Quant Puzzle #002: A Canonical Itô Integral Puzzle

Problem Statement

Solve the stochastic integral

$$\int_0^t W_s \, dW_s,$$

where W_t is standard Brownian motion.

Follow-Up Question

Find the distribution of

$$\int_0^1 W_s \, dW_s.$$

Core Ideas

Key Tools:

- Use Itô's lemma on $f(W_t) = \frac{W_t^2}{2}$ to derive a differential involving $W_t \, dW_t$, which rearranges directly into the desired integral.
- Recognize that the integral $\int_0^1 W_s \, dW_s$ is a function of W_1, and since $W_1 \sim \mathcal{N}(0,1)$, the integral's distribution can be derived by a change of variables from a standard normal.

Solution

We want to compute the stochastic integral

$$\int_0^t W_s \, dW_s$$

A naive application of standard calculus might suggest this equals $\frac{W_t^2}{2}$, since $\int x \, dx = \frac{x^2}{2}$. However, in stochastic calculus, this is incorrect due to the non-zero quadratic variation of Brownian motion. To get the correct expression, apply Itô's lemma to the function $f(W_t) = \frac{W_t^2}{2}$. Recall Itô's formula:

$$df(t, X_t) = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial x} \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \, (dX_t)^2$$

Using $X_t = W_t$ and $f(x) = \frac{x^2}{2}$, we get:

$$d\left(\frac{W_t^2}{2}\right) = W_t\,dW_t + \frac{1}{2}\,dt$$

Rearranging gives:

$$W_t\,dW_t = d\left(\frac{W_t^2}{2}\right) - \frac{1}{2}\,dt$$

Integrate both sides from 0 to t:

$$\int_0^t W_s\,dW_s = \frac{W_t^2}{2} - \frac{1}{2}\int_0^t ds = \frac{W_t^2 - t}{2}$$

This is the value of the integral. To find the distribution of $\int_0^1 W_s\,dW_s$, observe from above:

$$\int_0^1 W_s\,dW_s = \frac{W_1^2 - 1}{2}$$

Let $Z = W_1 \sim \mathcal{N}(0,1)$. Define:

$$X = \frac{Z^2 - 1}{2}$$

To find the distribution of X, compute its CDF:

$$\begin{aligned}
\mathbb{P}(X \le x) &= \mathbb{P}\left(\frac{Z^2 - 1}{2} \le x\right) \\
&= \mathbb{P}\left(Z^2 \le 2x + 1\right) \\
&= \mathbb{P}\left(-\sqrt{2x+1} \le Z \le \sqrt{2x+1}\right) \\
&= 2\Phi\left(\sqrt{2x+1}\right)
\end{aligned}$$

(where Φ is the standard normal CDF). Differentiate to get the density:

$$\begin{aligned}
f_X(x) &= \frac{d}{dx}\left[2\Phi\left(\sqrt{2x+1}\right)\right] \\
&= \frac{2}{\sqrt{2\pi}} \cdot \frac{d}{dx}\left(\int_0^{\sqrt{2x+1}} e^{-y^2/2}\,dy\right) \\
&= \frac{2}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{2x+1}{2}}}{2\sqrt{2x+1}} \cdot 2 \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-\frac{2x+1}{2}}}{\sqrt{2x+1}}
\end{aligned}$$

This is valid only when $2x + 1 > 0$, i.e. $x > -\frac{1}{2}$. So:

$$f_X(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{2x+1}{2}}}{\sqrt{2x+1}}, & x > -\frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

This is the distribution of $\int_0^1 W_s\,dW_s$.

Takeaways

This is a classic quant interview puzzle that tests:

- Knowing how and when to apply Itô's lemma, even in reverse.
- Comfort with working backwards from stochastic differentials to integrals.
- Ability to derive probability distributions via change-of-variable techniques.
- Competence in differentiating composite functions involving integrals.

