Let L be an arbitrary regular language over the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$. Prove that the following languages are also regular. Assume $\texttt{0}$-indexing.

You probably won't get to all of these but you'll have lots of practice on your own time and you can bring up anything difficult to understand in office hours.

1. $\textsc{FlipOdds}(L) := \{\mathit{flipOdds}(w) \mid w \in L\}$, where the function *flipOdds* inverts every odd-indexed bit in w. For example:

$$\mathit{flipOdds}(\texttt{000001111101010101}) = \texttt{010100101111111111}$$

2. $\textsc{FlipOdd}\texttt{1}\textsc{s}(L) := \{\mathit{flipOdd}\texttt{1}\mathit{s}(w) \mid w \in L\}$, where the function *flipOdd1* inverts every other $\texttt{1}$ bit of its input string, starting with the second $\texttt{1}$ (which would have a index of 1 in a 0-indexing scheme). For example:

$$\mathit{flipOdd}\texttt{1}\mathit{s}(\texttt{00001}\underline{\texttt{1}}\texttt{1}\underline{\texttt{1}}\texttt{10}\underline{\texttt{1}}\texttt{010}\underline{\texttt{1}}\texttt{01}) = \texttt{00001}\underline{\texttt{0}}\texttt{1}\underline{\texttt{0}}\texttt{10}\underline{\texttt{0}}\texttt{010}\underline{\texttt{0}}\texttt{01}$$

3. $\textsc{UnflipOdd}\texttt{1}\textsc{s}(L) := \{w \in \Sigma^* \mid \mathit{flipOdd}\texttt{1}\mathit{s}(w) \in L\}$, where the function *flipOdd1* is defined as in the previous problem.

4. $\mathsf{cycle}(L) := \{xy \mid x, y \in \Sigma^*, yx \in L\}$, The language that accepts the rotations of string from a regular language.

5. Prove that the language $\mathit{insert}\texttt{1}(L) := \{x\texttt{1}y \mid xy \in L\}$ is regular.

   Intuitively, $\mathit{insert}\texttt{1}(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one $\texttt{1}$. For example, if $L = \{\varepsilon, \texttt{OOK!}\}$, then $\mathit{insert}\texttt{1}(L) = \{\texttt{1}, \texttt{1OOK!}, \texttt{O1OK!}, \texttt{OO1K!}, \texttt{OOK1!}, \texttt{OOK!1}\}$.

Work on these later:

5. Prove that the language $delete\mathtt{1}(L) := \{xy \mid x\mathtt{1}y \in L\}$ is regular.

   Intuitively, $delete\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\mathtt{1}$. For example, if $L = \{\mathtt{101101}, \mathtt{00}, \varepsilon\}$, then $delete\mathtt{1}(L) = \{\mathtt{01101}, \mathtt{10101}, \mathtt{10110}\}$.

6. Consider the following recursively defined function on strings:

$$stutter(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet stutter(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   Intuitively, $stutter(w)$ doubles every symbol in w. For example:

   - $stutter(\mathtt{PRESTO}) = \mathtt{PPRREESSTTOO}$
   - $stutter(\mathtt{HOCUS\diamond POCUS}) = \mathtt{HHOOCCUUSS\diamond\diamond PPOOCCUUSS}$

   (a) Prove that the language $stutter^{-1}(L) := \{w \mid stutter(w) \in L\}$ is regular.

   (b) Prove that the language $stutter(L) := \{stutter(w) \mid w \in L\}$ is regular.

7. Consider the following recursively defined function on strings:

$$evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot evens(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

   Intuitively, $evens(w)$ skips over every other symbol in w. For example:

   - $evens(\mathtt{EXPELLIARMUS}) = \mathtt{XELAMS}$
   - $evens(\mathtt{AVADA\diamond KEDAVRA}) = \mathtt{VD\diamond EAR}$.

   (a) Prove that the language $evens^{-1}(L) := \{w \mid evens(w) \in L\}$ is regular.

   (b) Prove that the language $evens(L) := \{evens(w) \mid w \in L\}$ is regular.

You may find it helpful to imagine these transformations concretely on the following DFA for the language specified by the regular expression 00^*11^*.