Design Turing machines $M = (Q, \Sigma, \Gamma, \delta, \text{start}, \text{accept}, \text{reject})$ for each of the following tasks, either by listing the states Q, the tape alphabet Γ, and the transition function δ (in a table), or by drawing the corresponding labeled graph.

Each of these machines uses the input alphabet $\Sigma = \{\mathbf{1}, \#\}$; the tape alphabet Γ can be any superset of $\{\mathbf{1}, \#, \square, \triangleright\}$ where $\square$ is the blank symbol and $\triangleright$ is a special symbol marking the left end of the tape. Each machine should reject any input not in the form specified below.

The solutions below describe single-tape, single-head Turing machines. There are arguably simpler Turing machines that multiple tapes and/or multiple heads.

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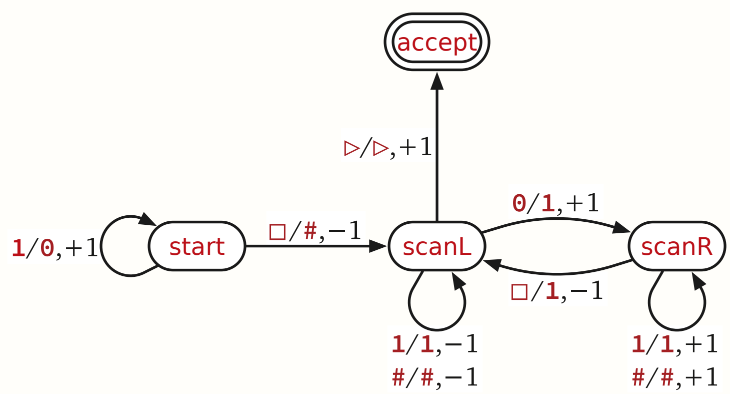
1. On input $\mathbf{1}^n$, for any non-negative integer n, write $\mathbf{1}^n\#\mathbf{1}^n$ on the tape and accept.

Solution: Our Turing machine M_1 uses the tape alphabet $\Gamma = \{\mathbf{0}, \mathbf{1}, \#, \square, \triangleright\}$ and the following states, in addition to accept and reject:

- start — Initialize the tape by replacing every **1** with **0**. When we find a blank, write **#** and start scanning left.
- scanL — Scan left for the rightmost **0**. If we find it, replace it with **1** and start scanning right. If we find $\triangleright$ instead, we're done; halt and accept.
- scanR — Scan right for the leftmost blank. When we find it, write **1** and start scanning left again.

Here is the transition graph of the machine. To simplify the drawing, we omit all transitions into the hidden reject state.



Here is the transition function; again, all unspecified transitions lead to the reject state.

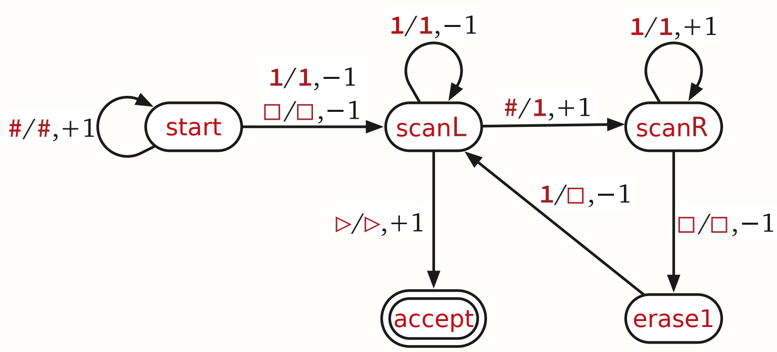
$\delta(\ p\ , a) = (\ q\ , b, \Delta)$	explanation
$\delta(\text{start}, \mathtt{1}) = (\text{start}, \mathtt{0}, +1)$	init phase: replace **1**s with **0**s
$\delta(\text{start}, \square) = (\text{scanL}, \mathtt{\#}, -1)$	finished init phase; write **#** and start scanning left
$\delta(\text{scanL}, \mathtt{1}) = (\text{scanL}, \mathtt{1}, -1)$	scan left to rightmost **0**
$\delta(\text{scanL}, \mathtt{\#}) = (\text{scanL}, \mathtt{\#}, -1)$	
$\delta(\text{scanL}, \mathtt{0}) = (\text{scanR}, \mathtt{1}, +1)$	found it; write **1** and start scanning right
$\delta(\text{scanL}, \triangleright) = (\text{accept}, \triangleright, +1)$	found start of tape instead; we're done!
$\delta(\text{scanR}, \mathtt{1}) = (\text{scanR}, \mathtt{1}, +1)$	main loop: scan right to leftmost $\square$
$\delta(\text{scanR}, \mathtt{\#}) = (\text{scanR}, \mathtt{\#}, +1)$	
$\delta(\text{scanR}, \square) = (\text{scanL}, \mathtt{1}, -1)$	found it; write **1** and start scanning left ■

2. On input $\#^n 1^m$, for any non-negative integers m and n, write 1^m on the tape and accept. In other words, delete all the #s, thereby shifting the 1s to the start of the tape.

Solution: Our machine M_2 repeatedly scans for the last # and replaces it with 1, then scans for the rightmost 1 and replaces it with a blank, until the search for the last # fails. We use the minimal tape alphabet $\Gamma = \{1, \#, \square, \triangleright\}$ and the following states, in addition to accept and reject:

- start — Scan right past all #s
- scanL — Scan left to the rightmost # or $\triangleright$. If we find #, replace it with 1; if we find $\triangleright$, we're done!
- scanR — Scan right to the leftmost $\square$ (just after the rightmost 1, if any).
- erase1 — Replace the rightmost 1 with $\square$.

Here is the transition graph of the machine. To simplify the drawing, we omit all transitions into the hidden reject state.



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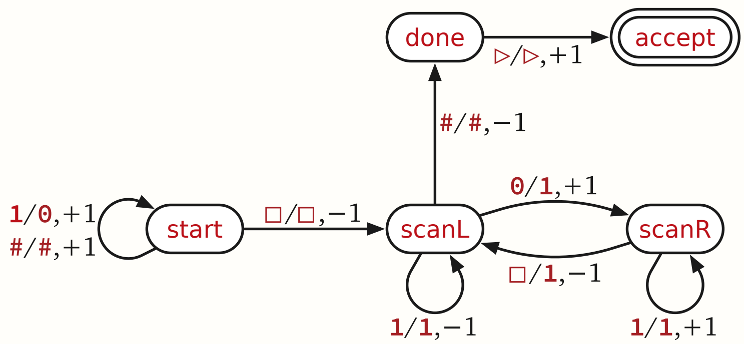
3. On input **#1**^{n}, for any non-negative integer n, write **#1**^{2n} on the tape and accept. *[Hint: Modify the Turing machine from problem 1.]*

Solution: Our machine M_3 mirrors M_1 with a few minor changes. First, we won't both writing a second **#** between the first and second copies of the input string; second, we treat the initial **#** as the de-facto beginning of the tape. Here are the states:

- start — Scan right for first blank, replacing **1**s with **0**s
- scanL — Scan left for rightmost **0**, replace with **1**
- scanR — Scan right for leftmost blank, replace with **1**
- done — Found the initial **#**; reset the head to the start position and accept

And here is the transition graph, as usual omitting transitions to reject.



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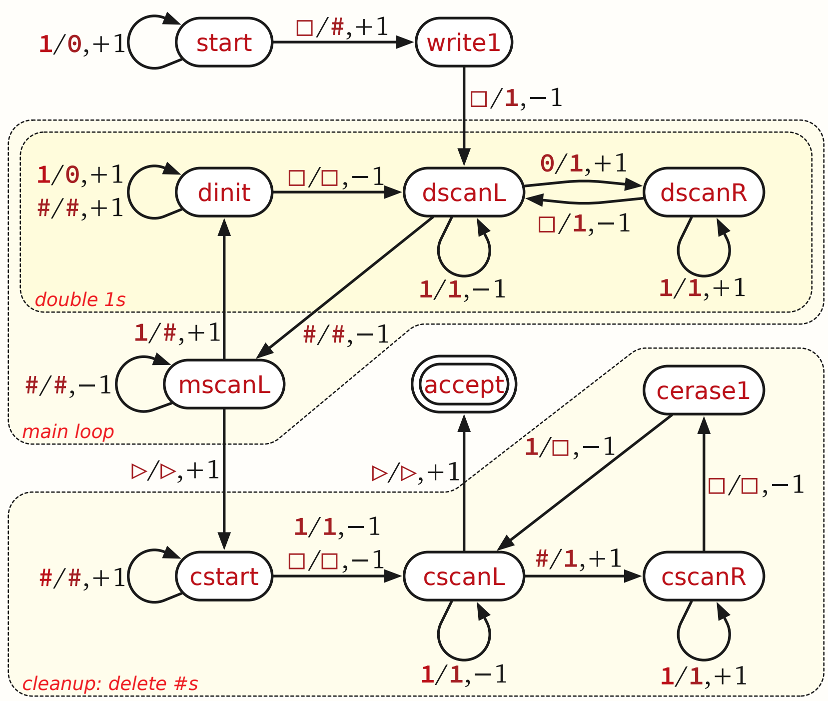
4. On input $\mathtt{1}^n$, for any non-negative integer n, write $\mathtt{1}^{2^n}$ on the tape and accept. *[Hint: Use the three previous Turing machines as subroutines.]*

Solution: Our machine M_4 works in several phases:

- Write **#1** at the end of the input string
- Repeatedly transform $\mathtt{1}^a\mathtt{\#}^b\mathtt{1}^c$ into $\mathtt{1}^{a-1}\mathtt{\#}^{b+1}\mathtt{1}^{2c}$ using a small modification of M_3 (which uses M_1 as a subroutine).
- When the initial string of **1**s is empty, remove all **#**s using M_2.

So here are the states:

- start: Scan right for a blank, and write **#**
- write1: Write **1** after **#** and start main loop
- three states from M_3 to double the number **1**s to the right of **#**s
- scanL1: scan left for rightmost **1** left of **#**s, replace with **#** and repeat main loop
- four states from M_2 to delete the **#**s



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