Let L be an arbitrary regular language over the alphabet $\Sigma = \{0, 1\}$. Prove that the following languages are also regular. (You probably won't get to all of these.)

1. $\textsc{FlipOdds}(L) := \{\mathit{flipOdds}(w) \mid w \in L\}$, where the function *flipOdds* inverts every odd-indexed bit in w. For example:

$$\mathit{flipOdds}(0000111101010101) = 1010010111111111$$

Solution: Let $M = (Q, s, A, \delta)$ be a DFA that accepts L. We construct a new DFA $M' = (Q', s', A', \delta')$ that accepts $\textsc{FlipOdds}(L)$ as follows.

Intuitively, M' receives some string $\mathit{flipOdds}(w)$ as input, restores every other bit to obtain w, and simulates M on the restored string w.

Each state (q, flip) of M' indicates that M is in state q, and we need to flip the next input bit if $\mathit{flip} = \textsc{True}$.

$$\begin{aligned} Q' &= Q \times \{\textsc{True}, \textsc{False}\} \\ s' &= (s, \textsc{True}) \\ A' &= \\ \delta'((q, \mathit{flip}), a) &= \end{aligned}$$

■

2. $\textsc{UnflipOdd1s}(L) := \{w \in \Sigma^* \mid \mathit{flipOdd1s}(w) \in L\}$, where the function *flipOdd1* inverts every other 1 bit of its input string, starting with the first 1. For example:

$$\mathit{flipOdd1s}(0000\underline{1}\underline{1}10\underline{1}010\underline{1}01) = 00000\underline{1}0\underline{1}00\underline{1}0\underline{0}01$$

Solution: Let $M = (Q, s, A, \delta)$ be a DFA that accepts L. We construct a new DFA $M' = (Q', s', A', \delta')$ that accepts $\textsc{UnflipOdd1s}(L)$ as follows.

Intuitively, M' receives some string w as input, flips every other 1 bit, and simulates M on the transformed string.

Each state (q, flip) of M' indicates that M is in state q, and we need to flip the next 1 bit of and only if $\mathit{flip} = \textsc{True}$.

$$\begin{aligned} Q' &= Q \times \{\textsc{True}, \textsc{False}\} \\ s' &= (s, \textsc{True}) \\ A' &= \\ \delta'((q, \mathit{flip}), a) &= \end{aligned}$$

■

3. FlipOdd1s$(L) := \{flipOdd1s(w) \mid w \in L\}$, where the function *flipOdd1* is defined as in the previous problem.

Solution: Let $M = (Q, s, A, \delta)$ be a DFA that accepts L. We construct a new **NFA** $M' = (Q', s', A', \delta')$ that accepts FlipOdd1s(L) as follows.

Intuitively, M' receives some string $flipOdd1s(w)$ as input, ***guesses*** which `0` bits to restore to `1`s, and simulates M on the restored string w. No string in FlipOdd1s(L) has two `1`s in a row, so if M' ever sees `11`, it rejects.

Each state $(q, flip)$ of M' indicates that M is in state q, and we need to flip a `0` bit before the next `1` if $flip =$ True.

$$Q' = Q \times \{\text{True}, \text{False}\}$$
$$s' = (s, \text{True})$$
$$A' =$$
$$\delta'((q, flip), a) =$$

■

4. cycle$(L) := \{xy | x, y \in \Sigma^*, yx \in L\}$, The language that accepts the rotations of string from a regular language.

Solution: The intuition here is that we want to traverse the DFA using any possible suffix of the string, then traverse the prefix before finally coming to rest in an accept state.

$$Q' = Q_{\text{cycle}} := Q \times Q \cup \{s_{\text{cycle}}, finish, over\}$$
$$s' = s_{\text{cycle}}$$
$$A'_{\text{cycle}} := \{finish\}$$
$$\delta'((q, flip), a) =$$

■

5. Prove that the language $\mathit{insert}\texttt{1}(L) := \{x\texttt{1}y \mid xy \in L\}$ is regular.

   Intuitively, $\mathit{insert}\texttt{1}(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one `1`. For example, if $L = \{\varepsilon, \texttt{OOK!}\}$, then $\mathit{insert}\texttt{1}(L) = \{\texttt{1}, \texttt{1OOK!}, \texttt{O1OK!}, \texttt{OO1K!}, \texttt{OOK1!}, \texttt{OOK!1}\}$.

Solution: Let $M = (\Sigma, Q, s, A, \delta)$ be a DFA that accepts L. We construct an **NFA** $M' = (\Sigma, Q', s', A', \delta')$ that accepts $\mathit{insert}\texttt{1}(L)$ as follows.

Intuitively, M' nondeterministically chooses a `1` in the input string to ignore, and simulates M running on the rest of the input string.

- The state (q, before) means (the simulation of) M is in state q and M' has not yet skipped over a `1`.
- The state (q, after) means (the simulation of) M is in state q and M' has already skipped over a `1`.

$$Q' = Q \times \{\mathit{before}, \mathit{after}\}$$

$$s' = (s, \mathit{before})$$

$$A' =$$

$$\delta'((q, \mathit{before}), a) =$$

$$\delta'((q, \mathit{after}), a) =$$

■

Work on these later:

5. Prove that the language $\mathit{delete}\texttt{1}(L) := \{xy \mid x\texttt{1}y \in L\}$ is regular.

   Intuitively, $\mathit{delete}\texttt{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\texttt{1}$. For example, if $L = \{\texttt{101101}, \texttt{00}, \varepsilon\}$, then $\mathit{delete}\texttt{1}(L) = \{\texttt{01101}, \texttt{10101}, \texttt{10110}\}$.

6. Consider the following recursively defined function on strings:

$$\mathit{stutter}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet \mathit{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   Intuitively, $\mathit{stutter}(w)$ doubles every symbol in w. For example:

   - $\mathit{stutter}(\texttt{PRESTO}) = \texttt{PPRREESSTTOO}$
   - $\mathit{stutter}(\texttt{HOCUS}\diamond\texttt{POCUS}) = \texttt{HHOOCCUUSS}\diamond\diamond\texttt{PPOOCCUUSS}$

   (a) Prove that the language $\mathit{stutter}^{-1}(L) := \{w \mid \mathit{stutter}(w) \in L\}$ is regular.

   (b) Prove that the language $\mathit{stutter}(L) := \{\mathit{stutter}(w) \mid w \in L\}$ is regular.

7. Consider the following recursively defined function on strings:

$$\mathit{evens}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot \mathit{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

   Intuitively, $\mathit{evens}(w)$ skips over every other symbol in w. For example:

   - $\mathit{evens}(\texttt{EXPELLIARMUS}) = \texttt{XELAMS}$
   - $\mathit{evens}(\texttt{AVADA}\diamond\texttt{KEDAVRA}) = \texttt{VD}\diamond\texttt{EAR}$.

   (a) Prove that the language $\mathit{evens}^{-1}(L) := \{w \mid \mathit{evens}(w) \in L\}$ is regular.

   (b) Prove that the language $\mathit{evens}(L) := \{\mathit{evens}(w) \mid w \in L\}$ is regular.

You may find it helpful to imagine these transformations concretely on the following DFA for the language specified by the regular expression 00^*11^*.