Prove that each of the following languages is ***not*** regular.

1. $\{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}$

Solution (verbose): Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{0}^i\mathtt{1}^i$.

Then $xz = \mathtt{0}^{2i}\mathtt{1}^i \in L$.

And $yz = \mathtt{0}^{i+j}\mathtt{1}^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^i$ and $\mathtt{0}^j$ are distinguished by the suffix $\mathtt{0}^i\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{i+j}\mathtt{1}^i \notin L$. Thus, the language $\mathtt{0}^*$ is an infinite fooling set for L. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^{2i}$ and $\mathtt{0}^{2j}$ are distinguished by the suffix $\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{2j}\mathtt{1}^i \notin L$. Thus, the language $(\mathtt{00})^*$ is an infinite fooling set for L. ■

2. $\{0^m 1^n \mid m \neq 2n\}$

Solution (verbose): Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \notin L$.

And $yz = 0^{i+j} 1^i \in L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i, because $0^{2i} 1^i \notin L$ but $0^{2j} 1^i \in L$. Thus, the language $(00)^*$ is an infinite fooling set for L. ■

3. $\left\{0^{2^n} \;\middle|\; n \geq 0\right\}$

Solution (verbose): Let $F = L = \left\{0^{2^n} \;\middle|\; n \geq 0\right\}$.

Let x and y be arbitrary elements of F.

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers x and y.

Let $z = 0^{2^i}$.

Then $xz = 0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$.

And $yz = 0^{2^j} 0^{2^i} = 0^{2^i + 2^j} \notin L$, because $i \neq j$

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings 0^{2^i} and 0^{2^j} are distinguished by the suffix 0^{2^i}, because $0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$ but $0^{2^j} 0^{2^i} = 0^{2^i + 2^j} \notin L$. Thus L itself is an infinite fooling set for L. ■

4. Strings over $\{\mathtt{0}, \mathtt{1}\}$ where the number of $\mathtt{0}$s is exactly twice the number of $\mathtt{1}$s.

Solution (verbose): Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{0}^i\mathtt{1}^i$.

Then $xz = \mathtt{0}^{2i}\mathtt{1}^i \in L$.

And $yz = \mathtt{0}^{i+j}\mathtt{1}^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^{2i}$ and $\mathtt{0}^{2j}$ are distinguished by the suffix $\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{2j}\mathtt{1}^i \notin L$. Thus, the language $(\mathtt{00})^*$ is an infinite fooling set for L. ■

Solution (closure properties): If L were regular, then the language

$$L \cap \mathtt{0}^*\mathtt{1}^* = \left\{\mathtt{0}^{2n}\mathtt{1}^n \;\middle|\; n \geq 0\right\}$$

would also be regular since regular languages are closed under intersection but we have seen in Problem 1 that $\left\{\mathtt{0}^{2n}\mathtt{1}^n \;\middle|\; n \geq 0\right\}$ is not regular.

Another solution based on closure properties. If L were regular, then the language

$$\big((\mathtt{0}+\mathtt{1})^* \setminus L\big) \cap \mathtt{0}^*\mathtt{1}^* = \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$ is not regular in problem 2. *[Yes, this proof would be worth full credit, either in homework or on an exam.]*

Note that the proofs based on closure properties relied on non-regularity of some previously known languages. One could also think of the proofs as allowing you to simplify the initial language to a more structured one which may be easier to work with. ■

5. Strings of properly nested parentheses `()`, brackets `[]`, and braces `{}`. For example, the string `([]){}` is in this language, but the string `([)]` is not, because the left and right delimiters don't match.

Solution (verbose): Let F be the language $\texttt{(}^*$.

Let x and y be arbitrary strings in F.

Then $x = \texttt{(}^i$ and $y = \texttt{(}^j$ for some non-negative integers $i \neq j$.

Let $z = \texttt{)}^i$.

Then $xz = \texttt{(}^i\texttt{)}^i \in L$.

And $yz = \texttt{(}^j\texttt{)}^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings $\texttt{(}^i$ and $\texttt{(}^j$ are distinguished by the suffix $\texttt{)}^i$, because $\texttt{(}^i\texttt{)}^i \in L$ but $\texttt{(}^i\texttt{)}^j \notin L$. Thus, the language $\texttt{(}^*$ is an infinite fooling set. ■

Solution (closure properties): If L were regular, then the language $L \cap \texttt{(}^*\texttt{)}^* = \{\texttt{(}^n\texttt{)}^n \mid n \geq 0\}$ would be regular. The language $\{\texttt{(}^n\texttt{)}^n \mid n \geq 0\}$ is the same as $\{\texttt{0}^n\texttt{1}^n \mid n \geq 0\}$ modulo changing the symbol names and is not regular from lecture. Thus L is not regular. ■

6. w, such that $|w| = \lceil k\sqrt{k} \rceil$, for some natural number k.

   Hint: since this one is more difficult, we'll even give you a fooling set that works: try $F = \{\mathtt{0}^{m^6} | m \geq 1\}$. We'll also provide a bound that can help: the difference between consecutive strings in the language, $\lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil$, is bounded above and below as follows
   $$1.5\sqrt{k} - 1 \leq \lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil \leq 1.5\sqrt{k} + 3$$
   All that's left is you need to carefully prove that F is a fooling set for L.

   Solution: HW Problem. ■

7. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0,1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution (verbose): Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = \#0^i$.

Then $xz = 0^i\#0^i \in L$.

And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings 0^i and 0^j are distinguished by the suffix $\#0^i$, because $0^i\#0^i \in L$ but $0^j\#0^i \notin L$. Thus, the language 0^* is an infinite fooling set. ■

Work on these later:

7. $\{\texttt{0}^{n^2} \mid n \geq 0\}$

Solution: Let x and y be distinct arbitrary strings in L.

Without loss of generality, $x = \texttt{0}^{i^2}$ and $y = \texttt{0}^{j^2}$ for some $i > j \geq 0$.

Let $z = \texttt{0}^{2i+1}$.

Then $xz = \texttt{0}^{i^2+2i+1} = \texttt{0}^{(i+1)^2} \in L$

On the other hand, $yz = \texttt{0}^{i^2+2j+1} \notin L$, because $i^2 < i^2 + 2j + 1 < (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that L is an infinite fooling set for L, so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in $\texttt{0}^*$.

Without loss of generality, $x = \texttt{0}^i$ and $y = \texttt{0}^j$ for some $i > j \geq 0$.

Let $z = \texttt{0}^{i^2+i+1}$.

Then $xz = \texttt{0}^{i^2+2i+1} = \texttt{0}^{(i+1)^2} \in L$.

On the other hand, $yz = \texttt{0}^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that $\texttt{0}^*$ is an infinite fooling set for L, so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in $\texttt{0000}^*$.

Without loss of generality, $x = \texttt{0}^i$ and $y = \texttt{0}^j$ for some $i > j \geq 3$.

Let $z = \texttt{0}^{i^2-i}$.

Then $xz = \texttt{0}^{i^2} \in L$.

On the other hand, $yz = \texttt{0}^{i^2-i+j} \notin L$, because

$$(i-1)^2 \;=\; i^2 - 2i + 1 \;<\; i^2 - i \;<\; i^2 - i + j \;<\; i^2.$$

(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.)

Thus, z distinguishes x and y.

We conclude that $\texttt{0000}^*$ is an infinite fooling set for L, so L cannot be regular. ■

8. $\{w \in (\mathtt{0} + \mathtt{1})^* \mid w \text{ is the binary representation of a perfect square}\}$

Solution: We design our fooling set around numbers of the form $(2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = \mathtt{10}^{k-2}\mathtt{10}^k\mathtt{1} \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = \mathtt{1}(\mathtt{00})^*\mathtt{1}$, and let x and y be arbitrary strings in F.

Then $x = \mathtt{10}^{2i-2}\mathtt{1}$ and $y = \mathtt{10}^{2j-2}\mathtt{1}$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = \mathtt{0}^{2i}\mathtt{1}$.

Then $xz = \mathtt{10}^{2i-2}\mathtt{10}^{2i}\mathtt{1}$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$, and therefore $xz \in L$.

On the other hand, $yz = \mathtt{10}^{2j-2}\mathtt{10}^{2i}\mathtt{1}$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}
(2^{i+j})^2 &= 2^{2i+2j} \\
&< \mathbf{2^{2i+2j} + 2^{2i+1} + 1} \\
&< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
&= (2^{i+j} + 1)^2.
\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L. Because F is infinite, L cannot be regular. ■