Quant Puzzle #001: Brownian Flip

Problem Statement

Suppose there is a Brownian Motion W_t. At time $t = 1$, $W_1 > 0$. What is the probability that at time $t = 2$, $W_2 < 0$?

Core Ideas

The following core ideas are used to solve this problem:

- Brownian motion has stationary, independent increments that are normally distributed.
- The sign of a Brownian increment is symmetric — positive and negative values are equally likely.
- To reach below zero, the downward increment must exceed the positive value at $t = 1$ in magnitude.

Solution

We have two time steps of equal length. The first runs from time $t = 0$ to time $t = 1$. The second runs from time $t = 1$ to $t = 2$. We can partition the Brownian motion into these two time intervals to create the Brownian increments $W_2 - W_1$ and $W_1 - W_0$. Brownian increments over arbitrary time intervals $t = x$ to $t = y$, $x \leq y$ are Normally distributed with zero mean and variance equal to $y - x$.

$$W_y - W_x \sim \mathcal{N}(0, y - x), \quad x \leq y$$

Also important is the fact that non-overlapping Brownian increments are independent of each other. Because the increment $W_2 - W_1$ occurs independently of the increment $W_1 - W_0$ it may very well be the case that $W_1 - W_0 > 0$ and $W_2 - W_1 < 0$. The occurrence of one outcome has no effect on the likelihood of the other occurring.

In order for $W_2 < 0$ we need $W_2 - W_1 < 0$ which, as we have a symmetric Normal distribution centered around a mean of 0, occurs with probability $\frac{1}{2}$. We must also have that $|W_2 - W_1| > |W_1 - W_0|$. Let $X = W_2 - W_1$ and $Y = W_1 - W_0$. This partitions into two cases:

1. $X > Y$ and $Y > 0$
2. $X < Y$ and $Y < 0$

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$$\begin{aligned}
\mathbb{P}(X > Y, Y > 0) &= \mathbb{P}(Y > 0)\mathbb{P}(X > Y) \\
&= \frac{1}{2}\mathbb{P}(X - Y > 0) \\
&= \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

And the same result holds for case 2 by identical reasoning. We see then that $\mathbb{P}(|W_2 - W_1| > |W_1 - W_0|) = \frac{1}{2}$. Putting it all together we have:

$$\begin{aligned}
\mathbb{P}(W_2 < 0) &= \mathbb{P}(\text{sgn}(W_2 - W_1) < 0) \cdot \mathbb{P}(|W_2 - W_1| > |W_1 - W_0|) \\
&= \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

which is the solution to this puzzle.

Takeaways

This is a classic quant brain-teaser. Simple to state yet subtle to solve. It tests knowledge of:

- Brownian increments and their independence
- The symmetry of the normal distribution
- Conditional probability without conditioning formulas

You'll see variations of this in interviews at hedge funds and trading firms.

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