

Quant Puzzle #003: Solving an Affine SDE with Multiplicative Noise.

Problem Statement

Solve the following Stochastic Differential Equation

$$dX_t = \mu dt + (aX_t + b)dW_t,$$

where W_t is a Brownian Motion.

Context & Key Insights

This affine SDE models systems where randomness depends linearly on the current state which is common in finance for volatility and interest rates. Because the noise can grow with the process, the system can become unstable or even explode, making its solution both challenging and instructive.

Core Ideas

Key Tools:

- Use the method of variation of constants to write the solution as a product of two simpler stochastic processes.
- Use Itô's Lemma to create an SDE for X_t in terms of its factors.
- Solve the resulting SDE by equating the drift and diffusion coefficients.

Solution

Summary: We solve this affine SDE using variation of constants. The process is decomposed into a product of two Itô processes, with coefficients reverse-engineered to match the original SDE. This results in a clean closed-form expression for X_t .

The SDE given in the problem statement is an Affine SDE. Any SDE of the form

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$

is an Affine SDE provided both $f(t, X_t)$ and $g(t, X_t)$ are linear in X_t . The method of variation of constants allows us to find a closed form expression for the Itô process X_t that solves the Affine SDE. For SDEs with non-affine f and g , closed form solutions are often not easily obtainable and simulation would be used instead.

The method of variation of constants

Given an SDE of the form

$$dX_t = [p_1(t) + p_2(t)X_t] dt + [q_1(t) + q_2(t)X_t] dW_t$$

We can solve it with the SDE equivalent of an integrating factor.

Step 1: Solving the Homogeneous version of the SDE

For now forget about $p_1(t)$ and $q_1(t)$, the simpler Homogeneous SDE

$$dY_t = p_2(t)Y_t dt + q_2(t)Y_t dW_t, \quad Y_0 = 1$$

can be solved explicitly as follows. We begin with an application of Itô's lemma:

$$d \log(Y_t) = \frac{dY_t}{Y_t} - \frac{(dY_t)^2}{2Y_t^2}$$

Given the dynamics of Y_t , we obtain:

$$d \log(Y_t) = \left(p_2(t) - \frac{q_2^2(t)}{2} \right) dt + q_2(t) dW_t$$

Integrating both sides gives:

$$\log(Y_t) = \left(p_2(t) - \frac{q_2^2(t)}{2} \right) t + q_2(t)W_t + A$$

where A is an arbitrary constant. Exponentiating:

$$Y_t = B \exp \left[\left(p_2(t) - \frac{q_2^2(t)}{2} \right) t + q_2(t)W_t \right]$$

Applying the initial condition, $Y_0 = 1$, allows us to solve for the arbitrary constant B . Thus, we simplify to:

$$Y_t = \exp \left[\left(p_2(t) - \frac{q_2^2(t)}{2} \right) t + q_2(t)W_t \right]$$

Step 2: Guess the Product Form

Now looking at the real SDE we realise we can create the solution X_t as a product of the solution of the Homogeneous version with another stochastic process. i.e.

$$X_t = Y_t \cdot Z_t$$

where Z_t is the solution of

$$dZ_t = \alpha(t)dt + \beta(t)dW_t, \quad Z_0 = X_0$$

and $\alpha(t)$, $\beta(t)$ are functions to be determined.

The structure of this solution is analogous to the expansion of brackets in ordinary algebra (think of expanding $(x+1)(x-3)$, you get cross terms which are linear and constant).

Step 3: Apply Itô's formula

Applying Itô's product rule we get

$$dX_t = d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t + dY_t dZ_t$$

group terms and then simply equate coefficients to solve for $\alpha(t)$, $\beta(t)$ using the previously computed value for Y_t where necessary to make dZ_t directly integrable. Having then solved for both Z_t , Y_t we can then piece together the solution for X_t rather trivially.

What we have done is solve the Homogeneous equation, then varied the constants to account for the extra terms, hence the name. By construction, the product $Y_t Z_t$ satisfies the original SDE and the form of Z_t is reverse-engineered to fix the mismatch cause by the terms $p_1(t)$ and $q_1(t)$.

Continuing the solution

To begin, we seek a solution of the form $X_t = Y_t Z_t$ where Y_t is the solution of $dY_t = aY_t dW_t$, $Y_0 = 1$ and Z_t is the solution of $dZ_t = \alpha_t dt + \beta_t dW_t$, $Z_0 = X_0$ and α_t , β_t are functions to be determined.

First we solve for Y_t .

$$d \log(Y_t) = -\frac{a^2}{2} dt + a dW_t$$

From which we see that

$$\begin{aligned} \log(Y_t) &= -\frac{a^2}{2} t + aW_t + A \\ Y_t &= B e^{-\frac{a^2}{2} t + aW_t} \end{aligned}$$

where A and B are arbitrary constants. Applying the initial conditions, $Y_0 = 1$, we see that

$$Y_t = e^{-\frac{a^2}{2} t + aW_t}$$

Returning now to X_t

$$\begin{aligned} dX_t &= (\alpha_t + a\beta_t) Y_t dt + (aY_t Z_t + \beta_t Y_t) dW_t \\ &= (\alpha_t + a\beta_t) Y_t dt + (aX_t + \beta_t Y_t) dW_t \end{aligned}$$

Equating coefficients

$$\beta_t = bY_t^{-1}, \quad \alpha_t = \mu Y_t^{-1} - a\beta_t$$

Solving for Z_t

$$\begin{aligned} dZ_t &= (\mu Y_t^{-1} - abY_t^{-1}) dt + bY_t^{-1} dW_t \\ &= e^{\frac{a^2}{2} t - aW_t} (\mu - ab) dt + e^{\frac{a^2}{2} t - aW_t} b dW_t \\ Z_t &= \int_0^t e^{\frac{a^2}{2} s - aW_s} (\mu - ab) ds + \int_0^t e^{\frac{a^2}{2} s - aW_s} b dW_s + X_0 \end{aligned}$$

We can now solve for X_t fully

$$X_t = e^{-\frac{a^2}{2} t + aW_t} \left(X_0 + \int_0^t e^{\frac{a^2}{2} s - aW_s} (\mu - ab) ds + \int_0^t e^{\frac{a^2}{2} s - aW_s} b dW_s \right)$$

which is the final answer for this puzzle.

Takeaways

This is one of the most technically demanding puzzles you may encounter in a Quant finance interview. Nonetheless solving this puzzle proves that you can:

- Apply Itô's formula and Itô's product rule.
- Work backwards from stochastic differentials to integrals.
- Apply the variation of constants method to solve any general affine SDE.
- Solve, for a closed form solution, any SDE for which a closed form solution can be found.

The ability to solve this, and explain your solution in an interview setting is a good indicator that you are among the brightest hopeful Quants.

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