Pre-lecture brain teaser

Write a (very simple) recursive algorithm that calcuates the Fibonnacci n^{th} number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

Indian Mathematician
Acharya Pingala in 200 BC

Named after Italian Mathematician
Leonardo of Pisa aka Fibonacci (1202)

0
1
1
2
3
5
8
13
21
34
55
⋮

ECE-374-B: Lecture 12 - Dynamic Programming I

Instructor: Abhishek Kumar Umrawal

February 29, 2024

University of Illinois at Urbana-Champaign

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Learning Objectives

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At the end of the lecture, you should be able to understand

- the concepts of the memoizationand dynamic programming,
- how to improve the time and space complexities of recursive algorithms using the above concepts,
- dynamic programming for the fibonacci numbers and longest increasing subsequence problem, and
- where and how to use dynamic programming to refine recursive algorithms.

Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

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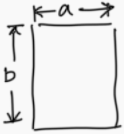
- Binet's formula: $F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}}$

  φ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
- $\lim_{n\to\infty} F(n+1)/F(n) = \varphi$



$\frac{b}{a} = \phi$

→ Most visually appealing photo frame / window / door, ...

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

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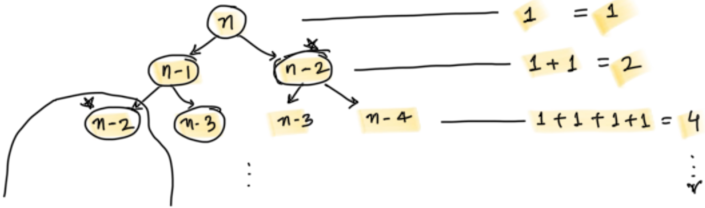
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + O(1)$$



$\Rightarrow$ # of leaves : $O(2^n)$

$\Rightarrow$ $T(n) = 1 \cdot O(2^n)$ Additions

$$T(n) \leq 1 + 2 + 4 + \cdots + 2^n$$

$$= O(2^n)$$

Exact bound:

$$T(n) = \Theta(\phi^n)$$

$$\phi = 1.6 < 2$$

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Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$.

The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

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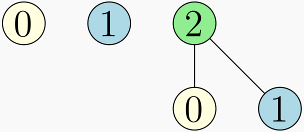

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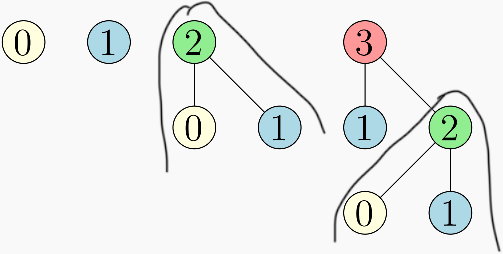

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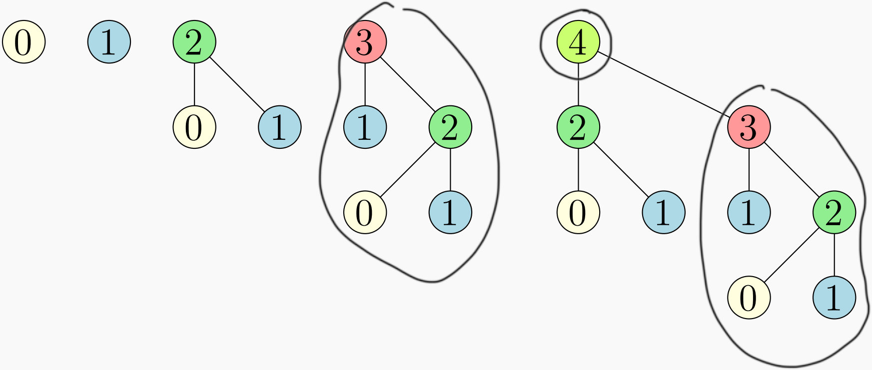

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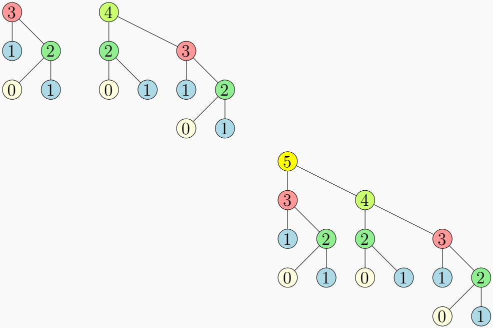

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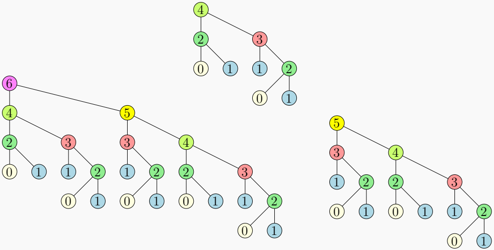

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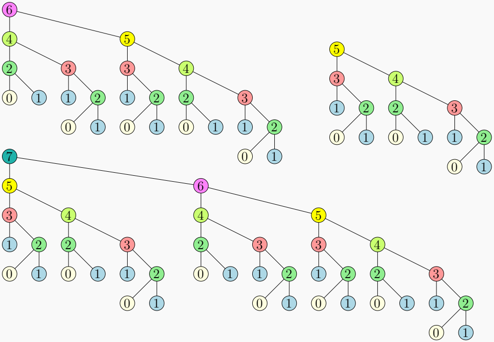

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
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    F[0] = 0
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        F[i] = F[i − 1] + F[i − 2]
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What is the running time of the algorithm?

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What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.

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Dynamic Programming: Finding a recursion that can be effectively/efficiently memorized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

Implicit vs. explicit memoization

Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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Fib(n):
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    if (Fib(n) was previously computed)
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```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

Implicit or automatic memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib(n):
        if (n = 0)
            return 0
        if (n = 1)
            return 1
        if (n is already in D)
            return value stored with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
        Store (n, val) in D
        return val
```

Use hash-table or a map to remember which values were already computed.

Explicit (not automatic) memoization

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.

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- Resulting code:

```
Fib(n):
        if (n = 0)
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        if (n = 1)
            return 1
        if (M[n] ≠ −1)  // M[n]:  stored value of Fib(n)
            return M[n]
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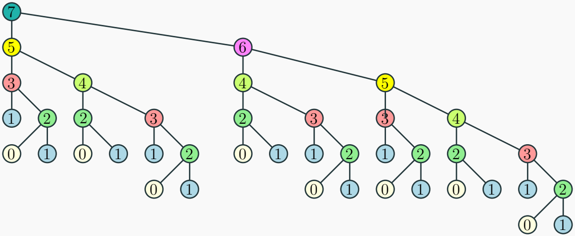
```

- Need to know upfront the number of sub-problems to allocate memory.

Recursion tree for the memorized Fib...



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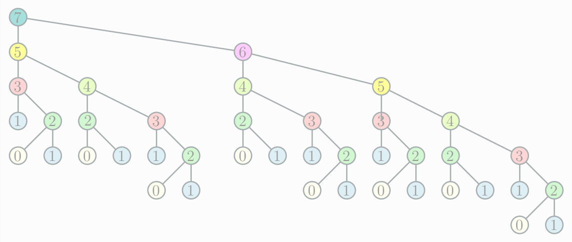

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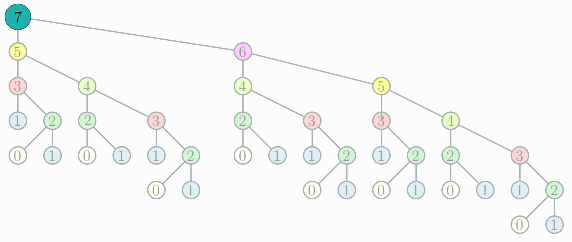

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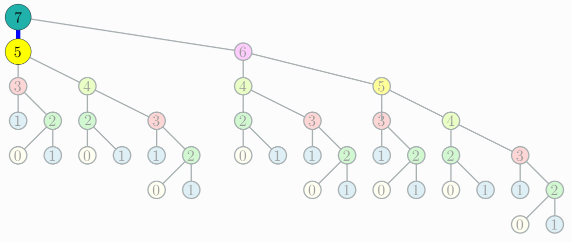

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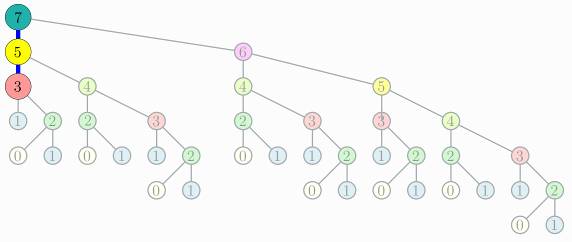

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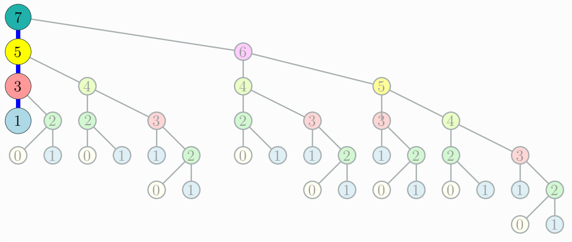

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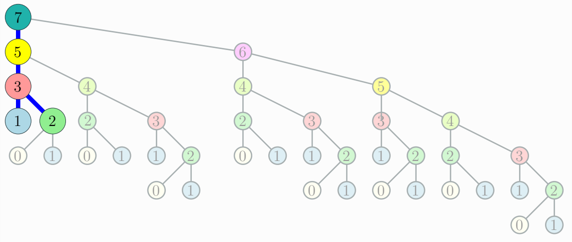

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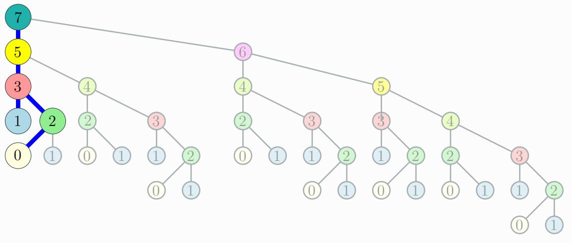

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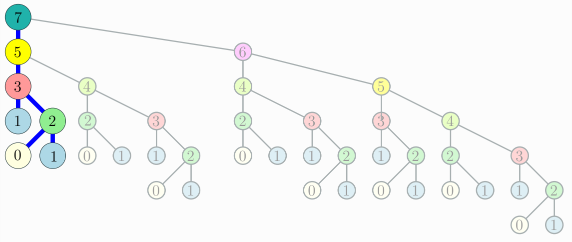

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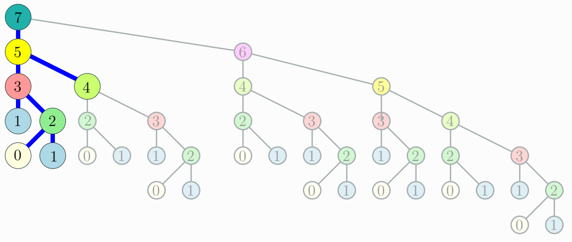

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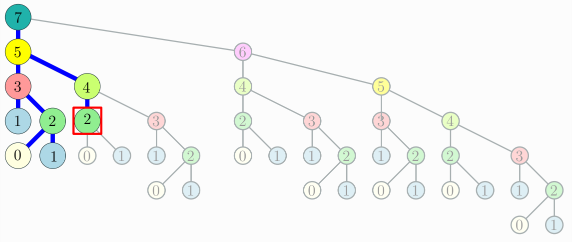

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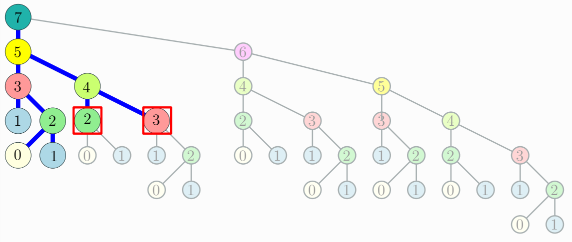

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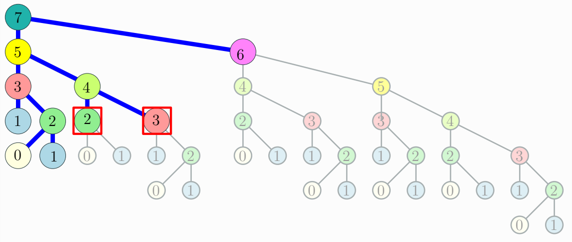

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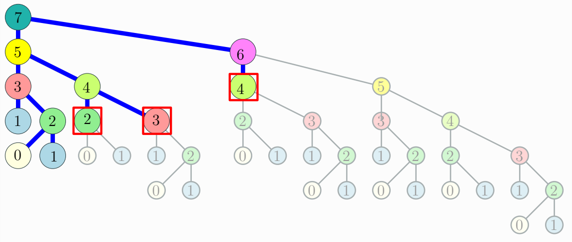

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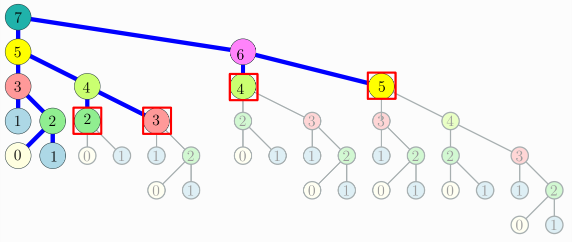

Implicit or automatic memoization

- Recursive version:

```
f(x_1, x_2, ..., x_d):
        CODE
```

- Recursive version with memoization:

```
g(x_1, x_2, ..., x_d):
        if f already computed for (x_1, x_2, ..., x_d) then
            return value already computed
        NEW_CODE
```

- NEW_CODE:
  - Replaces any "**return** α" with
  - Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

Explicit vs Implicit Memoization

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    - analyze problem ahead of time

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  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.
  - Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

Explicit/implicit memoization for Fibonacci

```
Init:   M[i] = −1,  i = 0, ..., n.

Fib(k):
    if (k = 0)
        return 0
    if (k = 1)
        return 1
    if (M[k] ≠ −1)
        return M[n]
    M[k] ⇐ Fib(k − 1) + Fib(k − 2)
    return M[k]
```

Explicit memoization

```
Init:   Init dictionary D

Fib(n):
    if (n = 0)
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    if (n = 1)
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    if (n is already in D)
        return value stored
            with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Implicit memoization

Dynamic programming

Removing the recursion by filling the table in the right order

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Fib(n):
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FibIter(n):
    if (n = 0) then
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```

Dynamic programming: Saving space!

Saving space. Do we need an array of n numbers? Not really.

```
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    return F[n]
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Dynamic programming – quick review

Dynamic Programming is smart recursion

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memoization

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memoization

+ filling the table in right order

+ removing recursion.

Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of <u>distinct</u> sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time <u>not counting</u> the time for its recursive calls.

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Suppose we memorize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

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Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$?

Analyzing memorized recursive function

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Q: What is an upper bound on the running time of memorized version of *foo*(x) if $|x| = n$? $O(A(n)B(n))$.

Longest Increasing Sub-sequence Revisited

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a sub-sequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Example

- Sequence: $6, 3, 5, 2, 7, 8, 1$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing subsequence of the first sequence: $2, 7, 8$.
- *Longest* Increasing subsequence of the first sequence: $3, 5, 7, 8$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

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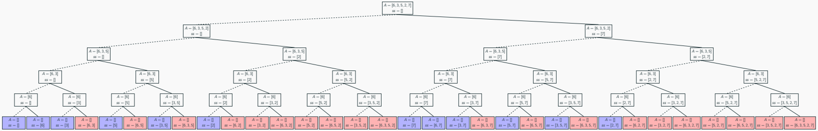
Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Naive Recursion Enumeration - State Tree



- This is just for [6,3,5,2,7]! (Tikz won't print larger trees)
- How many leafs are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?

Naive Recursion Enumeration - Code

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[0..n-1]$):

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Can we find a recursive algorithm for LIS?

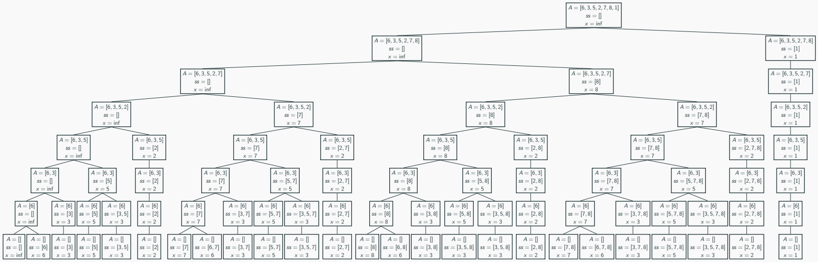
LIS($A[0..n-1]$):

- Case 1: Does not contain $A[n-1]$ in which case LIS($A[0..n-1]$) = LIS($A[0..(n-1)]$)
- Case 2: contains $A[n-1]$ in which case LIS($A[0..n-1]$) is not so clear.

Observation
For second case we want to find a subsequence in $A[1..(n-2)]$ that is restricted to numbers less than $A[n-1]$. This suggests that a more general problem is **LIS_smaller**($A[0..n-1], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$



Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i-1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i-1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
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Recursive Approach

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LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
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- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate?

Recursive Approach

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Naming sub-problems and recursive equation

After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$LIS(i,j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

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$LIS(i,j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $LIS(0,j) = 0$ for $1 \leq j \leq n+1$
Recursive relation:

- $LIS(i,j) = LIS(i-1,j)$ if $A[i] \geq A[j]$
- $LIS(i,j) = \max\{LIS(i-1,j), 1 + LIS(i-1,i)\}$ if $A[i] < A[j]$

Output: $LIS(n, n+1)$.

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
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Iterative algorithm

The dynamic program for longest increasing subsequence

LIS-Iterative($A[1..n]$):
    $A[n+1] = \infty$
    `int` $LIS[0..n-1, 0..n]$
    for $j = 0 \ldots n$) **if** `A[i]` $\leq$ `A[j]` **then** $LIS[0][j] = 1$

    for $i = 1 \ldots n-1$ **do**
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            if $(A[i] \geq A[j])$
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    `Return` $LIS[n, n+1]$

Running time: $O(n^2)$
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Running time: $O(n^2)$
Space: $O(n^2)$ Can be done in linear space. How?

Two comments

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Yes! See notes.

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$$A[1...7] = [6, 3, 5, 2, 7, 8, 1]$$

We know the LIS length (4) but how do we find the LIS itself?

$$LIS = [3, 5, 7, 8]$$

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Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

How to come up with dynamic programming algorithm: summary

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- Get rich!