1. You are given a list $D[n]$ of n words each of length k over an alphabet Σ in a language you don't know, although you are told that words are sorted in lexicographic order. Using $D[n]$, describe an algorithm to efficiently identify the order of the symbols in Σ. For example, given the alphabet $\Sigma = \{Q, X, Z\}$ and the list $D = \{QQZ, QZZ, XQZ, XQX, XXX\}$, your algorithm should return QZX. You may assume D always contains enough information to completely determine the order of the symbols. (Hint: use a graph structure, where each node represents one letter.)

Solution: Consider two words, $D[i]$, $D[i+1]$. Consider j such that $D[i][j] \neq D[i+1][j]$ and $\forall k < j$, $D[i][k] = D[i+1][k]$. That is, j is the index of the first different letter between $D[i]$ and $D[i+1]$. The comparison of $D[i][j]$ and $D[i+1][j]$ reveals the order between the two letters. Any further comparison of $D[i]$ and $D[j]$ would not help, since the following letters do not affect lexicographic order of $D[i]$ and $D[i+1]$. Also, for arbitrary x, y, z such that $x < y < z$, if you are given the comparisons of $D[x], D[y]$ and $D[y], D[z]$, then the comparison of $D[x], D[z]$ does not reveal any additional information about the order(Why? Let j, k be the first different index between $D[x], D[y]$ and $D[y], D[z]$ respectively. Reason about three cases: $j < k$, $j = k$, $j > k$). Therefore, the problem can be solved by constructing the following directed graph.

- $V = \{v \mid v \in \Sigma\}$
- $E = \{(u, v) \mid u, v \in \Sigma, u \neq v, \exists i, j \text{ s.t. } D[i][j] = u, D[i+1][j] = v, \forall k < j, D[i][k] = D[i+1][k]\}$

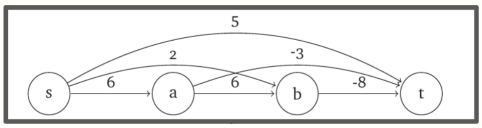
Note that for any edge $(u, v) \in E$, there is a corresponding pair of consecutive words $(D[i], D[i+1])$ such that if j is the index of the first different letter, then $D[i][j] = u$ and $D[i+1][j] = v$. This means for any edge (u, v), we know for sure that u comes before v in their language. Since there can be no cycle in the graph, it can be topologically sorted to obtain the order of symbols. The running time of the algorithm is $O(nk)$, since in worst case we should iterate over every symbol in D to construct the graph.

■

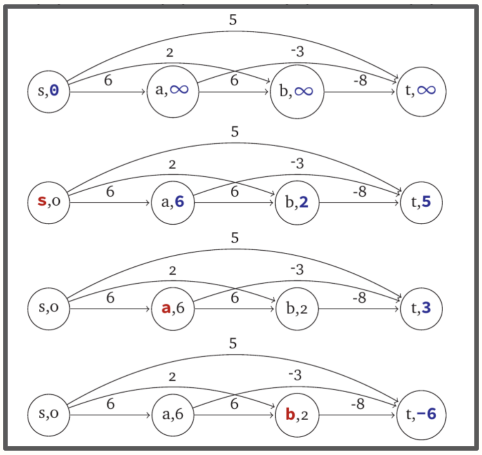
2. Given a directed-acyclic-graph ($G = (V, E)$) with integer (positive or negative) edge weights:

   (a) Give an algorithm to find the **shortest** path from a node s to a node t.

Solution: Because there are negative edge weights we cannot use a greedy method like Dijkstra's algorithm; however because the graph is directed and acyclic we can use a topological sort. Topologically sorting the graph means that every vertex can only reach vertices below them in the sort and cannot reach vertices above them in the sort.



This means after topologically sorting the graph we start at node s and compute the shortest path from node s to each of the nodes below it in sequential order. To do this we initialize the distance from s to all the other nodes as ∞ then we update the distance from node s to be the weight of the edges from s to the neighbors of s. Then we move on to the next sequential node say u, and look at its neighbor, say v. If the distance from s to u plus the edge weight from u to v is less than the current distance from s to v then we update the value. This repeats until we reach node t.



Let s and t be the nth and mth node in the topological sort and let $N(u)$ be the neighbor set of u. Then the algorithm is

```
DAGSP(V, E, s, t):
  (Vs, Es) ← TopSort(V, E)
  n ← index(s)
  m ← index(t)
  D[n] ← 0
  for k ← n + 1 to m
      D[k] ← ∞
  for i ← n to m − 1
      for v in N(v_i)
          j ← index(v)
          D[j] ← min{D[j], D[i] + Es[i][j]}
  return D[m]
```

A topological sort takes $\boldsymbol{O(V+E)}$ time and the for loops in the algorithm takes $\boldsymbol{O(V+E)}$. So the total running time is $\boldsymbol{O(V+E)}$. ■

(b) Give an algorithm to find the **longest** path from a node s to a node t.

Solution (Direct): To find the longest path from node s to node t we can do the same process as part a) but instead we initialize the values to $-\infty$ then take the maximum value.

```
DAGLP(V, E, s, t):
  (Vs, Es) ← TopSort(V, E)
  n ← index(s)
  m ← index(t)
  D[n] ← 0
  for k ← n + 1 to m
      D[k] ← −∞
  for i ← n to m − 1
      for v in N(v_i)
          j ← index(v)
          D[j] ← max{D[j], D[i] + Es[i][j]}
return D[m]
```

This has a running time of $\boldsymbol{O(V+E)}$.

■

Solution (Reduction): This problem can be reduced to the part a). Multiplying all of the edge weights by -1 then finding the shortest path then multiplying that value by -1 yields the longest path.

```
DAGLP(V, E, s, t):
  Et ← −1 ∗ E
  x ← DAGSP(V, Et, s, t)
  return −1 ∗ x
```

This has a running time of $\boldsymbol{O(V+E)}$.

■

3. You are given a directed graph $G = (V, E)$ with possibly negative weighted edges:

   (a) Give an algorithm that finds the shortest path of length at most k between two vertices u and v in $O(k(n+m))$ time.

Solution: One way to solve this problem is by using a modified version of the Bellman-Ford algorithm, which can handle negative edge weights. We can modify the Bellman-Ford algorithm to run at most k iterations instead of running it until convergence. In each iteration, we relax all the edges in the graph once.

We will use an array $dist$ that will finally have the shortest distance of each node from u. The elements in the array $dist$ will be all the vertices that have less than equal to k edges between u and itself. $dist[i]$ represents the distance of vertex i from u and the vertex can be reached from u with less than equal to k edges.

```
ShortestPathKEdges(G, u, v, k):
  n = Number of vertices in G
  dist = array of length n initialized to ∞
  dist[u] = 0
  for i = 1 to k
      for each edge(a, b) with weight w
          dist[b] = min(dist[b], dist[a] + w)
  return dist[v]
```

The time complexity of the algorithm is O(k(n+m)), where n is the number of vertices and m is the number of edges in the graph. ■

   (b) Give an algorithm that finds the shortest path of length exactly k between two vertices u and v in $O(k(n+m))$ time.

Solution: You would need to modify the recurrence of the Bellman-Ford algorithm:

$$d(v,k) = \min \begin{cases} \min_{u \in V}(d(u,k-1) + \ell(u,v)). \\ \cancel{d(v,k-1)} \end{cases}$$

By removing that part of the BF recurrence, you are effectively stopping the algorithm from considering path's that use less than k edges. ■