.

69. Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

a) $a = 2$ **b)** $a = 13/2$.

70. Find the value of b for which

$$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$

71. For what values of r does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

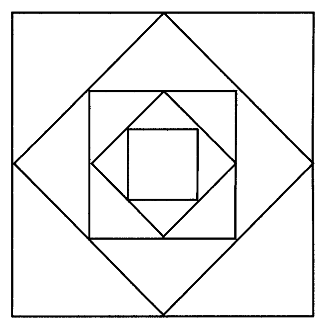
converge? Find the sum of the series when it converges.

72. Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums s_n is $ar^n/(1-r)$.

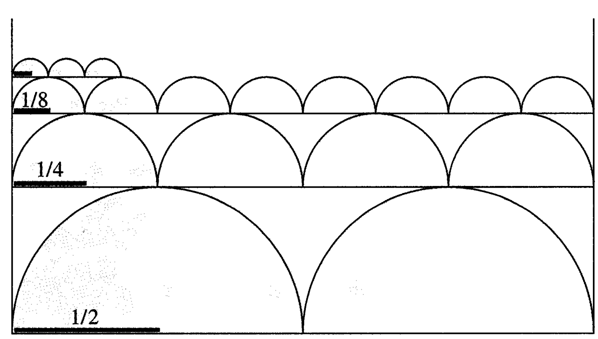
73. A ball is dropped from a height of 4 m. Each time it strikes the pavement after falling from a height of h meters it rebounds to a height of $0.75h$ meters. Find the total distance the ball travels up and down.

74. (*Continuation of Exercise 73.*) Find the total number of seconds the ball in Exercise 73 is traveling. (*Hint:* The formula $s = 4.9t^2$ gives $t = \sqrt{s/4.9}$.)

75. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



76. The accompanying figure shows the first three rows and part of the fourth row of a sequence of rows of semicircles. There are 2^n semicircles in the nth row, each of radius $1/2^n$. Find the sum of the areas of all the semicircles.

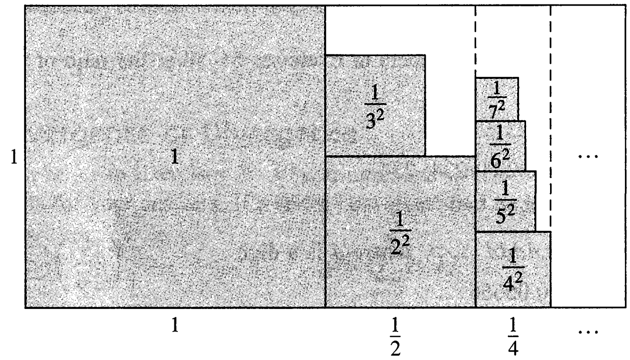


77. *Helga von Koch's snowflake curve.* Helga von Koch's snowflake (p. 167) is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

a) Find the length L_n of the nth curve C_n and show that $\lim_{n\to\infty} L_n = \infty$.

b) Find the area A_n of the region enclosed by C_n and calculate $\lim_{n\to\infty} A_n$.

78. The accompanying figure provides an informal proof that $\sum_{n=1}^{\infty}(1/n^2)$ is less than 2. Explain what is going on. (Source: "Convergence with Pictures" by P. J. Rippon, *American Mathematical Monthly,* Vol. 93, No. 6, 1986, pp. 476–78.)



8.4 The Integral Test for Series of Nonnegative Terms

Given a series $\sum a_n$, we have two questions:

1. Does the series converge?
2. If it converges, what is its sum?

Much of the rest of this chapter is devoted to the first question. But as a practical matter, the second question is just as important, and we will return to it later.

In this section and the next two, we study series that do not have negative terms. The reason for this restriction is that the partial sums of these series form nondecreasing sequences, and nondecreasing sequences that are bounded from above always converge (Theorem 1, Section 8.1). To show that a series of nonnegative terms converges, we need only show that its partial sums are bounded from above.

It may at first seem to be a drawback that this approach establishes the fact of convergence without producing the sum of the series in question. Surely it would be better to compute sums of series directly from formulas for their partial sums. But in most cases such formulas are not available, and in their absence we have to turn instead to the two-step procedure of first establishing convergence and then approximating the sum.

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots.$$

Since the partial sums form a nondecreasing sequence, the Nondecreasing Sequence Theorem (Theorem 1, Section 8.1) tells us that the series will converge if and only if the partial sums are bounded from above.

Corollary of Theorem 1

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Caution

Notice that the nth-Term Test for divergence does not detect the divergence of the harmonic series. The nth term, $1/n$, goes to zero, but the series still diverges.

Nicole Oresme (1320–1382)

The argument we use to show the divergence of the harmonic series was devised by the French theologian, mathematician, physicist, and bishop Nicole Oresme (pronounced "or-*rem*"). Oresme was a vigorous opponent of astrology, a dynamic preacher, an adviser of princes, a friend of King Charles V, a popularizer of science, and a skillful translator of Latin into French.

Oresme did not believe in Albert of Saxony's generally accepted model of free fall (Chapter 6, Additional Exercise 26) but preferred Aristotle's constant-acceleration model, the model that became popular among Oxford scholars in the 1330s and that Galileo eventually used three hundred years later.

EXAMPLE 1 *The harmonic series*

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is called the **harmonic series.** It diverges because there is no upper bound for its partial sums. To see why, group the terms of the series in the following way:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots.$$

The sum of the first two terms is 1.5. The sum of the next two terms is $1/3 + 1/4$, which is greater than $1/4 + 1/4 = 1/2$. The sum of the next four terms is $1/5 + 1/6 + 1/7 + 1/8$, which is greater than $1/8 + 1/8 + 1/8 + 1/8 = 1/2$. The sum of the next eight terms is $1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16$, which is greater than $8/16 = 1/2$. The sum of the next 16 terms is greater than $16/32 = 1/2$, and so on. In general, the sum of 2^n terms ending with $1/2^{n+1}$ is greater than $2^n/2^{n+1} = 1/2$. The sequence of partial sums is not bounded from above: If $n = 2^k$, the partial sum s_n is greater than $k/2$. The harmonic series diverges. ❑

The Integral Test

We introduce the Integral Test with a series that is related to the harmonic series, but whose nth term is $1/n^2$ instead of $1/n$.

EXAMPLE 2 Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots \tag{1}$$

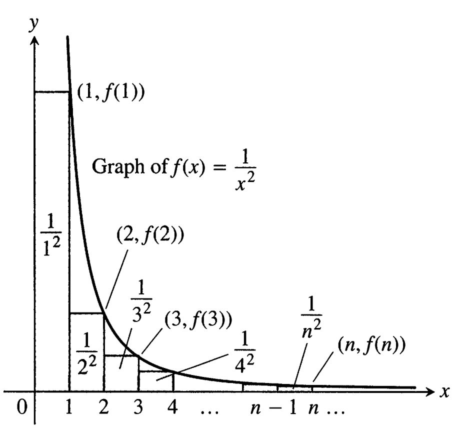
Solution We determine the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ by comparing it with $\int_1^{\infty}(1/x^2)\,dx$. To carry out the comparison, we think of the terms of the series as values of the function $f(x) = 1/x^2$ and interpret these values as the areas of rectangles under the curve $y = 1/x^2$.

As Fig. 8.12 shows,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2}\,dx \\ &< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx \\ &< 1 + 1 = 2. \end{aligned}$$

As in Section 7.6, Example 8, $\int_1^{\infty}(1/x^2)\,dx = 1$.

Thus the partial sums of $\sum_{n=1}^{\infty} 1/n^2$ are bounded from above (by 2) and the series converges. The sum of the series is known to be $\pi^2/6 \approx 1.64493$. ❑



8.12 Figure for the area comparisons in Example 2.

> **The Integral Test**
>
> Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Proof We establish the test for the case $N = 1$. The proof for general N is similar.

We start with the assumption that f is a decreasing function with $f(n) = a_n$ for every n. This leads us to observe that the rectangles in Fig. 8.13(a), which have areas $a_1, a_2, \ldots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n.$$

In Fig. 8.13(b) the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle, of area a_1, we see that

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx.$$

If we include a_1, we have

$$a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx.$$

Combining these results gives

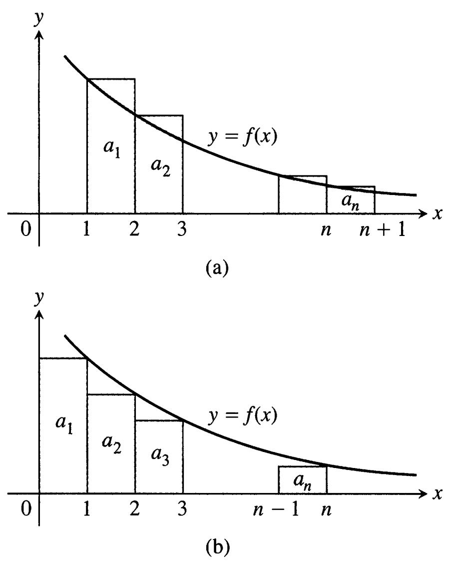
$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx. \tag{2}$$

If $\int_1^{\infty} f(x)\,dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^{\infty} f(x)\,dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite.

Hence the series and the integral are both finite or both infinite. ❑

Caution

The series and integral need not have the same value in the convergent case. As we saw in Example 2, $\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$ while $\int_1^{\infty}(1/x^2)\,dx = 1$.



8.13 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 *The p-series.* Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \tag{3}$$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx &= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b \\
&= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) \\
&= \frac{1}{1-p}(0-1) = \frac{1}{p-1},
\end{aligned}$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We have convergence for $p > 1$ but divergence for every other value of p. ❑

Exercises 8.4

Determining Convergence or Divergence

Which of the series in Exercises 1–30 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1. $\sum_{n=1}^\infty \frac{1}{10^n}$
2. $\sum_{n=1}^\infty e^{-n}$
3. $\sum_{n=1}^\infty \frac{n}{n+1}$
4. $\sum_{n=1}^\infty \frac{5}{n+1}$
5. $\sum_{n=1}^\infty \frac{3}{\sqrt{n}}$
6. $\sum_{n=1}^\infty \frac{-2}{n\sqrt{n}}$
7. $\sum_{n=1}^\infty -\frac{1}{8^n}$
8. $\sum_{n=1}^\infty \frac{-8}{n}$
9. $\sum_{n=2}^\infty \frac{\ln n}{n}$
10. $\sum_{n=2}^\infty \frac{\ln n}{\sqrt{n}}$
11. $\sum_{n=1}^\infty \frac{2^n}{3^n}$
12. $\sum_{n=1}^\infty \frac{5^n}{4^n+3}$
13. $\sum_{n=0}^\infty \frac{-2}{n+1}$
14. $\sum_{n=1}^\infty \frac{1}{2n-1}$
15. $\sum_{n=1}^\infty \frac{2^n}{n+1}$
16. $\sum_{n=1}^\infty \frac{1}{\sqrt{n}(\sqrt{n}+1)}$
17. $\sum_{n=2}^\infty \frac{\sqrt{n}}{\ln n}$
18. $\sum_{n=1}^\infty \left(1+\frac{1}{n}\right)^n$
19. $\sum_{n=1}^\infty \frac{1}{(\ln 2)^n}$
20. $\sum_{n=1}^\infty \frac{1}{(\ln 3)^n}$
21. $\sum_{n=3}^\infty \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$
22. $\sum_{n=1}^\infty \frac{1}{n(1+\ln^2 n)}$
23. $\sum_{n=1}^\infty n \sin\frac{1}{n}$
24. $\sum_{n=1}^\infty n \tan\frac{1}{n}$
25. $\sum_{n=1}^\infty \frac{e^n}{1+e^{2n}}$
26. $\sum_{n=1}^\infty \frac{2}{1+e^n}$
27. $\sum_{n=1}^\infty \frac{8\tan^{-1} n}{1+n^2}$
28. $\sum_{n=1}^\infty \frac{n}{n^2+1}$
29. $\sum_{n=1}^\infty \operatorname{sech} n$
30. $\sum_{n=1}^\infty \operatorname{sech}^2 n$

Theory and Examples

For what values of a, if any, do the series in Exercises 31 and 32 converge?

31. $\sum_{n=1}^\infty \left(\frac{a}{n+2} - \frac{1}{n+4}\right)$
32. $\sum_{n=3}^\infty \left(\frac{1}{n-1} - \frac{2a}{n+1}\right)$

33. **a)** Draw illustrations like those in Figs. 8.12 and 8.13 to show that the partial sums of the harmonic series satisfy the inequalities

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx \le 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
$$\le 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n.$$

b) There is absolutely no empirical evidence for the divergence of the harmonic series even though we know it diverges. The partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum s_n be today, assuming a 365-day year?

34. Are there any values of x for which $\sum_{n=1}^\infty (1/(nx))$ converges? Give reasons for your answer.

35. Is it true that if $\sum_{n=1}^\infty a_n$ is a divergent series of positive numbers then there is also a divergent series $\sum_{n=1}^\infty b_n$ of positive numbers with $b_n < a_n$ for every n? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

36. (*Continuation of Exercise 35*) Is there a "largest" convergent series of positive numbers? Explain.

37. *The Cauchy condensation test.* The Cauchy condensation test says: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \ge a_{n+1}$ for all n) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. For example, $\sum(1/n)$ diverges because $\sum 2^n \cdot (1/2^n) = \sum 1$ diverges. Show why the test works.

38. Use the Cauchy condensation test from Exercise 37 to show that

a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges;

b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

39. *Logarithmic p-series*

a) Show that

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

b) What implications does the fact in (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

40. (*Continuation of Exercise 39.*) Use the result in Exercise 39 to determine which of the following series converge and which diverge. Support your answer in each case.

a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ **b)** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

c) $\sum_{n=2}^{\infty} \frac{1}{n \ln (n^3)}$ **d)** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

41. *Euler's constant.* Graphs like those in Fig. 8.13 suggest that as n increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x}\, dx.$$

To explore this idea, carry out the following steps.

a) By taking $f(x) = 1/x$ in inequality (2), show that

$$\ln (n + 1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln (n + 1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

b) Show that

$$\frac{1}{n + 1} < \int_n^{n+1} \frac{1}{x}\, dx = \ln (n + 1) - \ln n,$$

and use this result to show that the sequence $\{a_n\}$ in part (a) is decreasing.

Since a decreasing sequence that is bounded from below converges (Exercise 41 in Section 8.1), the numbers a_n defined in (a) converge:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \to \gamma.$$

The number γ, whose value is 0.5772 . . ., is called *Euler's constant.* In contrast to other special numbers like π and e, no other expression with a simple law of formulation has ever been found for γ.

42. Use the integral test to show that

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.

8.5 Comparison Tests for Series of Nonnegative Terms

The key question in using Corollary 1 in the preceding section is how to determine in any particular instance whether the s_n's are bounded from above. Sometimes we can establish this by showing that each s_n is less than or equal to the corresponding partial sum of a series already known to converge.

EXAMPLE 1 The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{1}$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1+\sum_{n=0}^{\infty}\frac{1}{2^n}=1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots. \tag{2}$$

To see how this relationship leads to an upper bound for the partial sums of $\sum_{n=0}^{\infty}(1/(n!))$, let

$$s_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$$

and observe that, for each n,

$$s_n\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}<1+\sum_{n=0}^{\infty}\frac{1}{2^n}=1+\frac{1}{1-(1/2)}=3.$$

Thus the partial sums of $\sum_{n=0}^{\infty}(1/(n!))$ are all less than 3, so $\sum_{n=0}^{\infty}(1/(n!))$ converges.

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty}(1/(n!))$ does not mean that the series converges to 3. As we will see in Section 8.10, the series converges to e. ❑

The Direct Comparison Test

We established the convergence in Example 1 by comparing the terms of the given series with the terms of a series known to converge. This idea can be pursued further to yield a number of tests known as *comparison tests.*

Direct Comparison Test for Series of Nonnegative Terms

Let $\sum a_n$ be a series with no negative terms.

a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some integer N.

b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \ge d_n$ for all $n > N$, for some integer N.

Proof In part (a), the partial sums of $\sum a_n$ are bounded above by

$$M=a_1+a_2+\cdots+a_n+\sum_{n=N+1}^{\infty}c_n.$$

They therefore form a nondecreasing sequence with a limit $L \le M$.

In part (b), the partial sums of $\sum a_n$ are not bounded from above. If they were, the partial sums for $\sum d_n$ would be bounded by

$$M'=d_1+d_2+\cdots+d_N+\sum_{n=N+1}^{\infty}a_n$$

and $\sum d_n$ would have to converge instead of diverge. ❑

To apply the Direct Comparison Test to a series, we need not include the early terms of the series. We can start the test with any index N provided we include all the terms of the series being tested from there on.

EXAMPLE 2 Does the following series converge?

$$5+\frac{2}{3}+1+\frac{1}{7}+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{k!}+\cdots$$

Solution We ignore the first four terms and compare the remaining terms with those of the convergent geometric series $\sum_{n=1}^{\infty} 1/2^n$. We see that

$$\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots \leq \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots.$$

Therefore, the original series converges by the Direct Comparison Test. ❑

To apply the Direct Comparison Test, we need to have on hand a list of series whose convergence or divergence we know. Here is what we know so far:

Convergent series	**Divergent series**
Geometric series with $\lvert r\rvert < 1$	Geometric series with $\lvert r\rvert \geq 1$
Telescoping series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$	The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$
The series $\sum_{n=0}^{\infty} \frac{1}{n!}$	Any series $\sum a_n$ for which $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p > 1$	The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p \leq 1$

The Limit Comparison Test

We now introduce a comparison test that is particularly handy for series in which a_n is a rational function of n.

Suppose we wanted to investigate the convergence of the series

$$\textbf{a)}\ \sum_{n=2}^{\infty} \frac{2n}{n^2-n+1} \qquad \textbf{b)}\ \sum_{n=2}^{\infty} \frac{8n^3+100n^2+1000}{2n^6-n+5}.$$

In determining convergence or divergence, only the tails matter. And when n is very large, the highest powers in the numerator and denominator matter the most. So in (a), we might reason this way: For n large,

$$a_n = \frac{2n}{n^2-n+1}$$

behaves like $2n/n^2 = 2/n$. Since $\sum 1/n$ diverges, we expect $\sum a_n$ to diverge, too.

In (b) we might reason that for n large

$$a_n = \frac{8n^3+100n^2+1000}{2n^6-n+5}$$

will behave approximately like $(8n^3)/(2n^6) = 4/n^3$. Since $\sum 4/n^3$ converges (it is 4 times a convergent p-series), we expect $\sum a_n$ to converge, too.

Our expectations about $\sum a_n$ in each case are correct, as the following test shows.

Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof We will prove part (1). Parts (2) and (3) are left as Exercises 37 (a) and (b). Since $c/2 > 0$, there exists an integer N such that for all n

$$n > N \Rightarrow \left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}.$$

Limit definition with $\epsilon = c/2$, $L = c$, and a_n replaced by a_n/b_n

Thus, for $n > N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n.$$

If $\sum b_n$ converges, then $\sum(3c/2)b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum(c/2)b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test. ❑

EXAMPLE 3 Which of the following series converge, and which diverge?

a) $\dfrac{3}{4} + \dfrac{5}{9} + \dfrac{7}{16} + \dfrac{9}{25} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

b) $\dfrac{1}{1} + \dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{15} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

c) $\dfrac{1 + 2\ln 2}{9} + \dfrac{1 + 3\ln 3}{14} + \dfrac{1 + 4\ln 4}{21} + \cdots = \displaystyle\sum_{n=2}^{\infty} \frac{1 + n\ln n}{n^2 + 5}$

Solution

We could just as well have taken $b_n = 2/n$ but $1/n$ is simpler.

a) Let $a_n = (2n+1)/(n^2+2n+1)$. For n large, we expect a_n to behave like $2n/n^2 = 2/n$, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by part 1 of the Limit Comparison Test.

b) Let $a_n = 1/(2^n - 1)$. For n large, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2^n}{2^n - 1}\\ &= \lim_{n\to\infty} \frac{1}{1-(1/2^n)}\\ &= 1,\end{aligned}$$

$\sum a_n$ converges by part 1 of the Limit Comparison Test.

c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For n large, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we take $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5}\\ &= \infty,\end{aligned}$$

$\sum a_n$ diverges by part 3 of the Limit Comparison Test. ❑

EXAMPLE 4 Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution Because $\ln n$ grows more slowly than n^c for any positive constant c (Section 8.2, Exercise 69), we would expect to have

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for n sufficiently large. Indeed, taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}}\\ &= \lim_{n\to\infty} \frac{1/n}{(1/4)\,n^{-3/4}} && \text{l'Hôpital's rule}\\ &= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.\end{aligned}$$

Since $\sum b_n = \sum(1/n^{5/4})$ (a p-series with $p > 1$) converges, $\sum a_n$ converges by part 2 of the Limit Comparison Test. ❑

Exercises 8.5

Determining Convergence or Divergence

Which of the series in Exercises 1–36 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$
2. $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$
3. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$
4. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$
5. $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$
6. $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$
7. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$
9. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$
10. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$
11. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$
12. $\sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^3}$
13. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$
14. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$
15. $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$
16. $\sum_{n=1}^{\infty} \frac{1}{(1+\ln n)^2}$
17. $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$
18. $\sum_{n=1}^{\infty} \frac{1}{(1+\ln^2 n)}$
19. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$
20. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$
21. $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$
22. $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$
23. $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$
24. $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$
25. $\sum_{n=1}^{\infty} \sin \frac{1}{n}$
26. $\sum_{n=1}^{\infty} \tan \frac{1}{n}$
27. $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$
28. $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$
29. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$
30. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$
31. $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$
32. $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$
33. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$
34. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$
35. $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$
36. $\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$

Theory and Examples

37. Prove (a) Part 2 and (b) Part 3 of the Limit Comparison Test.
38. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} (a_n/n)$? Explain.
39. Suppose that $a_n > 0$ and $b_n > 0$ for $n \geq N$ (N an integer). If $\lim_{n\to\infty} (a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.
40. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.

CAS Exploration and Project

41. It is not yet known whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

a) Define the sequence of partial sums

$$s_k = \sum_{n=1}^{k} \frac{1}{n^3 \sin^2 n}.$$

What happens when you try to find the limit of s_k as $k \to \infty$? Does your CAS find a closed form answer for this limit?

b) Plot the first 100 points (k, s_k) for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?

c) Next plot the first 200 points (k, s_k). Discuss the behavior in your own words.

d) Plot the first 400 points (k, s_k). What happens when $k = 355$? Calculate the number 355/113. Explain from your calculation what happened at $k = 355$. For what values of k would you guess this behavior might occur again?

You will find an interesting discussion of this series in Chapter 72 of *Mazes for the Mind* by Clifford A. Pickover, St. Martin's Press, Inc., New York, 1992.

8.6 The Ratio and Root Tests for Series of Nonnegative Terms

Convergence tests that depend on comparing series with integrals or other series are called *extrinsic* tests. They are useful, but there are reasons to look for tests that do not require comparison. As a practical matter, we may not be able to find the series or functions we need to make a comparison work. And, in principle, all the information about a given series should be contained in its own terms. We therefore turn our attention to *intrinsic* tests—tests that depend only on the series at hand.

The Ratio Test

The first intrinsic test, the Ratio Test, measures the rate of growth (or decline) of a series by examining the ratio a_{n+1}/a_n. For a geometric series $\sum ar^n$, this rate is a constant $((ar^{n+1})/(ar^n) = r)$, and the series converges if and only if its ratio is less than 1 in absolute value. But even if the ratio is not constant, we may be able to find a geometric series for comparison, as in Example 1.

The series in Example 1 converges rapidly, as the following computer data suggest.

n	s_n
5	1.5492 06349
10	1.5702 89085
15	1.5707 83080
20	1.5707 95964
25	1.5707 96317
30	1.5707 96327
35	1.5707 96327

EXAMPLE 1 Let $a_1 = 1$ and let $a_{n+1} = \dfrac{n}{2n+1}a_n$ for all n. Does the series $\sum a_n$ converge?

Solution We begin by writing a few terms of the series:

$$a_1 = 1, \qquad a_2 = \frac{1}{3}a_1 = \frac{1}{3}, \qquad a_3 = \frac{2}{5}a_2 = \frac{1 \cdot 2}{3 \cdot 5}, \qquad a_4 = \frac{3}{7}a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}.$$

Each term is somewhat less than 1/2 the term before it, because $n/(2n+1)$ is less than 1/2. Therefore the terms of the series are less than or equal to the terms of the geometric series

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

which converges to 2. So our series also converges, and its sum is less than 2. The table in the margin shows how quickly the series converges to its known limit, $\pi/2$. ❑

The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

a) the series *converges* if $\rho < 1$,
b) the series *diverges* if $\rho > 1$ or ρ is infinite,
c) the test is *inconclusive* if $\rho = 1$.

In proving the Ratio Test, we will make a comparison with an appropriate geometric series as in Example 1, but when we *apply* the test there is no need for comparison.

Proof

a) $\boldsymbol{\rho < 1}$. Let r be a number between ρ and 1. Then the number $\epsilon = r - \rho$ is positive. Since

$$\frac{a_{n+1}}{a_n} \to \rho,$$

a_{n+1}/a_n must lie within ϵ of ρ when n is large enough, say for all $n \geq N$. In particular,

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \qquad \text{when } n \geq N.$$

That is,

$$\begin{aligned}
a_{N+1} &< ra_N, \\
a_{N+2} &< ra_{N+1} < r^2 a_N, \\
a_{N+3} &< ra_{N+2} < r^3 a_N, \\
&\vdots \\
a_{N+m} &< ra_{N+m-1} < r^m a_N.
\end{aligned}$$

These inequalities show that the terms of our series, after the Nth term, approach zero more rapidly than the terms in a geometric series with ratio $r < 1$. More precisely, consider the series $\sum c_n$, where $c_n = a_n$ for $n = 1, 2, \ldots, N$ and $c_{N+1} = ra_N, c_{N+2} = r^2 a_N, \ldots, c_{N+m} = r^m a_N, \ldots$. Now $a_n \le c_n$ for all n, and

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n &= a_1 + a_2 + \cdots + a_{N-1} + a_N + ra_N + r^2 a_N + \cdots \\
&= a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots).
\end{aligned}$$

The geometric series $1 + r + r^2 + \cdots$ converges because $|r| < 1$, so $\sum c_n$ converges. Since $a_n \le c_n$, $\sum a_n$ also converges.

b) $\mathbf{1 < \rho \le \infty}$**.** From some index M on,

$$\frac{a_{n+1}}{a_n} > 1 \qquad \text{and} \qquad a_M < a_{M+1} < a_{M+2} < \cdots.$$

The terms of the series do not approach zero as n becomes infinite, and the series diverges by the nth-Term Test.

c) $\mathbf{\rho = 1}$**.** The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

For $\sum_{n=1}^{\infty} \frac{1}{n}$: $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{1/(n+1)}{1/n} = \dfrac{n}{n+1} \to 1.$

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{1/(n+1)^2}{1/n^2} = \left(\dfrac{n}{n+1}\right)^2 \to 1^2 = 1.$

In both cases $\rho = 1$, yet the first series diverges while the second converges. ❑

The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to the nth power.

EXAMPLE 2 Investigate the convergence of the following series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

Solution

a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1.

This does *not* mean that 2/3 is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \to 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

c) If $a_n = 4^n n!n!/(2n)!$, then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!} \\ &= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \to 1. \end{aligned}$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. However, when we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges. ❑

The *n*th-Root Test

The convergence tests we have so far for $\sum a_n$ work best when the formula for a_n is relatively simple. But consider the following.

EXAMPLE 3 Let $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$ Does $\sum a_n$ converge?

Solution We write out several terms of the series:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \cdots. \end{aligned}$$

Clearly, this is not a geometric series. The nth term approaches zero as $n \to \infty$, so we do not know if the series diverges. The Integral Test does not look promising. The Ratio Test produces

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2n}, & n \text{ odd} \\[2ex] \dfrac{n+1}{2}, & n \text{ even.} \end{cases}$$

As $n \to \infty$, the ratio is alternately small and large and has no limit.

A test that will answer the question (the series converges) is the nth-Root Test. ❑

The nth-Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

a) the series *converges* if $\rho < 1$,
b) the series *diverges* if $\rho > 1$ or ρ is infinite,
c) the test is *inconclusive* if $\rho = 1$.

Proof

a) $\rho < 1$. Choose an $\epsilon > 0$ so small that $\rho + \epsilon < 1$. Since $\sqrt[n]{a_n} \to \rho$, the terms $\sqrt[n]{a_n}$ eventually get closer than ϵ to ρ. In other words, there exists an index $M \geq N$ such that

$$\sqrt[n]{a_n} < \rho + \epsilon \qquad \text{when } n \geq M.$$

Then it is also true that

$$a_n < (\rho + \epsilon)^n \qquad \text{for } n \geq M.$$

Now, $\sum_{n=M}^{\infty}(\rho+\epsilon)^n$, a geometric series with ratio $(\rho + \epsilon) < 1$, converges. By comparison, $\sum_{n=M}^{\infty} a_n$ converges, from which it follows that

$$\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{M-1} + \sum_{n=M}^{\infty} a_n$$

converges.

b) $1 < \rho \leq \infty$. For all indices beyond some integer M, we have $\sqrt[n]{a_n} > 1$, so that $a_n > 1$ for $n > M$. The terms of the series do not converge to zero. The series diverges by the nth-Term Test.

c) $\rho = 1$. The series $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^2)$ show that the test is not conclusive when $\rho = 1$. The first series diverges and the second converges, but in both cases $\sqrt[n]{a_n} \to 1$. ❑

EXAMPLE 3 (continued) Let $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$ Does $\sum a_n$ converge?

Solution We apply the nth-Root Test, finding that

$$\sqrt[n]{a_n} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even.} \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 8.2, Table 8.1), we have $\lim_{n\to\infty} \sqrt[n]{a_n} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges by the nth-Root Test. ❑

EXAMPLE 4 Which of the following series converges, and which diverges?

a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Solution

a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1}{2} < 1.$

b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{(\sqrt[n]{n})^2} \to \frac{2}{1} > 1.$ ❑

Exercises 8.6

Determining Convergence or Divergence

Which of the series in Exercises 1–26 converge, and which diverge? Give reasons for your answers. (When checking your answers, remember there may be more than one way to determine a series' convergence or divergence.)

1. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

2. $\sum_{n=1}^{\infty} n^2 e^{-n}$

3. $\sum_{n=1}^{\infty} n!\, e^{-n}$

4. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

5. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

6. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

7. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$

8. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

9. $\sum_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$

10. $\sum_{n=1}^{\infty} \left(1-\frac{1}{3n}\right)^n$

11. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

12. $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

13. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)$

14. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

15. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

16. $\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$

17. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

18. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

19. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

20. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

21. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

22. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

23. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

24. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$

25. $\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!}$

26. $\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 27–38 converge, and which diverge? Give reasons for your answers.

27. $a_1 = 2, \quad a_{n+1} = \frac{1+\sin n}{n} a_n$

28. $a_1 = 1, \quad a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$

29. $a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{3n-1}{2n+5} a_n$

30. $a_1 = 3, \quad a_{n+1} = \frac{n}{n+1} a_n$

31. $a_1 = 2, \quad a_{n+1} = \frac{2}{n} a_n$

32. $a_1 = 5, \quad a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$

33. $a_1 = 1, \quad a_{n+1} = \frac{1+\ln n}{n} a_n$

34. $a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n+\ln n}{n+10} a_n$

35. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$

36. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

37. $a_n = \frac{2^n n! n!}{(2n)!}$

38. $a_n = \frac{(3n)!}{n!(n+1)!(n+2)!}$

Which of the series in Exercises 39–44 converge, and which diverge? Give reasons for your answers.

39. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

40. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$

41. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

42. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

43. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$

44. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n+1)}$

Theory and Examples

45. Neither the Ratio nor the nth-Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

46. Show that neither the Ratio Test nor the nth-Root Test provides information about the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \qquad (p \text{ constant}).$$

47. Let $a_n = \begin{cases} n/2^n & \text{if } n \text{ is a prime number} \\ 1/2^n & \text{otherwise.} \end{cases}$

Does $\sum a_n$ converge? Give reasons for your answer.

8.7 Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series.**

Here are three examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \qquad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \qquad (2)$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \qquad (3)$$

Series (1), called the **alternating harmonic series,** converges, as we will see in a moment. Series (2), a geometric series with ratio $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series (3) diverges because the nth term does not approach zero.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test.

Theorem 8
The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N,
3. $u_n \to 0$.

Proof If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L. \tag{4}$$

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m\to\infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L. \tag{5}$$

Combining the results of (4) and (5) gives $\lim_{n\to\infty} s_n = L$ (Section 8.1, Exercise 53).

❑

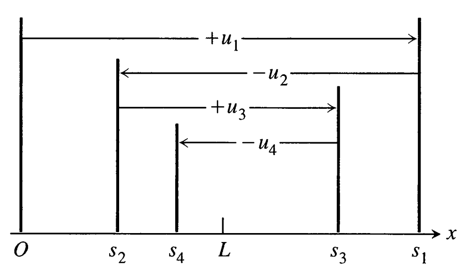
EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies the three requirements of Theorem 8 with $N = 1$; it therefore converges.

❑

A graphical interpretation of the partial sums (Fig. 8.14) shows how an alternating series converges to its limit L when the three conditions of Theorem 8 are satisfied with $N = 1$. (Exercise 63 asks you to picture the case $N > 1$.) Starting from the origin of the x-axis, we lay off the positive distance $s_1 = u_1$. To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2. Since $u_2 \leq u_1$, we do not back up any farther than the origin. We continue in this seesaw fashion, backing up or going forward as the signs in the series demand. But for $n \geq N$, each forward or backward step is shorter than (or at most the same size as) the preceding step, because $u_{n+1} \leq u_n$. And since the nth term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller. We oscillate across the limit L, and the amplitude of oscillation approaches zero. The limit L lies between any two successive sums s_n and s_{n+1} and hence differs from s_n by an amount less than u_{n+1}.



8.14 The partial sums of an alternating series that satisfies the hypotheses of Theorem 8 for $N = 1$ straddle the limit from the beginning.

Because

$$|L - s_n| < u_{n+1} \qquad \text{for } n \geq N,$$

we can make useful estimates of the sums of convergent alternating series.

Theorem 9

The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 8, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

We leave the verification of the sign of the remainder for Exercise 53.

EXAMPLE 2 We try Theorem 9 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} \;\vdots\; + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than 1/256. The sum of the first eight terms is 0.6640 625. The sum of the series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640\ 625 = 0.0026\ 04166\ 6\ldots$, is positive and less than $(1/256) = 0.0039\ 0625$. ❑

Absolute Convergence

Definition

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

The geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

converges absolutely because the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges. The alternating harmonic series does not converge absolutely. The corresponding series of absolute values is the (divergent) harmonic series.

Definition

A series that converges but does not converge absolutely **converges conditionally.**

The alternating harmonic series converges conditionally.

Absolute convergence is important for two reasons. First, we have good tests for convergence of series of positive terms. Second, if a series converges absolutely, then it converges. That is the thrust of the next theorem.

Caution

We can rephrase Theorem 10 to say that *every absolutely convergent series converges.* However, the converse statement is false: Many convergent series do not converge absolutely.

Theorem 10
The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof For each n,

$$-|a_n| \le a_n \le |a_n|, \qquad \text{so} \qquad 0 \le a_n + |a_n| \le 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ❑

EXAMPLE 3 For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely. ❑

EXAMPLE 4 For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ because $|\sin n| \le 1$ for every n. The original series converges absolutely; therefore it converges. ❑

EXAMPLE 5 *Alternating p-series*

If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \qquad p > 0$$

converges.

If $p > 1$, the series converges absolutely. If $0 < p \leq 1$, the series converges conditionally.

Conditional convergence: $\quad 1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \cdots$

Absolute convergence: $\quad 1 - \dfrac{1}{2^{3/2}} + \dfrac{1}{3^{3/2}} - \dfrac{1}{4^{3/2}} + \cdots$ ❑

Rearranging Series

Theorem 11

The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

(For an outline of the proof, see Exercise 60.)

EXAMPLE 6 As we saw in Example 3, the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + (-1)^{n-1}\frac{1}{n^2} + \cdots$$

converges absolutely. A possible rearrangement of the terms of the series might start with a positive term, then two negative terms, then three positive terms, then four negative terms, and so on: After k terms of one sign, take $k+1$ terms of the opposite sign. The first ten terms of such a series look like this:

$$1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \frac{1}{36} - \frac{1}{64} - \frac{1}{100} - \frac{1}{144} + \cdots.$$

The Rearrangement Theorem says that both series converge to the same value. In this example, if we had the second series to begin with, we would probably be glad to exchange it for the first, if we knew that we could. We can do even better: The sum of either series is also equal to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

(See Exercise 61.) ❑

Caution

If we rearrange infinitely many terms of a conditionally convergent series, we can get results that are far different from the sum of the original series.

The kind of behavior illustrated by this example is typical of what can happen with any conditionally convergent series. Moral: Add the terms of a conditionally convergent series in the order given.

EXAMPLE 7 *Rearranging the alternating harmonic series*

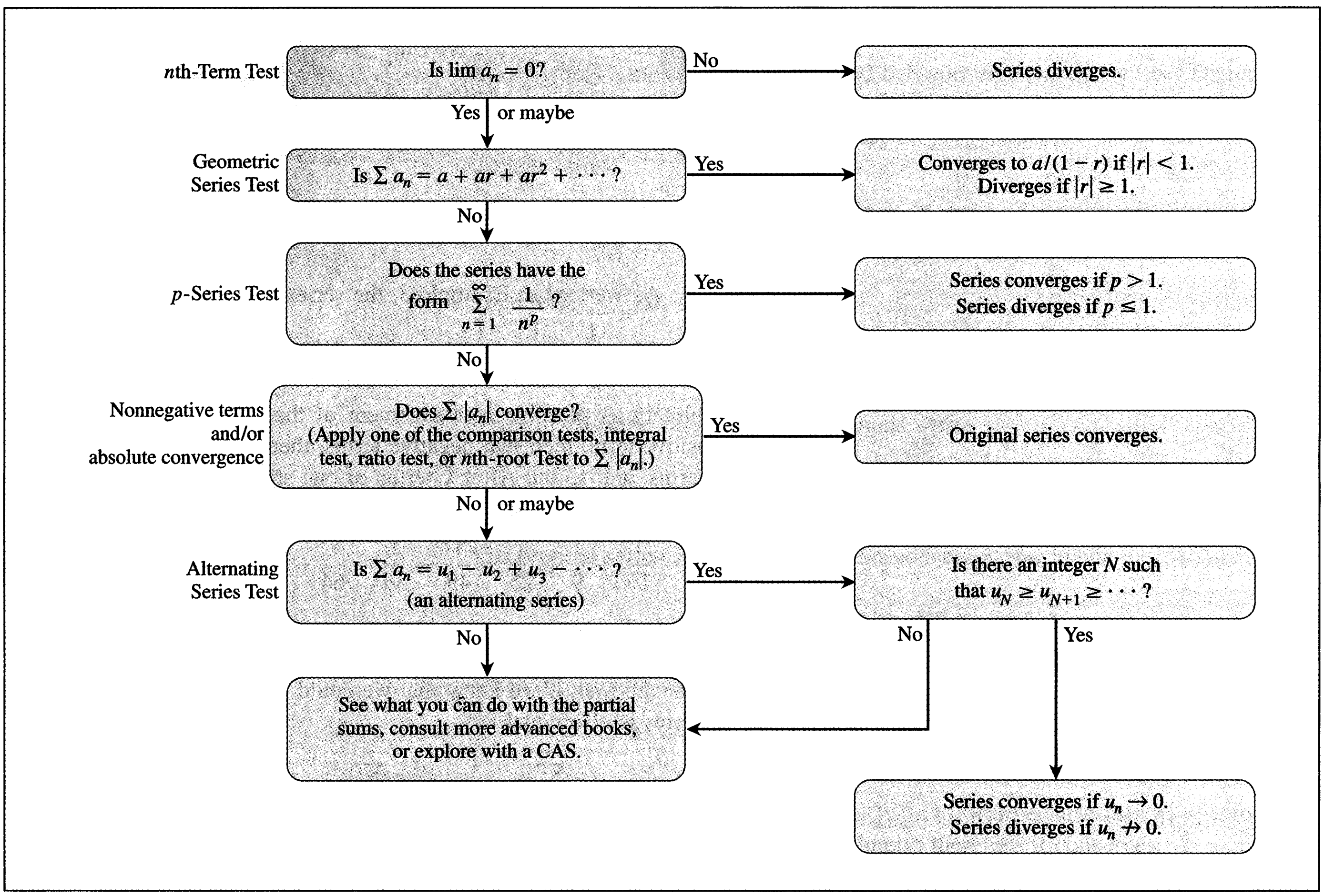
The alternating harmonic series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots$$

can be rearranged to diverge or to reach any preassigned sum.

a) *Rearranging* $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ *to diverge.* The series of terms $\sum[1/(2n-1)]$ diverges to $+\infty$ and the series of terms $\sum(-1/2n)$ diverges to $-\infty$. No matter how far out in the sequence of odd-numbered terms we begin, we can always add enough positive terms to get an arbitrarily large sum. Similarly, with the negative terms, no matter how far out we start, we can add enough consecutive even-numbered terms to get a negative sum of arbitrarily large absolute value. If we wished to do so, we could start adding odd-numbered terms until we had a sum greater than $+3$, say, and then follow that with enough consecutive negative terms to make the new total less than -4. We could then add enough positive terms to make the total greater than $+5$ and follow with consecutive unused negative terms to make a new total less than -6, and so on. In this way, we could make the swings arbitrarily large in either direction.

Flowchart 8.1 Procedure for Determining Convergence



b) *Rearranging* $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ *to converge to* 1. Another possibility is to focus on a particular limit. Suppose we try to get sums that converge to 1. We start with the first term, 1/1, and then subtract 1/2. Next we add 1/3 and 1/5, which brings the total back to 1 or above. Then we add consecutive negative terms until the total is less than 1. We continue in this manner: When the sum is less than 1, add positive terms until the total is 1 or more; then subtract (add negative) terms until the total is again less than 1. This process can be continued indefinitely. Because both the odd-numbered terms and the even-numbered terms of the original series approach zero as $n \to \infty$, the amount by which our partial sums exceed 1 or fall below it approaches zero. So the new series converges to 1. The rearranged series starts like this:

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}-\frac{1}{6}+\frac{1}{11}+\frac{1}{13}-\frac{1}{8}+\frac{1}{15}+\frac{1}{17}-\frac{1}{10}$$
$$+\frac{1}{19}+\frac{1}{21}-\frac{1}{12}+\frac{1}{23}+\frac{1}{25}-\frac{1}{14}+\frac{1}{27}-\frac{1}{16}+\cdots$$

❑

Exercises 8.7

Determining Convergence or Divergence

Which of the alternating series in Exercises 1–10 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}$

5. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$

7. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln n^2}$

8. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute Convergence

Which of the series in Exercises 11–44 converge absolutely, which converge, and which diverge? Give reasons for your answers.

11. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$

12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

13. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$

14. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

15. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$

17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$

18. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$

19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

20. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln(n^3)}$

21. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$

22. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$

23. $\sum_{n=1}^{\infty}(-1)^{n}n^2(2/3)^n$

24. $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt[n]{10})$

25. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\tan^{-1}n}{n^2+1}$

26. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

27. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$

28. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n-\ln n}$

29. $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$

30. $\sum_{n=1}^{\infty}(-5)^{-n}$

31. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$

32. $\sum_{n=2}^{\infty}(-1)^{n}\left(\frac{\ln n}{\ln n^2}\right)^n$

33. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$

34. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$

35. $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$

36. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$

37. $\sum_{n=1}^{\infty}(-1)^{n}\frac{(2n)!}{2^n n!\, n}$

38. $\sum_{n=1}^{\infty}(-1)^{n}\frac{(n!)^2 3^n}{(2n+1)!}$

39. $\sum_{n=1}^{\infty}(-1)^{n}(\sqrt{n+1}-\sqrt{n})$

40. $\sum_{n=1}^{\infty}(-1)^{n}(\sqrt{n^2+n}-n)$

41. $\sum_{n=1}^{\infty}(-1)^{n}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$

43. $\sum_{n=1}^{\infty}(-1)^n \operatorname{sech} n$

44. $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n$

Error Estimation

In Exercises 45–48, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

45. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ It can be shown that the sum is ln 2.

46. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$

47. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$ As you will see in Section 8.8, the sum is ln (1.01).

48. $\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n, \quad 0<t<1$

CALCULATOR Approximate the sums in Exercises 49 and 50 with an error of magnitude less than 5×10^{-6}.

49. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$ As you will see in Section 8.10, the sum is cos 1, the cosine of 1 radian.

50. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ As you will see in Section 8.10, the sum is e^{-1}.

Theory and Examples

51. a) The series

$$\frac{1}{3}-\frac{1}{2}+\frac{1}{9}-\frac{1}{4}+\frac{1}{27}-\frac{1}{8}+\cdots+\frac{1}{3^n}-\frac{1}{2^n}+\cdots$$

does not meet one of the conditions of Theorem 8. Which one?

b) Find the sum of the series in (a).

52. CALCULATOR The limit L of an alternating series that satisfies the conditions of Theorem 8 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n+s_{n+1}}{2}=s_n+\frac{1}{2}(-1)^{n+2}a_{n+1}$$

to estimate L. Compute

$$s_{20}+\frac{1}{2}\cdot\frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.6931\ldots$.

53. *The sign of the remainder of an alternating series that satisfies the conditions of Theorem 8.* Prove the assertion in Theorem 9 that whenever an alternating series satisfying the conditions of Theorem 8 is approximated with one of its partial sums, then the remainder (sum of the unused terms) has the same sign as the first unused term. (*Hint:* Group the remainder's terms in consecutive pairs.)

54. Show that the sum of the first $2n$ terms of the series

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\cdots$$

is the same as the sum of the first n terms of the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots.$$

Do these series converge? What is the sum of the first $2n+1$ terms of the first series? If the series converge, what is their sum?

55. Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

56. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}|a_n|.$$

57. Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then so does

a) $\sum_{n=1}^{\infty}(a_n+b_n)$ **b)** $\sum_{n=1}^{\infty}(a_n-b_n)$

c) $\sum_{n=1}^{\infty} ka_n$ (k any number)

58. Show by example that $\sum_{n=1}^{\infty} a_n b_n$ may diverge even if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

59. CALCULATOR In Example 7, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new arrangement with the first negative term, which is $-1/2$. Whenever you have a sum that is less than or equal to $-1/2$, start introducing positive terms, taken in order, until the new total is greater than $-1/2$. Then add negative terms until the total is less than or equal to $-1/2$ again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If s_n is the sum of the first n terms of your new series, plot the points (n, s_n) to illustrate how the sums are behaving.

60. *Outline of the proof of the Rearrangement Theorem (Theorem 11).*

a) Let ϵ be a positive real number, let $L=\sum_{n=1}^{\infty} a_n$, and let $s_k=\sum_{n=1}^{k} a_n$. Show that for some index N_1 and for some index $N_2 \ge N_1$,

$$\sum_{n=N_1}^{\infty}|a_n| < \frac{\epsilon}{2} \quad \text{and} \quad |s_{N_2}-L| < \frac{\epsilon}{2}.$$

Since all the terms $a_1, a_2, \ldots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is an index $N_3 \ge N_2$ such that if $n \ge N_3$, then $\left(\sum_{k=1}^{n} b_k\right) - s_{N_2}$ is at most a sum of terms a_m with $m \ge N_1$. Therefore, if $n \ge N_3$,

$$\left|\sum_{k=1}^{n} b_k - L\right| \le \left|\sum_{k=1}^{n} b_k - s_{N_2}\right| + |s_{N_2}-L|$$

$$\le \sum_{k=N_1}^{\infty}|a_k| + |s_{N_2}-L| < \epsilon.$$

b) The argument in (a) shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$. Now show that because $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} |b_n|$ converges to $\sum_{n=1}^{\infty} |a_n|$.

61. *Unzipping absolutely convergent series.*

a) Show that if $\sum_{n=1}^{\infty} |a_n|$ converges and

$$b_n = \begin{cases} a_n & \text{if } a_n \ge 0 \\ 0 & \text{if } a_n < 0, \end{cases}$$

then $\sum_{n=1}^{\infty} b_n$ converges.

b) Use the results in (a) to show likewise that if $\sum_{n=1}^{\infty} |a_n|$ converges and

$$c_n = \begin{cases} 0 & \text{if } a_n \ge 0 \\ a_n & \text{if } a_n < 0, \end{cases}$$

then $\sum_{n=1}^{\infty} c_n$ converges.

In other words, if a series converges absolutely, its positive terms form a convergent series, and so do its negative terms. Furthermore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n$$

because $b_n = (a_n + |a_n|)/2$ and $c_n = (a_n - |a_n|)/2$.

62. What is wrong here:

Multiply both sides of the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

by 2 to get

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \cdots.$$

Collect terms with the same denominator, as the arrows indicate, to arrive at

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

The series on the right-hand side of this equation is the series we started with. Therefore, $2S = S$, and dividing by S gives $2 = 1$. (Source: "Riemann's Rearrangement Theorem" by Stewart Galanor, *Mathematics Teacher,* Vol. 80, No. 8, 1987, pp. 675–81.)

63. Draw a figure similar to Fig. 8.14 to illustrate the convergence of the series in Theorem 8 when $N > 1$.

8.8 Power Series

Now that we can test infinite series for convergence we can study the infinite polynomials mentioned at the beginning of Section 8.3. We call these polynomials power series because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

Power Series and Convergence

We begin with the formal definition.

Definition

A **power series about $x = 0$** is a series of the form

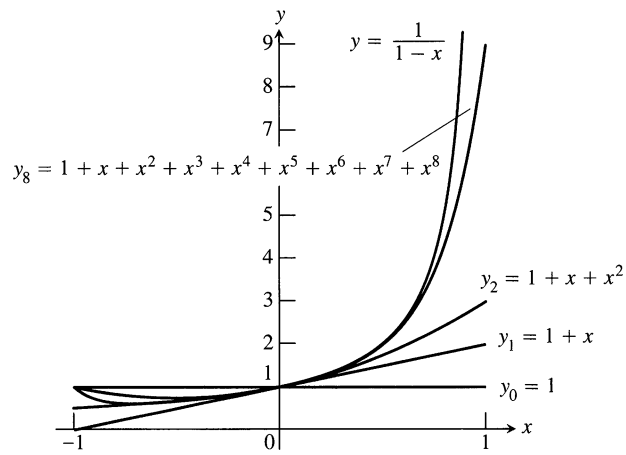
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

Equation (1) is the special case obtained by taking $a = 0$ in Eq. (2).

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.



8.15 The graphs of $f(x) = 1/(1 - x)$ and four of its polynomial approximations (Example 1).

EXAMPLE 1 Taking all the coefficients to be 1 in Eq. (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

❑

Up to now, we have used Eq. (3) as a formula for the sum of the series on the right. We now change the focus: We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need take only a few terms of the series to get a good approximation. As we move toward $x = 1$, or -1, we must take more terms. Figure 8.15 shows the graphs of $f(x) = 1/(1 - x)$, and the approximating polynomials $y_n = P_n(x)$ for $n = 0$, 1, 2, and 8.

EXAMPLE 2 The power series

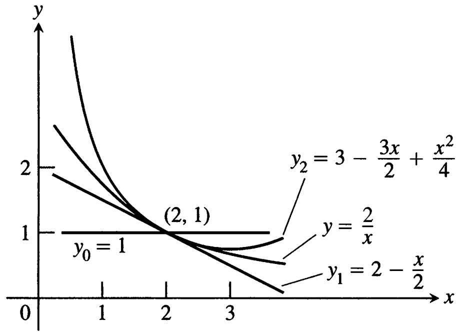
$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \tag{4}$$

matches Eq. (2) with $a = 2$, $c_0 = 1$, $c_1 = -1/2$, $c_2 = 1/4, \ldots, c_n = (-1/2)^n$. This is a geometric series with first term 1 and ratio $r = -\dfrac{x-2}{2}$. The series converges for $\left|\dfrac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \dfrac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$



8.16 **The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).**

Series (4) generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$$P_0(x) = 1$$

$$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2}$$

$$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4},$$

and so on (Fig. 8.16). ❑

EXAMPLE 3 For what values of x do the following power series converge?

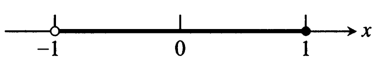
a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

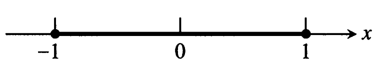
d) $\displaystyle\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the series in question.



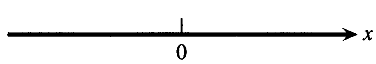
a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



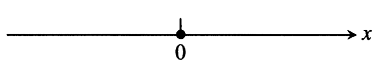
b) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



c) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ for every } x.$

The series converges absolutely for all x.



d) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty \text{ unless } x = 0.$

The series diverges for all values of x except $x = 0$. ❑

Example 3 illustrates how we usually test a power series for convergence, and the possible results.

How to Test a Power Series for Convergence

Step 1: Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a|<R \qquad \text{or} \qquad a-R<x<a+R.$$

Step 2: If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint, as in Examples 3(a) and (b). Use a Comparison Test, the Integral Test, or the Alternating Series Test.

Step 3: If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

To simplify the notation, Theorem 12 deals with the convergence of series of the form $\sum a_n x^n$. For series of the form $\sum a_n(x-a)^n$ we can replace $x-a$ by x' and apply the results to the series $\sum a_n(x')^n$.

Theorem 12
The Convergence Theorem for Power Series

If $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

converges for $x=c\neq 0$, then it converges absolutely for all $|x|<|c|$. If the series diverges for $x=d$, then it diverges for all $|x|>|d|$.

Proof Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$. Hence, there is an integer N such that $|a_n c^n|<1$ for all $n\geq N$. That is,

$$|a_n| < \frac{1}{|c|^n} \qquad \text{for } n \geq N. \tag{5}$$

Now take any x such that $|x|<|c|$ and consider

$$|a_0| + |a_1 x| + \cdots + |a_{N-1}\, x^{N-1}| + |a_N\, x^N| + |a_{N+1}\, x^{N+1}| + \cdots.$$

There are only a finite number of terms prior to $|a_N\, x^N|$, and their sum is finite. Starting with $|a_N\, x^N|$ and beyond, the terms are less than

$$\left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \cdots \tag{6}$$

because of (5). But the series in (6) is a geometric series with ratio $r=|x/c|$, which is less than 1, since $|x|<|c|$. Hence the series (6) converges, so the original series converges absolutely. This proves the first half of the theorem.

The second half of the theorem follows from the first. If the series diverges at $x=d$ and converges at a value x_0 with $|x_0|>|d|$, we may take $c=x_0$ in the first half of the theorem and conclude that the series converges absolutely at d. But the series cannot converge absolutely and diverge at one and the same time. Hence, if it diverges at d, it diverges for all $|x|>|d|$. ❑

The Radius and Interval of Convergence

The examples we have looked at, and the theorem we just proved, lead to the conclusion that a power series behaves in one of the following three ways.

Possible Behavior of $\sum c_n(x-a)^n$

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges absolutely for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

In case 1, the set of points at which the series converges is a finite interval, called the **interval of convergence.** We know from the examples that the interval can be open, half-open, or closed, depending on the particular series. But no matter which kind of interval it is, R is called the **radius of convergence** of the series, and $a + R$ is the least upper bound of the set of points at which the series converges. The convergence is absolute at every point in the interior of the interval. If a power series converges absolutely for all values of x, we say that its **radius of convergence is infinite.** If it converges only at $x = a$, the **radius of convergence is zero.**

Term-by-Term Differentiation

A theorem from advanced calculus says that a power series can be differentiated term by term at each interior point of its interval of convergence.

A word of caution

Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!\,x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n!\,\cos(n!\,x)}{n^2},$$

which diverges for all x.

Theorem 13

The Term-by-Term Differentiation Theorem

If $\sum c_n(x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a - R < x < a + R.$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$
$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$$

Solution

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$
$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1$$
$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$
$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1$$

❑

Term-by-Term Integration

Another advanced theorem states that a power series can be integrated term by term throughout its interval of convergence.

Theorem 14
The Term-by-Term Integration Theorem
Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R \ (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n (x-a)^{n+1}/(n+1)$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 *A series for* $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Notice that the original series in Example 5 converges at both endpoints of the original interval of convergence, but Theorem 13 can guarantee the convergence of the differentiated series only inside the interval.

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \tag{7}$$

In Section 8.11, we will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$. ❑

EXAMPLE 6 *A series for* $\ln(1+x)$, $-1 < x \le 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.\end{aligned}$$

It can also be shown that the series converges at $x = 1$ to the number ln 2, but that was not guaranteed by the theorem. ❑

Technology *Study of Series* Series are in many ways analogous to integrals. Just as the number of functions with explicit antiderivatives in terms of elementary functions is small compared to the number of integrable functions, the number of power series in x that agree with explicit elementary functions on x-intervals is small compared to the number of power series that converge on some x-interval. Graphing utilities can aid in the study of such series in much the same way that numerical integration aids in the study of definite integrals. The ability to study power series at particular values of x is built into most Computer Algebra Systems.

If a series converges rapidly enough, CAS exploration might give us an idea of the sum. For instance, in calculating the early partial sums of the series $\sum_{n=1}^{\infty}[1/(2^{n-1})]$ (Section 8.5, Example 3b), Maple returns $S_n = 1.6066\,95152$ for $31 \le n \le 200$. This suggests that the sum of the series is 1.6066 95152 to 10 digits. Indeed,

$$\sum_{n=201}^{\infty} \frac{1}{2^n - 1} = \sum_{n=201}^{\infty} \frac{1}{2^{n-1}(2 - (1/2^{n-1}))} < \sum_{n=201}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^{199}} < 1.25 \times 10^{-60}.$$

The remainder after 200 terms is negligible.

However, CAS and calculator exploration cannot do much for us if the series converges or diverges very slowly, and indeed can be downright misleading. For example, try calculating the partial sums of the series $\sum_{n=1}^{\infty}[1/(10^{10}n)]$. The terms are tiny in comparison to the numbers we normally work with and the partial sums, even for hundreds of terms, are miniscule. We might well be fooled into thinking that the series converges. In fact, it diverges, as we can see by writing it as $(1/10^{10})\sum_{n=1}^{\infty}(1/n)$.

We will know better how to interpret numerical results after studying error estimates in Section 8.10.

Multiplication of Power Series

Still another advanced theorem states that absolutely converging power series can be multiplied the way we multiply polynomials.

Theorem 15
The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0\, b_n + a_1\, b_{n-1} + a_2\, b_{n-2} + \cdots + a_{n-1}\, b_1 + a_n\, b_0 = \sum_{k=0}^{n} a_k\, b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

EXAMPLE 7 Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

by itself to get a power series for $1/(1-x)^2$, for $|x| < 1$.

Solution Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0}_{n+1 \text{ terms}}$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ ones}} = n + 1.$$

Then, by the Series Multiplication Theorem,

$$A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

is the series for $1/(1-x)^2$. The series all converge absolutely for $|x| < 1$.

Notice that Example 4 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

❑

Exercises 8.8

Intervals of Convergence

In Exercises 1–32, (a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

1. $\sum_{n=0}^{\infty} x^n$

2. $\sum_{n=0}^{\infty} (x+5)^n$

3. $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

4. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

6. $\sum_{n=0}^{\infty} (2x)^n$

7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\, 3^n}$

10. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

12. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

13. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

14. $\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

16. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}$

17. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

18. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n (n^2+1)}$

19. $\sum_{n=0}^{\infty} \frac{\sqrt{n}\, x^n}{3^n}$

20. $\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$

21. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

22. $\sum_{n=1}^{\infty} (\ln n)\, x^n$

23. $\sum_{n=1}^{\infty} n^n x^n$

24. $\sum_{n=0}^{\infty} n!(x-4)^n$

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$

26. $\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$

27. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ (Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 8.4, Exercise 39.)

28. $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$ (Get the information you need about $\sum 1/(n \ln n)$ from Section 8.4, Exercise 38.)

29. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$

30. $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

31. $\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}}$

32. $\sum_{n=0}^{\infty} \frac{(x-\sqrt{2})^{2n+1}}{2^n}$

In Exercises 33–38, find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

33. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

34. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

35. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2}-1\right)^n$

36. $\sum_{n=0}^{\infty} (\ln x)^n$

37. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$

38. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

Theory and Examples

39. For what values of x does the series

$$1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n (x-3)^n+\cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

40. If you integrate the series in Exercise 39 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

41. The series

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$$

converges to $\sin x$ for all x.

a) Find the first six terms of a series for $\cos x$. For what values of x should the series converge?
b) By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.
c) Using the result in (a) and series multiplication, calculate the first six terms of a series for $2\sin x \cos x$. Compare your answer with the answer in (b).

42. The series

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

converges to e^x for all x.

a) Find a series for $(d/dx)e^x$. Do you get the series for e^x? Explain your answer.
b) Find a series for $\int e^x dx$. Do you get the series for e^x? Explain your answer.
c) Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

43. The series

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

a) Find the first five terms of the series for $\ln|\sec x|$. For what values of x should the series converge?

b) Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?

c) Check your result in (b) by squaring the series given for $\sec x$ in Exercise 44.

44. The series for

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

a) Find the first five terms of a power series for the function $\ln|\sec x + \tan x|$. For what values of x should the series converge?

b) Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?

c) Check your result in (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 43.

45. *Uniqueness of convergent power series*

a) Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n. (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)

b) Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n.

46. *The sum of the series* $\sum_{n=0}^{\infty} (n^2/2^n)$. To find the sum of this series, express $1/(1-x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to 1/2. What do you get? (Source: David E. Dobbs' letter to the editor, *Illinois Mathematics Teacher,* Vol. 33, Issue 4, 1982, p. 27.)

47. *Convergence at endpoints.* Show by examples that the convergence of a power series at an endpoint of its interval of convergence may be either conditional or absolute.

48. Make up a power series whose interval of convergence is

a) $(-3, 3)$ b) $(-2, 0)$ c) $(1, 5)$.

8.9

Taylor and Maclaurin Series

This section shows how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions.

Series Representations

We know that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval I, can it be expressed as a power series on I? And if it can, what will its coefficients be?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots, \end{aligned}$$

with the nth derivative, for all n, being

$$f^{(n)}(x) = n!\,a_n + \text{ a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$\begin{aligned} f'(a) &= a_1, \\ f''(a) &= 1 \cdot 2a_2, \\ f'''(a) &= 1 \cdot 2 \cdot 3a_3, \end{aligned}$$

and, in general,

$$f^{(n)}(a) = n!\,a_n.$$

These formulas reveal a marvelous pattern in the coefficients of any power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ that converges to the values of f on I ("represents f on I," we say). If there *is* such a series (still an open question), then there is only one such series and its nth coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation, then the series must be

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \end{aligned} \tag{1}$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval I centered at $x = a$ and use it to generate the series in Eq. (1), will the series then converge to $f(x)$ at each x in the interior of I? The answer is maybe—for some functions it will but for other functions it will not, as we will see.

Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \end{aligned}$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2)$, $f'(2)$, $f''(2)$, ... Taking derivatives we get

$$\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2},\\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2},\\
f''(x) &= 2!\,x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3},\\
f'''(x) &= -3!\,x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4},\\
&\vdots & &\vdots\\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}$$

The Taylor series is

$$\begin{aligned}
f(2) &+ f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}}{n!}(x-2)^n + \cdots\\
&= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.
\end{aligned}$$

This is a geometric series with first term 1/2 and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1+(x-2)/2} = \frac{1}{2+(x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ❑

Taylor Polynomials

The linearization of a differentiable function f at a point a is the polynomial

$$P_1(x) = f(a) + f'(a)(x-a).$$

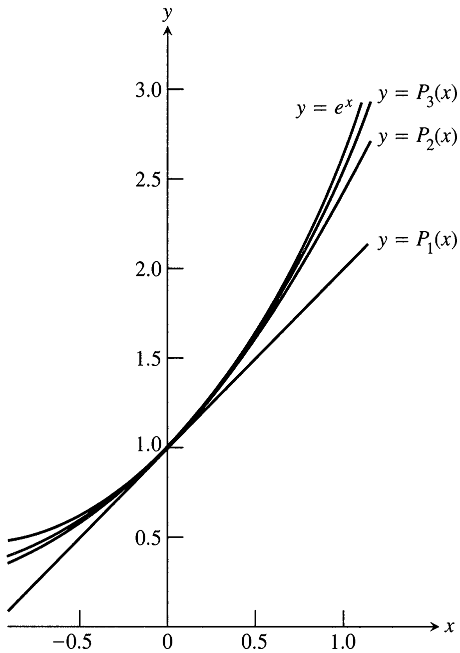
If f has derivatives of higher order at a, then it has higher order polynomial approximations as well, one for each available derivative. These polynomials are called the Taylor polynomials of f.

We speak of a Taylor polynomial of *order n* rather than *degree n* because $f^{(n)}(a)$ may be zero. The first two Taylor polynomials of $\cos x$ at $x = 0$, for example, are $P_0(x) = 1$ and $P_1(x) = 1$. The first order polynomial has degree zero, not one.

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order *n*** generated by f at $x = a$ is the polynomial

$$\begin{aligned}
P_n(x) = f(a) &+ f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\\
&+ \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\end{aligned}$$



8.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x,$

$P_2(x) = 1 + x + (x^2/2!)$, and

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Notice the very close agreement near the center $x = 0$.

Just as the linearization of f at $x = a$ provides the best linear approximation of f in the neighborhood of a, the higher order Taylor polynomials provide the best polynomial approximations of their respective degrees. (See Exercise 32.)

EXAMPLE 2 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since

$$f(x) = e^x, \qquad f'(x) = e^x, \qquad \ldots, \qquad f^{(n)}(x) = e^x, \qquad \ldots,$$

we have

$$f(0) = e^0 = 1, \qquad f'(0) = 1, \qquad \ldots, \qquad f^{(n)}(0) = 1, \qquad \ldots.$$

The Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

By definition, this is also the Maclaurin series for e^x. In Section 8.10 we will see that the series converges to e^x at every x.

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

See Fig. 8.17. ❑

EXAMPLE 3 Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

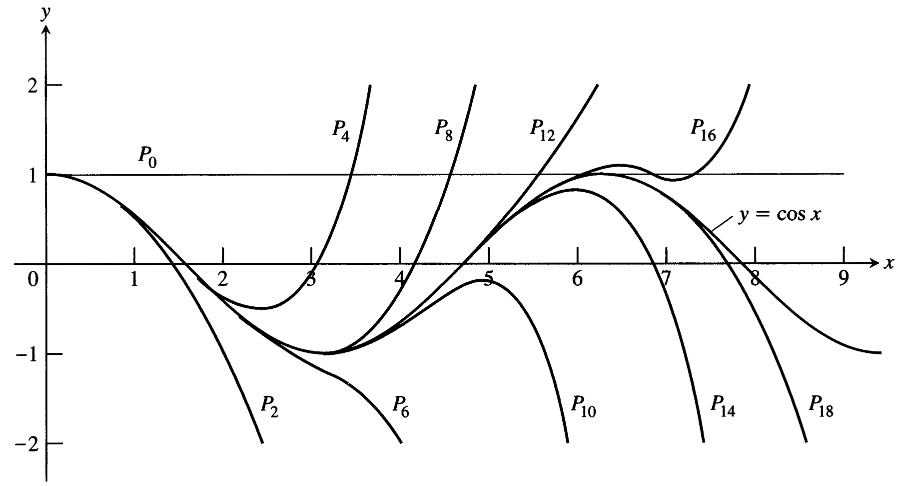
$$\begin{aligned} f(x) &= \cos x & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \end{aligned}$$



8.18 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} [(-1)^k x^{2k}/(2k)!]$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$.

By definition, this is also the Maclaurin series for $\cos x$. In Section 8.10, we will see that the series converges to $\cos x$ at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n+1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 8.18 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis. ❑

Infinitely differentiable functions that are represented by their Taylor series only at isolated points are, in practice, very rare.

Who invented Taylor series?

Brook Taylor (1685–1731) did not invent Taylor series, and Maclaurin series were not developed by Colin Maclaurin (1698–1746). James Gregory was already working with Taylor series when Taylor was only a few years old, and he published the Maclaurin series for $\tan x$, $\sec x$, $\tan^{-1} x$, and $\sec^{-1} x$ ten years before Maclaurin was born. Nicolaus Mercator discovered the Maclaurin series for $\ln(1 + x)$ at about the same time.

Taylor was unaware of Gregory's work when he published his book *Methodus incrementorum directa et inversa* in 1715, containing what we now call Taylor series. Maclaurin quoted Taylor's work in a calculus book he wrote in 1742. The book popularized series representations of functions and although Maclaurin never claimed to have discovered them, Taylor series centered at $x = 0$ became known as Maclaurin series. History evened things up in the end. Maclaurin, a brilliant mathematician, was the original discoverer of the rule for solving systems of equations that we call Cramer's rule.

EXAMPLE 4 *A function f whose Taylor series converges at every x but converges to f(x) only at x = 0*

It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

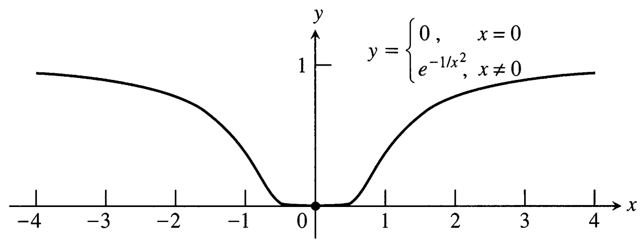
(Fig. 8.19) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots \\ &= 0 + 0 + \cdots + 0 + \cdots. \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. ❑

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?



8.19 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4).

2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.

Exercises 8.9

Finding Taylor Polynomials

In Exercises 1–8, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

1. $f(x) = \ln x, \quad a = 1$

2. $f(x) = \ln(1 + x), \quad a = 0$

3. $f(x) = 1/x, \quad a = 2$

4. $f(x) = 1/(x + 2), \quad a = 0$

5. $f(x) = \sin x, \quad a = \pi/4$

6. $f(x) = \cos x, \quad a = \pi/4$

7. $f(x) = \sqrt{x}, \quad a = 4$

8. $f(x) = \sqrt{x + 4}, \quad a = 0$

Finding Maclaurin Series

Find the Maclaurin series for the functions in Exercises 9–20.

9. e^{-x}

10. $e^{x/2}$

11. $\dfrac{1}{1 + x}$

12. $\dfrac{1}{1 - x}$

13. $\sin 3x$

14. $\sin \dfrac{x}{2}$

15. $7 \cos(-x)$

16. $5 \cos \pi x$

17. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

18. $\sinh x = \dfrac{e^x - e^{-x}}{2}$

19. $x^4 - 2x^3 - 5x + 4$

20. $(x + 1)^2$

Finding Taylor Series

In Exercises 21–28, find the Taylor series generated by f at $x = a$.

21. $f(x) = x^3 - 2x + 4, \quad a = 2$

22. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

23. $f(x) = x^4 + x^2 + 1, \quad a = -2$

24. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

25. $f(x) = 1/x^2, \quad a = 1$

26. $f(x) = x/(1 - x), \quad a = 0$

27. $f(x) = e^x, \quad a = 2$

28. $f(x) = 2^x, \quad a = 1$

Theory and Examples

29. Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x - a) + \frac{(x - a)^2}{2!} + \cdots \right].$$

30. *(Continuation of Exercise 29.)* Find the Taylor series generated by e^x at $x = 1$. Compare your answer with the formula in Exercise 29.

31. Let $f(x)$ have derivatives through order n at $x = a$. Show that the Taylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = a$.

32. *Of all polynomials of degree $\leq n$, the Taylor polynomial of order n gives the best approximation.* Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$ is a polynomial of degree n with constant coefficients $b_0, \cdots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

a) $E(a) = 0$ The approximation error is zero at $x = a$.

b) $\lim_{x \to a} \dfrac{E(x)}{(x-a)^n} = 0,$ The error is negligible when compared to $(x-a)^n$.

then

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x-a)^n$.

Quadratic Approximations

The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the **quadratic approximation** of f at $x = a$. In Exercises 33–38, find the (a) linearization (Taylor polynomial of order 1) and (b) quadratic approximation of f at $x = 0$.

33. $f(x) = \ln(\cos x)$
34. $f(x) = e^{\sin x}$
35. $f(x) = 1/\sqrt{1-x^2}$
36. $f(x) = \cosh x$
37. $f(x) = \sin x$
38. $f(x) = \tan x$

8.10 Convergence of Taylor Series; Error Estimates

This section addresses the two questions left unanswered by Section 8.9:

1. When does a Taylor series converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

Taylor's Theorem

We answer these questions with the following theorem.

Theorem 16

Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ or on $[b, a]$, and $f^{(n)}$ is differentiable on (a, b) or on (b, a), then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's theorem is a generalization of the Mean Value Theorem (Exercise 39). There is a proof of Taylor's theorem at the end of this section.

When we apply Taylor's theorem, we usually want to hold a fixed and treat b as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x. Here is how the theorem reads with this change.

Corollary to Taylor's Theorem
Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

When we state Taylor's theorem this way, it says that for each x in I,

$$f(x) = P_n(x) + R_n(x).$$

Pause for a moment to think about how remarkable this equation is. For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I.

Equation (1) is called **Taylor's formula.** The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

EXAMPLE 1 *The Maclaurin series for* e^x

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

Polynomial from Section 8.9, Example 2

and

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$. Thus,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \le 0,$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \qquad \text{when } x > 0.$$

Finally, because

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \qquad \text{for every } x,$$

Section 8.2

$\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x.

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots.}$$

❑

Estimating the Remainder

It is often possible to estimate $R_n(x)$ as we did in Example 1. This method of estimation is so convenient that we state it as a theorem for future reference.

Theorem 17

The Remainder Estimation Theorem

If there are positive constants M and r such that $|f^{(n+1)}(t)| \le Mr^{n+1}$ for all t between a and x, inclusive, then the remainder term $R_n(x)$ in Taylor's theorem satisfies the inequality

$$|R_n(x)| \le M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

If these conditions hold for every n and all the other conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

In the simplest examples, we can take $r = 1$ provided f and all its derivatives are bounded in magnitude by some constant M. In other cases, we may need to consider r. For example, if $f(x) = 2\cos(3x)$, each time we differentiate we get a factor of 3 and r needs to be greater than 1. In this particular case, we can take $r = 3$ along with $M = 2$.

We are now ready to look at some examples of how the Remainder Estimation Theorem and Taylor's theorem can be used together to settle questions of convergence. As you will see, they can also be used to determine the accuracy with which a function is approximated by one of its Taylor polynomials.

EXAMPLE 2 *The Maclaurin series for* sin *x*

Show that the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

Solution The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \qquad \text{and} \qquad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ and $r = 1$ to obtain

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $(|x|^{2k+2}/(2k+2)!) \to 0$ as $k \to \infty$, whatever the value of x, $R_{2k+1}(x) \to 0$, and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{3}$$

❑

EXAMPLE 3 *The Maclaurin series for* cos *x*

Show that the Maclaurin series for $\cos x$ converges to $\cos x$ for every value of x.

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 8.9, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ and $r = 1$ gives

$$|R_{2k}(x)| \le 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x.

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{4}$$

❑

EXAMPLE 4 *Finding a Maclaurin series by substitution*

Find the Maclaurin series for $\cos 2x$.

Solution We can find the Maclaurin series for $\cos 2x$ by substituting $2x$ for x in the Maclaurin series for $\cos x$:

$$\cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \qquad \text{Eq. (4) with } 2x \text{ for } x$$

$$= 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}x^{2k}}{(2k)!}.$$

Eq. (4) holds for $-\infty < x < \infty$, implying that it holds for $-\infty < 2x < \infty$, so the newly created series converges for all x. Exercise 45 explains why the series is in fact the Maclaurin series for $\cos 2x$. ❑

EXAMPLE 5 *Finding a Maclaurin series by multiplication*

Find the Maclaurin series for $x \sin x$.

Solution We can find the Maclaurin series for $x \sin x$ by multiplying the Maclaurin series for $\sin x$ (Eq. 3) by x:

$$x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots.$$

The new series converges for all x because the series for $\sin x$ converges for all x. Exercise 45 explains why the series is the Maclaurin series for $x \sin x$. ❑

Truncation Error

The Maclaurin series for e^x converges to e^x for all x. But we still need to decide how many terms to use to approximate e^x to a given degree of accuracy. We get this information from the Remainder Estimation Theorem.

EXAMPLE 6 Calculate e with an error of less than 10^{-6}.

Solution We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n+1)!} \qquad \text{for some } c \text{ between 0 and 1.}$$

For the purposes of this example, we assume that we know that $e < 3$. Hence, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By experiment we find that $1/9! > 10^{-6}$, while $3/10! < 10^{-6}$. Thus we should take $(n+1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.7182\ 82.$$

❑

EXAMPLE 7 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution Here we can take advantage of the fact that the Maclaurin series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem (Section 8.7), the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than

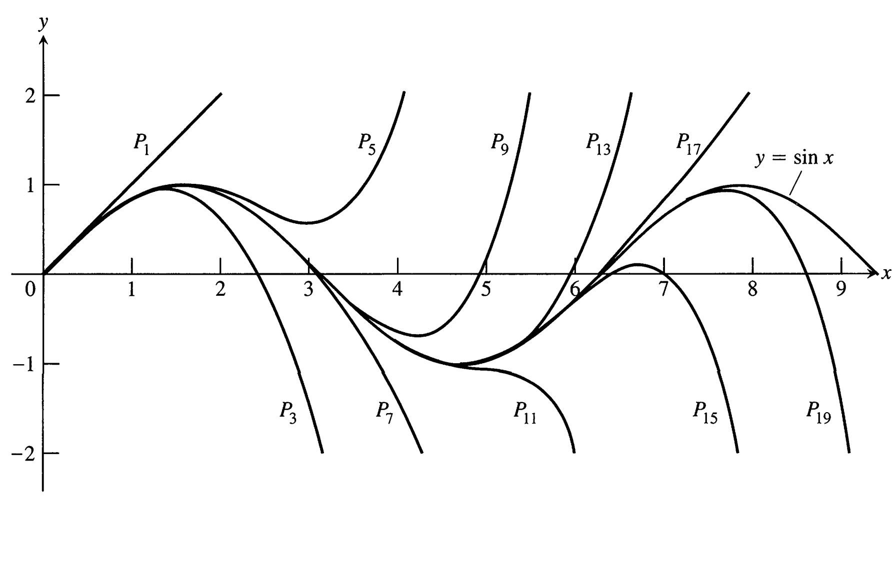
$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \qquad \text{or} \qquad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$ Rounded down, to be safe

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive because then $x^5/120$ is positive.

Figure 8.20 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $-1 \le x \le 1$.



8.20 The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \to \infty$.

You might wonder how the estimate given by the Remainder Estimation Theorem compares with the one just obtained from the Alternating Series Estimation Theorem. If we write

$$\sin x = x - \frac{x^3}{3!} + R_3,$$

then the Remainder Estimation Theorem gives

$$|R_3| \le 1 \cdot \frac{|x|^4}{4!} = \frac{|x|^4}{24},$$

which is not as good. But if we recognize that $x - (x^3/3!) = 0 + x + 0x^2 - (x^3/3!) + 0x^4$ is the Taylor polynomial of order 4 as well as of order 3, then

$$\sin x = x - \frac{x^3}{3!} + 0 + R_4,$$

and the Remainder Estimation Theorem with $M = r = 1$ gives

$$|R_4| \le 1 \cdot \frac{|x|^5}{5!} = \frac{|x|^5}{120}.$$

This is what we had from the Alternating Series Estimation Theorem. ❑

Combining Taylor Series

On the intersection of their intervals of convergence, Taylor series can be added, subtracted, and multiplied by constants, and the results are once again Taylor series. The Taylor series for $f(x) + g(x)$ is the sum of the Taylor series for $f(x)$ and $g(x)$ because the nth derivative of $f + g$ is $f^{(n)} + g^{(n)}$, and so on. Thus we obtain the Maclaurin series for $(1 + \cos 2x)/2$ by adding 1 to the Maclaurin series for $\cos 2x$ and dividing the combined results by 2, and the Maclaurin series for $\sin x + \cos x$ is the term-by-term sum of the Maclaurin series for $\sin x$ and $\cos x$.

✲ Euler's Formula

As you may recall, a complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. If we substitute $x = i\theta$ (θ real) in the Maclaurin series for e^x and use the relations

$$i^2 = -1, \qquad i^3 = i^2 i = -i, \qquad i^4 = i^2 i^2 = 1, \qquad i^5 = i^4 i = i,$$

and so on, to simplify the result, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

This does not *prove* that $e^{i\theta} = \cos\theta + i\sin\theta$ because we have not yet defined what it means to raise e to an imaginary power. But it does say how to define $e^{i\theta}$ to be consistent with other things we know.

One of the amazing consequences of Euler's formula is the equation

$$e^{i\pi} = -1.$$

When written in the form $e^{i\pi} + 1 = 0$, this equation combines the five most important constants in mathematics.

Definition

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. **(5)**

Equation (5), called **Euler's formula,** enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any complex number $a + bi$.

A Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$. The proof for $a > b$ is nearly the same.

The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and its first n derivatives match the function f and its first n derivatives at $x = a$. We do not disturb that matching if we add another term of the form $K(x-a)^{n+1}$, where K is any constant, because such a term and its first n derivatives are all equal to zero at $x = a$. The new function

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}$$

and its first n derivatives still agree with f and its first n derivatives at $x = a$.

We now choose the particular value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b-a)^{n+1}, \qquad \text{or} \qquad K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}. \tag{6}$$

With K defined by Eq. (6), the function

$$F(x) = f(x) - \phi_n(x)$$

measures the difference between the original function f and the approximating function ϕ_n for each x in $[a, b]$.

We now use Rolle's theorem (Section 3.2). First, because $F(a) = F(b) = 0$ and both F and F' are continuous on $[a, b]$, we know that

$$F'(c_1) = 0 \qquad \text{for some } c_1 \text{ in } (a, b).$$

Next, because $F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$, we know that

$$F''(c_2) = 0 \qquad \text{for some } c_2 \text{ in } (a, c_1).$$

Rolle's theorem, applied successively to F'', F''', …, $F^{(n-1)}$ implies the existence of

$$\begin{array}{lll} c_3 & \text{in } (a, c_2) & \text{such that } F'''(c_3) = 0, \\ c_4 & \text{in } (a, c_3) & \text{such that } F^{(4)}(c_4) = 0, \\ \vdots & & \\ c_n & \text{in } (a, c_{n-1}) & \text{such that } F^{(n)}(c_n) = 0. \end{array}$$

Finally, because $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n), and $F^{(n)}(a) = F^{(n)}(c_n) = 0$, Rolle's theorem implies that there is a number c_{n+1} in (a, c_n) such that

$$F^{(n+1)}(c_{n+1}) = 0. \tag{7}$$

If we differentiate $F(x) = f(x) - P_n(x) - K(x-a)^{n+1}$ a total of $n + 1$ times, we get

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!K. \tag{8}$$

Equations (7) and (8) together give

$$K = \frac{f^{(n+1)}(c)}{(n+1)!} \quad \text{for some number } c = c_{n+1} \text{ in } (a, b). \tag{9}$$

Equations (6) and (9) give

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

This concludes the proof. ❑

Exercises 8.10

Maclaurin Series by Substitution

Use substitution (as in Example 4) to find the Maclaurin series of the functions in Exercises 1–6.

1. e^{-5x} **2.** $e^{-x/2}$ **3.** $5\sin(-x)$

4. $\sin\left(\frac{\pi x}{2}\right)$ **5.** $\cos\sqrt{x}$ **6.** $\cos(x^{3/2}/\sqrt{2})$

More Maclaurin Series

Find Maclaurin series for the functions in Exercises 7–18.

7. xe^x **8.** $x^2 \sin x$ **9.** $\frac{x^2}{2} - 1 + \cos x$

10. $\sin x - x + \frac{x^3}{3!}$ **11.** $x\cos\pi x$ **12.** $x^2\cos(x^2)$

13. $\cos^2 x$ (*Hint:* $\cos^2 x = (1+\cos 2x)/2$.)

14. $\sin^2 x$ **15.** $\frac{x^2}{1-2x}$ **16.** $x\ln(1+2x)$

17. $\frac{1}{(1-x)^2}$ **18.** $\frac{2}{(1-x)^3}$

Error Estimates

19. For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.

20. If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large, or too small? Give reasons for your answer.

21. How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$?

22. The estimate $\sqrt{1+x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.

23. The approximation $e^x = 1 + x + (x^2/2)$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

24. (*Continuation of Exercise 23.*) When $x < 0$, the series for e^x is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing e^x by $1 + x + (x^2/2)$ when $-0.1 < x < 0$. Compare your estimate with the one you obtained in Exercise 23.

25. Estimate the error in the approximation $\sinh x = x + (x^3/3!)$ when $|x| < 0.5$. (*Hint:* Use R_4, not R_3.)

26. When $0 \le h \le 0.01$, show that e^h may be replaced by $1 + h$ with an error of magnitude no greater than 0.6% of h. Use $e^{0.01} = 1.01$.

27. For what positive values of x can you replace $\ln(1+x)$ by x with an error of magnitude no greater than 1% of the value of x?

28. You plan to estimate $\pi/4$ by evaluating the Maclaurin series for $\tan^{-1} x$ at $x = 1$. Use the Alternating Series Estimation Theorem to determine how many terms of the series you would have to add to be sure the estimate is good to 2 decimal places.

29. a) Use the Maclaurin series for $\sin x$ and the Alternating Series Estimation Theorem to show that

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1, \quad x \neq 0.$$

b) GRAPHER Graph $f(x) = (\sin x)/x$ together with the functions $y = 1 - (x^2/6)$ and $y = 1$ for $-5 \le x \le 5$. Comment on the relationships among the graphs.

30. a) Use the Maclaurin series for $\cos x$ and the Alternating Series Estimation Theorem to show that

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}, \quad x \neq 0.$$

(This is the inequality in Section 1.2, Exercise 46.)

b) GRAPHER Graph $f(x) = (1 - \cos x)/x^2$ together with $y = (1/2) - (x^2/24)$ and $y = 1/2$ for $-9 \le x \le 9$. Comment on the relationships among the graphs.

Finding and Identifying Maclaurin Series

Each of the series in Exercises 31–34 is the value of the Maclaurin series of a function $f(x)$ at some point. What function and what point? What is the sum of the series?

31. $(0.1) - \dfrac{(0.1)^3}{3!} + \dfrac{(0.1)^5}{5!} - \cdots + \dfrac{(-1)^k(0.1)^{2k+1}}{(2k+1)!} + \cdots$

32. $1 - \dfrac{\pi^2}{4^2 \cdot 2!} + \dfrac{\pi^4}{4^4 \cdot 4!} - \cdots + \dfrac{(-1)^k(\pi)^{2k}}{4^{2k} \cdot (2k!)} + \cdots$

33. $\dfrac{\pi}{3} - \dfrac{\pi^3}{3^3 \cdot 3} + \dfrac{\pi^5}{3^5 \cdot 5} - \cdots + \dfrac{(-1)^k \pi^{2k+1}}{3^{2k+1}(2k+1)} + \cdots$

34. $\pi - \dfrac{\pi^2}{2} + \dfrac{\pi^3}{3} - \cdots + (-1)^{k-1}\dfrac{\pi^k}{k} + \cdots$

35. Multiply the Maclaurin series for e^x and $\sin x$ together to find the first five nonzero terms of the Maclaurin series for $e^x \sin x$.

36. Multiply the Maclaurin series for e^x and $\cos x$ together to find the first five nonzero terms of the Maclaurin series for $e^x \cos x$.

37. Use the identity $\sin^2 x = (1 - \cos 2x)/2$ to obtain the Maclaurin series for $\sin^2 x$. Then differentiate this series to obtain the Maclaurin series for $2 \sin x \cos x$. Check that this is the series for $\sin 2x$.

38. (*Continuation of Exercise 37.*) Use the identity $\cos^2 x = \cos 2x + \sin^2 x$ to obtain a power series for $\cos^2 x$.

Theory and Examples

39. *Taylor's theorem and the Mean Value Theorem.* Explain how the Mean Value Theorem (Section 3.2, Theorem 4) is a special case of Taylor's theorem.

40. *Linearizations at inflection points* (*Continuation of Section 3.7, Exercise 63*). Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = a$, then the linearization of f at $x = a$ is also the quadratic approximation of f at $x = a$. This explains why tangent lines fit so well at inflection points.

41. *The (second) second derivative test.* Use the equation

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c_2)}{2}(x - a)^2$$

to establish the following test.

Let f have continuous first and second derivatives and suppose that $f'(a) = 0$. Then

a) f has a local maximum at a if $f'' \le 0$ throughout an interval whose interior contains a;

b) f has a local minimum at a if $f'' \ge 0$ throughout an interval whose interior contains a.

42. *A cubic approximation.* Use Taylor's formula with $a = 0$ and $n = 3$ to find the standard cubic approximation of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the error in the approximation when $|x| \le 0.1$.

43. a) Use Taylor's formula with $n = 2$ to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ (k a constant).

b) If $k = 3$, for approximately what values of x in the interval $[0, 1]$ will the error in the quadratic approximation be less than $1/100$?

44. *Improving approximations to π.*

a) Let P be an approximation of π accurate to n decimals. Show that $P + \sin P$ gives an approximation correct to $3n$ decimals. (*Hint:* Let $P = \pi + x$.)

b) Try it with a calculator.

45. *The Maclaurin series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $\sum_{n=0}^{\infty} a_n x^n$.* A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $c > 0$ has a Maclaurin series that converges to the function at every point of $(-c, c)$. Show this by showing that the Maclaurin series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $\sum_{n=0}^{\infty} a_n x^n$ itself.

An immediate consequence of this is that series like

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

and

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots,$$

obtained by multiplying Maclaurin series by powers of x, as well as series obtained by integration and differentiation of convergent power series, are themselves the Maclaurin series generated by the functions they represent.

46. *Maclaurin series for even functions and odd functions* (*Continuation of Section 8.8, Exercise 45*). Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-c, c)$. Show that

a) If f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the series for f contains only even powers of x.

b) If f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the series for f contains only odd powers of x.

47. *Taylor polynomials of periodic functions*

a) Show that every continuous periodic function $f(x)$, $-\infty < x < \infty$, is bounded in magnitude by showing that there exists a positive constant M such that $|f(x)| \le M$ for all x.

b) Show that the graph of every Taylor polynomial of positive degree generated by $f(x) = \cos x$ must eventually move away from the graph of $\cos x$ as $|x|$ increases. You can see this in Fig. 8.18. The Taylor polynomials of $\sin x$ behave in a similar way (Fig. 8.20).

48. GRAPHER

a) Graph the curves $y = (1/3) - (x^2)/5$ and $y = (x - \tan^{-1} x)/x^3$ together with the line $y = 1/3$.

b) Use a Maclaurin series to explain what you see. What is

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} \,?$$

Euler's Formula

49. Use Eq. (5) to write the following powers of e in the form $a + bi$.

a) $e^{-i\pi}$ **b)** $e^{i\pi/4}$ **c)** $e^{-i\pi/2}$

50. *Euler's identities.* Use Eq. (5) to show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

51. Establish the equations in Exercise 50 by combining the formal Maclaurin series for $e^{i\theta}$ and $e^{-i\theta}$.

52. Show that

a) $\cosh i\theta = \cos\theta$, **b)** $\sinh i\theta = i\sin\theta$.

53. By multiplying the Maclaurin series for e^x and $\sin x$, find the terms through x^5 of the Maclaurin series for $e^x \sin x$. This series is the imaginary part of the series for

$$e^x \cdot e^{ix} = e^{(1+i)x}.$$

Use this fact to check your answer. For what values of x should the series for $e^x \sin x$ converge?

54. When a and b are real, we define $e^{(a+ib)x}$ with the equation

$$e^{(a+ib)x} = e^{ax} \cdot e^{ibx} = e^{ax}(\cos bx + i\sin bx).$$

Differentiate the right-hand side of this equation to show that

$$\frac{d}{dx}e^{(a+ib)x} = (a+ib)e^{(a+ib)x}.$$

Thus the familiar rule $(d/dx)e^{kx} = ke^{kx}$ holds for k complex as well as real.

55. Use the definition of $e^{i\theta}$ to show that for any real numbers θ, θ_1, and θ_2,

a) $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$,
b) $e^{-i\theta} = 1/e^{i\theta}$.

56. Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. Use this fact to evaluate

$$\int e^{ax}\cos bx\,dx \quad \text{and} \quad \int e^{ax}\sin bx\,dx$$

from

$$\int e^{(a+ib)x}\,dx = \frac{a-ib}{a^2+b^2}e^{(a+ib)x} + C,$$

where $C = C_1 + iC_2$ is a complex constant of integration.

CAS Explorations and Projects—Linear, Quadratic, and Cubic Approximations

Taylor's formula with $n = 1$ and $a = 0$ gives the linearization of a function at $x = 0$. With $n = 2$ and $n = 3$ we obtain the standard quadratic and cubic approximations. In these exercises we explore the errors associated with these approximations. We seek answers to two questions:

a) For what values of x can the function be replaced by each approximation with an error less than 10^{-2}?
b) What is the maximum error we could expect if we replace the function by each approximation over the specified interval?

Using a CAS, perform the following steps to aid in answering questions (a) and (b) for the functions and intervals in Exercises 57–62.

Step 1: Plot the function over the specified interval.

Step 2: Find the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ at $x = 0$.

Step 3: Calculate the $(n+1)$st derivative $f^{(n+1)}(c)$ associated with the remainder term for each Taylor polynomial. Plot the derivative as a function of c over the specified interval and estimate its maximum absolute value, M.

Step 4: Calculate the remainder $R_n(x)$ for each polynomial. Using the estimate M from step 3 in place of $f^{(n+1)}(c)$, plot $R_n(x)$ over the specified interval. Then estimate the values of x that answer question (a).

Step 5: Compare your estimated error with the actual error $E_n(x) = |f(x) - P_n(x)|$ by plotting $E_n(x)$ over the specified interval. This will help answer question (b).

Step 6: Graph the function and its three Taylor approximations together. Discuss the graphs in relation to the information discovered in steps 4 and 5.

57. $f(x) = \dfrac{1}{\sqrt{1+x}}$, $|x| \le \dfrac{3}{4}$

58. $f(x) = (1+x)^{3/2}$, $-\dfrac{1}{2} \le x \le 2$

59. $f(x) = \dfrac{x}{x^2+1}$, $|x| \le 2$

60. $f(x) = (\cos x)(\sin 2x)$, $|x| \le 2$

61. $f(x) = e^{-x}\cos 2x$, $|x| \le 1$

62. $f(x) = e^{x/3}\sin 2x$, $|x| \le 2$

8.11 Applications of Power Series

This section introduces the binomial series for estimating powers and roots and shows how series are sometimes used to approximate the solution of an initial value problem, to evaluate nonelementary integrals, and to evaluate limits that lead to indeterminate forms. We provide a self-contained derivation of the Maclaurin series for $\tan^{-1} x$ and conclude with a reference table of frequently used series.

The Binomial Series for Powers and Roots

The Maclaurin series generated by $f(x) = (1+x)^m$, when m is constant, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots. \quad (1)$$

This series, called the **binomial series,** converges absolutely for $|x| < 1$. To derive the series, we first list the function and its derivatives:

$$\begin{aligned} f(x) &= (1+x)^m \\ f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} \\ f'''(x) &= m(m-1)(m-2)(1+x)^{m-3} \\ &\vdots \\ f^{(k)}(x) &= m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}. \end{aligned}$$

We then evaluate these at $x = 0$ and substitute into the Maclaurin series formula to obtain the series in (1).

If m is an integer greater than or equal to zero, the series stops after $(m+1)$ terms because the coefficients from $k = m+1$ on are zero.

If m is not a positive integer or zero, the series is infinite and converges for $|x| < 1$. To see why, let u_k be the term involving x^k. Then apply the Ratio Test for absolute convergence to see that

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \to |x| \qquad \text{as } k \to \infty.$$

Our derivation of the binomial series shows only that it is generated by $(1+x)^m$ and converges for $|x| < 1$. The derivation does not show that the series converges to $(1+x)^m$. It does, but we assume that part without proof.

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad (2)$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

EXAMPLE 1 If $m = -1$,

$$\binom{-1}{1} = -1, \quad \binom{-1}{2} = \frac{-1(-2)}{2!} = 1,$$

and

$$\binom{-1}{k} = \frac{-1(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k \left(\frac{k!}{k!}\right) = (-1)^k.$$

With these coefficient values, Eq. (2) becomes the geometric series

$$(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots. \quad \square$$

EXAMPLE 2 We know from Section 3.7, Example 1, that $\sqrt{1-x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, i.e., } |x| \text{ large.} \quad \square$$

Power Series Solutions of Differential Equations and Initial Value Problems

When we cannot find a relatively simple expression for the solution of an initial value problem or differential equation, we try to get information about the solution in other ways. One way is to try to find a power series representation for the solution. If we can do so, we immediately have a source of polynomial approximations of the solution, which may be all that we really need. The first example (Example 3) deals with a first order linear differential equation that could be solved with the methods of Section 6.11. The example shows how, not knowing this, we can solve the equation with power series. The second example (Example 4) deals with an equation that cannot be solved by previous methods.

EXAMPLE 3 Solve the initial value problem

$$y' - y = x, \qquad y(0) = 1.$$

Solution We assume that there is a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n + \cdots. \tag{3}$$

Our goal is to find values for the coefficients a_k that make the series and its first derivative

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots \tag{4}$$

satisfy the given differential equation and initial condition. The series $y' - y$ is the difference of the series in Eqs. (3) and (4):

$$\begin{aligned} y' - y = {} & (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots \\ & + (na_n - a_{n-1})x^{n-1} + \cdots. \end{aligned} \tag{5}$$

If y is to satisfy the equation $y' - y = x$, the series in (5) must equal x. Since power series representations are unique, as you saw if you did Exercise 45 in Section 8.8, the coefficients in Eq. (5) must satisfy the equations

$$\begin{aligned} a_1 - a_0 &= 0 && \text{Constant terms} \\ 2a_2 - a_1 &= 1 && \text{Coefficients of } x \\ 3a_3 - a_2 &= 0 && \text{Coefficients of } x^2 \\ &\vdots && \quad\vdots \\ na_n - a_{n-1} &= 0 && \text{Coefficients of } x^{n-1} \\ &\vdots && \quad\vdots \end{aligned}$$

We can also see from Eq. (3) that $y = a_0$ when $x = 0$, so that $a_0 = 1$ (this being the initial condition). Putting it all together, we have

$$a_0 = 1, \qquad a_1 = a_0 = 1, \qquad a_2 = \frac{1 + a_1}{2} = \frac{1+1}{2} = \frac{2}{2},$$

$$a_3 = \frac{a_2}{3} = \frac{2}{3 \cdot 2} = \frac{2}{3!}, \qquad \ldots, \qquad a_n = \frac{a_{n-1}}{n} = \frac{2}{n!}, \qquad \ldots$$

Substituting these coefficient values into the equation for y (Eq. 3) gives

$$\begin{aligned} y &= 1 + x + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} + \cdots + 2 \cdot \frac{x^n}{n!} + \cdots \\ &= 1 + x + 2\underbrace{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)}_{\text{the Maclaurin series for } e^x - 1 - x} \\ &= 1 + x + 2(e^x - 1 - x) = 2e^x - 1 - x. \end{aligned}$$

The solution of the initial value problem is $y = 2e^x - 1 - x$.

As a check, we see that

$$y(0) = 2e^0 - 1 - 0 = 2 - 1 = 1$$

and

$$y' - y = (2e^x - 1) - (2e^x - 1 - x) = x. \qquad \square$$

EXAMPLE 4 Find a power series solution for

$$y'' + x^2 y = 0. \tag{6}$$

Solution We assume that there is a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{7}$$

and find what the coefficients a_k have to be to make the series and its second derivative

$$y'' = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots \tag{8}$$

satisfy Eq. (6). The series for $x^2 y$ is x^2 times the right-hand side of Eq. (7):

$$x^2 y = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots + a_n x^{n+2} + \cdots. \tag{9}$$

The series for $y'' + x^2 y$ is the sum of the series in Eqs. (8) and (9):

$$\begin{aligned} y'' + x^2 y &= 2a_2 + 6a_3 x + (12a_4 + a_0)x^2 + (20a_5 + a_1)x^3 \\ &\quad + \cdots + (n(n-1)a_n + a_{n-4})x^{n-2} + \cdots. \end{aligned} \tag{10}$$

Notice that the coefficient of x^{n-2} in Eq. (9) is a_{n-4}. If y and its second derivative y'' are to satisfy Eq. (6), the coefficients of the individual powers of x on the right-hand side of Eq. (10) must all be zero:

$$2a_2 = 0, \qquad 6a_3 = 0, \qquad 12a_4 + a_0 = 0, \qquad 20a_5 + a_1 = 0, \tag{11}$$

and for all $n \geq 4$,

$$n(n-1)a_n + a_{n-4} = 0. \tag{12}$$

We can see from Eq. (7) that

$$a_0 = y(0), \qquad a_1 = y'(0).$$

In other words, the first two coefficients of the series are the values of y and y' at $x = 0$. The equations in (11) and the recursion formula in (12) enable us to evaluate all the other coefficients in terms of a_0 and a_1.

The first two of Eqs. (11) give

$$a_2 = 0, \qquad a_3 = 0.$$

Equation (12) shows that if $a_{n-4} = 0$, then $a_n = 0$; so we conclude that

$$a_6 = 0, \qquad a_7 = 0, \qquad a_{10} = 0, \qquad a_{11} = 0,$$

and whenever $n = 4k + 2$ or $4k + 3$, a_n is zero. For the other coefficients we have

$$a_n = \frac{-a_{n-4}}{n(n-1)}$$

so that

$$a_4 = \frac{-a_0}{4 \cdot 3}, \qquad a_8 = \frac{-a_4}{8 \cdot 7} = \frac{a_0}{3 \cdot 4 \cdot 7 \cdot 8}$$

$$a_{12} = \frac{-a_8}{11 \cdot 12} = \frac{-a_0}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12}$$

and

$$a_5 = \frac{-a_1}{5 \cdot 4}, \qquad a_9 = \frac{-a_5}{9 \cdot 8} = \frac{a_1}{4 \cdot 5 \cdot 8 \cdot 9}$$

$$a_{13} = \frac{-a_9}{12 \cdot 13} = \frac{-a_1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13}.$$

The answer is best expressed as the sum of two separate series—one multiplied by a_0, the other by a_1:

$$y = a_0\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \frac{x^{12}}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} + \cdots\right)$$
$$+ a_1\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \frac{x^{13}}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} + \cdots\right).$$

Both series converge absolutely for all x, as is readily seen by the ratio test. ❑

Evaluating Nonelementary Integrals

Maclaurin series can be used to express nonelementary integrals in terms of series.

Integrals like $\int \sin x^2\, dx$ arise in the study of the diffraction of light.

EXAMPLE 5 Express $\displaystyle\int \sin x^2\, dx$ as a power series.

Solution From the series for sin x we obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots.$$ ❑

EXAMPLE 6 Estimate $\displaystyle\int_0^1 \sin x^2\, dx$ with an error of less than 0.001.

Solution From the indefinite integral in Example 5,

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots.$$

The series alternates, and we find by experiment that

$$\frac{1}{11 \cdot 5!} \approx 0.0007\ 6$$

is the first term to be numerically less than 0.001. The sum of the preceding two terms gives

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms we could estimate

$$\int_0^1 \sin x^2\, dx \approx 0.3102\ 68$$

with an error of less than 10^{-6}. With only one term beyond that we have

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.3102\ 68303,$$

with an error of about 1.08×10^{-9}. To guarantee this accuracy with the error formula for the trapezoidal rule would require using about 8,000 subintervals. ❑

Arctangents

In Section 8.8, Example 5, we found a series for $\tan^{-1} x$ by differentiating to get

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

and integrating to get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

However, we did not prove the term-by-term integration theorem on which this conclusion depended. We now derive the series again by integrating both sides of the finite formula

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}, \tag{13}$$

in which the last term comes from adding the remaining terms as a geometric series with first term $a = (-1)^{n+1} t^{2n+2}$ and ratio $r = -t^2$. Integrating both sides of Eq. (13) from $t = 0$ to $t = x$ gives

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R(n, x),$$

where

$$R(n, x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt.$$

The denominator of the integrand is greater than or equal to 1; hence

$$|R(n, x)| \le \int_0^{|x|} t^{2n+2}\, dt = \frac{|x|^{2n+3}}{2n+3}.$$

We take this route instead of finding the Maclaurin series directly because the formulas for the higher order derivatives of $\tan^{-1} x$ are unmanageable.

If $|x| \le 1$, the right side of this inequality approaches zero as $n \to \infty$. Therefore $\lim_{n\to\infty} R(n, x) = 0$ if $|x| \le 1$ and

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1.$$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| \le 1} \tag{14}$$

When we put $x = 1$ in Eq. (14), we get **Leibniz's formula:**

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

This series converges too slowly to be a useful source of decimal approximations of π. It is better to use a formula like

$$\pi = 48 \tan^{-1}\frac{1}{18} + 32 \tan^{-1}\frac{1}{57} - 20 \tan^{-1}\frac{1}{239},$$

which uses values of x closer to zero.

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

EXAMPLE 7 Evaluate $\lim_{x\to 1} \dfrac{\ln x}{x-1}$.

Solution We represent $\ln x$ as a Taylor series in powers of $x-1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x-1$ in the series for $\ln x$ in Section 8.8, Example 6. Either way, we obtain

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots,$$

from which we find that

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1. \qquad \square$$

EXAMPLE 8 Evaluate $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$.

Solution The Maclaurin series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} &= \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)\\ &= -\frac{1}{2}.\end{aligned} \qquad \square$$

If we apply series to calculate $\lim_{x\to 0}((1/\sin x) - (1/x))$, we not only find the limit successfully but also discover an approximation formula for $\csc x$.

EXAMPLE 9 Find $\lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

Solution

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \frac{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \cdots\right)} = x\,\frac{\frac{1}{3!} - \frac{x^2}{5!} + \cdots}{1 - \frac{x^2}{3!} + \cdots}.$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\left(x\,\frac{\frac{1}{3!} - \frac{x^2}{5!} + \cdots}{1 - \frac{x^2}{3!} + \cdots}\right) = 0.$$

From the quotient on the right, we can see that if $|x|$ is small, then

$$\frac{1}{\sin x} - \frac{1}{x} \approx x \cdot \frac{1}{3!} = \frac{x}{6} \qquad \text{or} \qquad \csc x \approx \frac{1}{x} + \frac{x}{6}.$$ ❑

Frequently Used Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\ln\frac{1+x}{1-x} = 2\tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

(Continued)

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots$$
$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \qquad |x| < 1,$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

Note: To write the binomial series compactly, it is customary to define $\binom{m}{0}$ to be 1 and to take $x^0 = 1$ (even in the usually excluded case where $x = 0$), yielding $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$. If m is a *positive integer,* the series terminates at x^m and the result converges for all x.

Exercises 8.11

Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1–10.

1. $(1+x)^{1/2}$ **2.** $(1+x)^{1/3}$ **3.** $(1-x)^{-1/2}$

4. $(1-2x)^{1/2}$ **5.** $\left(1+\frac{x}{2}\right)^{-2}$ **6.** $\left(1-\frac{x}{2}\right)^{-2}$

7. $(1+x^3)^{-1/2}$ **8.** $(1+x^2)^{-1/3}$

9. $\left(1+\frac{1}{x}\right)^{1/2}$ **10.** $\left(1-\frac{2}{x}\right)^{1/3}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1+x)^4$ **12.** $(1+x^2)^3$

13. $(1-2x)^3$ **14.** $\left(1-\frac{x}{2}\right)^4$

Initial Value Problems

Find series solutions for the initial value problems in Exercises 15–32.

15. $y' + y = 0, \quad y(0) = 1$ **16.** $y' - 2y = 0, \quad y(0) = 1$

17. $y' - y = 1, \quad y(0) = 0$ **18.** $y' + y = 1, \quad y(0) = 2$

19. $y' - y = x, \quad y(0) = 0$ **20.** $y' + y = 2x, \quad y(0) = -1$

21. $y' - xy = 0, \quad y(0) = 1$ **22.** $y' - x^2y = 0, \quad y(0) = 1$

23. $(1-x)y' - y = 0, \quad y(0) = 2$

24. $(1+x^2)y' + 2xy = 0, \quad y(0) = 3$

25. $y'' - y = 0, \quad y'(0) = 1$ and $y(0) = 0$

26. $y'' + y = 0, \quad y'(0) = 0$ and $y(0) = 1$

27. $y'' + y = x, \quad y'(0) = 1$ and $y(0) = 2$

28. $y'' - y = x, \quad y'(0) = 2$ and $y(0) = -1$

29. $y'' - y = -x, \quad y'(2) = -2$ and $y(2) = 0$

30. $y'' - x^2y = 0, \quad y'(0) = b$ and $y(0) = a$

31. $y'' + x^2y = x, \quad y'(0) = b$ and $y(0) = a$

32. $y'' - 2y' + y = 0, \quad y'(0) = 1$ and $y(0) = 0$

Approximations and Nonelementary Integrals

CALCULATOR In Exercises 33–36, use series to estimate the integrals' values with an error of magnitude less than 10^{-3}. (The answer section gives the integrals' values rounded to 5 decimal places.)

33. $\int_0^{0.2} \sin x^2 \, dx$ **34.** $\int_0^{0.2} \frac{e^{-x}-1}{x} \, dx$

35. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}} \, dx$ **36.** $\int_0^{0.25} \sqrt[3]{1+x^2} \, dx$

CALCULATOR Use series to approximate the values of the integrals in Exercises 37–40 with an error of magnitude less than 10^{-8}. (The answer section gives the integrals' values rounded to 10 decimal places.)

37. $\int_0^{0.1} \frac{\sin x}{x} \, dx$ **38.** $\int_0^{0.1} e^{-x^2} \, dx$

39. $\int_0^{0.1} \sqrt{1+x^4} \, dx$ **40.** $\int_0^{1} \frac{1-\cos x}{x^2} \, dx$

41. Estimate the error if $\cos t^2$ is approximated by $1 - \frac{t^4}{2} + \frac{t^8}{4!}$ in the integral $\int_0^1 \cos t^2 \, dt$.

42. Estimate the error if $\cos\sqrt{t}$ is approximated by $1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!}$ in the integral $\int_0^1 \cos\sqrt{t}\,dt$.

In Exercises 43–46, find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

43. $F(x) = \int_0^x \sin t^2\,dt,\quad [0, 1]$

44. $F(x) = \int_0^x t^2 e^{-t^2}\,dt,\quad [0, 1]$

45. $F(x) = \int_0^x \tan^{-1} t\,dt,\quad$ a) $[0, 0.5]$ b) $[0, 1]$

46. $F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt,\quad$ a) $[0, 0.5]$ b) $[0, 1]$

Indeterminate Forms

Use series to evaluate the limits in Exercises 47–56.

47. $\lim_{x\to 0} \frac{e^x - (1+x)}{x^2}$

48. $\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

49. $\lim_{t\to 0} \frac{1 - \cos t - (t^2/2)}{t^4}$

50. $\lim_{\theta\to 0} \frac{\sin\theta - \theta + (\theta^3/6)}{\theta^5}$

51. $\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$

52. $\lim_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y}$

53. $\lim_{x\to\infty} x^2(e^{-1/x^2} - 1)$

54. $\lim_{x\to\infty} (x+1)\sin\frac{1}{x+1}$

55. $\lim_{x\to 0} \frac{\ln(1+x^2)}{1 - \cos x}$

56. $\lim_{x\to 2} \frac{x^2 - 4}{\ln(x-1)}$

Theory and Examples

57. Replace x by $-x$ in the Maclaurin series for $\ln(1+x)$ to obtain a series for $\ln(1-x)$. Then subtract this from the Maclaurin series for $\ln(1+x)$ to show that for $|x| < 1$,

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

58. How many terms of the Maclaurin series for $\ln(1+x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}? Give reasons for your answer.

59. According to the Alternating Series Estimation Theorem, how many terms of the Maclaurin series for $\tan^{-1} 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3}? Give reasons for your answer.

60. Show that the Maclaurin series for $f(x) = \tan^{-1} x$ diverges for $|x| > 1$.

61. CALCULATOR About how many terms of the Maclaurin series for $\tan^{-1} x$ would you have to use to evaluate each term on the right-hand side of the equation

$$\pi = 48\tan^{-1}\frac{1}{18} + 32\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239}$$

with an error of magnitude less than 10^{-6}? In contrast, the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ to $\pi^2/6$ is so slow that even 50 terms will not yield two-place accuracy.

62. Integrate the first three nonzero terms of the Maclaurin series for $\tan t$ from 0 to x to obtain the first three nonzero terms of the Maclaurin series for $\ln\sec x$.

63. a) Use the binomial series and the fact that

$$\frac{d}{dx}\sin^{-1} x = (1 - x^2)^{-1/2}$$

to generate the first four nonzero terms of the Maclaurin series for $\sin^{-1} x$. What is the radius of convergence?

b) Use your result in (a) to find the first five nonzero terms of the Maclaurin series for $\cos^{-1} x$.

64. a) Find the first four nonzero terms of the Maclaurin series for

$$\sinh^{-1} x = \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

b) CALCULATOR Use the first *three* terms of the series in (a) to estimate $\sinh^{-1} 0.25$. Give an upper bound for the magnitude of the estimation error.

65. Obtain the Maclaurin series for $1/(1+x)^2$ from the series for $-1/(1+x)$.

66. Use the Maclaurin series for $1/(1-x^2)$ to obtain a series for $2x/(1-x^2)^2$.

67. CAS The English mathematician Wallis discovered the formula

$$\frac{\pi}{4} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot\cdots}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot\cdots}.$$

Find π to 2 decimal places with this formula.

68. CALCULATOR Construct a table of natural logarithms $\ln n$ for $n = 1, 2, 3, \ldots, 10$ by using the formula in Exercise 57, but taking advantage of the relationships $\ln 4 = 2\ln 2$, $\ln 6 = \ln 2 + \ln 3$, $\ln 8 = 3\ln 2$, $\ln 9 = 2\ln 3$, and $\ln 10 = \ln 2 + \ln 5$ to reduce the job to the calculation of relatively few logarithms by series. Start by using the following values for x in Exercise 57:

$$\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}.$$

69. Integrate the binomial series for $(1-x^2)^{-1/2}$ to show that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\frac{x^{2n+1}}{2n+1}.$$

70. *Series for* $\tan^{-1} x$ *for* $|x| > 1$. Derive the series

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots,\quad x > 1$$

$$\tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots,\quad x < -1,$$

by integrating the series

$$\frac{1}{1+t^2} = \frac{1}{t^2}\cdot\frac{1}{1+(1/t^2)} = \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots$$

in the first case from x to ∞ and in the second case from $-\infty$ to x.

71. *The value of* $\sum_{n=1}^{\infty} \tan^{-1}(2/n^2)$
    a) Use the formula for the tangent of the difference of two angles to show that
    $$\tan(\tan^{-1}(n+1) - \tan^{-1}(n-1)) = \frac{2}{n^2}$$

CHAPTER 8 QUESTIONS TO GUIDE YOUR REVIEW

1. What is an infinite sequence? What does it mean for such a sequence to converge? to diverge? Give examples.
2. What uses can be found for subsequences? Give examples.
3. What is a nondecreasing sequence? Under what circumstances does such a sequence have a limit? Give examples.
4. What theorems are available for calculating limits of sequences? Give examples.
5. What theorem sometimes enables us to use l'Hôpital's rule to calculate the limit of a sequence? Give an example.
6. What six sequence limits are likely to arise when you work with sequences and series?
7. What is Picard's method for solving the equation $f(x) = 0$? Give an example.
8. What is an infinite series? What does it mean for such a series to converge? to diverge? Give examples.
9. What is a geometric series? When does such a series converge? diverge? When it does converge, what is its sum? Give examples.
10. Besides geometric series, what other convergent and divergent series do you know?
11. What is the nth-Term Test for Divergence? What is the idea behind the test?
12. What can be said about term-by-term sums and differences of convergent series? about constant multiples of convergent and divergent series?
13. What happens if you add a finite number of terms to a convergent series? a divergent series? What happens if you delete a finite number of terms from a convergent series? a divergent series?
14. How do you reindex a series? Why might you want to do this?
15. Under what circumstances will an infinite series of nonnegative terms converge? diverge? Why study series of nonnegative terms?
16. What is the Integral Test? What is the reasoning behind it? Give an example of its use.
17. When do p-series converge? diverge? How do you know? Give examples of convergent and divergent p-series.
18. What are the Direct Comparison Test and the Limit Comparison Test? What is the reasoning behind these tests? Give examples of their use.
19. What are the Ratio and Root Tests? Do they always give you the information you need to determine convergence or divergence? Give examples.
20. What is an alternating series? What theorem is available for determining the convergence of such a series?
21. How can you estimate the error involved in approximating the sum of an alternating series with one of the series' partial sums? What is the reasoning behind the estimate?
22. What is absolute convergence? conditional convergence? How are the two related?
23. What do you know about rearranging the terms of an absolutely convergent series? of a conditionally convergent series? Give examples.
24. What is a power series? How do you test a power series for convergence? What are the possible outcomes?
25. What are the basic facts about
    a) term-by-term differentiation of power series?
    b) term-by-term integration of power series?
    c) multiplication of power series?

    Give examples.
26. What is the Taylor series generated by a function $f(x)$ at a point $x = a$? What information do you need about f to construct the series? Give an example.
27. What is a Maclaurin series?
28. Does a Taylor series always converge to its generating function? Explain.

29. What are Taylor polynomials? Of what use are they?

30. What is Taylor's formula? What does it say about the errors involved in using Taylor polynomials to approximate functions? In particular, what does Taylor's formula say about the error in a linearization? a quadratic approximation?

31. What is the binomial series? On what interval does it converge? How is it used?

32. How can you sometimes use power series to solve initial value problems?

33. How can you sometimes use power series to estimate the values of nonelementary definite integrals?

34. What are the Maclaurin series for $1/(1-x)$, $1/(1+x)$, e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $\ln[(1+x)/(1-x)]$, and $\tan^{-1} x$? How do you estimate the errors involved in replacing these series with their partial sums?

CHAPTER 8 PRACTICE EXERCISES

Convergent or Divergent Sequences

Which of the sequences whose nth terms appear in Exercises 1–18 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 1 + \dfrac{(-1)^n}{n}$
2. $a_n = \dfrac{1-(-1)^n}{\sqrt{n}}$
3. $a_n = \dfrac{1-2^n}{2^n}$
4. $a_n = 1 + (0.9)^n$
5. $a_n = \sin \dfrac{n\pi}{2}$
6. $a_n = \sin n\pi$
7. $a_n = \dfrac{\ln(n^2)}{n}$
8. $a_n = \dfrac{\ln(2n+1)}{n}$
9. $a_n = \dfrac{n + \ln n}{n}$
10. $a_n = \dfrac{\ln(2n^3+1)}{n}$
11. $a_n = \left(\dfrac{n-5}{n}\right)^n$
12. $a_n = \left(1 + \dfrac{1}{n}\right)^{-n}$
13. $a_n = \sqrt[n]{\dfrac{3^n}{n}}$
14. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$
15. $a_n = n(2^{1/n} - 1)$
16. $a_n = \sqrt[n]{2n+1}$
17. $a_n = \dfrac{(n+1)!}{n!}$
18. $a_n = \dfrac{(-4)^n}{n!}$

Convergent Series

Find the sums of the series in Exercises 19–24.

19. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$
20. $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$
22. $\displaystyle\sum_{n=3}^{\infty} \frac{-8}{(4n-3)(4n+1)}$
23. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$
24. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Convergent or Divergent Series

Which of the series in Exercises 25–40 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
26. $\displaystyle\sum_{n=1}^{\infty} \frac{-5}{n}$
27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$
29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$
30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$
32. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$
33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$
34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$
35. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$
36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$
37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$
38. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$
39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$
40. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Power Series

In Exercises 41–50, (a) find the series' radius and interval of convergence. Then identify the values of x for which the series converges (b) absolutely and (c) conditionally.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\,3^n}$
42. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$
43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$
44. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$
45. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$
46. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

47. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

48. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

49. $\sum_{n=1}^{\infty} (\operatorname{csch} n)\, x^n$

50. $\sum_{n=1}^{\infty} (\coth n)\, x^n$

Maclaurin Series

Each of the series in Exercises 51–56 is the value of the Maclaurin series of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

51. $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$

52. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$

53. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$

54. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$

55. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$

56. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots$

$$+ (-1)^{n-1} \frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \cdots$$

Find Maclaurin series for the functions in Exercises 57–64.

57. $\frac{1}{1-2x}$

58. $\frac{1}{1+x^3}$

59. $\sin \pi x$

60. $\sin \frac{2x}{3}$

61. $\cos (x^{5/2})$

62. $\cos \sqrt{5x}$

63. $e^{(\pi x/2)}$

64. e^{-x^2}

Taylor Series

In Exercises 65–68, find the first four nonzero terms of the Taylor series generated by f at $x = a$.

65. $f(x) = \sqrt{3 + x^2}$ at $x = -1$

66. $f(x) = 1/(1 - x)$ at $x = 2$

67. $f(x) = 1/(x + 1)$ at $x = 3$

68. $f(x) = 1/x$ at $x = a > 0$

Initial Value Problems

Use power series to solve the initial value problems in Exercises 69–76.

69. $y' + y = 0, \quad y(0) = -1$

70. $y' - y = 0, \quad y(0) = -3$

71. $y' + 2y = 0, \quad y(0) = 3$

72. $y' + y = 1, \quad y(0) = 0$

73. $y' - y = 3x, \quad y(0) = -1$

74. $y' + y = x, \quad y(0) = 0$

75. $y' - y = x, \quad y(0) = 1$

76. $y' - y = -x, \quad y(0) = 2$

Nonelementary Integrals

Use series to approximate the values of the integrals in Exercises 77–80 with an error of magnitude less than 10^{-8}. (The answer section gives the integrals' values rounded to 10 decimal places.)

77. $\int_0^{1/2} e^{-x^3}\, dx$

78. $\int_0^1 x \sin (x^3)\, dx$

79. $\int_0^{1/2} \frac{\tan^{-1} x}{x}\, dx$

80. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}}\, dx$

Indeterminate Forms

In Exercises 81–86:

a) Use power series to evaluate the limit.

b) GRAPHER Then use a grapher to support your calculation.

81. $\lim_{x\to 0} \frac{7 \sin x}{e^{2x} - 1}$

82. $\lim_{\theta\to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin \theta}$

83. $\lim_{t\to 0} \left(\frac{1}{2 - 2\cos t} - \frac{1}{t^2}\right)$

84. $\lim_{h\to 0} \frac{(\sin h)/h - \cos h}{h^2}$

85. $\lim_{z\to 0} \frac{1 - \cos^2 z}{\ln (1 - z) + \sin z}$

86. $\lim_{y\to 0} \frac{y^2}{\cos y - \cosh y}$

87. Use a series representation of $\sin 3x$ to find values of r and s for which

$$\lim_{x\to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s\right) = 0.$$

88. **a)** Show that the approximation $\csc x \approx 1/x + x/6$ in Section 8.11, Example 9, leads to the approximation $\sin x \approx 6x/(6 + x^2)$.

b) GRAPHER EXPLORATION Compare the accuracies of the approximations $\sin x \approx x$ and $\sin x \approx 6x/(6 + x^2)$ by comparing the graphs of $f(x) = \sin x - x$ and $g(x) = \sin x - (6x/(6 + x^2))$. Describe what you find.

Theory and Examples

89. **a)** Show that the series

$$\sum_{n=1}^{\infty} \left(\sin \frac{1}{2n} - \sin \frac{1}{2n+1}\right)$$

converges.

b) CALCULATOR Estimate the magnitude of the error involved in using the sum of the sines through $n = 20$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

90. **a)** Show that the series $\sum_{n=1}^{\infty} \left(\tan \frac{1}{2n} - \tan \frac{1}{2n+1}\right)$ converges.

b) CALCULATOR Estimate the magnitude of the error in using the sum of the tangents through $-\tan (1/41)$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

91. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} x^n.$$

92. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}{4 \cdot 9 \cdot 14 \cdot \cdots \cdot (5n-1)} (x-1)^n.$$

93. Find a closed-form formula for the nth partial sum of the series $\sum_{n=2}^{\infty} \ln(1-(1/n^2))$ and use it to determine the convergence or divergence of the series.

94. Evaluate $\sum_{k=2}^{\infty} (1/(k^2-1))$ by finding the limit as $n \to \infty$ of the series' nth partial sum.

95. a) Find the interval of convergence of the series

$$y = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{(3n)!} x^{3n} + \cdots.$$

b) Show that the function defined by the series satisfies a differential equation of the form

$$\frac{d^2y}{dx^2} = x^a y + b$$

and find the values of the constants a and b.

96. a) Find the Maclaurin series for the function $x^2/(1+x)$.

b) Does the series converge at $x = 1$? Explain.

97. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

98. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

99. Prove that the sequence $\{x_n\}$ and the series $\sum_{k=1}^{\infty}(x_{k+1} - x_k)$ both converge or both diverge.

100. Prove that $\sum_{n=1}^{\infty}(a_n/(1+a_n))$ converges if $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges.

101. (*Continuation of Section 3.8, Exercise 25.*) If you did Exercise 25 in Section 3.8, you saw that in practice Newton's method stopped too far from the root of $f(x) = (x-1)^{40}$ to give a useful estimate of its value, $x = 1$. Prove that nevertheless, for any starting value $x_0 \neq 1$, the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ of approximations generated by Newton's method really does converge to 1.

102. a) Suppose that $a_1, a_2, a_3, \ldots, a_n$ are positive numbers satisfying the following conditions:

i) $a_1 \geq a_2 \geq a_3 \geq \cdots$;

ii) the series $a_2 + a_4 + a_8 + a_{16} + \cdots$ diverges.

Show that the series

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

diverges.

b) Use the result in (a) to show that

$$1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

103. Suppose you wish to obtain a quick estimate for the value of $\int_0^1 x^2 e^x \, dx$. There are several ways to do this.

a) Use the trapezoidal rule with $n = 2$ to estimate $\int_0^1 x^2 e^x \, dx$.

b) Write out the first three nonzero terms of the Maclaurin series for $x^2 e^x$ to obtain the fourth Maclaurin polynomial $P(x)$ for $x^2 e^x$. Use $\int_0^1 P(x)\, dx$ to obtain another estimate for $\int_0^1 x^2 e^x \, dx$.

c) The second derivative of $f(x) = x^2 e^x$ is positive for all $x > 0$. Explain why this enables you to conclude that the trapezoidal rule estimate obtained in (a) is too large. (*Hint:* What does the second derivative tell you about the graph of a function? How does this relate to the trapezoidal approximation of the area under this graph?)

d) All the derivatives of $f(x) = x^2 e^x$ are positive for $x > 0$. Explain why this enables you to conclude that all Maclaurin polynomial approximations to $f(x)$ for x in $[0, 1]$ will be too small. (*Hint:* $f(x) = P_n(x) + R_n(x)$.)

e) Use integration by parts to evaluate $\int_0^1 x^2 e^x \, dx$.

CHAPTER 8 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Convergence or Divergence

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 1–4 converge, and which diverge? Give reasons for your answers.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2)^{n+(1/2)}}$ **2.** $\displaystyle\sum_{n=1}^{\infty} \frac{(\tan^{-1} n)^2}{n^2+1}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \tanh n$ **4.** $\displaystyle\sum_{n=2}^{\infty} \frac{\log_n(n!)}{n^3}$

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 5–8 converge, and which diverge? Give reasons for your answers.

5. $a_1 = 1, \quad a_{n+1} = \dfrac{n(n+1)}{(n+2)(n+3)} a_n$

(*Hint:* Write out several terms, see which factors cancel, and then generalize.)

6. $a_1 = a_2 = 7, \quad a_{n+1} = \dfrac{n}{(n-1)(n+1)} a_n$ if $n \ge 2$

7. $a_1 = a_2 = 1, \quad a_{n+1} = \dfrac{1}{1+a_n}$ if $n \ge 2$

8. $a_n = 1/3^n$ if n is odd, $\quad a_n = n/3^n$ if n is even

Choosing Centers for Taylor Series

Taylor's formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

expresses the value of f at x in terms of the values of f and its derivatives at $x = a$. In numerical computations, we therefore need f to be a point where we know the values of f and its derivatives. We also need a to be close enough to the values of f we are interested in to make $(x-a)^{n+1}$ so small we can neglect the remainder.

In Exercises 9–14, what Taylor series would you choose to represent the function near the given value of x? (There may be more than one good answer.) Write out the first four nonzero terms of the series you choose.

9. $\cos x$ near $x = 1$ **10.** $\sin x$ near $x = 6.3$

11. e^x near $x = 0.4$ **12.** $\ln x$ near $x = 1.3$

13. $\cos x$ near $x = 69$ **14.** $\tan^{-1} x$ near $x = 2$

Theory and Examples

15. Let a and b be constants with $0 < a < b$. Does the sequence $\{(a^n + b^n)^{1/n}\}$ converge? If it does converge, what is the limit?

16. Find the sum of the infinite series

$$1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \cdots.$$

17. Evaluate

$$\sum_{n=0}^{\infty} \int_n^{n+1} \frac{1}{1+x^2}\,dx.$$

18. Find all values of x for which

$$\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)(2x+1)^n}$$

converges absolutely.

19. *Generalizing Euler's constant.* Figure 8.21 shows the graph of a positive twice-differentiable decreasing function f whose second derivative is positive on $(0, \infty)$. For each n, the number A_n is the area of the lunar region between the curve and the line segment joining the points $(n, f(n))$ and $(n+1, f(n+1))$.

a) Use the figure to show that $\sum_{n=1}^{\infty} A_n < (1/2)(f(1) - f(2))$.

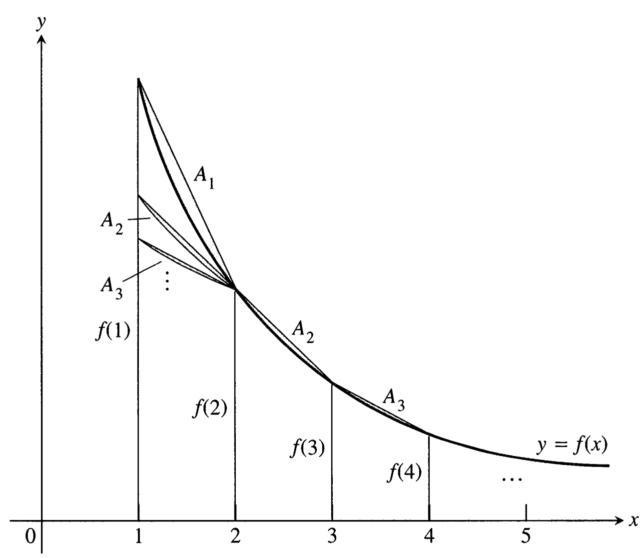
b) Then show the existence of

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \frac{1}{2}(f(1) + f(n)) - \int_1^n f(x)\,dx\right].$$

c) Then show the existence of

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right].$$

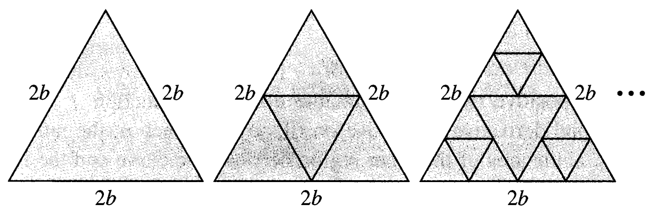
If $f(x) = 1/x$, the limit in (c) is Euler's constant (Section 8.4, Exercise 41). (Source: "Convergence with Pictures" by P. J. Rippon, *American Mathematical Monthly,* Vol. 93, No. 6, 1986, pp. 476–78.)



8.21 The figure for Exercise 19.

20. This exercise refers to the "right side up" equilateral triangle with sides of length $2b$ in the accompanying figure. "Upside down" equilateral triangles are removed from the original triangle as the sequence of pictures suggests. The sum of the areas removed from the original triangle forms an infinite series.

a) Find this infinite series.
b) Find the sum of this infinite series and hence find the total area removed from the original triangle.
c) Is every point on the original triangle removed? Explain why or why not.



21. CAS EXPLORATION

a) Does the value of

$$\lim_{n\to\infty}\left(1-\frac{\cos(a/n)}{n}\right)^n, \quad a \text{ constant},$$

appear to depend on the value of a? If so, how?

b) Does the value of

$$\lim_{n\to\infty}\left(1-\frac{\cos(a/n)}{bn}\right)^n, \quad a \text{ and } b \text{ constant}, b\neq 0,$$

appear to depend on the value of b? If so, how?

c) Use calculus to confirm your findings in (a) and (b).

22. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then

$$\sum_{n=1}^{\infty}\left(\frac{1+\sin(a_n)}{2}\right)^n$$

converges.

23. Find a value for the constant b that will make the radius of convergence of the power series

$$\sum_{n=2}^{\infty}\frac{b^n x^n}{\ln n}$$

equal to 5.

24. How do you know that the functions $\sin x$, $\ln x$, and e^x are not polynomials? Give reasons for your answer.

25. Find the value of a for which the limit

$$\lim_{x\to 0}\frac{\sin(ax)-\sin x - x}{x^3}$$

is finite and evaluate the limit.

26. Find values of a and b for which

$$\lim_{x\to 0}\frac{\cos(ax)-b}{2x^2} = -1.$$

27. *Raabe's (or Gauss's) test.* The following test, which we state without proof, is an extension of the Ratio Test.

Raabe's test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive constants and there exist constants C, K, and N such that

$$\frac{u_n}{u_{n+1}} = 1+\frac{C}{n}+\frac{f(n)}{n^2}, \tag{1}$$

where $|f(n)|<K$ for $n\geq N$, then $\sum_{n=1}^{\infty} u_n$ converges if $C>1$ and diverges if $C\leq 1$.

Show that the results of Raabe's test agree with what you know about the series $\sum_{n=1}^{\infty}(1/n^2)$ and $\sum_{n=1}^{\infty}(1/n)$.

28. (*Continuation of Exercise 27.*) Suppose that the terms of $\sum_{n=1}^{\infty} u_n$ are defined recursively by the formulas

$$u_1 = 1, \quad u_{n+1} = \frac{(2n-1)^2}{(2n)(2n+1)}u_n.$$

Apply Raabe's test to determine whether the series converges.

29. If $\sum_{n=1}^{\infty} a_n$ converges, and if $a_n \neq 1$ and $a_n > 0$ for all n,

a) Show that $\sum_{n=1}^{\infty} a_n^2$ converges.
b) Does $\sum_{n=1}^{\infty} a_n/(1-a_n)$ converge? Explain.

30. (*Continuation of Exercise 29.*) If $\sum_{n=1}^{\infty} a_n$ converges, and if $1 > a_n > 0$ for all n, show that $\sum_{n=1}^{\infty}\ln(1-a_n)$ converges. (*Hint:* First show that $|\ln(1-a_n)|\leq a_n/(1-a_n)$.)

31. *Nicole Oresme's theorem.* Prove Nicole Oresme's theorem that

$$1+\frac{1}{2}\cdot 2+\frac{1}{4}\cdot 3+\cdots+\frac{n}{2^{n-1}}+\cdots = 4.$$

(*Hint*: Differentiate both sides of the equation $1/(1-x) = 1+\sum_{n=1}^{\infty} x^n$.)

32. a) Show that

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{x^n} = \frac{2x^2}{(x-1)^3}$$

for $|x|>1$ by differentiating the identity

$$\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}$$

twice, multiplying the result by x, and then replacing x by $1/x$.

b) CALCULATOR Use part (a) to find the real solution greater than 1 of the equation

$$x = \sum_{n=1}^{\infty}\frac{n(n+1)}{x^n}.$$

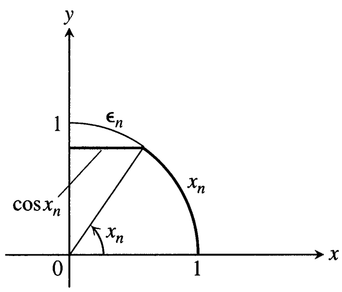
33. *A fast estimate of $\pi/2$.* As you saw if you did Exercise 29 in Section 8.1, the sequence generated by starting with $x_0 = 1$

and applying the recursion formula $x_{n+1} = x_n + \cos x_n$ converges rapidly to $\pi/2$. To explain the speed of the convergence, let $\epsilon_n = (\pi/2) - x_n$. (See the accompanying figure.) Then

$$\begin{aligned}\epsilon_{n+1} &= \frac{\pi}{2} - x_n - \cos x_n \\ &= \epsilon_n - \cos\left(\frac{\pi}{2} - \epsilon_n\right) \\ &= \epsilon_n - \sin \epsilon_n \\ &= \frac{1}{3!}\left(\epsilon_n\right)^3 - \frac{1}{5!}\left(\epsilon_n\right)^5 + \cdots.\end{aligned}$$

Use this equality to show that

$$0 < \epsilon_{n+1} < \frac{1}{6}\left(\epsilon_n\right)^3.$$



34. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, can anything be said about the convergence of $\sum_{n=1}^{\infty} \ln(1 + a_n)$? Give reasons for your answer.

35. *Quality control*

a) Differentiate the series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

to obtain a series for $1/(1-x)^2$.

b) In one throw of two dice, the probability of getting a roll of 7 is $p = 1/6$. If you throw the dice repeatedly, the probability that a 7 will appear for the first time at the nth throw is $q^{n-1}p$, where $q = 1 - p = 5/6$. The expected number of throws until a 7 first appears is $\sum_{n=1}^{\infty} nq^{n-1}p$. Find the sum of this series.

c) As an engineer applying statistical control to an industrial operation, you inspect items taken at random from the assembly line. You classify each sampled item as either "good" or "bad." If the probability of an item's being good is p and of an item's being bad is $q = 1 - p$, the probability that the first bad item found is the nth one inspected is $p^{n-1}q$. The average number inspected up to and including the first bad item found is $\sum_{n=1}^{\infty} np^{n-1}q$. Evaluate this sum, assuming $0 < p < 1$.

36. *Expected value.* Suppose that a random variable X may assume the values 1, 2, 3, ..., with probabilities $p_1, p_2, p_3, \ldots$, where p_k is the probability that X equals k ($k = 1, 2, 3, \ldots$). Suppose also that $p_k \geq 0$ and that $\sum_{k=1}^{\infty} p_k = 1$. The **expected value** of X, denoted by $E(X)$, is the number $\sum_{k=1}^{\infty} kp_k$, provided the series converges. In each of the following cases, show that $\sum_{k=1}^{\infty} p_k = 1$ and find $E(X)$ if it exists. (*Hint:* See Exercise 35.)

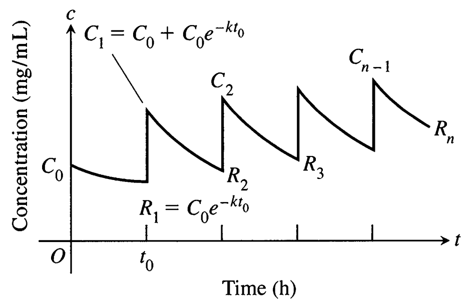
a) $p_k = 2^{-k}$ **b)** $p_k = \dfrac{5^{k-1}}{6^k}$

c) $p_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$

37. *Safe and effective dosage.* The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. Doses may therefore need to be repeated periodically to keep the concentration from dropping below some particular level. One model for the effect of repeated doses gives the residual concentration just before the $(n+1)$st dose as

$$R_n = C_0\, e^{-kt_0} + C_0\, e^{-2kt_0} + \cdots + C_0\, e^{-nkt_0},$$

where $C_0 =$ the change in concentration achievable by a single dose (mg/ml), $k =$ the *elimination constant* (h^{-1}), and $t_0 =$ time between doses (h). See Fig. 8.22.



8.22 One possible effect of repeated doses on the concentration of a drug in the bloodstream.

a) Write R_n in closed form as a single fraction, and find $R = \lim_{n\to\infty} R_n$.

b) Calculate R_1 and R_{10} for $C_0 = 1$ mg/ml, $k = 0.1\ \mathrm{h}^{-1}$, and $t_0 = 10$ h. How good an estimate of R is R_{10}?

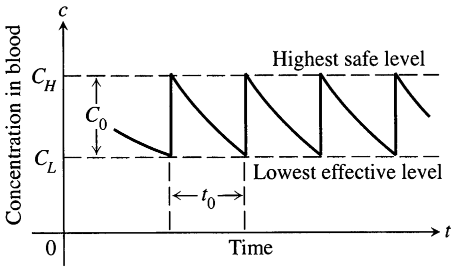
c) If $k = 0.01\ \mathrm{h}^{-1}$ and $t_0 = 10$ h, find the smallest n such that $R_n > (1/2)R$.

(Source: *Prescribing Safe and Effective Dosage,* B. Horelick and S. Koont, COMAP, Inc., Lexington, MA.)

38. (*Continuation of Exercise 37.*) If a drug is known to be ineffective below a concentration C_L and harmful above some higher concentration C_H, one needs to find values of C_0 and t_0 that will produce a concentration that is safe (not above C_H) but effective (not below C_L). See Fig. 8.23. We therefore want to find values for C_0 and t_0 for which

$$R = C_L \quad \text{and} \quad C_0 + R = C_H.$$

Thus $C_0 = C_H - C_L$. When these values are substituted in the



8.23 Safe and effective concentrations of a drug. C_0 is the change in concentration produced by one dose; t_0 is the time between doses.

equation for R obtained in part (a) of Exercise 37, the resulting equation simplifies to

$$t_0 = \frac{1}{k} \ln \frac{C_H}{C_L}.$$

To reach an effective level rapidly, one might administer a "loading" dose that would produce a concentration of C_H mg/ml. This could be followed every t_0 hours by a dose that raises the concentration by $C_0 = C_H - C_L$ mg/ml.

a) Verify the preceding equation for t_0.

b) If $k = 0.05\,\text{h}^{-1}$ and the highest safe concentration is e times the lowest effective concentration, find the length of time between doses that will assure safe and effective concentrations.

c) Given $C_H = 2$ mg/ml, $C_L = 0.5$ mg/ml, and $k = 0.02\,\text{h}^{-1}$, determine a scheme for administering the drug.

d) Suppose that $k = 0.2\,\text{h}^{-1}$ and that the smallest effective concentration is 0.03 mg/ml. A single dose that produces a concentration of 0.1 mg/ml is administered. About how long will the drug remain effective?

39. *An infinite product.* The infinite product

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots$$

is said to converge if the series

$$\sum_{n=1}^{\infty} \ln(1+a_n),$$

obtained by taking the natural logarithm of the product, converges. Prove that the product converges if $a_n > -1$ for every n and if $\sum_{n=1}^{\infty} |a_n|$ converges. (*Hint:* Show that

$$|\ln(1+a_n)| \le \frac{|a_n|}{1-|a_n|} \le 2|a_n|$$

when $|a_n| < 1/2$.)

40. If p is a constant, show that the series

$$1 + \sum_{n=3}^{\infty} \frac{1}{n \cdot \ln n \cdot [\ln(\ln n)]^p}$$

(a) converges if $p > 1$, (b) diverges if $p \le 1$. In general, if $f_1(x) = x$, $f_{n+1}(x) = \ln(f_n(x))$, and n takes on the values 1, 2, 3, ..., we find that $f_2(x) = \ln x$, $f_3(x) = \ln(\ln x)$, and so on. If $f_n(a) > 1$, then

$$\int_a^{\infty} \frac{dx}{f_1(x) f_2(x) \cdots f_n(x)(f_{n+1}(x))^p}$$

converges if $p > 1$ and diverges if $p \le 1$.

41. a) Prove the following theorem: If $\{c_n\}$ is a sequence of numbers such that every sum $t_n = \sum_{k=1}^{n} c_k$ is bounded, then the series $\sum_{n=1}^{\infty} c_n/n$ converges and is equal to $\sum_{n=1}^{\infty} t_n/(n(n+1))$.

Outline of proof: Replace c_1 by t_1 and c_n by $t_n - t_{n-1}$ for $n \ge 2$. If $s_{2n+1} = \sum_{k=1}^{2n+1} c_k/k$, show that

$$\begin{aligned} s_{2n+1} &= t_1\left(1-\frac{1}{2}\right) + t_2\left(\frac{1}{2}-\frac{1}{3}\right) \\ &\quad + \cdots + t_{2n}\left(\frac{1}{2n}-\frac{1}{2n+1}\right) + \frac{t_{2n+1}}{2n+1} \\ &= \sum_{k=1}^{2n} \frac{t_k}{k(k+1)} + \frac{t_{2n+1}}{2n+1}. \end{aligned}$$

Because $|t_k| < M$ for some constant M, the series

$$\sum_{k=1}^{\infty} \frac{t_k}{k(k+1)}$$

converges absolutely and s_{2n+1} has a limit as $n \to \infty$. Finally, if $s_{2n} = \sum_{k=1}^{2n} c_k/k$, then $s_{2n+1} - s_{2n} = c_{2n+1}/(2n+1)$ approaches zero as $n \to \infty$ because $|c_{2n+1}| = |t_{2n+1} - t_{2n}| < 2M$. Hence the sequence of partial sums of the series $\sum c_k/k$ converges and the limit is $\sum_{k=1}^{\infty} t_k/(k(k+1))$.

b) Show how the foregoing theorem applies to the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

c) Show that the series

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$$

converges. (After the first term, the signs are two negative, two positive, two negative, two positive, and so on in that pattern.)

42. *The convergence of* $\sum_{n=1}^{\infty}[(-1)^{n-1}x^n]/n$ *to* $\ln(1+x)$ *for* $-1 < x \le 1$

a) Show by long division or otherwise that

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \frac{(-1)^{n+1} t^{n+1}}{1+t}.$$

b) By integrating the equation of part (a) with respect to t from 0 to x, show that

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &\quad + (-1)^n \frac{x^{n+1}}{n+1} + R_{n+1} \end{aligned}$$

where

$$R_{n+1} = (-1)^{n+1} \int_0^x \frac{t^{n+1}}{1+t}\, dt.$$

c) If $x \geq 0$, show that

$$|R_{n+1}| \leq \int_0^x t^{n+1}\, dt = \frac{x^{n+2}}{n+2}.$$

(*Hint*: As t varies from 0 to x,

$$1 + t \geq 1 \quad \text{and} \quad t^{n+1}/(1+t) \leq t^{n+1},$$

and

$$\left| \int_0^x f(t)\, dt \right| \leq \int_0^x \left| f(t) \right| dt.)$$

d) If $-1 < x < 0$, show that

$$\left| R_{n+1} \right| \leq \left| \int_0^x \frac{t^{n+1}}{1-|x|}\, dt \right| = \frac{|x|^{n+2}}{(n+2)(1-|x|)}.$$

(*Hint*: If $x < t \leq 0$, then $|1+t| \geq 1 - |x|$ and

$$\left| \frac{t^{n+1}}{1+t} \right| \leq \frac{|t|^{n+1}}{1-|x|}.)$$

e) Use the foregoing results to prove that the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots$$

converges to $\ln(1+x)$ for $-1 < x \leq 1$.