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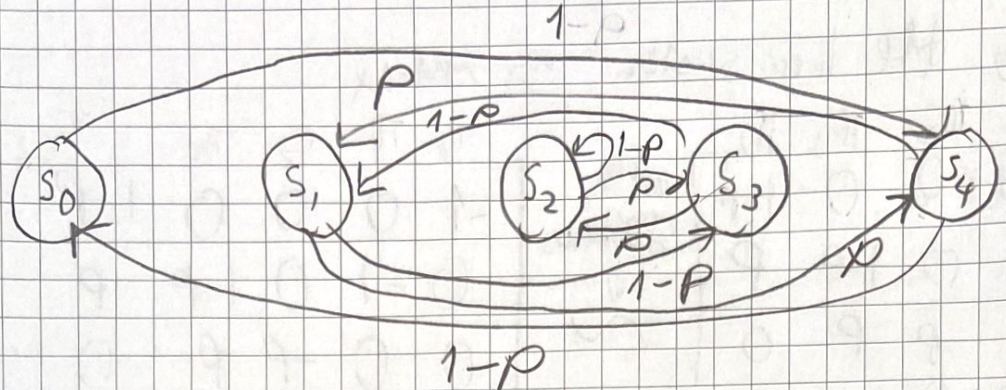
Q1)

a)



transition matrix

$$P = \begin{array}{c|ccccc} & S_0 & S_1 & S_2 & S_3 & S_4 \\ \hline S_0 & 0 & 0 & 0 & 0 & 1 \\ S_1 & 0 & 0 & 0 & 1-p & p \\ S_2 & 0 & 0 & 1-p & p & 0 \\ S_3 & 0 & 1-p & p & 0 & 0 \\ S_4 & 1-p & p & 0 & 0 & 0 \end{array}$$

$S = \{ S_i : i = 0, 1, 2, 3, 4 \}$

i : # of umbrellas at the current location

b) $\pi = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4]$

$\Rightarrow \pi P = \pi \ \& \ \sum_{i=0}^{4} \pi_i = 1$

$$[\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4] \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 1-p & p & 0 \\ 0 & 1-p & p & 0 & 0 \\ 1-p & p & 0 & 0 & 0 \end{bmatrix} = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4]$$

$\Rightarrow$

$\pi_4 (1-p) = \pi_0$ $\qquad$ $\pi_0 + p\pi_1 = \pi_4$

$\pi_3 (1-p) + p\pi_4 = \pi_1$

$\pi_2 (1-p) + p\pi_3 = \pi_2$

$\pi_1 (1-p) + p\pi_2 = \pi_3$

KeskinColor

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• modelling the linear system as matrix

$$
\begin{array}{ccccc}
\pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 \\
\end{array}
\qquad\qquad\qquad
\begin{array}{ccccc}
\pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 \\
\end{array}
$$

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 & 1-p \\
0 & -1 & 0 & 1-p & p \\
0 & 0 & -p & p & 0 \\
0 & 1-p & p & -1 & 0 \\
1 & p & 0 & 0 & -1
\end{bmatrix}
\underset{\downarrow}{\overset{\text{apply row operations}}{\sim}}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 1-p \\
0 & -1 & 0 & 1-p & p \\
0 & 0 & -p & p & 0 \\
0 & 0 & 0 & p^2-p & -p^2+p \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\underbrace{\pi_3}_{k}(p^2-p) + \underbrace{\pi_4}_{k}(-p^2+p) = 0$$

$$\Rightarrow \boxed{\pi_4 = k} \Rightarrow \pi_3 = k \Rightarrow \pi_2 = k \Rightarrow$$

$$\pi_1 = (1-p)\overbrace{\pi_3}^{k} + p\overbrace{\pi_4}^{k}$$

$$= k - \cancel{pk} + \cancel{pk} = k$$

$$\pi_0 = (1-p)k \Rightarrow [(1-p)k,\ k,\ k,\ k,\ k]$$

$$5k - pk = 1 \Rightarrow k(5-p) = 1 \Rightarrow k = \frac{1}{5-p}$$

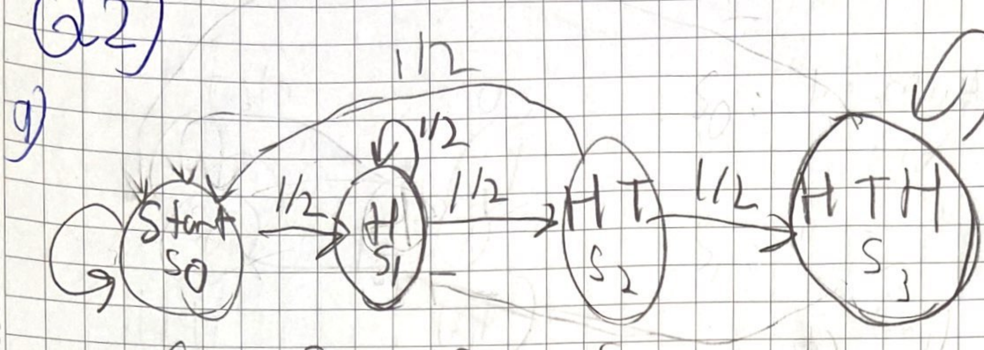
$$
\begin{array}{ccccccc}
 & & 0 & 1 & 2 & 3 & 4 \\
\pi & = & \Big[\ \frac{1-p}{5-p} & \frac{1}{5-p} & \frac{1}{5-p} & \frac{1}{5-p} & \frac{1}{5-p}\ \Big]
\end{array}
\Rightarrow \text{Unique}
$$

$$P(\text{I walk under rain}) = \pi_0 \cdot \overbrace{p}^{\text{probability that it rains}}$$

$$= \frac{1-p}{5-p} \cdot p = \boxed{\frac{p-p^2}{5-p}}$$

Q2)

a)



$$\begin{array}{c|cccc} & S_0 & S_1 & S_2 & S_3 \\ \hline S_0 & 1/2 & 1/2 & 0 & 0 \\ S_1 & 0 & 1/2 & 1/2 & 0 \\ S_2 & 1/2 & 0 & 0 & 1/2 \\ S_3 & 0 & 0 & 0 & 1 \end{array}$$

(transition matrix)

S_0: Initial state
no coin is flipped
or T flipped

S_1: H flipped

S_2: T flipped just after H. (HT)

S_3: H flipped after S_2. (HTH)

b) Let $F(S_i)$ be the expected number of steps to reach S_3 (HTH) starting from S_i.

we have

$$F(S_2) = \frac{1}{2}(1) + \frac{1}{2}\left(1 + F(S_0)\right) = 1 + \frac{1}{2}F_0$$

$$F(S_1) = \frac{1}{2}\left(1 + F(S_2)\right) + \frac{1}{2}\left(1 + F(S_1)\right) = 1 + \frac{1}{2}F(S_1) + \frac{1}{2}F(S_2$$

$$F(S_0) = \frac{1}{2}\left(1 + F(S_0)\right) + \frac{1}{2}\left(1 + F(S_1)\right) = 1 + \frac{1}{2}F(S_0) + \frac{1}{2}F(S_1$$

Solving these equations we have:

$$\boxed{F(S_0) = 10}$$

Q3)

Initial Observations

- $X_0 \Rightarrow$ can take values in $\{1, \dots N\}$ independent of other variables as stated in the question

- $X_1 = f(X_0, w_0) \rightarrow$ Can take values in $\{1, \dots N\}$

- X_1 can be dependent on X_0 and w_0. If I know X_0 and w_0 I know X_1.

- X_0 and X_1 are not dependent on previous stages, X_0 is independent of anything, X_1 is depended on current stage that is X_0.

Generalization

- $X_n = f(X_{n-1}, w_{n-1}) \rightarrow X$, $X = \{1, \dots N\}$

$X_{n-1} = f(X_{n-2}, w_{n-2}) \rightarrow X$

$\vdots$

$X_1 = f(X_0, w_0) \rightarrow X$

$\Rightarrow (X_0, w_0) \rightarrow X_1 \Rightarrow (X_1, w_1) \rightarrow X_2 \cdots \Rightarrow (X_{n-1}, w_{n-1}) \rightarrow X_n$

$\Rightarrow (X_n, w_n) \rightarrow X_{n+1}$

- Assume that the current stage is (X_{n-1}, w_{n-1}) then we can find X_n if we know the current stage. Previous step (X_{n-2}, w_{n-2}) not needed

Yes!

Moreover $(X_0, w_0) \cdots (X_{n-2}, w_{n-2})$ (Past states) and $(X_{n+1}, w_{n+1}) \cdots (X_\infty, w_\infty)$ (future states) are independent conditioned on (X_n, w_n) (Current State) $\Rightarrow$ MC

Yes!