

Prove that each of the following languages is *not* regular.

1.  $\{0^{2n}1^n \mid n \geq 0\}$

**Solution (verbose):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^i1^i$ .

Then  $xz = 0^{2i}1^i \in L$ .

And  $yz = 0^{i+j}1^i \notin L$ , because  $i + j \neq 2i$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise):** For all non-negative integers  $i \neq j$ , the strings  $0^i$  and  $0^j$  are distinguished by the suffix  $0^i1^i$ , because  $0^{2i}1^i \in L$  but  $0^{i+j}1^i \notin L$ . Thus, the language  $0^*$  is an infinite fooling set for  $L$ . ■

**Solution (concise, different fooling set):** For all non-negative integers  $i \neq j$ , the strings  $0^{2i}$  and  $0^{2j}$  are distinguished by the suffix  $1^i$ , because  $0^{2i}1^i \in L$  but  $0^{2j}1^i \notin L$ . Thus, the language  $(00)^*$  is an infinite fooling set for  $L$ . ■

2.  $\{0^m 1^n \mid m \neq 2n\}$

**Solution (verbose):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^i 1^i$ .

Then  $xz = 0^{2i} 1^i \notin L$ .

And  $yz = 0^{i+j} 1^i \in L$ , because  $i + j \neq 2i$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise, different fooling set):** For all non-negative integers  $i \neq j$ , the strings  $0^{2i}$  and  $0^{2j}$  are distinguished by the suffix  $1^i$ , because  $0^{2i} 1^i \notin L$  but  $0^{2j} 1^i \in L$ . Thus, the language  $(00)^*$  is an infinite fooling set for  $L$ . ■

3.  $\{0^{2^n} \mid n \geq 0\}$

**Solution (verbose):** Let  $F = L = \{0^{2^n} \mid n \geq 0\}$ .

Let  $x$  and  $y$  be arbitrary elements of  $F$ .

Then  $x = 0^{2^i}$  and  $y = 0^{2^j}$  for some non-negative integers  $x$  and  $y$ .

Let  $z = 0^{2^i}$ .

Then  $xz = 0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$ .

And  $yz = 0^{2^j} 0^{2^i} = 0^{2^i+2^j} \notin L$ , because  $i \neq j$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise):** For any non-negative integers  $i \neq j$ , the strings  $0^{2^i}$  and  $0^{2^j}$  are distinguished by the suffix  $0^{2^i}$ , because  $0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$  but  $0^{2^j} 0^{2^i} = 0^{2^i+2^j} \notin L$ . Thus  $L$  itself is an infinite fooling set for  $L$ . ■

4. Strings over  $\{0, 1\}$  where the number of 0s is exactly twice the number of 1s.

**Solution (verbose):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^i 1^i$ .

Then  $xz = 0^{2i} 1^i \in L$ .

And  $yz = 0^{i+j} 1^i \notin L$ , because  $i + j \neq 2i$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise, different fooling set):** For all non-negative integers  $i \neq j$ , the strings  $0^{2i}$  and  $0^{2j}$  are distinguished by the suffix  $1^i$ , because  $0^{2i} 1^i \in L$  but  $0^{2j} 1^i \notin L$ . Thus, the language  $(00)^*$  is an infinite fooling set for  $L$ . ■

**Solution (closure properties):** If  $L$  were regular, then the language

$$L \cap 0^* 1^* = \{0^{2n} 1^n \mid n \geq 0\}$$

would also be regular since regular languages are closed under intersection but we have seen in Problem 1 that  $\{0^{2n} 1^n \mid n \geq 0\}$  is not regular.

Another solution based on closure properties. If  $L$  were regular, then the language

$$((0 + 1)^* \setminus L) \cap 0^* 1^* = \{0^m 1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that  $\{0^m 1^n \mid m \neq 2n\}$  is not regular in problem 2. [Yes, this proof would be worth full credit, either in homework or on an exam.]

Note that the proofs based on closure properties relied on non-regularity of some previously known languages. One could also think of the proofs as allowing you to simplify the initial language to a more structured one which may be easier to work with. ■

5. Strings of properly nested parentheses  $()$ , brackets  $[]$ , and braces  $\{\}$ . For example, the string  $([])\{\}$  is in this language, but the string  $([])]$  is not, because the left and right delimiters don't match.

**Solution (verbose):** Let  $F$  be the language  $()^*$ .

Let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = ()^i$  and  $y = ()^j$  for some non-negative integers  $i \neq j$ .

Let  $z = )^i$ .

Then  $xz = ()^i)^i \in L$ .

And  $yz = ()^j)^i \notin L$ , because  $i \neq j$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise):** For any non-negative integers  $i \neq j$ , the strings  $()^i$  and  $()^j$  are distinguished by the suffix  $)^i$ , because  $()^i)^i \in L$  but  $()^j)^i \notin L$ . Thus, the language  $()^*$  is an infinite fooling set. ■

**Solution (closure properties):** If  $L$  were regular, then the language  $L \cap ()^* = \{()^n)^n \mid n \geq 0\}$  would be regular. The language  $\{()^n)^n \mid n \geq 0\}$  is the same as  $\{0^n 1^n \mid n \geq 0\}$  modulo changing the symbol names and is not regular from lecture. Thus  $L$  is not regular. ■

6.  $w$ , such that  $|w| = \lceil k\sqrt{k} \rceil$ , for some natural number  $k$ .

Hint: since this one is more difficult, we'll even give you a fooling set that works: try  $F = \{0^m \mid m \geq 1\}$ . We'll also provide a bound that can help: the difference between consecutive strings in the language,  $\lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil$ , is bounded above and below as follows

$$1.5\sqrt{k} - 1 \leq \lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil \leq 1.5\sqrt{k} + 3$$

All that's left is you need to carefully prove that  $F$  is a fooling set for  $L$ .

**Solution:** HW Problem. ■

7. Strings of the form  $w_1 \# w_2 \# \dots \# w_n$  for some  $n \geq 2$ , where each substring  $w_i$  is a string in  $\{0, 1\}^*$ , and some pair of substrings  $w_i$  and  $w_j$  are equal.

**Solution (verbose):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = \#0^i$ .

Then  $xz = 0^i \# 0^i \in L$ .

And  $yz = 0^j \# 0^i \notin L$ , because  $i \neq j$ .

Thus,  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (concise):** For any non-negative integers  $i \neq j$ , the strings  $0^i$  and  $0^j$  are distinguished by the suffix  $\#0^i$ , because  $0^i \# 0^i \in L$  but  $0^j \# 0^i \notin L$ . Thus, the language  $0^*$  is an infinite fooling set. ■

Work on these later:

7.  $\{\mathbf{0}^{n^2} \mid n \geq 0\}$

**Solution:** Let  $x$  and  $y$  be distinct arbitrary strings in  $L$ .

Without loss of generality,  $x = \mathbf{0}^{2i+1}$  and  $y = \mathbf{0}^{2j+1}$  for some  $i > j \geq 0$ .

Let  $z = \mathbf{0}^{i^2}$ .

Then  $xz = \mathbf{0}^{i^2+2i+1} = \mathbf{0}^{(i+1)^2} \in L$

On the other hand,  $yz = \mathbf{0}^{i^2+2j+1} \notin L$ , because  $i^2 < i^2 + 2j + 1 < (i+1)^2$ .

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $L$  is an infinite fooling set for  $L$ , so  $L$  cannot be regular. ■

**Solution:** Let  $x$  and  $y$  be distinct arbitrary strings in  $\mathbf{0}^*$ .

Without loss of generality,  $x = \mathbf{0}^i$  and  $y = \mathbf{0}^j$  for some  $i > j \geq 0$ .

Let  $z = \mathbf{0}^{i^2+i+1}$ .

Then  $xz = \mathbf{0}^{i^2+2i+1} = \mathbf{0}^{(i+1)^2} \in L$ .

On the other hand,  $yz = \mathbf{0}^{i^2+i+j+1} \notin L$ , because  $i^2 < i^2 + i + j + 1 < (i+1)^2$ .

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $\mathbf{0}^*$  is an infinite fooling set for  $L$ , so  $L$  cannot be regular. ■

**Solution:** Let  $x$  and  $y$  be distinct arbitrary strings in  $\mathbf{0000}^*$ .

Without loss of generality,  $x = \mathbf{0}^i$  and  $y = \mathbf{0}^j$  for some  $i > j \geq 3$ .

Let  $z = \mathbf{0}^{i^2-i}$ .

Then  $xz = \mathbf{0}^{i^2} \in L$ .

On the other hand,  $yz = \mathbf{0}^{i^2-i+j} \notin L$ , because

$$(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.$$

(The first inequality requires  $i \geq 2$ , and the second  $j \geq 1$ .)

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $\mathbf{0000}^*$  is an infinite fooling set for  $L$ , so  $L$  cannot be regular. ■

8.  $\{w \in (\mathbf{0} + \mathbf{1})^* \mid w \text{ is the binary representation of a perfect square}\}$

**Solution:** We design our fooling set around numbers of the form  $(2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = \mathbf{10}^{k-2}\mathbf{10}^k\mathbf{1} \in L$ , for any integer  $k \geq 2$ . The argument is somewhat simpler if we further restrict  $k$  to be even.

Let  $F = \mathbf{1}(\mathbf{00})^*\mathbf{1}$ , and let  $x$  and  $y$  be arbitrary strings in  $F$ .

Then  $x = \mathbf{10}^{2i-2}\mathbf{1}$  and  $y = \mathbf{10}^{2j-2}\mathbf{1}$ , for some positive integers  $i \neq j$ .

Without loss of generality, assume  $i < j$ . (Otherwise, swap  $x$  and  $y$ .)

Let  $z = \mathbf{0}^{2i}\mathbf{1}$ .

Then  $xz = \mathbf{10}^{2i-2}\mathbf{10}^{2i}\mathbf{1}$  is the binary representation of  $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$ , and therefore  $xz \in L$ .

On the other hand,  $yz = \mathbf{10}^{2j-2}\mathbf{10}^{2i}\mathbf{1}$  is the binary representation of  $2^{2i+2j} + 2^{2i+1} + 1$ . Simple algebra gives us the inequalities

$$\begin{aligned} (2^{i+j})^2 &= 2^{2i+2j} \\ &< 2^{2i+2j} + 2^{2i+1} + 1 \\ &< 2^{2(i+j)} + 2^{i+j+1} + 1 \\ &= (2^{i+j} + 1)^2. \end{aligned}$$

So  $2^{2i+2j} + 2^{2i+1} + 1$  lies between two consecutive perfect squares, and thus is not a perfect square, which implies that  $yz \notin L$ .

We conclude that  $F$  is a fooling set for  $L$ . Because  $F$  is infinite,  $L$  cannot be regular. ■