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Formulate a **language** that describes the above problem.

ECE-374-B: Lecture 2 - Regular Languages

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This is an example of a regular language which we'll be discussing today.

Strings

Alphabet

An alphabet is a **finite** set of symbols.

Examples of alphabets:

- $\Sigma = \{0, 1\}$,
- $\Sigma = \{a, b, c, \ldots, z\}$,
- ASCII.
- UTF8.
- $\Sigma = \{\langle \text{moveforward} \rangle, \langle \text{moveback} \rangle, \langle \text{moveleft} \rangle, \langle \text{moveright} \rangle\}$

String Definition

Definition

1. A string/word over Σ is a **finite sequence** of symbols over Σ. For example, '0101001', '*string*', '$\langle \text{moveback} \rangle \langle \text{rotate90} \rangle$'
2. $x \cdot y \equiv xy$ is the concatenation of two strings
3. The length of a string w (denoted by $|w|$) is the number of symbols in w. For example, $|101| = 3$, $|\epsilon| = 0$
4. For integer $n \geq 0$, Σ^n is set of all strings over Σ of length n. Σ^* is the set of all strings over Σ.
5. Σ^* set of all strings of all lengths including empty string.

Question: $\{'a',' c'\}^* =$

Emptiness

- ϵ is a string containing no symbols. It is not a set
- $\{\epsilon\}$ is a set containing one string: the empty string. It is a set, not a string.
- $\emptyset$ is the empty set. It contains no strings.

Question: What is $\{\emptyset\}$

Concatenation and properties

- If x and y are strings then xy denotes their concatenation.
- Concatenation defined recursively :
  - $xy = y$ if $x = \epsilon$
  - $xy = a(wy)$ if $x = aw$
- xy sometimes written as $x \bullet y$.
- concatenation is associative: $(uv)w = u(vw)$ hence write $uvw \equiv (uv)w = u(vw)$
- **not** commutative: uv not necessarily equal to vu
- The identity element is the empty string ϵ:

$$\epsilon u = u\epsilon = u.$$

Substrings, prefixes, Suffixes

Definition
v is substring of $w \iff$ there exist strings x, y such that $w = xvy$.

- If $x = \epsilon$ then v is a prefix of w
- If $y = \epsilon$ then v is a suffix of w

Subsequence

A subsequence of a string $w[1...n]$ is either a subsequence of $w[2...n]$ or $w[1]$ followed by a subsequence of $w[2...n]$.

Example
kapa is a sub-sequence of *knapsack*

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Question: How many sub-sequences are there in a string $|w| = 5$?

String exponent

Definition
If w is a string then w^n is defined inductively as follows:

$w^n = \epsilon$ if $n = 0$

$w^n = ww^{n-1}$ if $n > 0$

Question: $(blah)^3 =$.

Rapid-fire questions -strings

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. What is Σ^0?

2. How many elements are there in Σ^n?

3. If $|u| = 2$ and $|v| = 3$ then what is $|u \bullet v|$?

4. Let u be an arbitrary string in Σ^*. What is ϵu? What is $u\epsilon$?

Languages

Languages

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

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Standard set operations apply to languages.

- For languages A, B the concatenation of A, B is $AB = \{xy \mid x \in A, y \in B\}$.
- For languages A, B, their union is $A \cup B$, intersection is $A \cap B$, and difference is $A \setminus B$ (also written as $A - B$).
- For language $A \subseteq \Sigma^*$ the complement of A is $\bar{A} = \Sigma^* \setminus A$.

Set Concatenation

Definition
Given two sets X and Y of strings (over some common alphabet Σ) the concatenation of X and Y is

$$XY = \{xy \mid x \in X, y \in Y\} \tag{2}$$

Question: $X = \{\mathit{fido}, \mathit{rover}, \mathit{spot}\}$, $Y = \{\mathit{fluffy}, \mathit{tabby}\} \implies XY = .$

Σ^* and languages

Definition

1. Σ^n is the set of all strings of length n. Defined inductively:
   $\Sigma^n = \{\epsilon\}$ if $n = 0$
   $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
2. $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ is the set of all finite length strings
3. $\Sigma^+ = \cup_{n \geq 1} \Sigma^n$ is the set of non-empty strings.

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Question: Does Σ^* have strings of infinite length?

Rapid-Fire questions - Languages

Problem
Consider languages over $\Sigma = \{0, 1\}$.

1. *What is $\emptyset^0$?*
2. *If $|L| = 2$, then what is $|L^4|$?*
3. *What is $\emptyset^*$, $\{\epsilon\}^*$, ϵ^*?*
4. *For what L is L^* finite?*
5. *What is $\emptyset^+$?*
6. *What is $\{\epsilon\}^+$, ϵ^+?*

Terminology Review

Let's review what we learned.

- A character(a, b, c, x) is a unit of information represented by a symbol: (letters, digits, whitespace)
- A alphabet(Σ) is a set of characters
- A string(w) is a sequence of characters
- A language(A, B, C, L) is a set of strings

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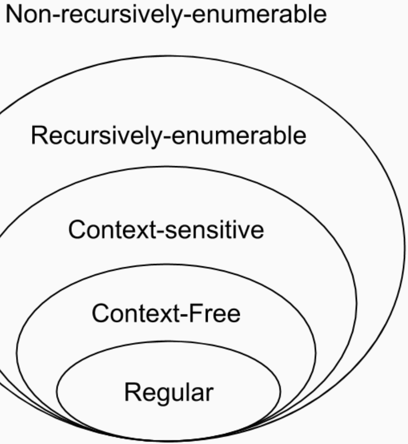
- A language(A, B, C, L) is a set of strings
- A grammar(G) is a set of rules that defines the strings that belong to a language

Chomsky Hierarchy



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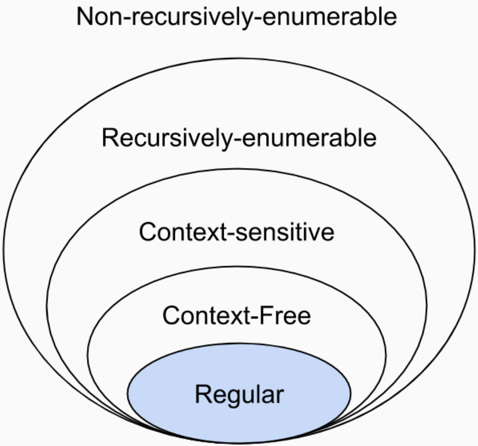

Regular Languages

Theorem (Kleene's Theorem)

A language is regular if and only if it can be obtained from finite languages by applying the three operations:

- *Union*
- *Concatenation*
- *Repetition*

a finite number of times.

Regular Languages

A class of simple but useful languages.

The set of regular languages over some alphabet Σ is defined inductively.

Base Case

- $\emptyset$ is a regular language.
- $\{\epsilon\}$ is a regular language.
- $\{a\}$ is a regular language for each $a \in \Sigma$. Interpreting a as string of length 1.

Regular Languages

Inductive step:

We can build up languages using a few basic operations:

- If L_1, L_2 are regular then $L_1 \cup L_2$ is regular.
- If L_1, L_2 are regular then $L_1 L_2$ is regular.
- If L is regular, then $L^* = \cup_{n \geq 0} L^n$ is regular.
  The $\cdot^*$ operator name is Kleene star.
- If L is regular, then so is $\overline{L} = \Sigma^* \setminus L$.

Regular languages are closed under operations of union, concatenation and Kleene star.

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

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Lemma
Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

Have basic operations to build regular languages.

Important: Any language generated by a finite sequence of such operations is regular.

Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Regular Languages

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Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Note:Kleene star (repetition) is a **single** operation!

Regular Languages - Example

Example: The language $L_{01} = 0^i 1^j |$ for all $i, j \geq 0$ is regular:

Rapid-fire questions - regular languages

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4. $L_4 = \{w \in \{0, 1\}^* \mid w \text{ has at most } 2 \text{ 1s}\}$. L_4 is regular. T/F?

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him [1].

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabet Σ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- a denote the language $\{a\}$.

Inductive cases: If $\mathbf{r_1}$ and $\mathbf{r_2}$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r_1} + \mathbf{r_2})$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r_1} \bullet \mathbf{r_2}) = r_1 \bullet r_2 = (\mathbf{r_1}\mathbf{r_2})$ denotes the language $R_1 R_2$
- $(\mathbf{r_1})^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	**Regular Expressions**
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	$\mathbf{a}$ denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1} \cdot \mathbf{r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression $\mathbf{r}$, $L(\mathbf{r})$ is the language denoted by $\mathbf{r}$. Multiple regular expressions can denote the same language!
  Example: $(0+1)$ and $(1+0)$ denotes same language $\{0,1\}$

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  Example: $rst = (rs)t = r(st)$,
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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{rr}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

Some examples of regular expressions

Interpreting regular expressions

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3. $\mathbf{0}^*+(\mathbf{0}^*\mathbf{1}\mathbf{0}^*\mathbf{1}\mathbf{0}^*\mathbf{1}\mathbf{0}^*)^*$:   with number of 1's divisible by 3
4. $(\epsilon+\mathbf{1})(\mathbf{01})^*(\epsilon+\mathbf{0})$:

Creating regular expressions

1. All strings that end in 1011?

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2. All strings except 11?
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4. All strings that do not contain the substring 10?

Tying everything together

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with an input alphabet $\Sigma_i = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x. We knwo the language used to describe it is:

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Formulate the regular expression which describes the above language:

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Formulate the regular expression which describes the above language: $\Sigma = \{0, 1, `\cdot', `|'\}$

$$r_{AND_N} = \underbrace{(\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* 0 (\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}|0\text{”}} + \overbrace{(\text{“}1\cdot\text{”})^* \text{“}|1\text{”}}^{\text{all output 1 instances}}$$

Regular expressions in programming

One last expression....

Bit strings with odd number of 0s and 1s

Bit strings with odd number of 0s and 1s

The regular expression is

$$\big(00 + 11\big)^* (01 + 10) \Big(00 + 11 + (01 + 10)(00 + 11)^* (01 + 10)\Big)^*$$

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00+11)^*(01+10)\Big(00+11+(01+10)(00+11)^*(01+10)\Big)^*$$

(Solved using techniques to be presented in the following lectures...)