Decidability II

Sides based on material by Kani, Erickson, Chekuri, et. al.

All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Reduction

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Lemma: A language X reduces to a language Y, if one can construct a *TM* decider for X using an oracle ORAC_Y for Y.

We will also denote this by $X \implies Y$.

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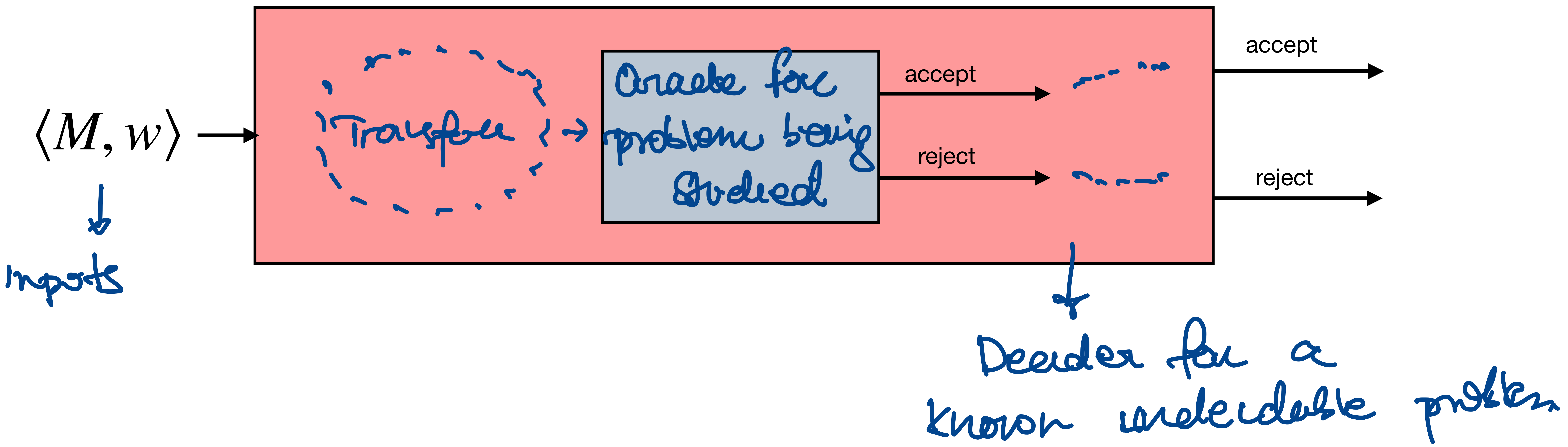
- Assume L is decided by a TM M. Create a decider for ***known*** undecidable problem X *using* M.
  - Results in decider for X (i.e., A_{TM}).
- Contradiction since X is known not to be decidable. Thus, L must be not decidable.

Key diagram

A picture is worth atleast a few slides of proofs.



Reduction implies decidability

Lemma: Let X and Y be two languages, and assume that $X \implies Y$. Then if Y is decidable then X is decidable.

Reduction implies decidability

oracle for
Y exist.

Lemma: Let X and Y be two languages, and assume that $X \implies Y$. Then if Y is decidable then X is decidable.

Proof: Since X reduces to Y, there is a procedure $T_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. Since Y is decidable, let T be a decider for Y (i.e., a program or a *TM*). We replace the calls to the oracle in $T_{X|Y}$ with calls to T. The resulting program T_X is a decider and its language is X. Thus X is decidable (or more formally *TM* decidable).

The contrapositive …

Lemma: Let X and Y be two languages, and assume that $X \implies Y$. Then if X is undecidable then Y is undecidable.

Halting

The halting problem

Define the language of all pairs $\langle M, w \rangle$ such that M halts on w as:

$$A_{\text{HALT}} = \{\langle M, w \rangle \,|\, M \text{ is a } TM \text{ and } M \text{ stops on } w\}$$

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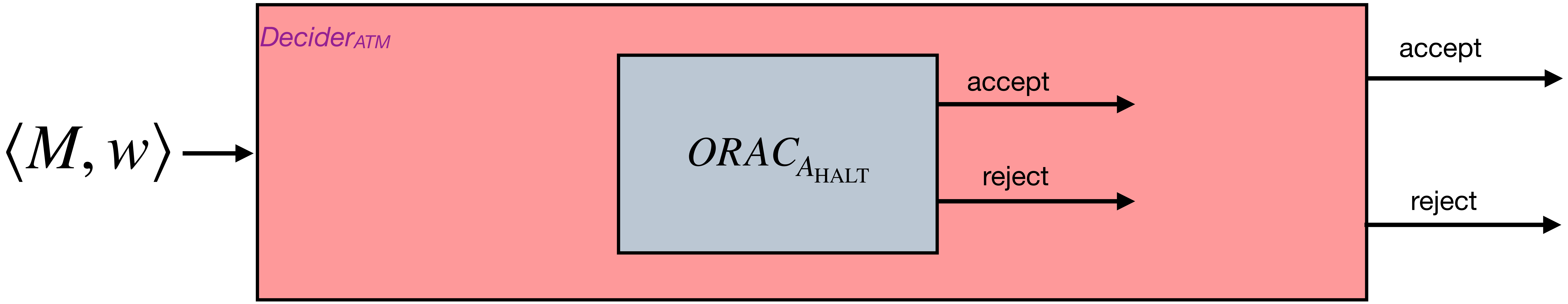
"Similar" to language we already know to be undecidable:

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a } TM \text{ and } \underline{M \text{ accepts}}\ w\}$$

One way to proving that Halting is undecidable...

Lemma: The language A_{TM} reduces to A_{HALT}. Namely, given an oracle for A_{HALT} one can build a decider (that uses this oracle) for A_{TM}.



One way to proving that Halting is undecidable...

$A_{TM} \Longrightarrow A_{HALT}$

Proof: Let $ORAC_{A_{\mathrm{HALT}}}$ be the given oracle for A_{HALT}. We build the following decider for A_{TM}.

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Proof: Let $ORAC_{A_{\text{HALT}}}$ be the given oracle for A_{HALT}. We build the following decider for A_{TM}: = <M,w> s.t. M accepts w

```
AnotherDecider-A_TM(⟨M,w⟩):
  res ← ORAC_Halt(⟨M,w⟩)
  // if M does not halt on w then reject
  if res = reject then
      halt and reject
  // M halts on w since res = accept.
  // Simulating M on w terminates in finite time.
  res₂ ← Simulate M on w
  return res₂
```

Oracle tells you machine will halt

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  // Simulating M on w terminates in finite time.
  res2 ← Simulate M on w
  return res2
```

This procedure always returns and as such its a decider for A_{TM}. → problem, A_{TM} is undecidable

The Halting problem is not decidable

Theorem: The language A_{HALT} is not decidable.

The Halting problem is not decidable

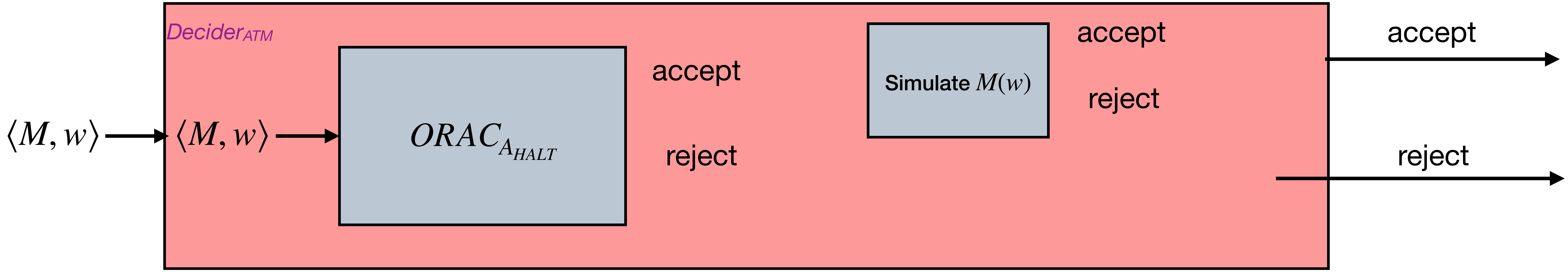
Idea: Reduce

$A_{TM} \Rightarrow A_{HALT}$

Theorem: The language A_{HALT} is not decidable.

Proof: Assume, for the sake of contradiction, that A_{HALT} is decidable. As such, there is a TM, denoted by TM_{HALT}, that is a decider for A_{HALT}. We can use TM_{HALT} as an implementation of an oracle for A_{HALT}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable which is contradiction. Therefore it must be the case that A_{HALT} is undecidable.

The same proof by figure...



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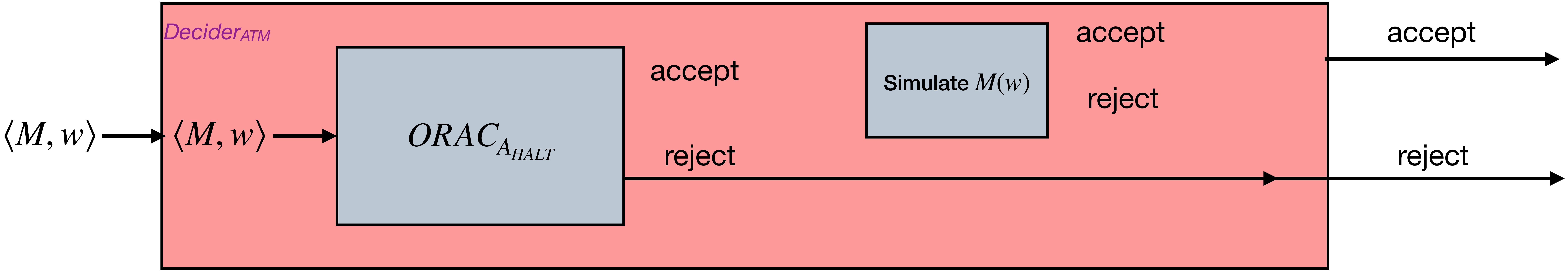

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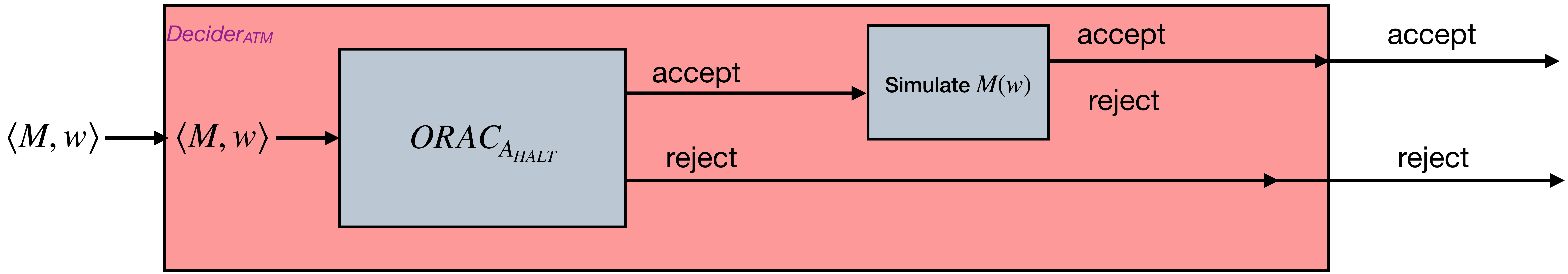

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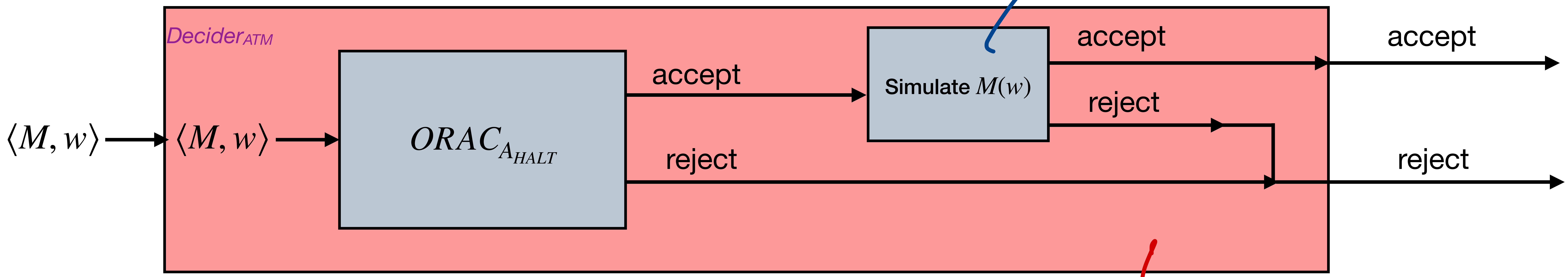
valid move.



This now a decider for ATM

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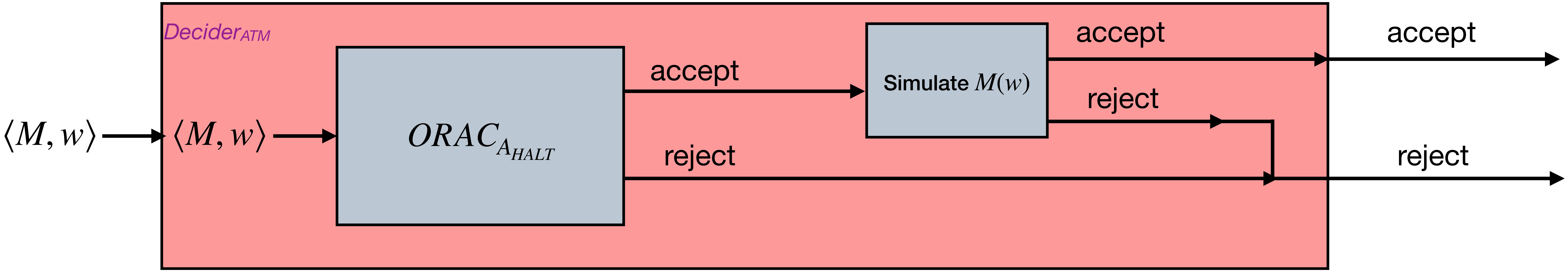

... if A_{HALT} is decidable, then A_{TM} is decidable, which is impossible!

Emptiness

The language of empty languages

- Let $E_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M) = \emptyset\ \}$ and let TM_{ETM} be a decider for E_{TM}.

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$A_{TM} \Rightarrow E_{TM}$.

- Let $E_{TM} = \{ \langle M \rangle \mid M \text{ is a } TM \text{ and } L(M) = \emptyset \}$ and let TM_{ETM} be a decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.

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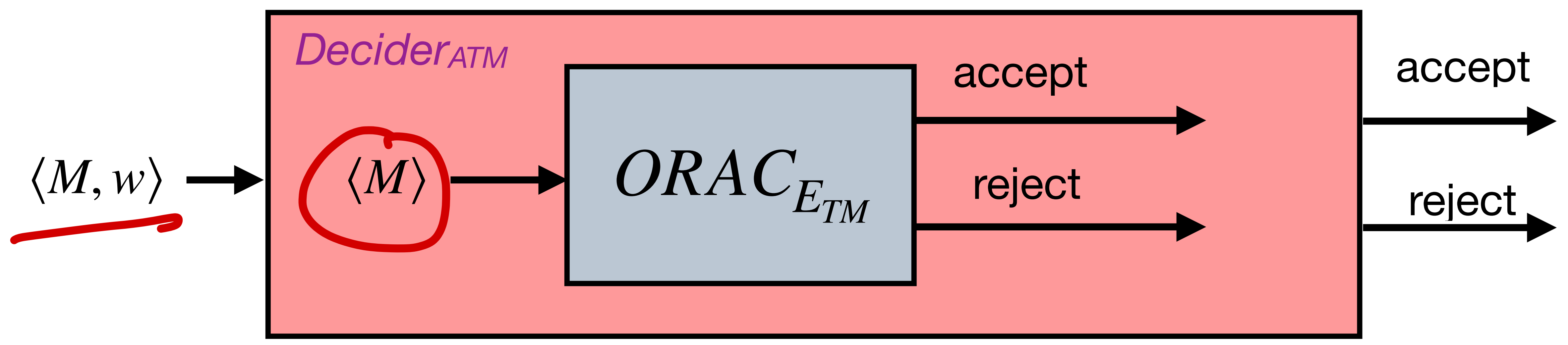

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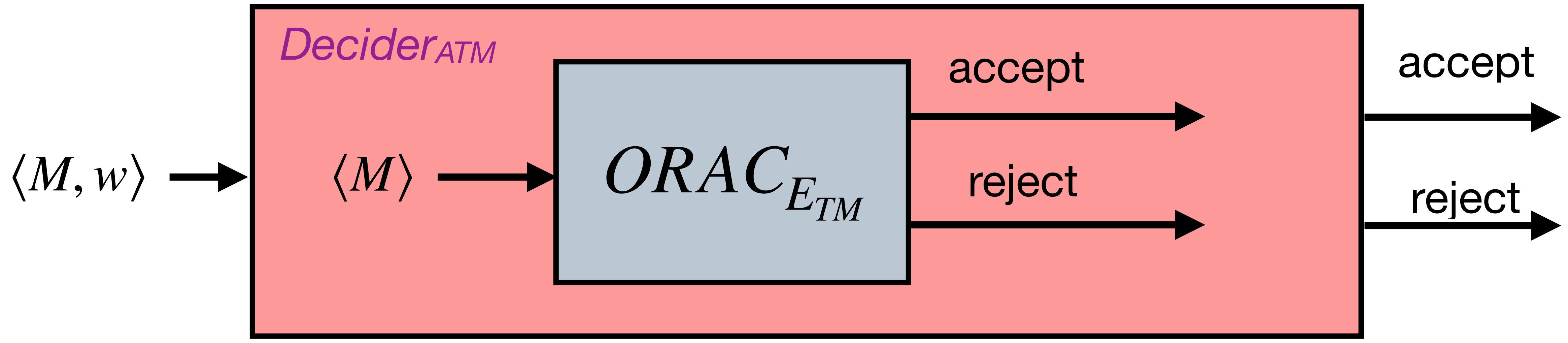
- Decider for A_{TM}: Given M and w decide whether M accepts w.

- Need to somehow make the second input (w) disappear



Embedding strings

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- Let EmbedString$\left(\langle M \rangle, w\right)$ take as input two strings $\langle M \rangle$ and w, and outputs a string encoding (TM) $\langle M_w \rangle$.

$\langle M_w \rangle, x$

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Question: What is $L(M_w)$?

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Question: What is $L(M_w)$?

Since M_w ignores any input … language M_w is either Σ^* or $\varnothing$. It is Σ^* if M accepts w, and it is $\varnothing$ if M does not accept w.

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- Build decider AnotherDecider-ATM for A_{TM}:

```
AnotherDecider-A_TM(⟨M, w⟩):
  ⟨M_w⟩ ← EmbedString(⟨M⟩, w)
  r ← TM_ETM(⟨M_w⟩)
  if r = accept then
       return reject
  // TM_ETM(⟨M_w⟩) rejected its input
  return accept
```

$A_{TM} := \{ \langle M, w \rangle, \; M \text{ accepts } w \}$

… is undecidable.

Consider the possible behavior of AnotherDecider-A_{TM} on the input $\langle M, w \rangle$.

- If TM_{ETM} accepts $\langle M_w \rangle$, then $L\langle M_w \rangle$ is empty. This implies that M does not accept w. As such, AnotherDecider-A_{TM} rejects its input $\langle M, w \rangle$.

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Remember M_w if it accepts so then X (input to M_w) can be anything.

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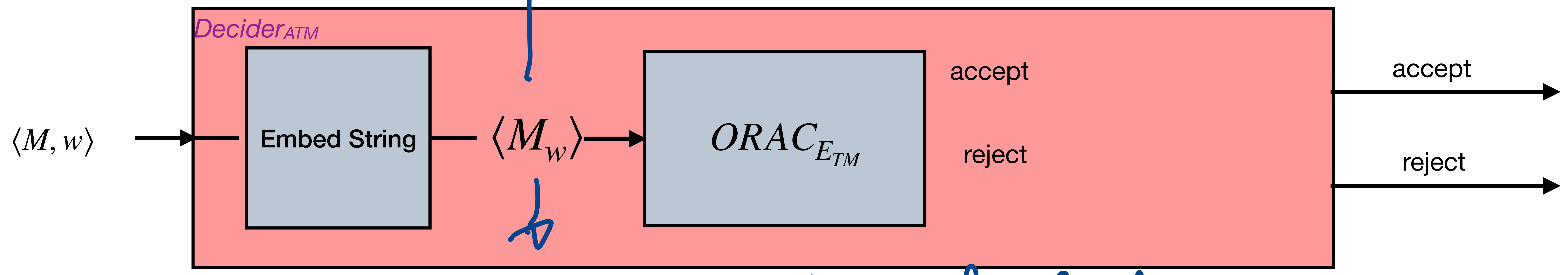
$\Longrightarrow$ AnotherDecider-A_{TM} is a decider for A_{TM}

But A_{TM} is undecidable … so the assumption that E_{TM} is decidable must be false.

Emptiness is undecidable via diagram

$L(M_w)$ is either Σ^* or $\emptyset$



running M on w not guaranteed to stop

NOTE: AnotherDecider-A_{TM} never actually runs the code for M_w. It hands the code to a function TM_{ETM} which analyzes what the code would do if run*. So it does not matter that M_w might go into an infinite loop.

M,w
M_w' w

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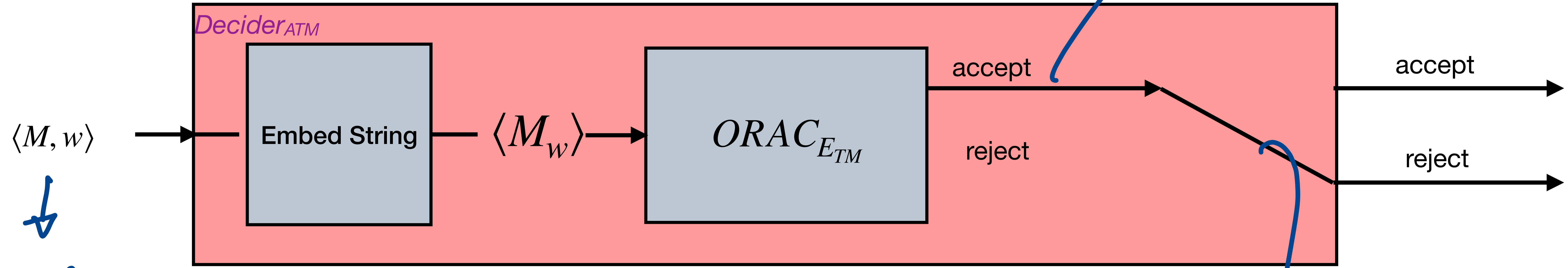
L(M_w) is "empty"
(∵ M didn't accept w)



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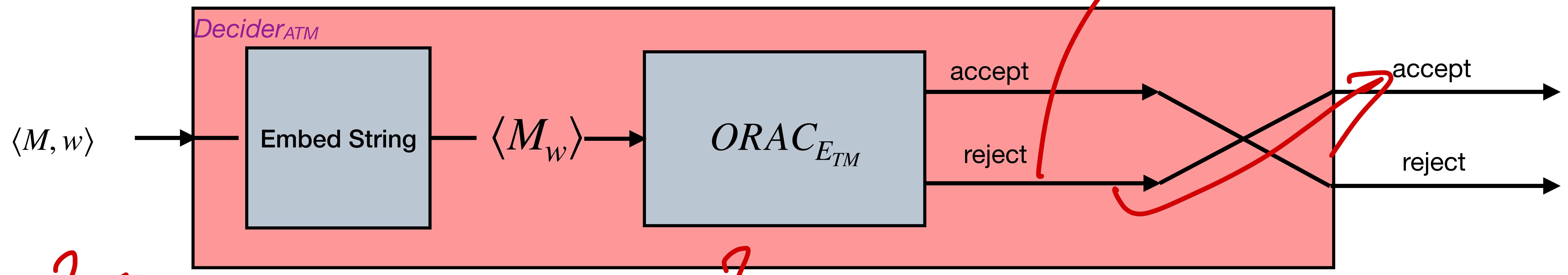
oracle said $L(M_w)$ not empty ($\because$ it is Σ^*)



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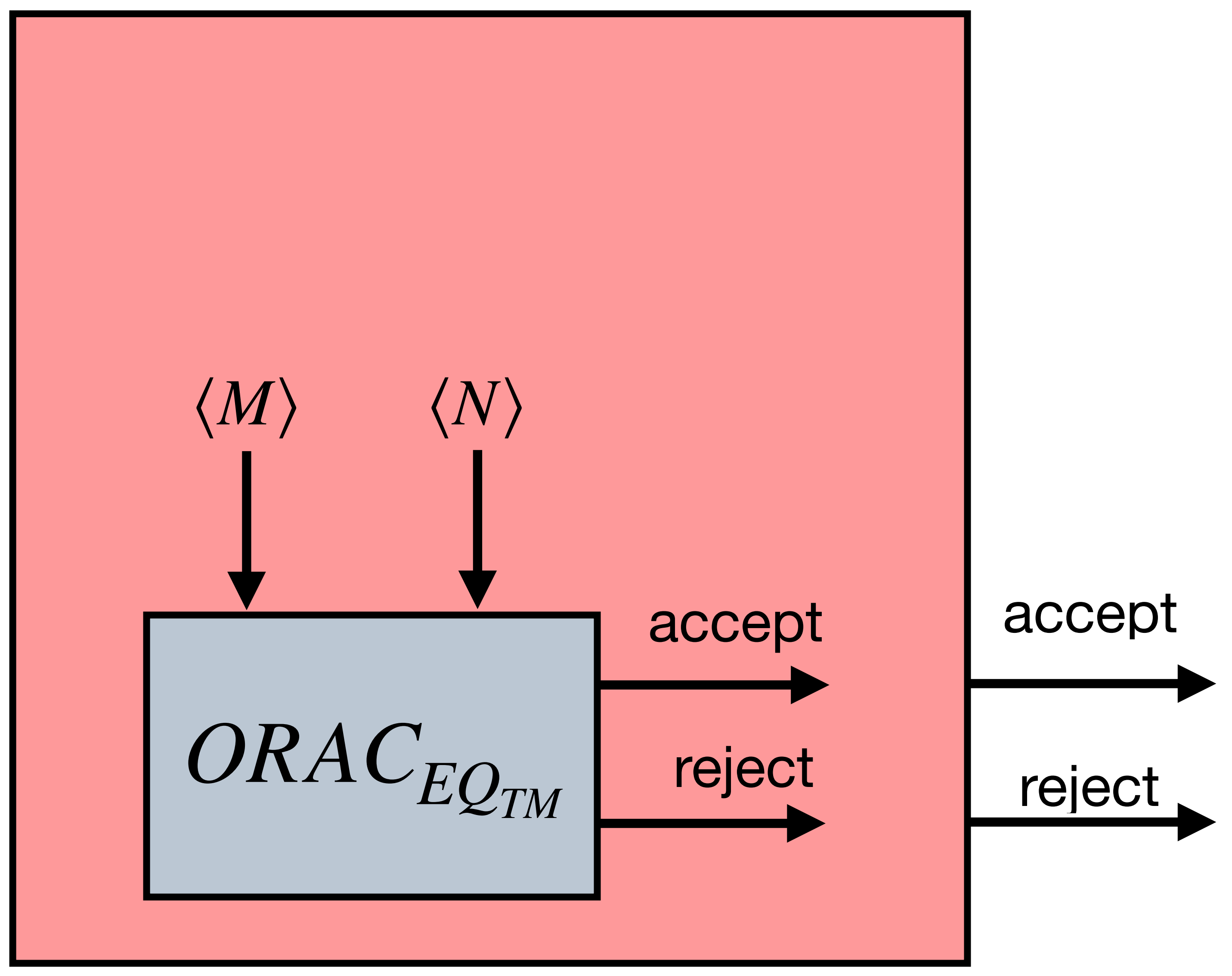

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need to reduce problem of whether $L(M)$ is empty to that of (if $L(M) = L(N)$)

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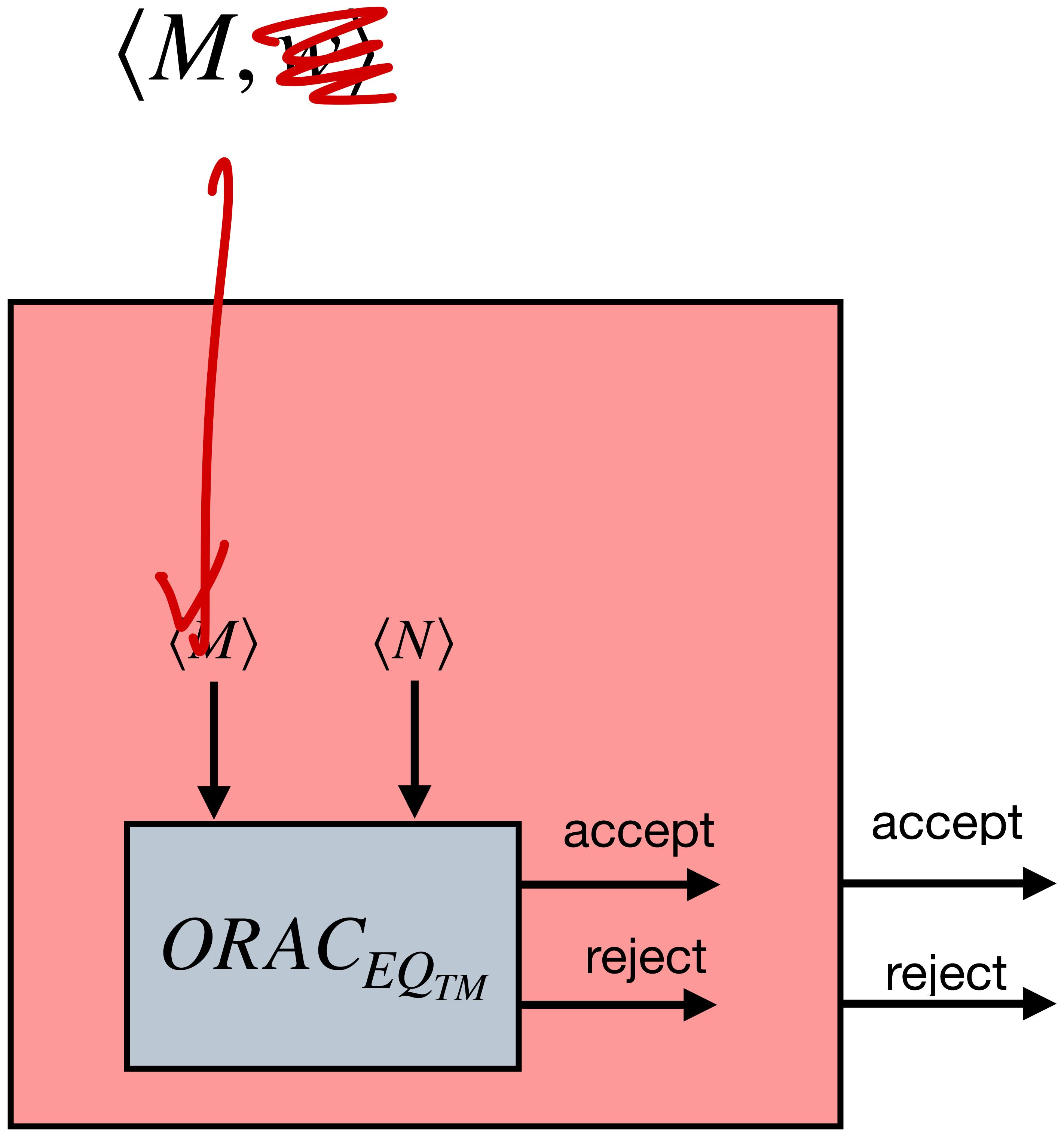

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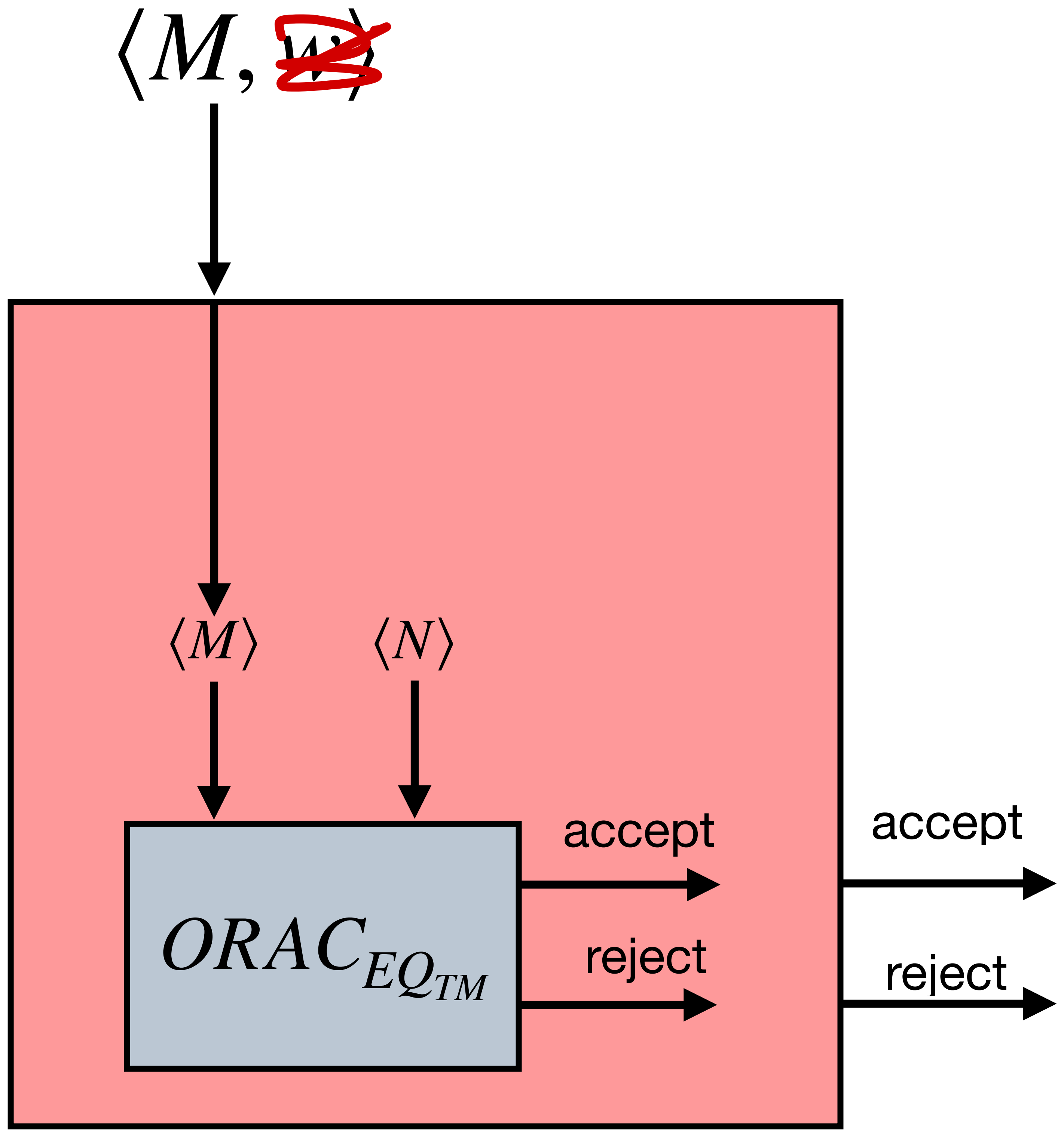

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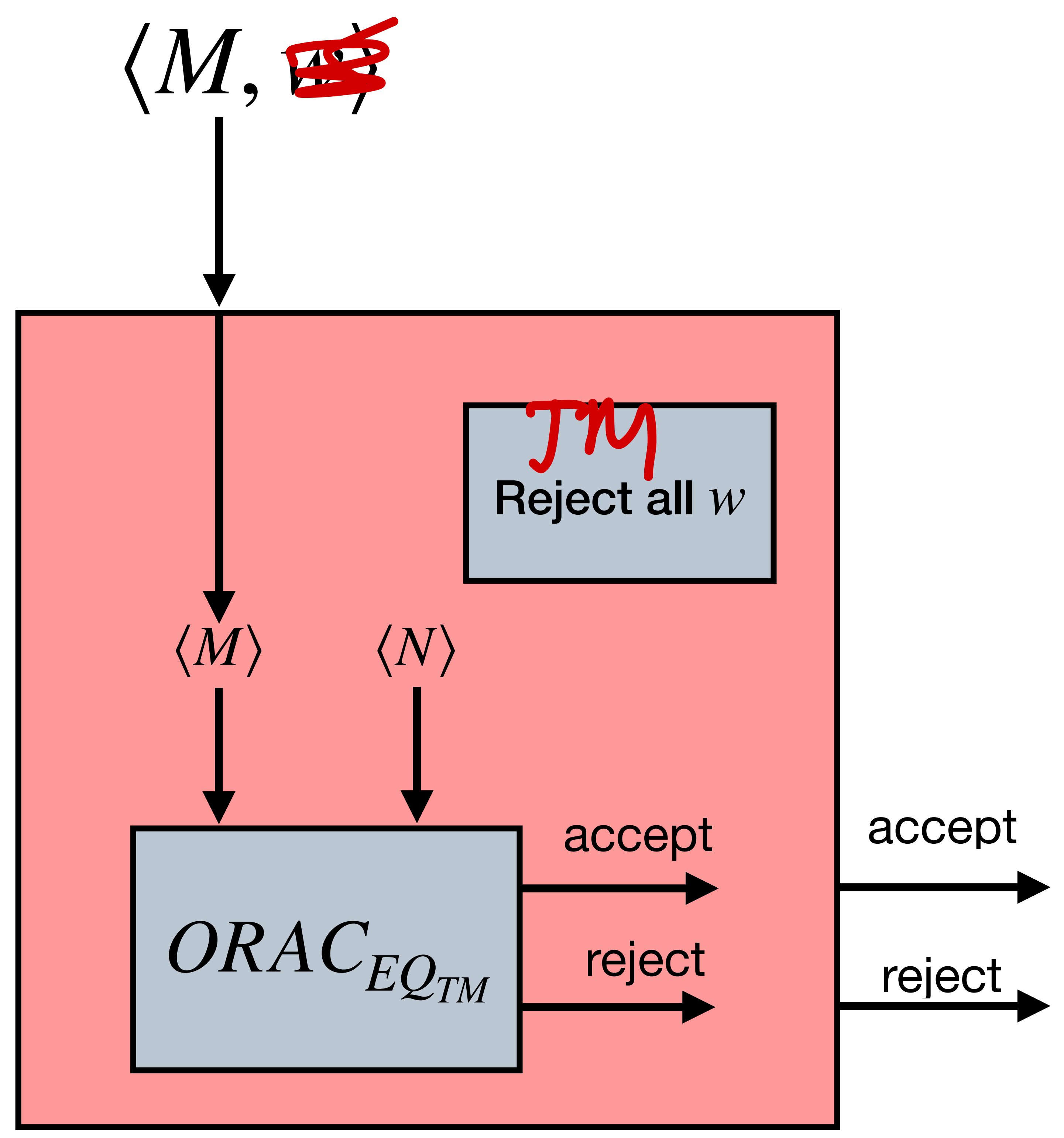

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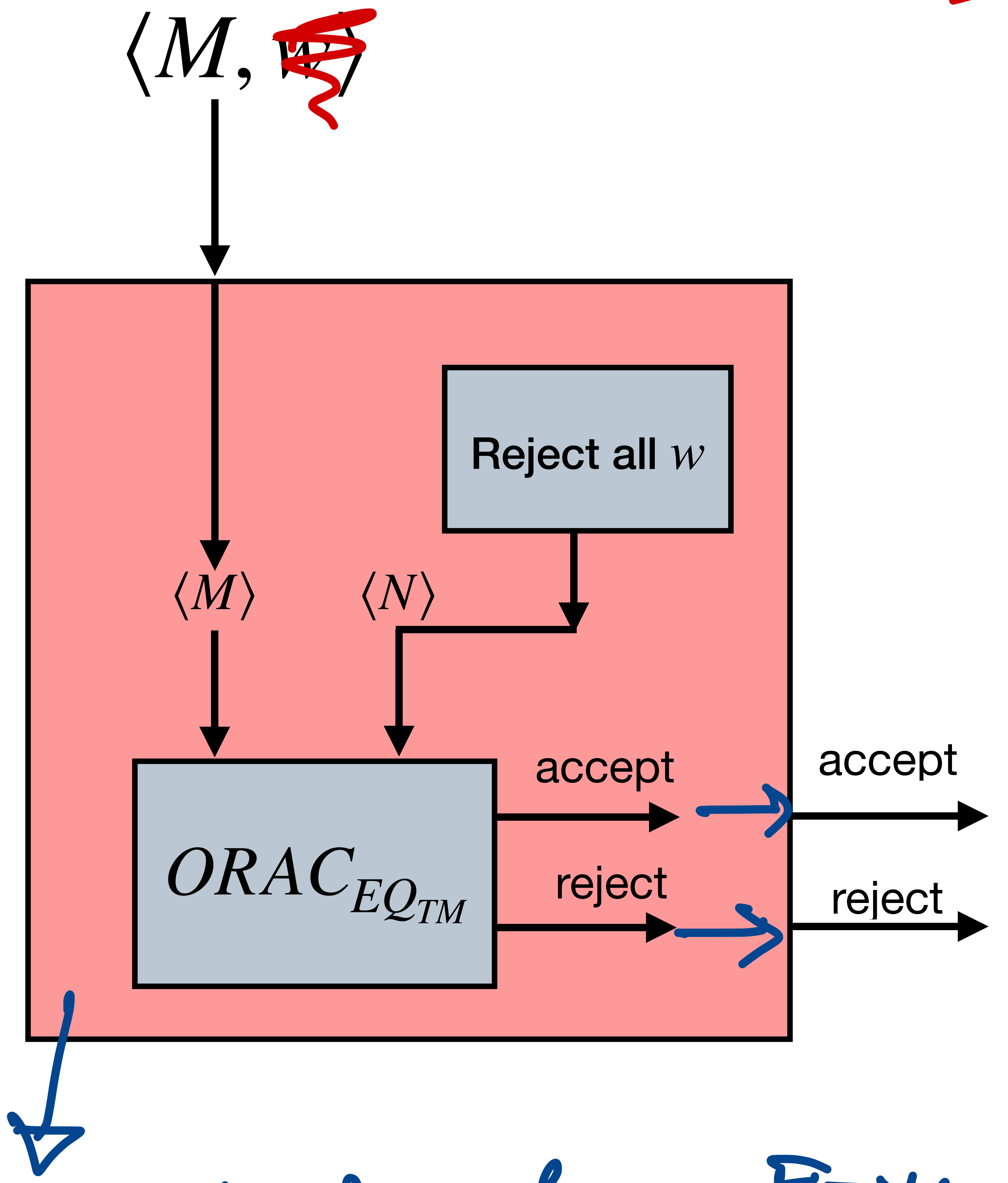

now a decider for ETM which showed previously is undecidable

$(\Sigma, Q, q_0, \delta, A)$

DFAs

DFAs are empty?

an encoding of A

$$E_{DFA} = \{\langle A\rangle \mid A \text{ is a DFA and } \underline{L(A) = \emptyset}\}$$

What does the above language describe?

encodings of DFA's accepting no strings.

Is the language above decidable?

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What does the above language describe?

Is the language above decidable?

Lemma

The language E_{DFA} is decidable.

Proof

Unlike in the previous cases, we can directly build a decider (DeciderEmptyDFA) for E_{DFA}.

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Unlike in the previous cases, we can directly build a decider (DeciderEmptyDFA) for E_{DFA}.

1. Input = $\langle A \rangle$
2. Mark start state of A as visited. → keep $M = 2$ ...3
3. Repeat until no new states get marked:
    - Mark any state that has a transition coming into it from any state that is already marked.

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4. If no accept state is marked, then accept.

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3. Repeat until no new states get marked:
   - Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, then accept.
5. Otherwise, then reject.

Equal DFAs

DFAs are equal?

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$$EQ_{DFA} = \{\langle A, b \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

What does the above language describe?

Is the language above decidable?

Lemma

The language E_{DFA} is decidable.

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One way: Ensure A, B are "minimal", and encoded the same way $\langle A \rangle, \langle B \rangle$ .... then compare.

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Can we show this using reductions?

Equal DFA trick I

Need a way to determine if there any strings in one language and not the other...

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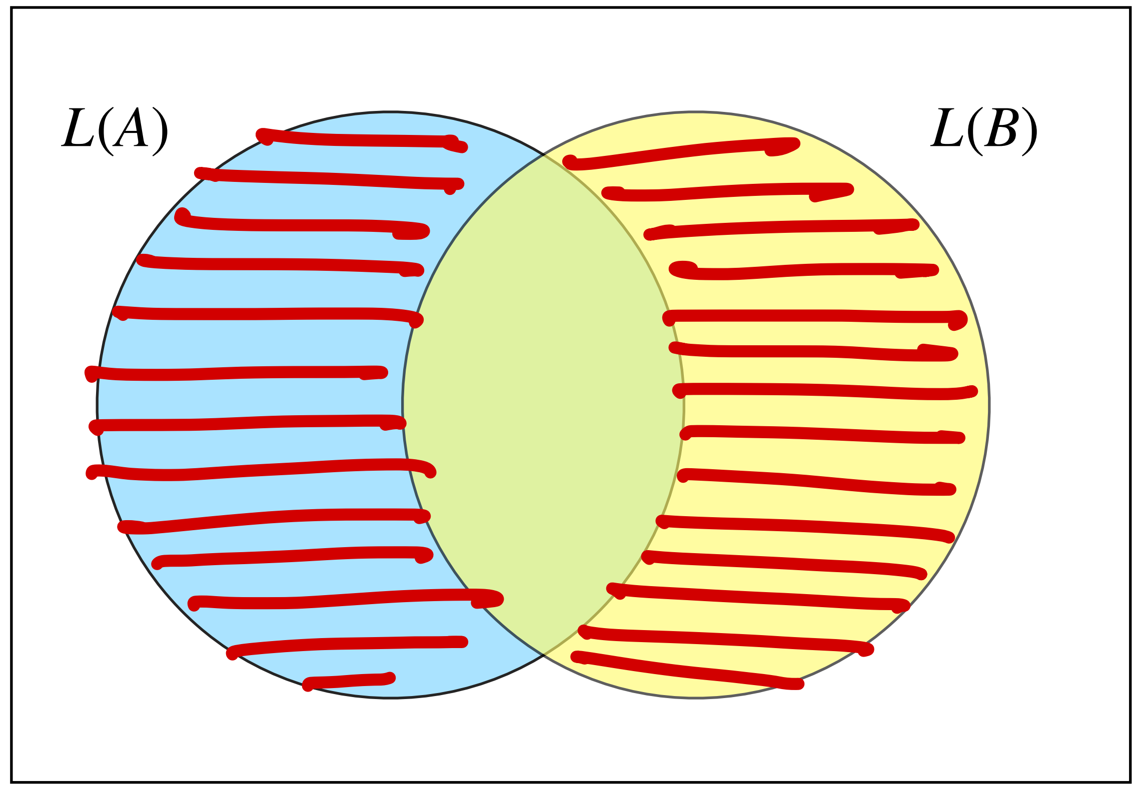


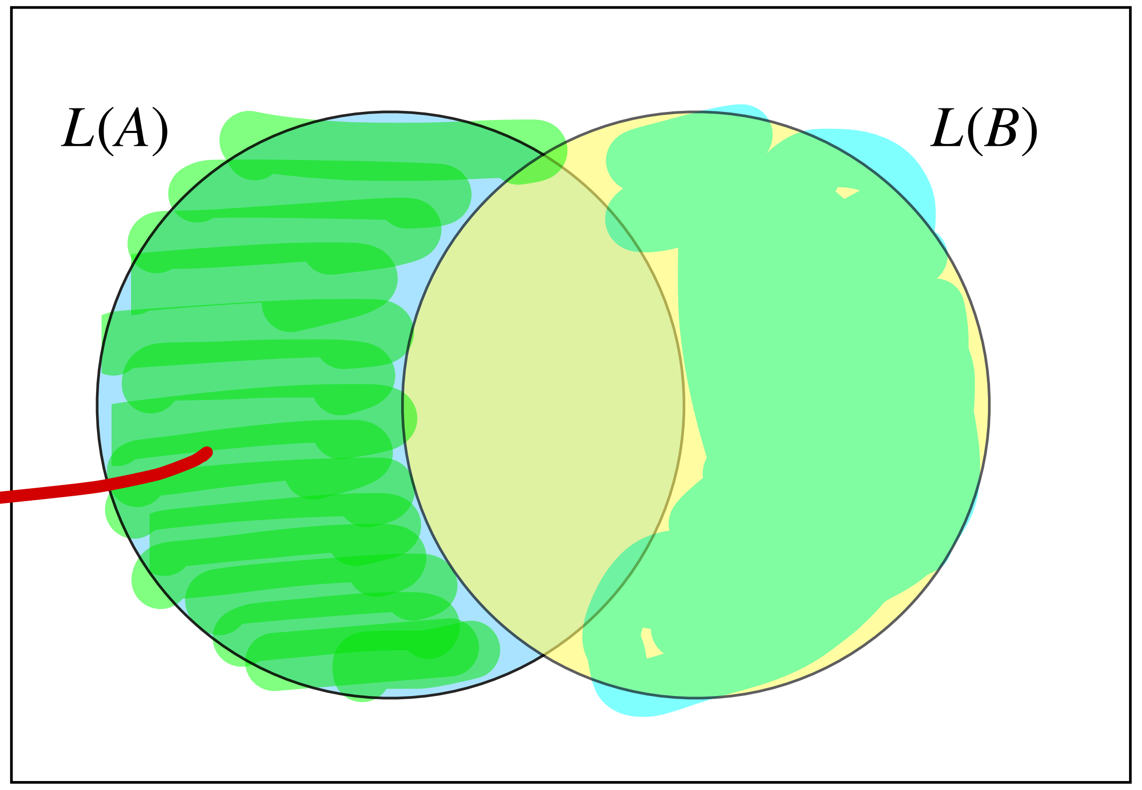
Equal DFA trick I

$\left(L(A) \cup L(B) \right) \setminus \left(L(A) \cap L(B) \right)$

not known to prove regularity

Need a way to determine if there any strings in one language and not the other...

$L(A) \cap \overline{L(B)}$



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This is known as the symmetric difference. Given A, B, can create a new DFA (call it C) which represents the symmetric difference of L_A and L_B.

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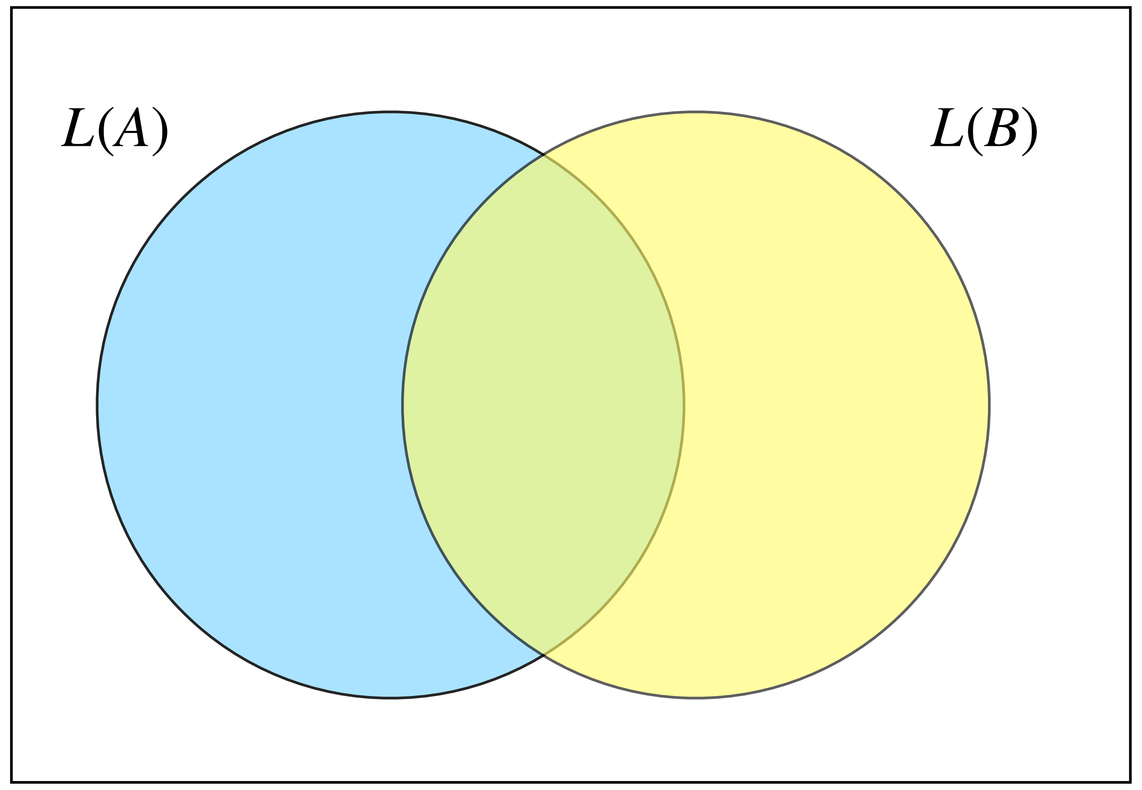

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$\emptyset \cup \emptyset$

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Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \varnothing$
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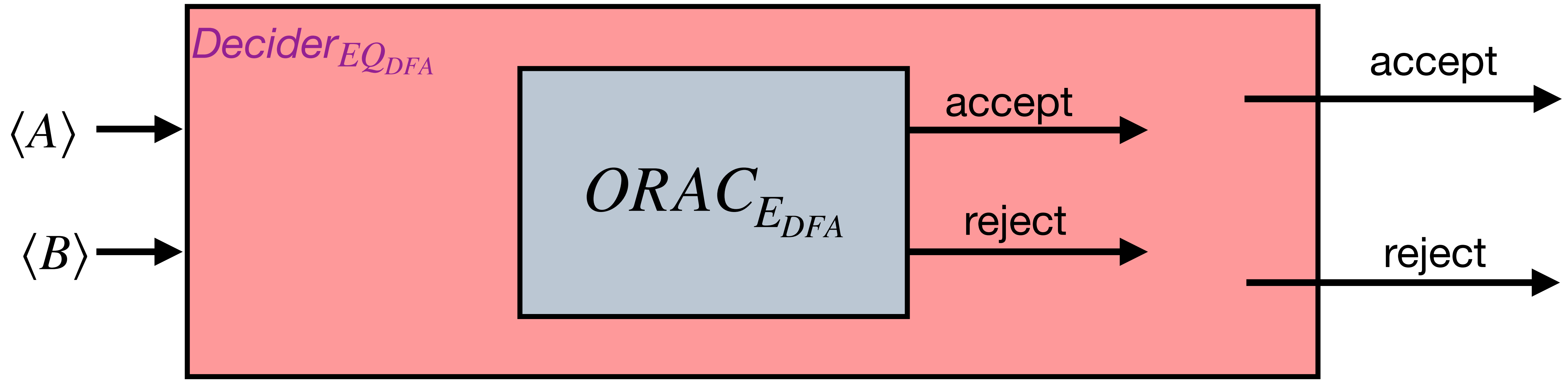

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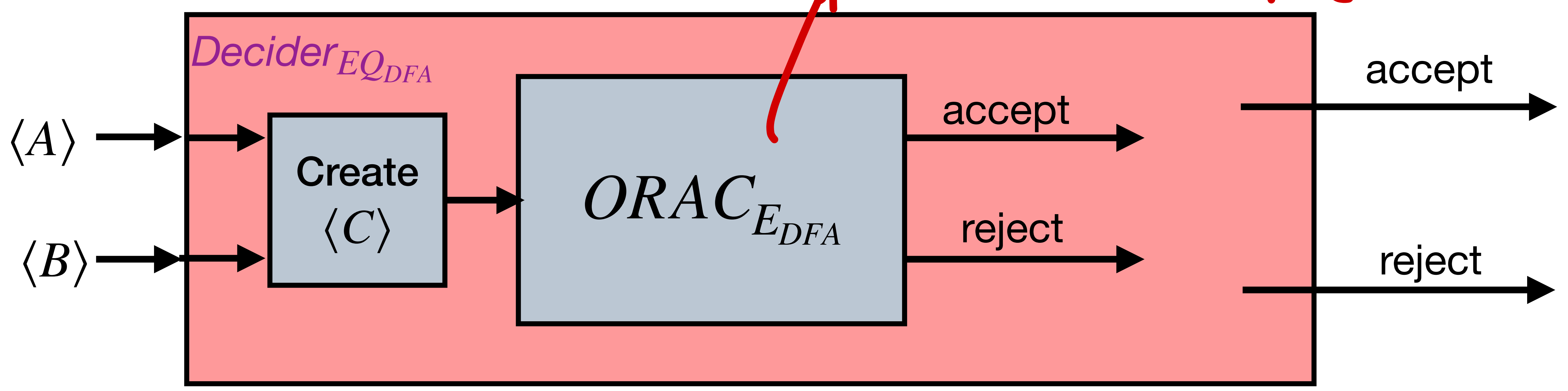

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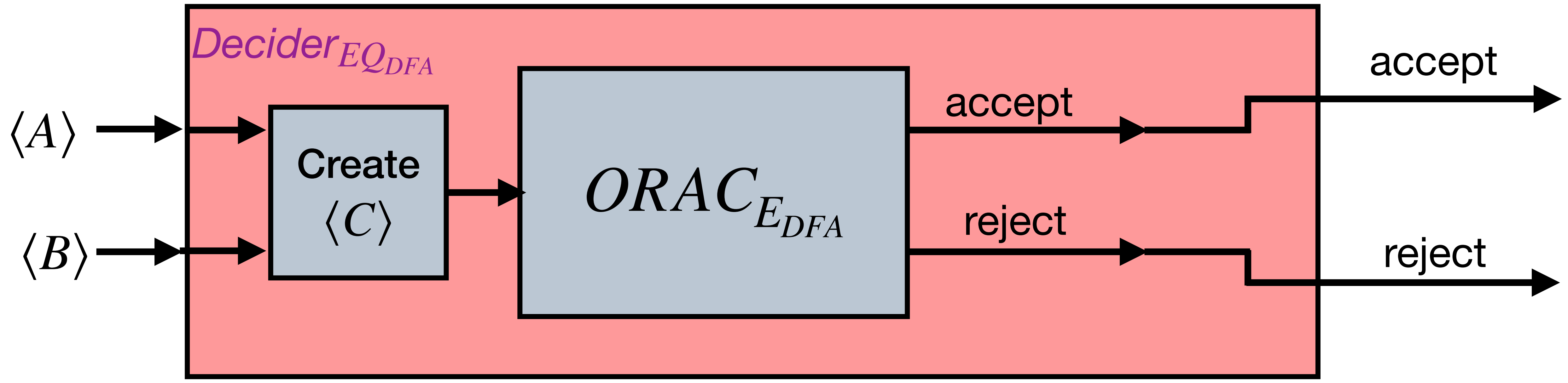

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- DeciderRegL/OracRegL$_{TM}$: Assume *TM* decider for Regular$_{TM}$.
- Reduction from halting requires to turn problem about deciding whether a *TM* M accepts w (i.e., is $w \in A_{TM}$) into a problem about whether some *TM* accepts a regular set of strings.

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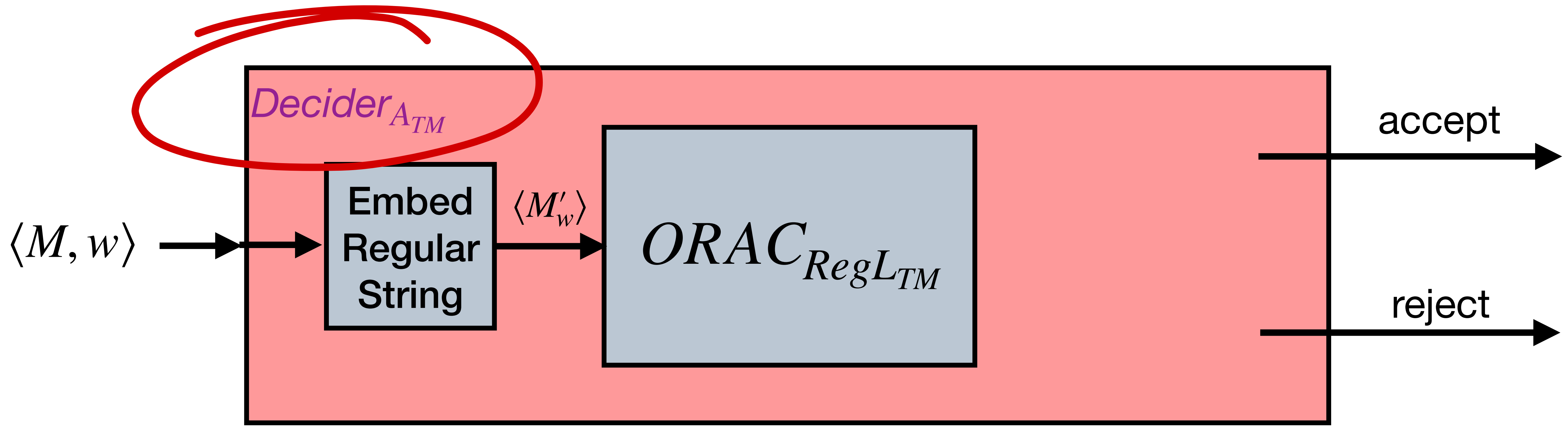
  - Assume EmbedRegularString: program with input $\langle M \rangle$ and w, and outputs $\langle M'_w \rangle$, encoding the program M'_w.

IsRegular reduction



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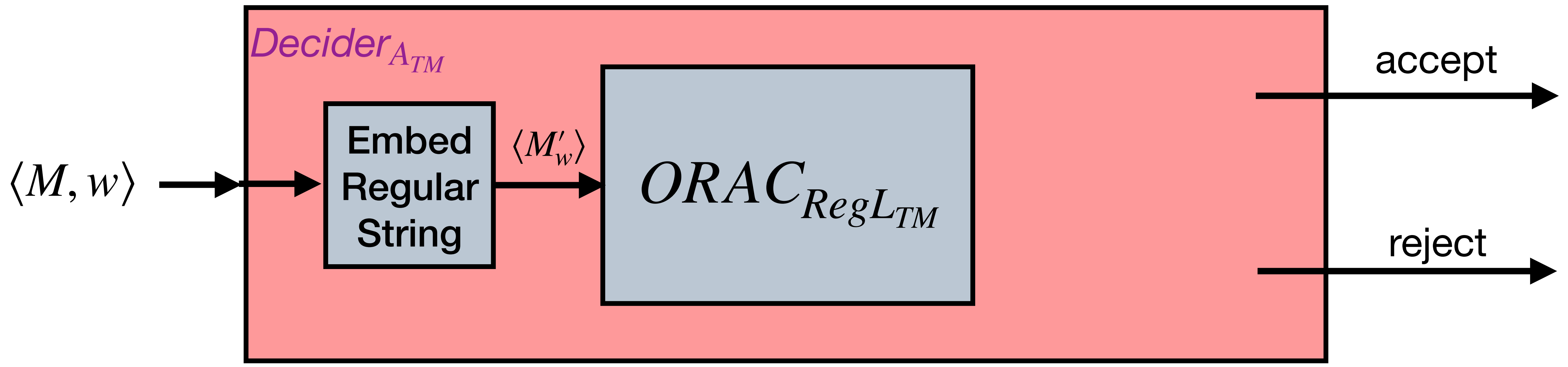

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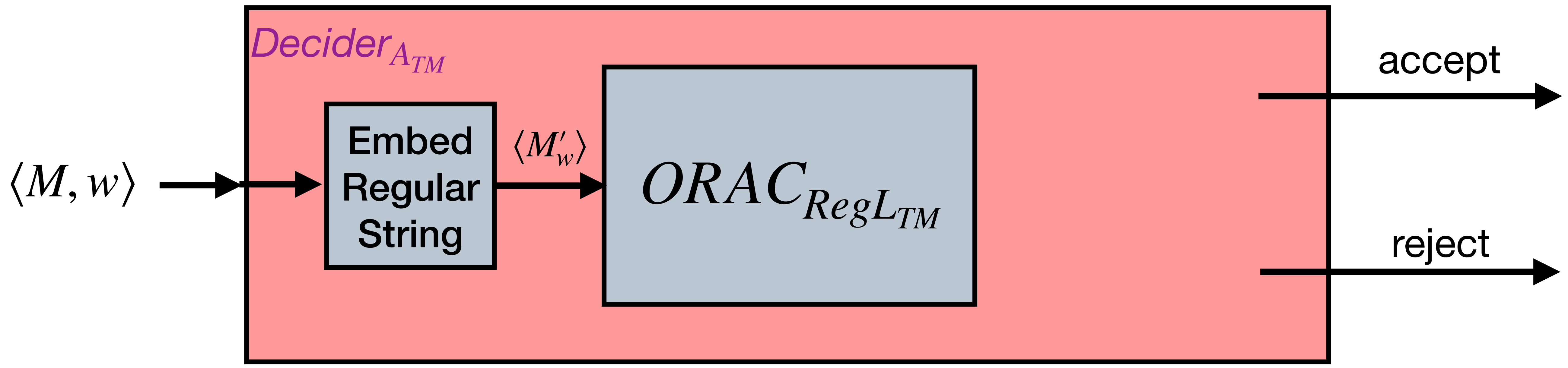

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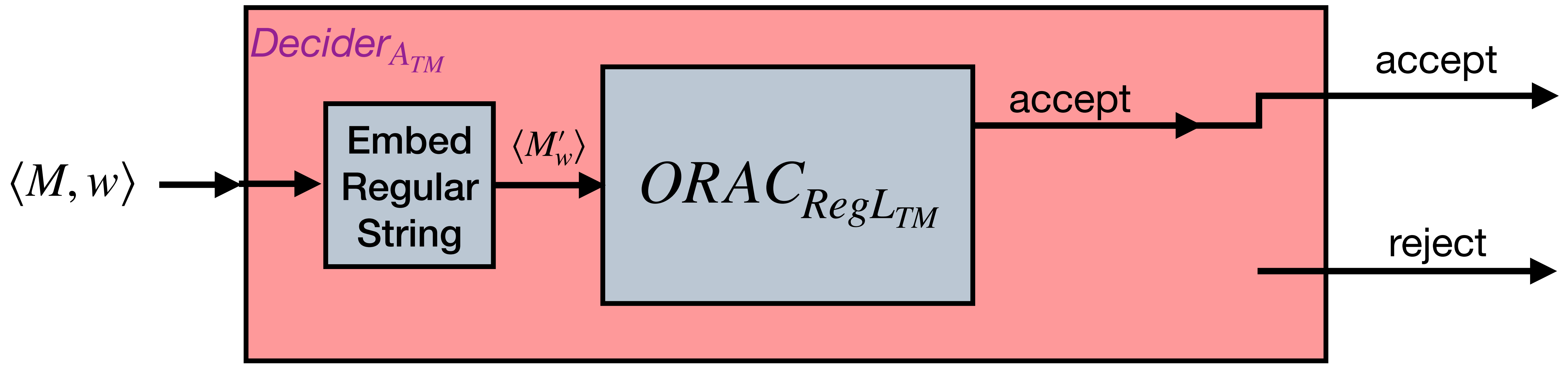

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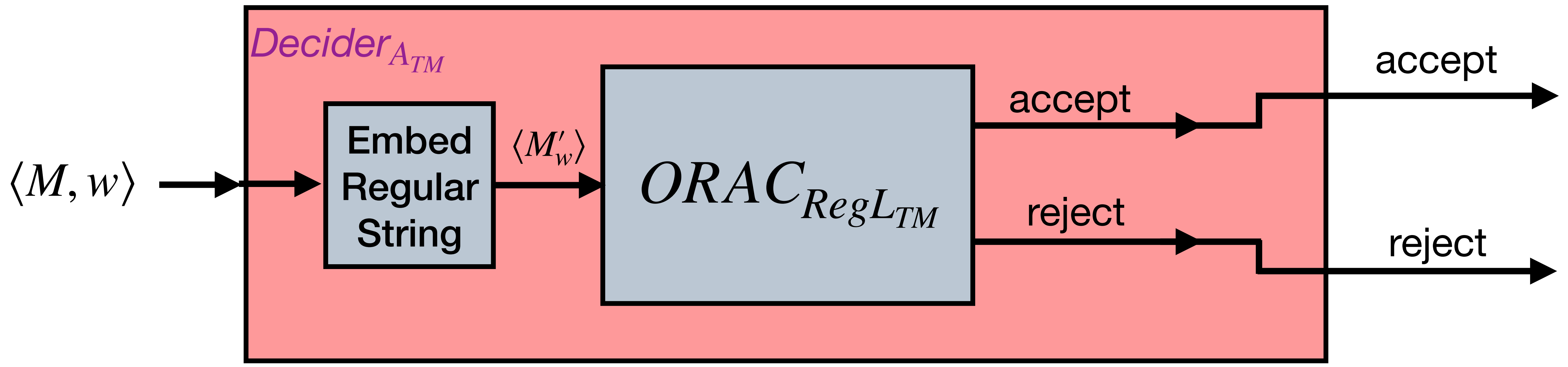

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- If OracRegLTM rejects $\Longrightarrow L(M'_w)$ is not regular $\Longrightarrow L(M'_w) = a^n b^n$ $\Longrightarrow$ M does not accept $w \Longrightarrow$ AnotherDecider-A_{TM} should reject $\langle M, w \rangle$

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Then L is a undecidable.

T’is the …

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