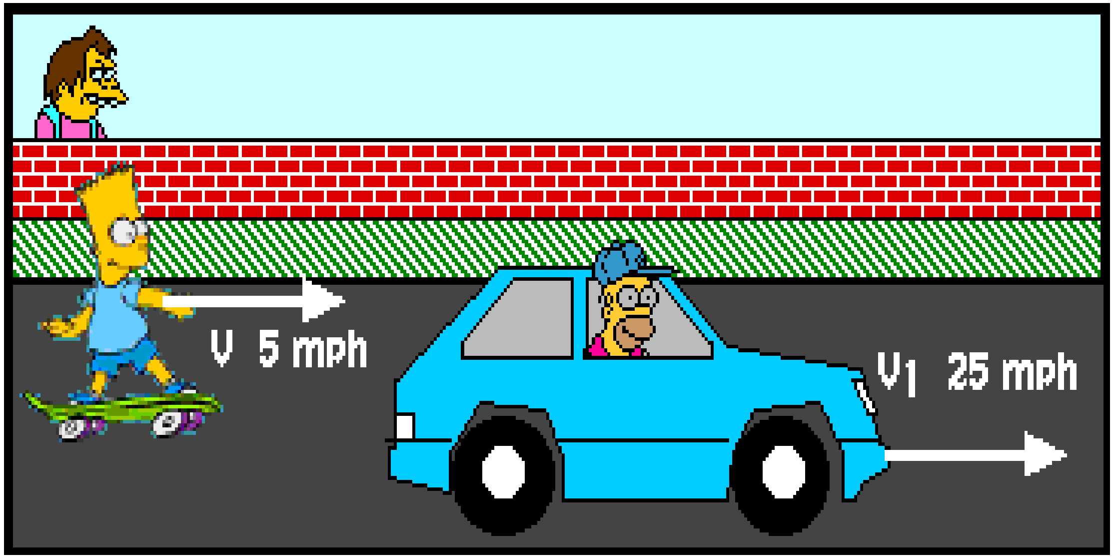
Module - 4

RELATIVITY

What does "relative" mean?

This meant that the measurement of motion depended on the relative velocity and position of the observer.



Relativity means its Einstein's theory

Which says that time and space are not absolute.

(Not absolute means changing from different frame of references)

The theory of relativity is actually two theories, that Albert Einstein came up with in the early 1900s.

One is called **"special" relativity** and the other is called "**general" relativity**.

We will talk mostly about **special relativity** here.

"special" relativity

Special relativity applies to all physical phenomena in absence of gravity

"general" relativity.

It applies to all physical phenomena in present of gravity.

Inertial frame and non-inertial frame

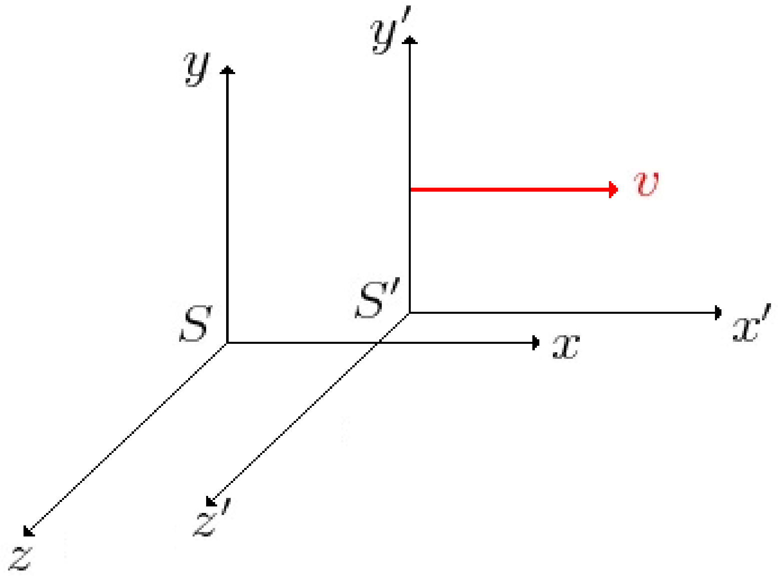
Inertial frame

A frame which is stationary and moving with constant velocity and in which all newton’s laws are valid called inertial frame of reference.

Non inertial frame

A frame which accelerating and in which all newton’s laws are not valid called non inertial frame of reference.

Galilean transformation (Newtonian transformation)



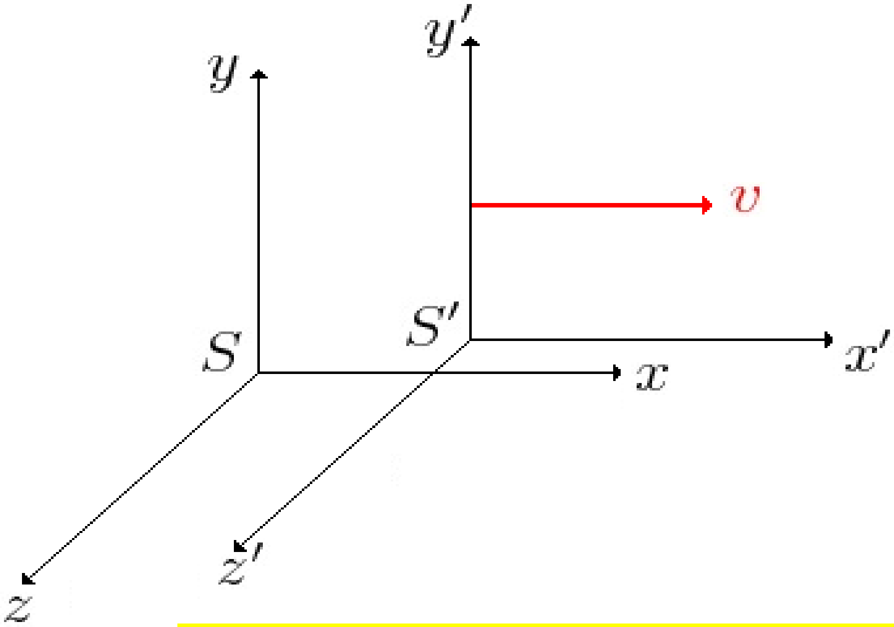
Suppose there are two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure).

In S, we have the co-ordinates $\{x, y, z, t\}$ and

In S' we have the co-ordinates $\{x', y', z', t'\}$

S' is moving with respect to S with velocity **v** (as measured in S) in the x direction.

The clocks in both systems were synchronised at time t=0 and they run at the same rate.



We have the intuitive relationships

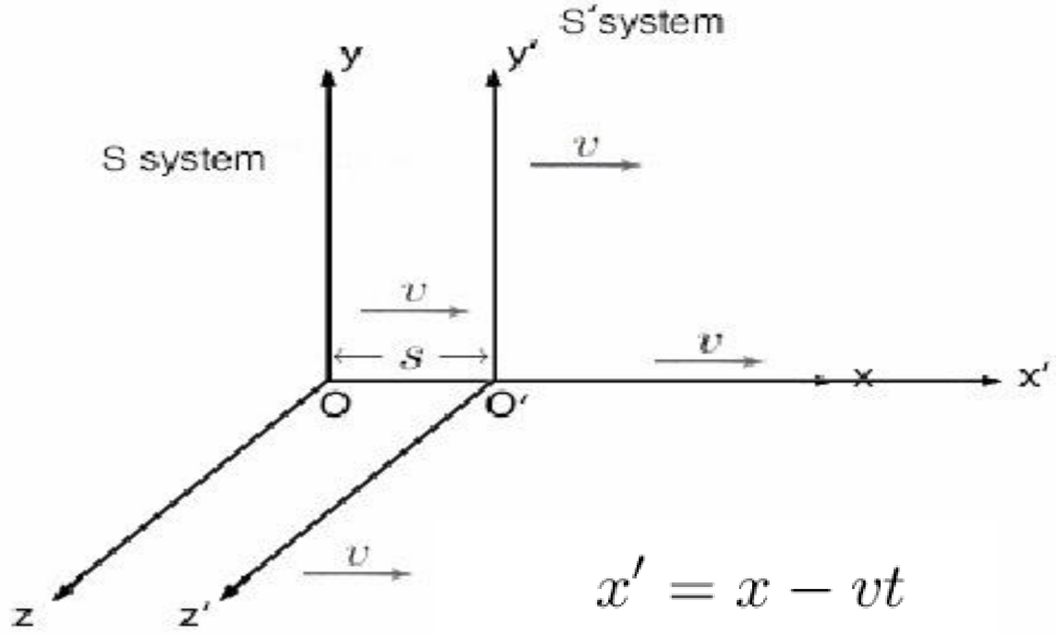
$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

This set of equations is known as the Galilean Transformation.

They enable us to relate a measurement in one inertial reference frame to another.

For example, suppose we measure the velocity of a vehicle moving in the in x --direction in system S, and we want to know what would be the velocity of the vehicle in S'.

$$v'_x = \frac{dx'}{dt'} = \frac{d(x - vt)}{dt} = v_x - v$$



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilean transformation equation

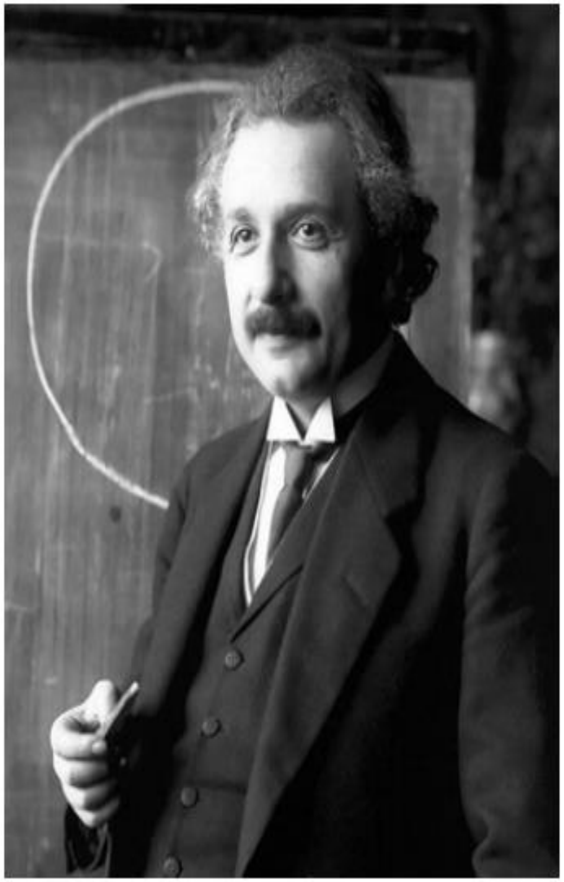
$$\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}$$

Inverse Galilean transformation equation

Einstein's Postulates:

(E1) ALL the laws of physics are the same in every inertial frame of reference.

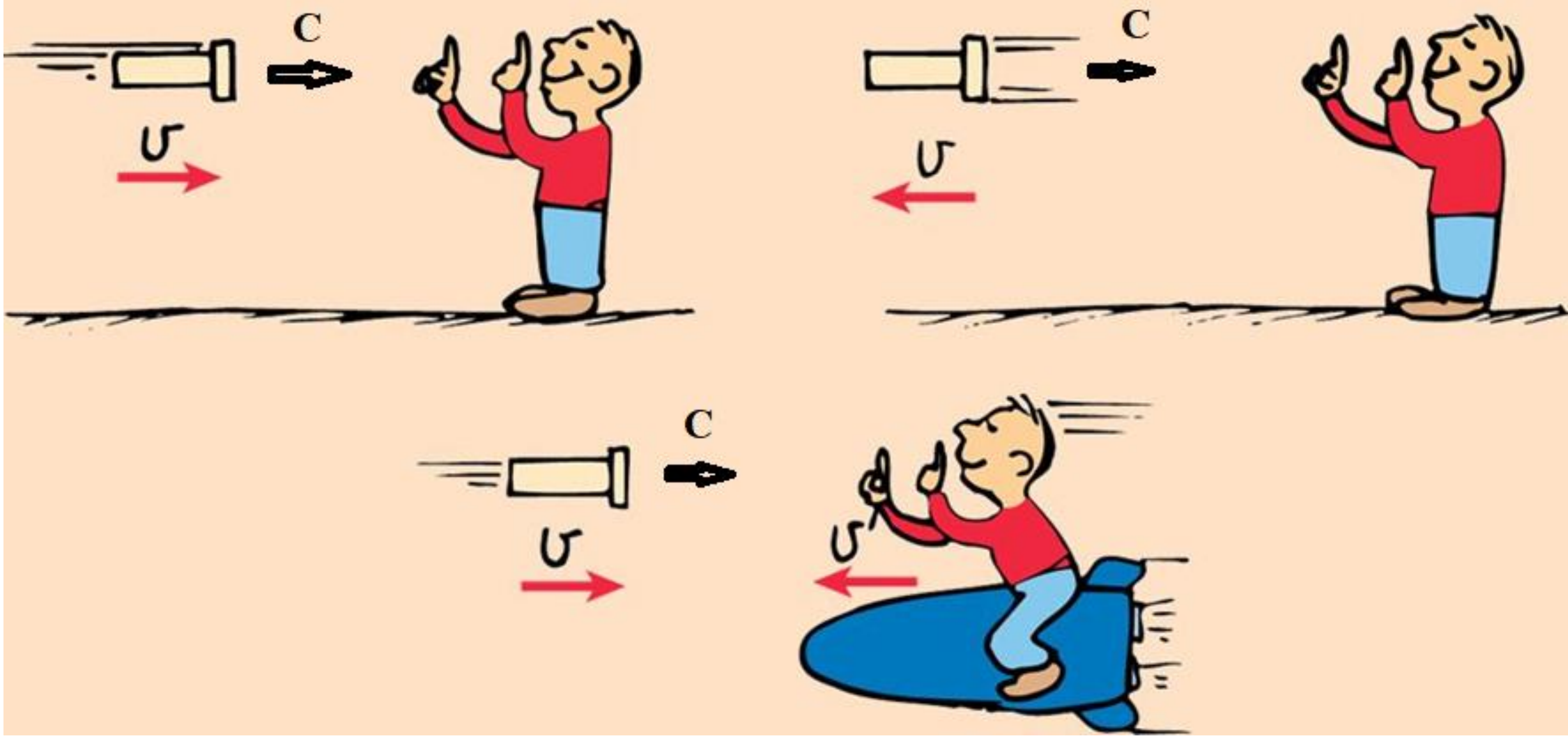
(E2) The speed of light is independent of the motion of its source.



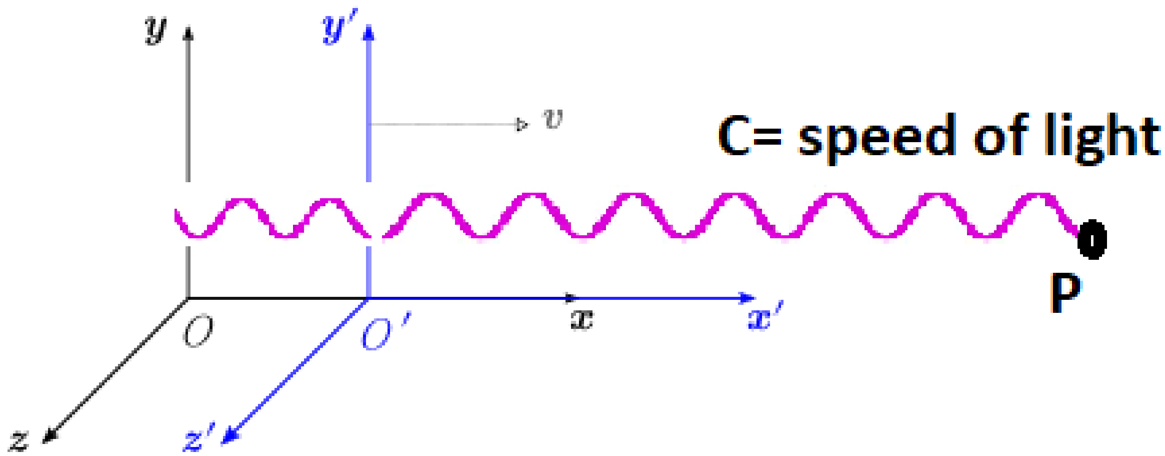
The Second Postulate of Special Relativity

2nd Postulate of S.R.:

speed of light is **constant regardless** of the speed of the flashlight or **observer**.



Lorentz transformation of space and time



Consider light is emitted from rest frame with speed C

From rest frame point of view the speed of light is as original as C

From moving frame point of view the speed of light is C’ =C – v

It means the speed of light as per moving point of view is less than the speed of light in rest frame.

This is what happens as per Galilean transformation but this not correct as per the second postulate of special theory of relativity and by Michelson Morley experiment the speed of light is always same at all condition.

To resolve this problem we will have to study the **Lorentz transformation**.

When light reaches at the point P from stationary and moving frame point of view and as per second postulate of speed of light. The speed of light is same

$X = ct$

$X' = ct'$

By Galilean transformation co-ordinates the coordinate of point P from frame S' w.r.t. frame S is

$X' = X - vt$

$Y' = y$

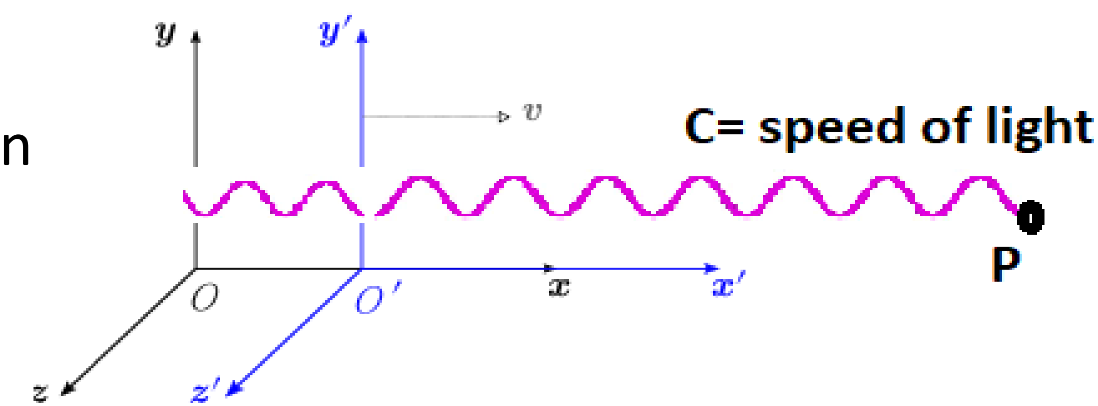
$Z' = z$

By its inverse transformation

$X = x' + vt'$

$Y = y'$

$Z = z'$



To improve the GT, we will multiply the constant γ

$X' = \gamma\,(X - vt)$ $\quad\quad\quad$ $X = \gamma\,(x' + vt')$

We need to find the value of constant $\boldsymbol{\gamma}$ Multiply the above two equation

$X'X = \gamma\,(X - vt)\,\gamma\,(x' + vt')$

$X'X = \gamma^2\,\{xx' + xvt' - vtx' - v^2 tt'\}$

But

$X = ct$

$X' = ct'$

$Ct\,ct' = \gamma^2\,\{\,Ct\,ct' - Ct\,vt' + vt\,ct' - (v)^2 tt'\,\}$

$C^2 tt' = \gamma^2\,\{\,C^2\,tt' - \cancel{Ct\,vt'} + \cancel{vt\,ct'} - (v)^2 tt'\,\}$

$C^2 tt' = \gamma^2\,\{\,C^2 - (v)^2\,\}\,tt'$

$C^2 \cancel{tt'} = \gamma^2\,\{\,C^2 - (v)^2\,\}\,\cancel{tt'}$

$C^2 = \gamma^2\,\{\,C^2 - (v)^2\,\}$

$$\gamma^2 = \frac{c^2}{c^2 - v^2}$$

$$\gamma = \sqrt{\frac{c^2}{c^2 - v^2}}$$

$$\gamma = \frac{c}{\sqrt{c^2 - v^2}}$$

$$\gamma = \frac{c}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Very important term for Lorentz transformation

To improve the GT, we will multiply the constant γ

$$X' = \gamma\,(X - vt)$$

$$X' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,(x - vt)$$

Using the equation

$$X' = ct'$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,(x - vt) = ct'$$

But $x = ct$

$$t = \frac{x}{c}$$

Putting above values in below equation

$$\frac{(x - vt)}{\sqrt{1-\frac{v^2}{c^2}}} = ct'$$

$$\frac{(ct - v\frac{x}{c})}{\sqrt{1-\frac{v^2}{c^2}}} = ct'$$

Taking C common in numerator

$$\frac{c\,(t - v\frac{x}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} = c\,t'$$

$$\frac{c\,(t - v\frac{x}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} = c\,t' \qquad \frac{(t - v\frac{x}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} = t'$$

$$\boldsymbol{\frac{(t - x\frac{v}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} = t'}$$

Measurement of space and time depends on the frame reference of observer (it means they are no longer absolute)

Lorentz transformation equation for space and time

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

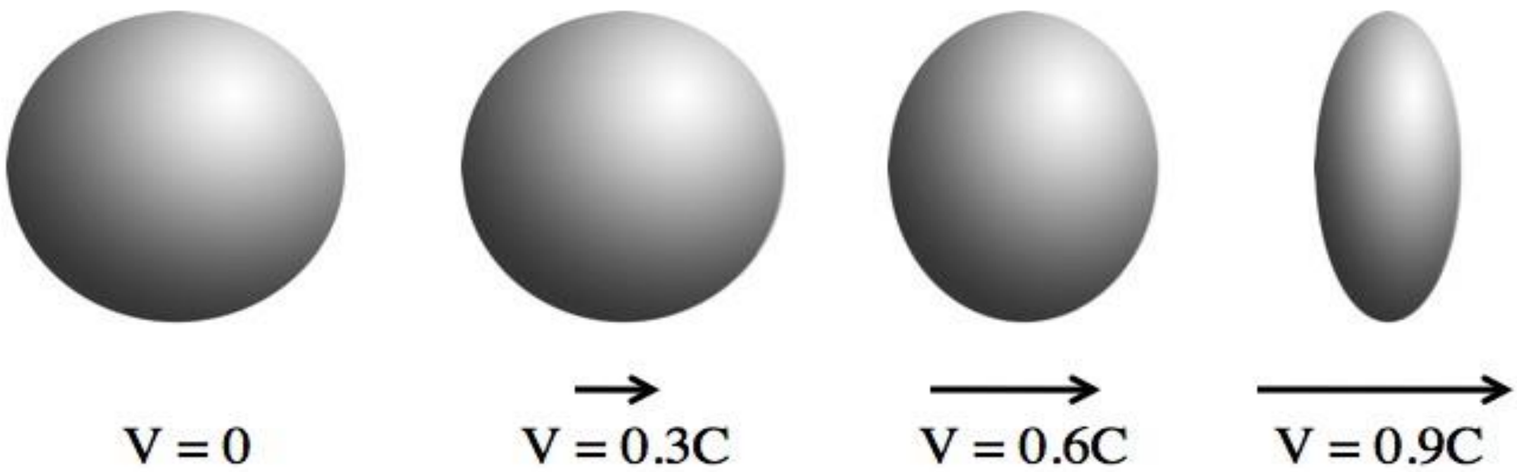
$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

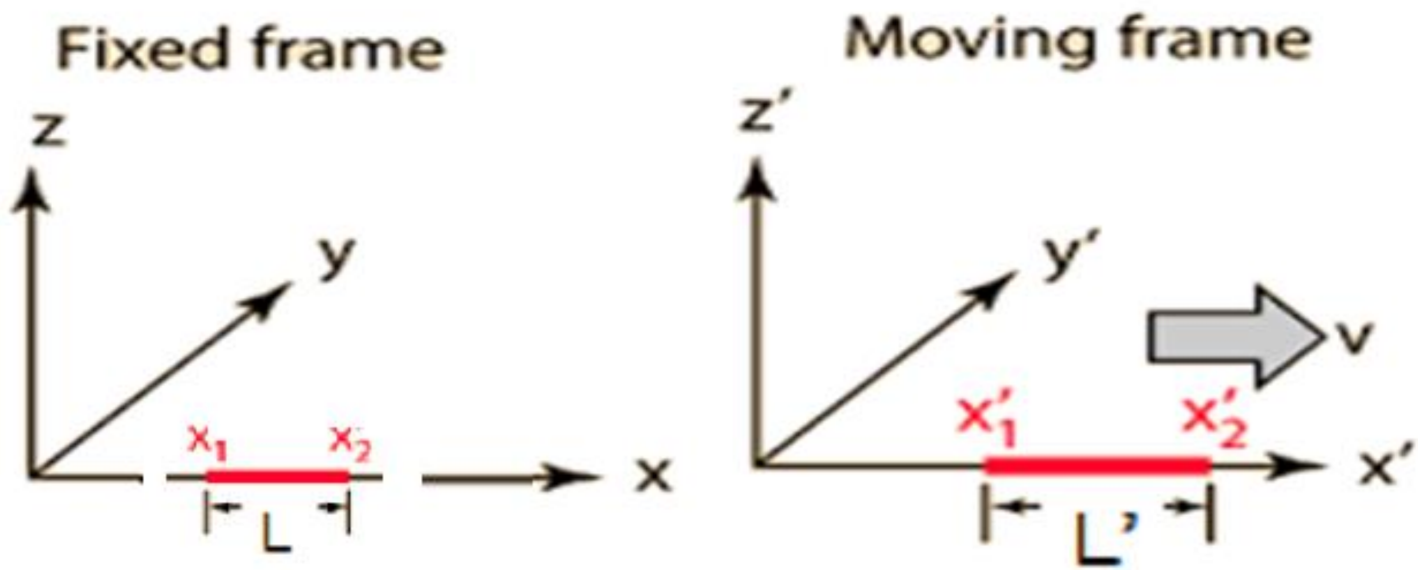
Length contraction

According to special theory of relativity, the length of moving object decreases than the length of an object at rest. This process is called length contraction.

Let us see the following example



Derivation



Consider two frames of references S and S' in which frame S' is moving with speed v.

The length of rod measured by as per **rest frame** L = $x_1 - x_2$.

The length of rod is at rest as per **moving frame** L_0 = $x'_1 - x'_2$

From Lorentz transformation

$$X' = \frac{(x-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$$

We can write,

$$X_1' = \frac{(x_1-vt)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad X_2' = \frac{(x_2-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$$

The length of rod as per moving frame $L_0 = x'_2 - x'_1 = \frac{(x_2-vt)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{(x_1-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$

$$= \frac{(x_2-vt)-(x_1-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{x_2-x_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$L_0 = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

So length of rod moving with velocity v is reduced by factor $\sqrt{1-\frac{v^2}{c^2}}$ in direction of motion.

Case (I)

When v is very small as compared to c (v<< c) then $\frac{v^2}{c^2}$ term is negligible.

So, L = L_0

Case (II)

When v is comparable to c

$\sqrt{1-\frac{v^2}{c^2}}$ This value will be less than 1

So, L < L_0

The length of a rod is found to be half of its length when at rest. What is the speed of the rod relative to the observer?

Data : $L = \frac{L_o}{2}$

Formula : $L = L_o\sqrt{1-(v^2/c^2)}$

$$\frac{L_o}{2} = L_o\sqrt{1-(v^2/c^2)}$$

$$\sqrt{1-(v^2/c^2)} = \frac{1}{2}$$

$$\therefore \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$\frac{v^2}{c^2} = \frac{3}{4} \qquad \therefore \quad v = \frac{\sqrt{3}}{2}c = 0.866\ c$$

A 1 m long rod is moving along its length with a velocity 0.6 c. Calculate its length as it appears to an observer on the earth.

Data **:** $v = 0.6\ c,\ L_o = 1\ m.$

Formula **:** $$L = L_o \sqrt{1-(v^2/c^2)}$$

Calculations **:** $$L = 1\sqrt{1-(0.6)^2} = 0.8\ m$$

A rod has a length of 2 m. Find its length when it is carried in a rocket with a speed of 0.9 e.

Data : $L_o = 2$ m, $v = 0.9$ c

Formula : $L = L_o \sqrt{1-(v^2/c^2)}$

Calculations : $L = 2\sqrt{1-(0.9)^2}$

$\therefore$ $L = 0.872$ m

What is the length of a meter stick moving parallel to its length when its mass is 3/2 times of its rest mass?

Data : $L_o = 1\ \text{m},\ m = \frac{3}{2} m_o.$

Formula : $L = L_o \sqrt{1-(v^2/c^2)},\quad m = \dfrac{m_o}{\sqrt{1-(v^2/c^2)}}$

Calculations : $\sqrt{1-(v^2/c^2)} = \dfrac{m_o}{m}$

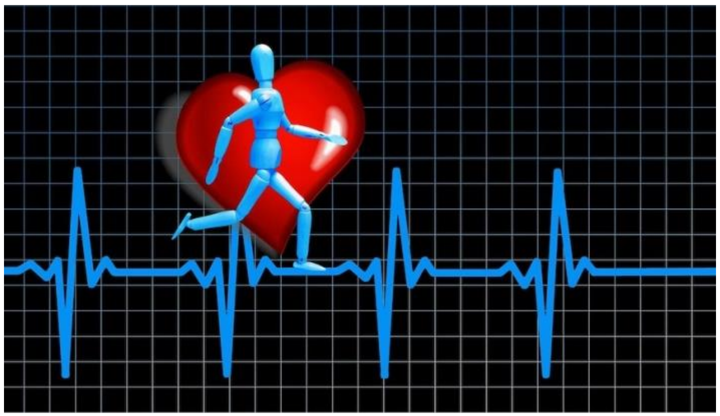
$$L = L_o \frac{m_o}{m} = 1 \times \frac{m_o}{(3/2)\, m_o} = \frac{2}{3}\ \text{m}$$

$$L = 0.67\ \text{m}$$

Time dilation

Speed of light is always constant irrespective of motion of frames, that why it's **c**onstant is represented by symbol **c**

Let us discuss some amazing thing



If we run then heart beats increases and if we walk then heart beats decreases.

But with time something different happens
If we run slowly then time runs faster and if we run faster than time runs slower

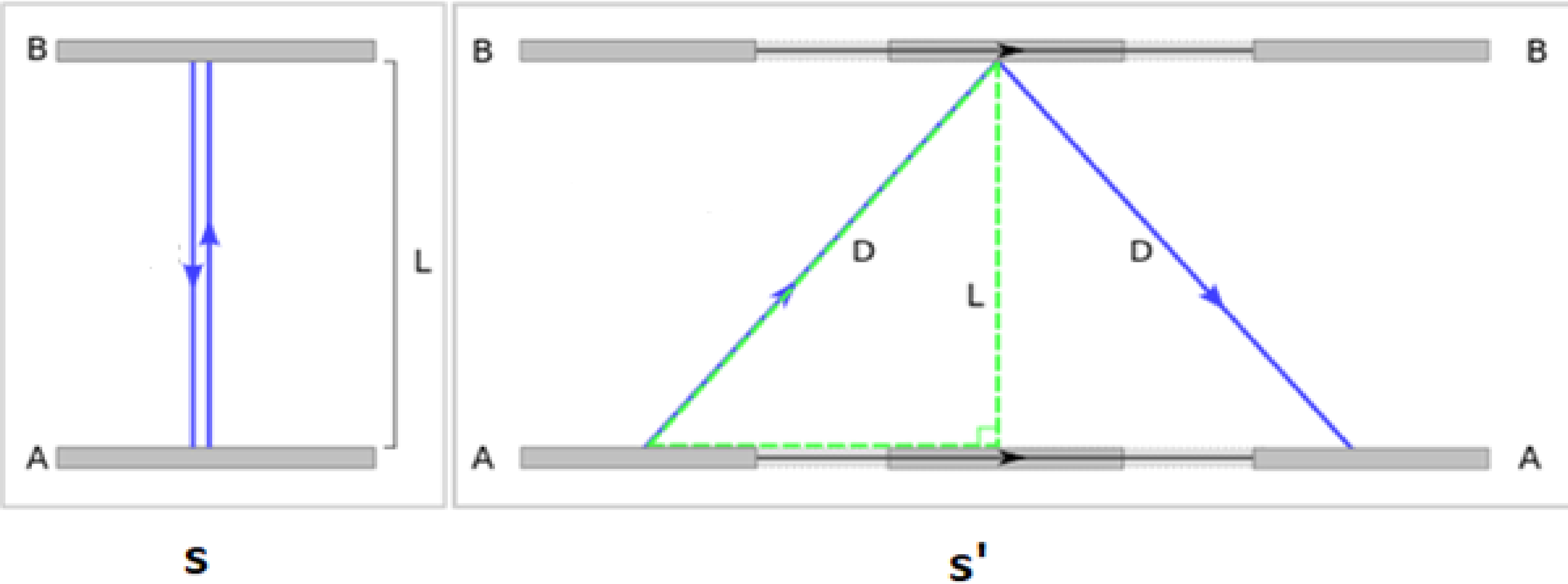
Definition of Time Dilation

Time slows down by a factor $\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ this phenomenon is known as time dilation.

Before to understand time dilation let us understand what time is.

Time (Clock)

A clock is something that repeats itself in a particular ways



Consider two frames s and s’

Frame S is rest and S’ is moving with constant speed V

For stationary clock the distance covered by light is less

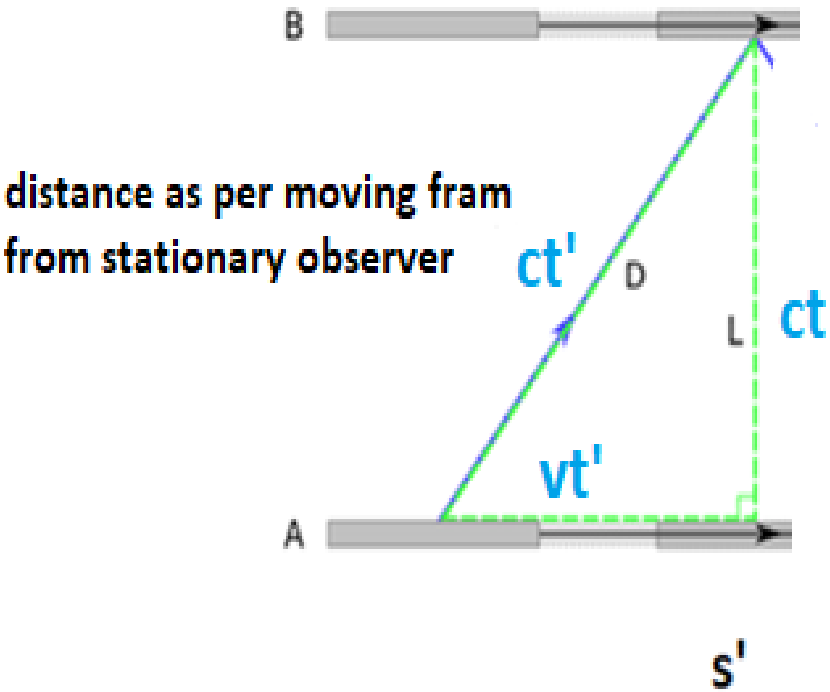
C = $\frac{2L}{t}$

For moving frame the distance covered by light is more

C = $\frac{2D}{t'}$

As per special theory of relativity **speed of light is constant irrespective of motion of frame**

Therefor in case of moving frame distance D increases but to maintain the speed of light constant the time has to increase (i.e. dilation of time)



distance as per stationary fram

Using Pythagoras theorem

$(ct')^2 = (ct)^2 + (vt')^2$

$C^2 t'^2 = c^2 t^2 + v^2 t'^2$

Using Pythagoras theorem

$$(ct')^2 = (ct)^2 + (vt')^2$$

$$C^2 t'^2 = c^2 t^2 + v^2 t'^2$$

$$C^2 t'^2 - v^2 t'^2 = c^2 t^2$$

$$(C^2 - v^2) t'^2 = c^2 t^2$$

$$t'^2 = \frac{c^2 t^2}{c^2 - v^2}$$

$$t'^2 = \frac{c^2 t^2}{c^2 (1 - \frac{v^2}{c^2})}$$

$$t'^2 = \frac{t^2}{(1 - \frac{v^2}{c^2})}$$

$$t' = \frac{t}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

t = is proper time which is measured in the rest frame

The above equation verifies that the time interval in moving frame appears to be increased by factor $\frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}}$ when it is measured by an observer in the rest frame

What is the velocity of π mesons whose observed mean life is 2.5×10^{-7} sec. The proper mean life of these π mesons is 2.5×10^{-8} sec.

Data : $T_o = 2.5 \times 10^{-8}$ sec., $T = 2.5 \times 10^{-7}$ sec.

Formula :

$$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$\sqrt{1-(v^2/c^2)} = \frac{T_o}{T}$$

$$v^2 = c^2\left[1-\left(\frac{T_o}{T}\right)^2\right]$$

$$v^2 = c^2\left[1-\left(\frac{2.5\times10^{-8}}{2.5\times10^{-7}}\right)^2\right]$$

$$v = 0.995\ c$$

In the laboratory, the lifetime of a particle moving with speed 2.8×10^8 m/sec is found to be 2×10^{-7} sec. Calculate the proper life time of the particle.

Data : $v = 2 \times 10^{-7}$ sec, $T = 2.8 \times 10^8$ m/sec.

Formula : $T = \dfrac{T_o}{\sqrt{1-(v^2/c^2)}}$; T = Life time measured, T_o = Proper lifetime.

Calculations :

$$T_o = T\sqrt{1-(v^2/c^2)}$$

$$= 2 \times 10^{-7}\sqrt{1-\left(\frac{2.8\times10^8}{3\times10^8}\right)^2}$$

$$T_o = 7.18 \times 10^{-7} \text{ sec.}$$

A certain process requires 10^{-6} sec. to occur in an atom at rest in laboratory. How much time will this process require to an observer in the laboratory, when the atom is moving with a speed of 5×10^{7} m/sec ?

Data : $T_o = 10^{-6}$ sec., $v = 5 \times 10^{7}$ m/sec.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : $$T = \frac{10^{-6}}{\sqrt{1-\left(\frac{5\times10^{7}}{3\times10^{8}}\right)^{2}}} = 1.014 \times 10^{-6} \text{ sec.}$$

Ans. : Time = 1.014×10^{-6} sec.

The mean life of a meson is 2×10^{-8} sec. Calculate the mean life of a meson moving with a velocity of 0.8 c.

Data : $T_o = 2 \times 10^{-8}$ sec., $v = 0.8$ c.

Formula :
$$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :
$$T = \frac{2 \times 10^{-8}}{\sqrt{1-\left(\frac{0.8\,c}{c}\right)^2}} = \frac{2 \times 10^{-8}}{\sqrt{1-(0.8)^2}}$$

$$T = 3.33 \times 10^{-8} \text{ sec.}$$

A clock keeps correct time on the earth. It is put on the space ship moving uniformly with a speed of 10^8 m/sec. How many hours does it appear to lose per day?

Data : $T = 24$ hrs as measured in the space ship

T_o = the time observed by an observe on the earth.

$v = 10^8$ m/sec.

Formula :
$$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :
$$T_o = T\sqrt{1-\frac{v^2}{c^2}} = 24\sqrt{1-\left(\frac{10^{-8}}{3\times10^{-8}}\right)^2}$$

$$T_o = 24\times\frac{2\sqrt{2}}{3} = 22.63 \text{ sec.}$$

Time lost per day $= 24 - 22.63 = 1.37$ hr

Mass and energy relation

In classical mechanics, the mass of a particle is not dependent on velocity

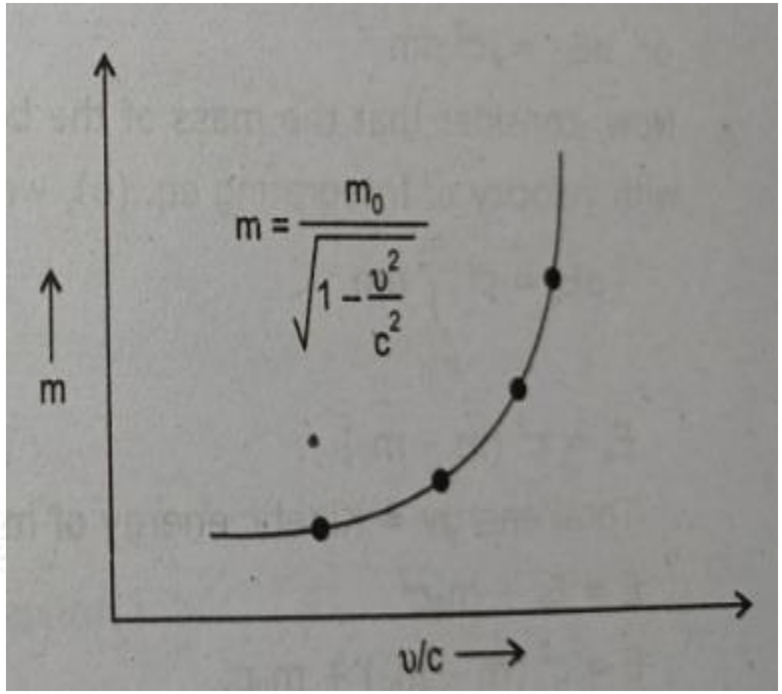
But in Einstein special theory of relativity, the mass of a moving object depends upon its velocity and it is given by,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where:

m	is relativistic mass
m_0	is rest mass
v	is m's velocity
c	is 'constant' vacuum speed of light

Figure 1 - Relativistic mass

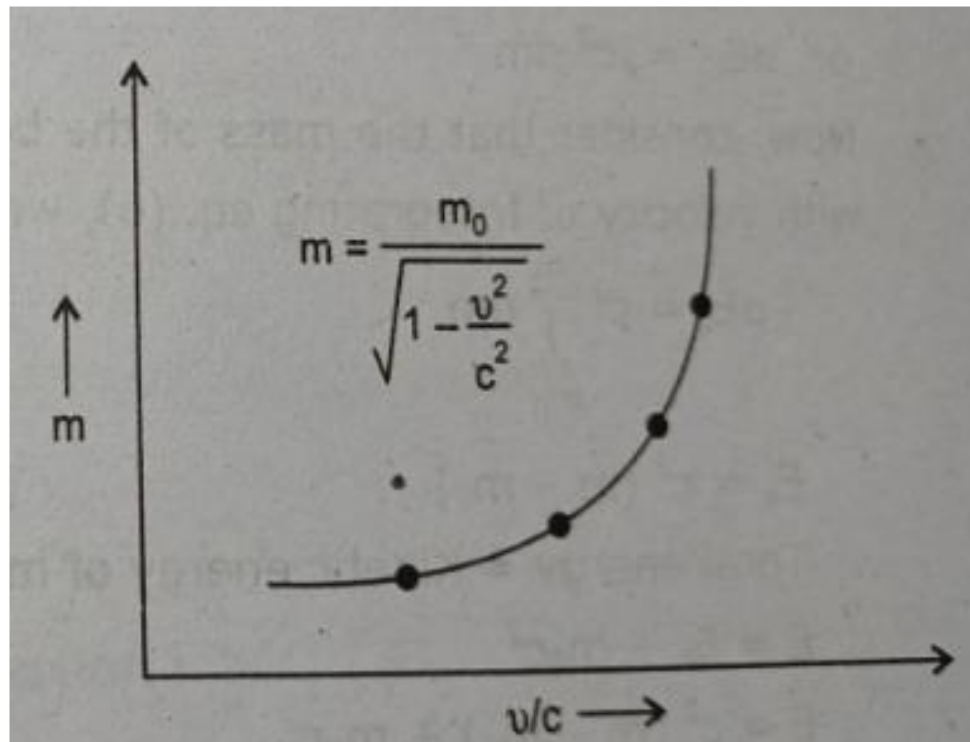


The relative mass reaches to infinity when it moves with speed of light.

$\therefore$ if $\upsilon = c$

$$m = \frac{m_0}{\sqrt{1 - \frac{c^2}{c^2}}}$$

$$m = \frac{m_0}{0} = \propto$$



The most famous relationship Einstein obtained from the postulates of special relativity is mass and energy relation **E=mC^2**.

This equation states that mass can be converted into energy and energy can be converted into mass.

If no force acts on the object and the object starts from rest then all the work becomes KE

W = KE= F.S = F. ds

F = $\frac{dp}{dt}$ = $\frac{d(mv)}{dt}$ (by newtons second law)

According to relativity mass m and velocity v, both are variables

F= m $\frac{dv}{dt}$ + v $\frac{dm}{dt}$

W = KE = F. ds = m $\frac{dv}{dt}$ + v $\frac{dm}{dt}$

= m $\frac{dv}{dt}$ ds + v $\frac{dm}{dt}$ ds

= m $\frac{ds}{dt}$ dv + v $\frac{ds}{dt}$ dm

= m v dv + v v dm

W = KE = m v dv + v^2 dm --------(1)

According to relativistic mass

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$m\sqrt{1-\frac{v^2}{c^2}} = m_0$$

Squaring both side

$$m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$$

$$c^2m^2 - m^2v^2 = m_0^2c^2$$

Differentiating both the sides

$$c^2 2m\,dm - (m^2 2v dv + v^2 2m dm) = 0$$

$$m_0 \ and\ c\ is\ constant$$

On solving

$c^2\,dm = (mvdv + v^2dm)$------(2)

W = KE = $c^2\ dm$

When body is at rest and acquires a velocity v, mass get changed from m_0 to m

$$\int KE = \int_{m_0}^{m} c^2 \text{ dm}$$

KE= c^2(m-m_0)

KE = m c^2- $m_0\ c^2$ (m c^2= Total energy, $m_0\ c^2$ =rest energy)

m c^2= KE + $m_0\ c^2$

This result shows that KE of an object is equal to increase in its mass due to its relative motion

Total energy = KE + $m_0\ c^2$

E= KE + E_0

rest mass energy E_0 = $m_0\ c^2$