ECE-374-B: Lecture 1 - Logistics and Strings/Languages

Lecturer: Nickvash Kani

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University of Illinois at Urbana Champaign

Course Administration

Instructional Staff

- Instructor:
  - Section B: Nickvash Kani and Yi Lu
- Teaching Assistants:
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  - Junyeob Lim
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- Undergraduate Course Assistants
  - Max Kopinsky
- Office hours: See course webpage
- Contacting us: Use private notes on Piazza to reach course staff. Direct email only for sensitive or confidential information.

Section A vs B

This semester, the two sections will be run completely independently.

- Different lectures.
- Different homeworks, quizzes, exams.
- Different grading policies.

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Section B will be in-person only.

Online resources

- Webpage: General information, announcements, homeworks, quizzes, course policies `https://canvas.illinois.edu/courses/13237`
- Gradescope: Written homework submission and grading, regrade requests
- Piazza: Announcements, online questions and discussion, contacting course staff (via private notes)

See course webpage for links

Important: check Piazza/course web page at least once each day

Grading Policy: Overview

- Quizzes: 4%

- Two assigned on Wednesday and due the following Wednesday.
- Covers topics from the week assigned.
- Get full marks as long as quiz is attempted.
- Approximately 20 quizzes assigned. Only need to complete 16 for full marks in final grade.

Grading Policy: Overview

- Quizzes: 4%
- Homeworks: 21%

- There will be approximately 9 HWs with 3 questions each.
- Hoemworks need to be submitted on Gradescope.
- Only the top 21 question grades will be considered with the

Grading Policy: Overview

- Quizzes: 4%
- Homeworks: 21%
- Midterm/Final exams: 75% (~~2~~3 × 2~~1~~5%)

Exam dates:

- Midterm 1: Thurs, Sep 23, 2:00pm–3:15pm
- Midterm 2: Thurs, Nov 4, 2:00pm–3:15pm
- Midterm 3: Thurs, Dec 2, 2:00pm–3:15pm
- Final: Thurs, Dec 14, 13:30––16:30

One exam will be dropped Drop policies should eliminate need for conflict exams.

Discussion Sessions/Labs

- 50min problem solving session led by TAs
- Two times a week
- Go to your assigned discussion section
- Bring pen and paper!

Advice

- Attend lectures, please ask plenty of questions.
- Attend discussion sessions.
- Don't skip homework and don't copy homework solutions. Each of you should think about <u>all</u> the problems on the home work - do not divide and conquer.
- Start homework early! Your mind needs time to think.
- Study regularly and keep up with the course.
- This is a course on problem solving. Solve as many as you can! Books/notes have plenty.
- This is also a course on providing rigourous proofs of correctness. Refresh your 173 background on proofs.
- Ask for help promptly. Make use of office hours/Piazza.

Miscellaneous

Please contact instructors if you need special accommodations.

Lectures are being taped. See course webpage.

Over-arching course questions

High-Level Questions

This course introduces three distinct fields of computer science research:

- Computational complexity.
  - Given infinite time and a certain machine, is it possible to solve a given problem.
- Algorithms
  - Given a deterministic Turing machine, how fast can we solve certain problems.
- Limits of computation.
  - Are there tasks that our computers cannot do and how do we identify these problems?

Course Structure

Course divided into three parts:

Complexity

- Basic automata theory: finite state machines, regular languages, hint of context free languages/grammars, Turing Machines

Algorithms

- Algorithms and algorithm design techniques
- Undecidability and NP-Completeness, reductions to prove intractability of problems

Week	Tuesday Lecture	Wed Lab	Thursday Lecture	Fri Lab
Aug 23-27	Adminis trivia and course goals Introduction and history ; strings [Sariel's Videos, Lec 1] [B: video, slides]	String induction [Jeff's induction notes , Chandra's induction notes] [solutions]	Languages and regular expressions [Sariel's Videos, Lec 2] [B: video, slides, scribble]	Regular expressions [solutions]
Aug 30 - Sep 3	DFAs: intuition, definitions, closure properties [Automata Tutor , JFLAP , Mahesh's DFA notes , Sariel's Videos, Lec 3] [B: video, slides, scribble]	DFA construction [solutions]	Non-Determinism, NFAs [Sariel's Videos, Lec 4] [B: video, slides, scribble]	DFA product construction [solutions]
Sep 6–10	Equivalence of DFAs, NFAs, and regular expressions [Sariel's Videos, Lec 5] [B: video, slides, scribble]	Regex to NFA to DFA (to Regex) [solutions]	Fooling Sets and Proving Non-Regularity [Mahesh's DFA notes , Fall 2015 TAs' Fooling Sets Notes , Sariel's Videos, Lec 6] [B: video, slides, scribble]	Proving Non-Regularity [solutions]
Sep 13–17	Context-free languages and grammars [Sariel's Videos, Lec 7] [B: video, slides, scribble]	Context-free grammars [solutions]	Turing machines: history, formal definitions, examples, variations [Sariel's Videos, Lec 8] [B: video, slides, scribble]	Turing Machines [solutions]
Sep 20 – 24	Universal Turing machines [Sariel's Videos, Lec 8] [B: video, slides, scribble]	Midterm 1 Review	**Midterm 1 – Thursday, Sep 23 13:30–16:30 [Skill set]**	No Instruction
Sep 27 – Oct 1	Reductions & Recursion [Sariel's Videos, Lec 10 , Notes on Solving Recurrences] **[B: video, slides, scribbles]**	Hint: Binary search [solutions]	Divide and conquer: Selection, Karatsuba [Sariel's Videos, Lec 11] [B: video,slides, scribbles]	Divide and Conquer [solutions]
Oct 4 – 8	Backtracking [Sariel's Videos, Lec 12] [B: video, slides, scribbles]	Backtracking [solutions]	Dynamic programming [Sariel's Videos, Lec 13] [B: video, slides, scribbles]	Dynamic programming [solutions]
Oct 11 – 15	More Dynamic programming [Sariel's Videos, Lec 14] [B: video, slides, scribbles]	More Dynamic programming [solutions]	Graphs, Basic Search [Chandra's Graph notes , Sariel's Videos, Lec 15] [B: video, slides, scribbles]	Even more DP [solutions]
Oct 18 – 22	Directed Graphs, DFS, DAGs and Topological Sort [Chandra's Graph notes , Sariel's Videos, Lec 16] [B: video, slides, scribbles]	Graph Modeling [solutions]	Shortest Paths: BFS and Dijkstra [Chandra's Graph notes , Sariel's Videos, Lec 17] [B: video, slides, scribbles]	Shortest Paths [solutions]
Oct 25 – 29	Bellman-Ford, Dynamic Programming on DAGs [Chandra's Graph notes , Sariel's Videos, Lec 18] [B: video, slides, scribbles]	More Shortest Paths [solutions]	MST Algorithms [B: video, slides, scribbles]	MST [solutions]
Nov 1 – 5	Reductions [Sariel's Videos, Lec 21] [B: video, slides, scribbles]	Midterm 2 Review	**Midterm 2 (Recursion/DP/Graph Algorithms) – Thursday, Nov 4 13:30–16:30 [Skill set]**	Reductions [solutions]
Nov 8 – 12	**SAT, NP and NP-Hardness** **[Sariel's Videos, Lec 22-24]** **[B: video, slides, scribbles]**	**NP-hardness reductions** **[solutions]**	More NP-Hardness [Sariel's Videos, Lec 23-24] [B: video, slides, scribbles]	More NP-Hardness [solutions]
Nov 15 – 19	Undecidability 1 [Sariel's Videos, Lec 9] [B: video, slides, scribbles]	Undecidability reductions [solutions]	Undecidability 2 [Sariel's Videos, Lec 9] [B: video, slides, scribbles]	No instruction
Thanksgiving Break (Nov 20-28). Give thanks.				
Nov 29 – Dec 3	*Optional review for Midterm 3* [B: video, slides, scribble]	*Optional Review for midterm 3*	**Midterm 3 (Reductions/P-NP/Decidability) – Thursday, Dec 2 14:00–15:15 [Skill set]**	
Dec 6 – Dec 10	Wrap-up, closing remarks *Optional review for Final Exam* [B: video, slides, scribbles]	*Optional Review for final exam*	**Reading Day** **ICES Forms Due**	

Final Exam – Tuesday, Dec 14 13:30–16:30 [Skill set]

Goals

- Algorithmic thinking
- Learn/remember some basic tricks, algorithms, problems, ideas
- Understand/appreciate limits of computation (intractability)
- Appreciate the importance of algorithms in computer science and beyond (engineering, mathematics, natural sciences, social sciences, …)

Formal languages and complexity (The Blue Weeks!)

Why Languages?

First 5 weeks devoted to language theory.

Why Languages?

First 5 weeks devoted to language theory.

But why study languages?

Multiplying Numbers

Consider the following problem:

Problem Given two n-digit numbers x and y, compute their product.

Grade School Multiplication
Compute "partial product" by multiplying each digit of y with x and adding the partial products.

$$
\begin{array}{r}
3141 \\
\times 2718 \\
\hline
25128 \\
3141 \\
21987 \\
6282 \\
\hline
8537238
\end{array}
$$

Time analysis of grade school multiplication

- Each partial product: $\Theta(n)$ time
- Number of partial products: $\leq n$
- Adding partial products: n additions each $\Theta(n)$ (Why?)
- Total time: $\Theta(n^2)$
- Is there a faster way?

Fast Multiplication

- $O(n^{1.58})$ time [Karatsuba 1960] disproving Kolmogorov's belief that $\Omega(n^2)$ is best possible
- $O(n \log n \log \log n)$ [Schonhage-Strassen 1971].
  Conjecture: $O(n \log n)$ time possible
- $O(n \log n \cdot 2^{O(\log^* n)})$ time [Furer 2008]
- $O(n \log n)$ [Harvey-van der Hoeven 2019]

Can we achieve $O(n)$? No lower bound beyond trivial one!

Equivalent Complexity

Does this mean multiplication is as complex as another problem that has a $O(n \log n)$ algorithm like sorting/QuickSort?

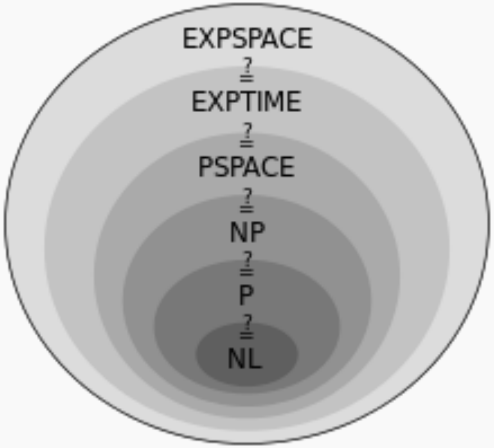
Equivalent Complexity

Does this mean multiplication is as complex as another problem that has a $O(n \log n)$ algorithm like sorting/QuickSort? How do we compare? The two problems have:

- Different inputs (two numbers vs n-element array)
- Different outputs (a number vs n-element array)
- Different entropy characteristics (from a information theory perspective)

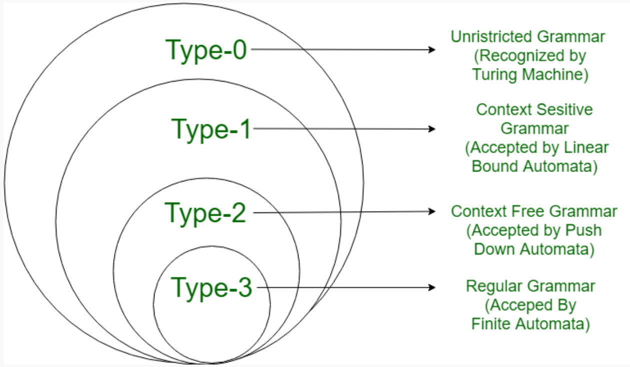
Languages, Problems and Algorithms ... oh my! II

An algorithm has a runtime complexity.



Languages, Problems and Algorithms ... oh my! III

A problem has a complexity class!



Problems do not have run-time since a problem $\neq$ the algorithm used to solve it. *Complexity classes are defined differently.*

How do we compare problems? What if we just want to know if a problem is "computable".

Definition

1. An algorithm is a step-by-step way to solve a problem.
2. A problem is some question that we'd like answered given some input. It should be a decision problem of the form "Does a given input fulfill property X."
3. A Language is a set of strings. Given a alphabet, Σ a language is a subset of Σ^*

Algorithms, Problems and Languages ... oh my! I

Definition

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3. A Language is a set of strings. Given a alphabet, Σ a language is a subset of Σ^* A language is a formal realization of this problem. For problem X, the corresponding language is:

   L = {w | w is the encoding of an input y to problem X and the answer to input y for a problem X is "YES" }
   A decision problem X is "YES" is the string is in the language.

Language of multiplication

How do we define the multiplication problem as a language?

Define L as language where inputs are separated by comma and output is separated by |.

Machine accepts a x*y=z if "x*y|z" is in L. Rejects otherwise.

$$\left\{ \begin{matrix} 1*1=1 \\ 1*2=2 \\ 2*2=4 \end{matrix} \right\} = L_{MULT}$$

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$\mathbb{N} \times \mathbb{N} \mid \text{Solution}$

$$L_{MULT2} = \left\{ \begin{matrix} 1 \times 1|1, & 1 \times 2|2, & 1 \times 3|3, \dots \\ 2 \times 1|2, & 2 \times 2|4, & 2 \times 3|6, \dots \\ \vdots & \vdots & \vdots \\ n \times 1|n, & n \times 2|2n, & n \times 3|3n, \dots \end{matrix} \right\} \tag{1}$$

$1 \times 1 | 1$ ✓

$1 \times 1 | 2$ ✗

M_{MULT}

Language of sorting

We do the same thing for sorting.

Define L as language where inputs are separated by comma and output is separated by |.

Machine accepts a $[i_1, i_2, \ldots] = sort(\{i_1, i_2, \ldots\})$ if "x[]|z[]" is in L. Rejects otherwise.

Sort$_2$

Language of sorting

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$$L_{Sort2} = \begin{Bmatrix} 1,1|1,1 & 1,2|1,2 & 1,3|1,3,\ldots \\ 2,1|1,2, & 2,2|2,2, & 2,3|2,3,\ldots \\ \vdots & \vdots & \vdots \\ n,1|1,n, & n,2|2,n, & n,3|3,n,\ldots \end{Bmatrix} \tag{2}$$

Language of sorting

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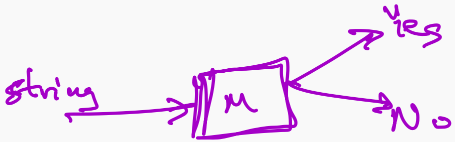

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If the same type of machine can recognize both languages, then that gives us an upperbound top their hardness.

How do we formulate languages?

Strings

Alphabet

An alphabet is a **finite** set of symbols.

$\Sigma = \{ \text{[illegible]}, \text{'x'}, \text{'1'} \}$

Examples of alphabets:

- $\Sigma = \{0, 1\}$,
- $\Sigma = \{a, b, c, \ldots, z\}$,
- ASCII.
- UTF8.
- $\Sigma = \{\langle\text{moveforward}\rangle, \langle\text{moveback}\rangle, \langle\text{moveleft}\rangle, \langle\text{moveright}\rangle\}$

Definition

1. A string/word over Σ is a finite sequence of symbols over Σ. For example, '0101001', '*string*', '$\langle\text{moveback}\rangle\langle\text{rotate90}\rangle$'
2. $x \cdot y \equiv xy$ is the concatenation of two strings "cat" · "dog" = "catdog"
3. The length of a string w (denoted by $|w|$) is the number of symbols in w. For example, $|101| = 3$, $|\epsilon| = 0$
4. For integer $n \geq 0$, Σ^n is set of all strings over Σ of length n. Σ^* is the set of all strings over Σ. $\Sigma = \{a \ldots z\}$ $\Sigma^3 = \{aaa, cat, dog, \ldots\}$
5. Σ^* set of all strings of all lengths including empty string.
Kleene star
$\Sigma^* = \{\Sigma^0 + \Sigma^1 + \Sigma^2 + \Sigma^3 + \ldots\}$

Question: $\{'a', 'c'\}^* =$ ϵ, a, c, aa, ac, ca, cc, aaa. . . .

Emptiness

- ϵ is a string containing no symbols. It is not a set
- $\{\epsilon\}$ is a set containing one string: the empty string. It is a set, not a string.
- $\emptyset$ is the empty set. It contains no strings.

Question: What is $\{\emptyset\}$ $= \{\{\}\}$

$\emptyset$

Concatenation and properties

- If x and y are strings then xy denotes their concatenation.
- Concatenation defined recursively :
  - $xy = y$ if $x = \epsilon$ base
  - $xy = a(wy)$ if $x = aw$
- xy sometimes written as $x \bullet y$.
- concatenation is associative: $(uv)w = u(vw)$ hence write $uvw \equiv (uv)w = u(vw)$
- **not** commutative: uv not necessarily equal to vu
- The identity element is the empty string ϵ:

$$\epsilon u = u\epsilon = u.$$

Substrings, prefixes, Suffixes

Definition
v is substring of w $\iff$ there exist strings x, y such that $w = xvy$.

x,y can be ϵ

- If $x = \epsilon$ then v is a prefix of w
- If $y = \epsilon$ then v is a suffix of w

Subsequence

A subsequence of a string $w[1...n]$ is either a subsequence of $w[2...n]$ or $w[1]$ followed by a subsequence of $w[2...n]$.

Example
kapa is a supsequence of *knapsack*

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Question: How many sub-sequences are there in a string $|w| = 5$?

a b o-ut

1

ab t

aot

a

ab

2^5

String exponent

Definition
If w is a string then w^n is defined inductively as follows:

$w^n = \epsilon$ if $n = 0$

$w^n = ww^{n-1}$ if $n > 0$

Question: $(blah)^3 =$. blah blah blah

Rapid-fire questions -strings

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. What is Σ^0? ϵ
2. How many elements are there in Σ^n? 2^n
3. If $|u| = 2$ and $|v| = 3$ then what is $|u \bullet v|$? 5
4. Let u be an arbitrary string in Σ^*. What is ϵu? What is $u\epsilon$? u

Languages

Languages

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

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Standard set operations apply to languages.

- For languages A, B the concatenation of A, B is $AB = \{xy \mid x \in A, y \in B\}$.
- For languages A, B, their union is $A \cup B$, intersection is $A \cap B$, and difference is $A \setminus B$ (also written as $A - B$).
- For language $A \subseteq \Sigma^*$ the complement of A is $\bar{A} = \Sigma^* \setminus A$.

Set Concatenation

Definition
Given two sets X and Y of strings (over some common alphabet Σ) the concatenation of X and Y is

$$XY = \{xy \mid x \in X, y \in Y\} \tag{3}$$

Question: $X = \{fido, rover, spot\}$, $Y = \{fluffy, tabby\} \implies$
$XY = .$ $\left| \{ \text{fido fluffy} \quad \ldots \quad \text{spot tabby} \} \right| = 6$

Definition

1. Σ^n is the set of all strings of length n. Defined inductively:
   $\Sigma^n = \{\epsilon\}$ if $n = 0$
   $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
2. $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ is the set of all finite length strings
3. $\Sigma^+ = \cup_{n \geq 1} \Sigma^n$ is the set of non-empty strings. avoids empty string

Definition

A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Question: Does Σ^* have strings of infinite length? NO!!

Problem
Consider languages over $\Sigma = \{0, 1\}$.

1. *What is $\emptyset^0$?* $\{\epsilon\}$
2. *If $|L| = 2$, then what is $|L^4|$?* 16
3. *What is $\emptyset^*$, $\{\epsilon\}^*$, ϵ^*?* $= \emptyset^0 \cup \emptyset^1 \cdots$ $\{\epsilon\}$
4. *For what L is L^* finite?* $\{\epsilon\}$, $\emptyset$
5. *What is $\emptyset^+$, $\{\epsilon\}^+$, ϵ^+?*

$\{\}$ $\{\epsilon\}$ $\{\epsilon\}$

$\{\epsilon\}^* = \{\epsilon\}^0 \cup \{\epsilon\}^1 \cup$

$\epsilon\epsilon\epsilon = \epsilon$

Terminology Review

Let's review what we learned.

- A character(a, b, c, x) is a unit of information represented by a symbol: (letters, digits, whitespace)
- A alphabet(Σ) is a set of characters
- A string(w) is a sequence of characters
- A language(A, B, C, L) is a set of strings

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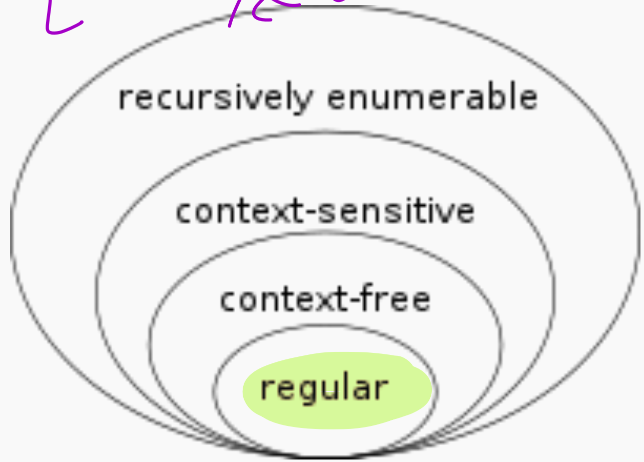
- A string(w) is a sequence of characters
- A language(A, B, C, L) is a set of strings
- A grammar(G) is a set of rules that defines the strings that belong to a language

Languages: easiest, easy, hard, really hard, really^n hard

- Regular languages.
    - Regular expressions.
    - DFA: Deterministic finite automata.
    - NFA: Non-deterministic finite automata.
    - Languages that are not regular.
- Context free languages (stack).
- Turing machines: Decidable languages.
- TM Undecidable/unrecognizable languages (halting theorem).

$L^* = L^0 \cup L^1 \cup L^2 \cup L^3$

$L^+ = \cancel{L^0} \cup \cdots \cup \ldots$



$\{\}^1 = \{\}$

$\underbrace{\{\}}_{n} \cdot \underbrace{\{w_1, \ldots, w_m\}}_{m} = \underbrace{\{\quad\}}_{n \times m}$

$\emptyset^1 = \emptyset \cdot \{\epsilon\} = \emptyset$

Induction on strings

Inductive proofs on strings

Inductive proofs on strings and related problems follow inductive definitions.

Definition
The reverse w^R of a string w is defined as follows:

- $w^R = \epsilon$ if $w = \epsilon$
- $w^R = x^R a$ if $w = ax$ for some $a \in \Sigma$ and string x

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Theorem
Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Example: $(dog \bullet cat)^R = (cat)^R \bullet (dog)^R = tacgod$.

Principle of mathematical induction

Induction is a way to prove statements of the form $\forall n \geq 0, P(n)$ where $P(n)$ is a statement that holds for integer n.

Example: Prove that $\sum_{i=0}^{n} i = n(n+1)/2$ for all n.

Induction template:

- **Base case:** Prove $P(0)$
- **Induction hypothesis:** Let $k > 0$ be an **arbitrary** integer. Assume that $P(n)$ holds for any $n \leq k$.
- **Induction Step:** Prove that $P(n)$ holds, for $n = k + 1$.

Structured induction

- Unlike simple cases we are working with...
- ...induction proofs also work for more complicated “structures”.
- Such as strings, tuples of strings, graphs etc.
- See class notes on induction for details.

Proving the theorem

Theorem
Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Proof: by induction.

On what?? $|uv| = |u| + |v|$?

$|u|$?

$|v|$?

What does it mean "induction on $|u|$"?

By induction on |u|

Theorem
Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Proof by induction on $|u|$ means that we are proving the following.

Base case: Let u be an arbitrary string of length 0. $u = \epsilon$ since there is only one such string. Then

$$(uv)^R = (\epsilon v)^R = v^R = v^R \epsilon = v^R \epsilon^R = v^R u^R$$

By induction on $|u|$

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Induction hypothesis: $\forall n \geq 0$, for any string u of length n:

For all strings $v \in \Sigma^*$, $(uv)^R = v^R u^R$.

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Theorem
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Induction hypothesis: $\forall n \geq 0$, for any string u of length n:

For all strings $v \in \Sigma^*$, $(uv)^R = v^R u^R$.

No assumption about v, hence statement holds for all $v \in \Sigma^*$.

Inductive step

- Let u be an arbitrary string of length $n > 0$. Assume inductive hypothesis holds for all strings w of length $< n$.
- Since $|u| = n > 0$ we have $u = ay$ for some string y with $|y| < n$ and $a \in \Sigma$.
- Then

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- Then

$$(uv)^R \quad =$$

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- Since $|u| = n > 0$ we have $u = ay$ for some string y with $|y| < n$ and $a \in \Sigma$.
- Then

$$\begin{aligned}
(uv)^R &= ((ay)v)^R \\
&= (a(yv))^R \\
&= (yv)^R a^R \\
&= (v^R y^R) a^R \\
&= v^R (y^R a^R) \\
&= v^R (ay)^R \\
&= v^R u^R
\end{aligned}$$

Another example!

Theorem
Prove that for any strings x and y, $|xy| = |x| + |y|$