Quant Puzzle #004: An Insightful ODE

Problem Statement

Find $f(x)$ if

$$f'(x) = f(x)\,(1 - f(x))$$

Context & Key Insights

This is essentially the logistic equation, a cornerstone in modelling constrained growth. This problem is representative of what you might encounter when modelling population dynamics and could be used in an interview setting as a test of mathematical dexterity and clarity of explanation.

Core Ideas

Key Tools:

- Use separation of variables ($\frac{dy}{dx} \equiv y'$).
- Use Partial Fractions to simplify the integrand of the resulting integral to a form amenable to direct integration.

Solution

Let $y = f(x)$ for notational simplicity. It is easy to see that:

$$\frac{y'}{y(1-y)} = 1$$

Integrating...

$$\int \frac{y'}{y(1-y)} dx = \int 1 dx$$

Note that $\frac{dy}{dx} \equiv y'$ implies that $dy = y'dx$. This is the basis behind the technique of separation of variables. Using A, B and C to denote the arbitrary constants of integration we have

$$\int \frac{1}{y(1-y)} dy = x + A$$

We recognise the denominator of the integrand as a product of two linear factors. Therefore we can use partial fractions to simplify the integrand, making it amenable to direct integration.

$$\int \frac{1}{y(1-y)} dy = \int \frac{1}{y} + \frac{1}{1-y} dy$$

Now that we have something we can integrate we can proceed.

$$\log|y| - \log|1-y| = x + B$$

Using the laws of logarithms we can simplify this to

$$\log\left|\frac{y}{1-y}\right| = x + B$$

Exponentiating and recalling that the exponential function is a purely positive valued function for any input (allowing us to drop the modulus operator) we arrive at

$$\frac{y}{1-y} = Ce^x$$

Rearranging for y we get

$$y = \frac{Ce^x}{1+Ce^x}$$

Given the absence of initial or boundary conditions, this is the limit of how much of a solution we can give. We cannot solve for specific values of the arbitrary constant C. We can however comment on the asymptotic behaviour of the solution which tends to 1 as $x \to \infty$.

Takeaways

This is a subtle ODE whose difficulty stems from two tricks, one less obvious than the other. These techniques are used to simplify your problem and are:

- The use of the separation of variables identity $\frac{dy}{dx} \equiv y'$.
- The technique of partial fractions.

This problem is classified as Easy, since a mathematically literate postgraduate should be able to solve it by applying standard ODE techniques. The only challenge lies in spotting separation of variables and partial fractions quickly.

More Editorials: https://github.com/UIronsMaths/Quant-Puzzle-Editorials

