Module -3

Electrodynamics

Introduction

Electrodynamics is branch of Physics that deals with rapidly changing electric and magnetic fields. These fields are interdependent and can not be separated from each other. Maxwell's equations are heart of Electrodynamics.

Studying electrodynamics is important as it forms basis of communication systems. It is used in antenna design, waveguides and satellite communications.

What is field?

A function that describes behavior of a physical quantity in certain space is **Field**. Electric field, gravitational field, temperature field etc.

The space or region in which each point is affected by a force is called field

Eg object falls on ground because they are affected by gravitational field

What is scalar field?

A scalar field is a function that gives us a single value of some variable for every point in space.

Example: Temperature, Humidity, Pressure, Potential, etc.

What is Vector field?

A vector is a quantity which has both a magnitude and a direction in space

Examples: Velocity of fluid (air) in motion, Fields of Electric or Magnetic forces, etc.

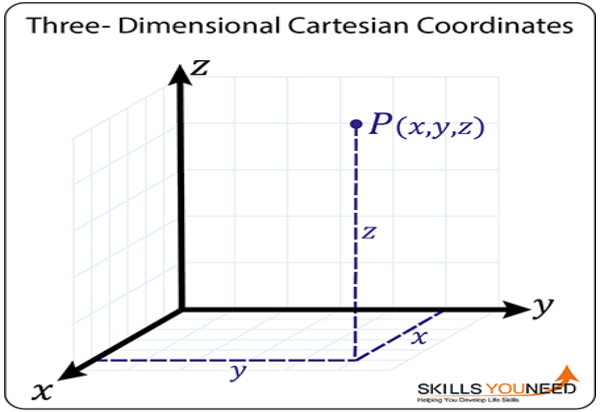
What is coordinate system?

Coordinate systems are used to describe the position of an object in space

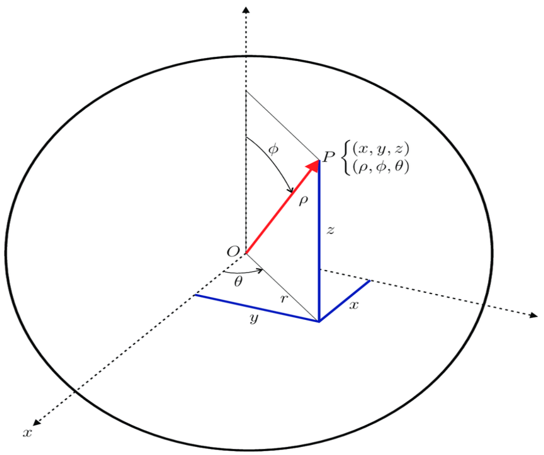
Three major coordinate systems used in the study of physics are :

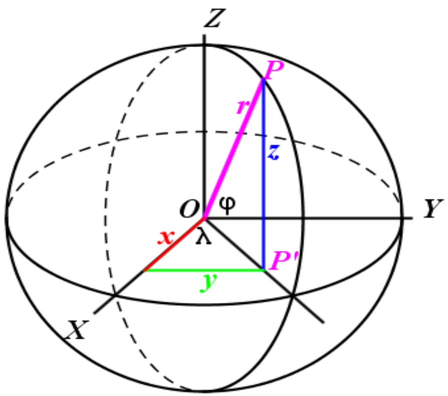
Rectangular (Cartesian)
Spherical.
Cylindrical.

Rectangular (Cartesian)

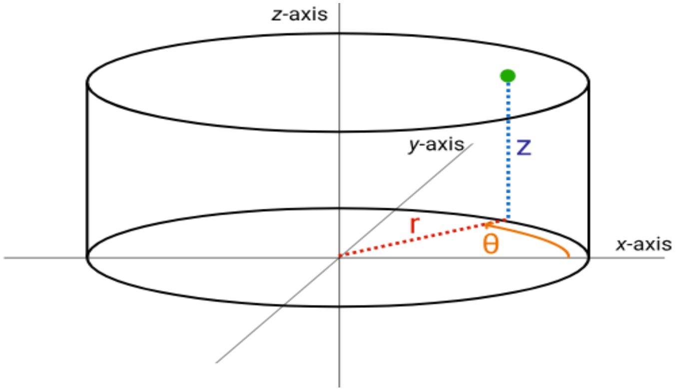


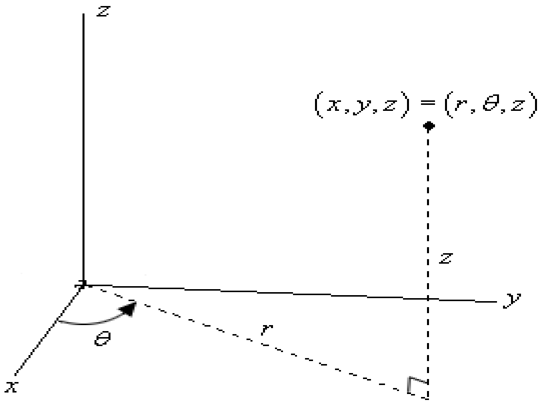
Spherical.





Cylindrical.





Differential Operator Del

Del, is an operator used in vector calculus, as a vector differential operator. It is represented by the symbol **∇**. When applied to a function defined on a one-dimensional domain, it denotes its standard derivative.

Depending on the way it is applied to a field, it denotes the **gradient,** the **divergence** or the **curl** (rotation) of a field.

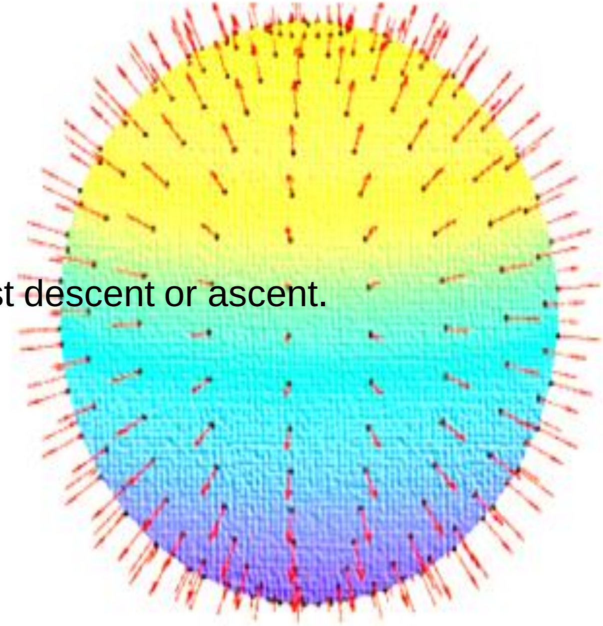
Its value is different in different coordinate systems. In Cartesian system it is

$$\nabla = \mathbf{i}\frac{\partial}{\partial \mathbf{x}} + \mathbf{j}\frac{\partial}{\partial \mathbf{y}} + \mathbf{k}\frac{\partial}{\partial \mathbf{z}}$$

Gradient of a Scalar

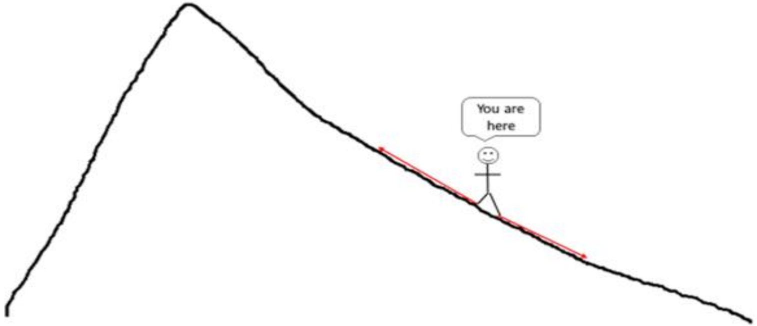
Gradient gives change in scalar quantity as one moves from one point to another. It is vector quantity. Mathematically, gradient of potential is,

$$\nabla V = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \right) V$$



In 3D, Gradients are surface normal to particular points.

In 2D, Gradient are tangents representing the direction of steepest descent or ascent.



From the definitions of Potential (V), Electric field (**E**) it can be shown that, the negative gradient of potential gives Electric field (**E**). Thus,

$$dV = -(\mathbf{E})\,(\mathbf{i}dx) \quad \mathbf{Or}$$

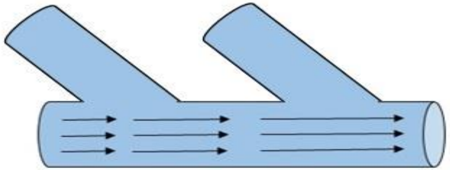
$$\mathbf{E} = -\left(\mathbf{i}\,\frac{d}{dx}\right)V = \boldsymbol{\nabla}\boldsymbol{V}$$

The negative sign indicates that the electric field points out in direction of decrease in potential energy

Divergence of Vector

Consider water flowing through a large pipe. Now, it has smaller pipes joined to it. If, as the water flows, more water is added along the way by the smaller pipes. Hence, the mass flow rate increases as the water flows.

On the other hand, in case of leakages in the pipe. the mass flow rate decreases as it flows out.

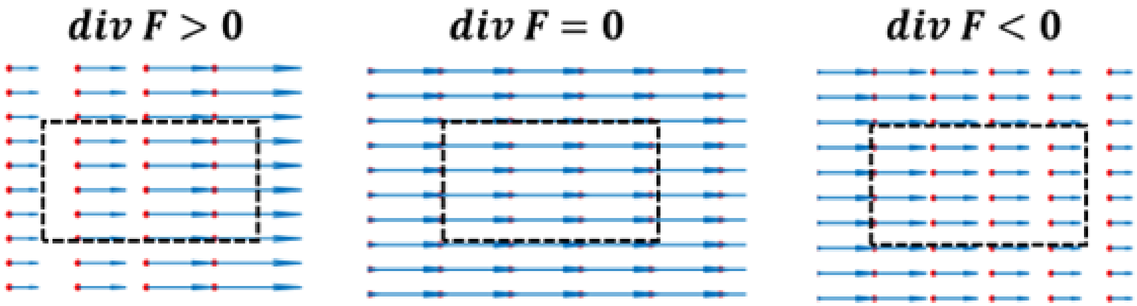


This change in the flow rate through the pipe, is divergence. Mathematically,

$$\nabla . \mathbf{E} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) . \left(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z\right)$$

Divergence give only change in magnitude. It is a scalar quantity

Physical meaning of Divergence



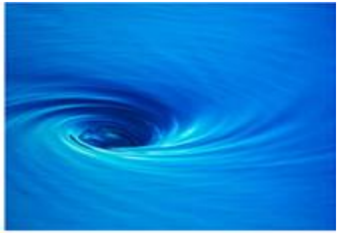
When the initial flow rate is less than the final flow rate, divergence is positive (divergence > 0). If the two quantities are same, divergence is zero. If the initial flow rate is greater than the final flow rate divergence is negative (divergence <0).

Curl of a Vector

Curl is a measure of how much a vector field circulates or rotates about a given point. Mathematically,

$$\nabla \times \mathbf{E} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z\right)$$

Consider water flowing down the drain. Here the curl of velocity of water describes its local rotation. The rotation has direction and is about direction of motion.



These rotational flows are denoted and measured by Curl. It is a vector quantity. For counter-clockwise rotation curl is positive. For clock-wise rotation curl is negative.

Obviously, As for point charge the electric field lines are along radius the **Curl of static electric field is always zero. Or,**

$$\nabla \times \boldsymbol{E} = 0$$

Numerical Compute $\mathrm{div}\vec{F}$ and $\mathrm{curl}\vec{F}$ for $\vec{F} = x^2y\,\vec{i} - \left(z^3 - 3x\right)\vec{j} + 4y^2\vec{k}$.

$$\mathrm{div}\vec{F} = \nabla\,.\,\vec{F}$$

$$= \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}\left(3x - z^3\right) + \frac{\partial}{\partial z}\left(4y^2\right) = \boxed{2xy}$$

$$\mathrm{curl}\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 3x - z^3 & 4y^2 \end{vmatrix} = \frac{\partial}{\partial y}\left(4y^2\right)\vec{i} + \frac{\partial}{\partial z}\left(x^2y\right)\vec{j} + \frac{\partial}{\partial x}\left(3x - z^3\right)\vec{k} - \frac{\partial}{\partial y}\left(x^2y\right)\vec{k} - \frac{\partial}{\partial x}\left(4y^2\right)\vec{j} - \frac{\partial}{\partial z}\left(3x - z^3\right)\vec{i}$$

$$= 8y\vec{i} + 3\vec{k} - x^2\vec{k} + 3z^2\vec{i}$$

$$= \boxed{\left(8y + 3z^2\right)\vec{i} + \left(3 - x^2\right)\vec{k}}$$

Compute $\mathrm{div}\vec{F}$ and $\mathrm{curl}\vec{F}$ for $\vec{F} = \left(3x + 2z^2\right)\,\vec{i} + \dfrac{x^3y^2}{z}\vec{j} - (z - 7x)\,\vec{k}$.

$$\mathrm{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x + 2z^2) + \frac{\partial}{\partial y}\left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z}(7x - z) = \boxed{2 + \frac{2x^3y}{z}}$$

$$\mathrm{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x + 2z^2 & \dfrac{x^3y^2}{z} & 7x - z \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(7x - z)\,\vec{i} + \frac{\partial}{\partial z}(3x + 2z^2)\,\vec{j} + \frac{\partial}{\partial x}\left(\frac{x^3y^2}{z}\right)\vec{k}$$

$$- \frac{\partial}{\partial y}(3x + 2z^2)\,\vec{k} - \frac{\partial}{\partial x}(7x - z)\,\vec{j} - \frac{\partial}{\partial z}\left(\frac{x^3y^2}{z}\right)\vec{i}$$

$$= 4z\vec{j} + \frac{3x^2y^2}{z}\vec{k} - 7\vec{j} + \frac{x^3y^2}{z^2}\vec{i}$$

$$= \boxed{\frac{x^3y^2}{z^2}\vec{i} + (4z - 7)\,\vec{j} + \frac{3x^2y^2}{z}\vec{k}}$$

Divergence of curl is zero (Proof)

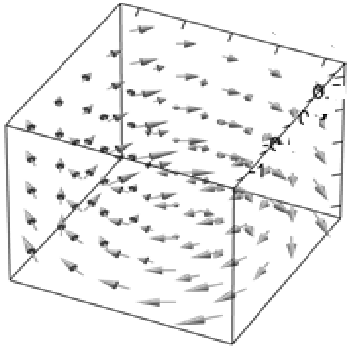
$$\text{To show } \nabla.(\nabla \times V) = 0$$

$$\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_x & V_z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix}\right.$$

$$= \frac{\partial}{\partial x}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix} - \frac{\partial}{\partial y}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_x & V_z \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = 0$$



Divergence of curl is zero it means rotating something in box will never come out or come in the box.

Numeric

Find the divergence of vector function $\vec{A} = x^2\,\hat{i} + x^2y^2\,\hat{j} + 24\,x^2y^2z^3\,\hat{k}$.

(M.U. Nov. 2018) (3 m)

Solution :

Divergence of $\vec{A} = \vec{\nabla} \cdot \vec{A}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x^2\,\hat{i} + x^2y^2\,\hat{j} + 24\,x^2y^2z^3\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial z}(24\,x^2y^2z^3)$$

$$= 2x + 2xy^2 + 72\,x^2\,y^2\,z^2$$

Result : $\vec{\nabla} \cdot \vec{A} = 2x\,(1 + y^2 + 36\,x\,y^2\,z^2)$

Numerical

Find the divergence of the vector field $\vec{F} = x^2 y\,\hat{i} - (z^3 - 3x)\,\hat{j} + 4y^2\,\hat{k}$.

(M.U. May 2019) (5 m)

Solution :

$$\text{Divergence of } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[x^2 y\,\hat{i} - (z^3 - 3x)\,\hat{j} + 4y^2\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial y}(z^3 - 3x) + \frac{\partial}{\partial z}(4y^2)$$

$$= 2xy - 0 - 0 = 2xy$$

Result : $\vec{\nabla} \cdot \vec{F} = 2xy.$

Numerical

If $\vec{A} = x^2 z\,\hat{i} - 2y^2 z^2\,\hat{j} + xy^2 z\,\hat{k}$, find $\vec{\nabla} \cdot \vec{A}$ at point (1, – 1, 1).

(M.U. May 2017) (5 m)

Solution :

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x^2 z\,\hat{i} - 2y^2 z^2\,\hat{j} + xy^2 z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 z) - \frac{\partial}{\partial y}(2y^2 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 4yz^2 + xy^2$$

At point (1, – 1, 1), $\vec{\nabla} \cdot \vec{A} = 7.$

Numerical

Calculate the divergence of the vector field $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 2z\,\hat{k}$.

Solution :

$$\text{Divergence of } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(3x\,\hat{i} + 4y\,\hat{j} + 2z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(2z)$$

$$= 3 + 4 + 2 = 9$$

Ans. : $\vec{\nabla} \cdot \vec{F} = 9$

Numerical

A vector field is given as $\vec{F} = y\hat{i} + (x^2 + y^2)\hat{j} + (yz + zx)\hat{k}$. Find (i) Div. $\vec{F}$, and (ii) Curl $\vec{F}$.

Solution :

(i) Div. $\vec{F} = \vec{\nabla} \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(y\hat{i} + (x^2 + y^2)\hat{j} + (yz + zx)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x^2 + y^2) + \frac{\partial}{\partial z}(yz + zx)$$

(ii) Curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 + y^2 & yz + zx \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(yz + zx) - \frac{\partial}{\partial z}(x^2 + y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(yz + zx)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y)\right]$$

$$= \hat{i}\,z + \hat{j}(-z) + \hat{k}(2x - 1)$$

$$\therefore \text{ Curl } \vec{F} = z\hat{i} - z\hat{j} + (2x - 1)\hat{k}$$

Numerical

Find the divergence and curl of a vector $\vec{A} = x^2y\,\hat{i} + (x-y)\,\hat{k}$.

We have $\vec{A} = x^2y\,\hat{i} + (x-y)\,\hat{k}$.

$$\text{Div}\,\vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot[x^2y\,\hat{i} + (x-y)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial z}(x-y) = 2xy$$

$$\text{Curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 0 & x-y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x-y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(x-y)\right] + \hat{k}\left[-\frac{\partial}{\partial y}(x^2y)\right]$$

$$= \hat{i}(-1) + \hat{j}(-1) + \hat{k}(x^2) = -\hat{i} - \hat{j} - x^2\hat{k}$$

Ans. : $\text{Div}\,\vec{A} = 2xy$

$\text{Curl}\,\vec{A} = -\hat{i} - \hat{j} - x^2\hat{k}$

Numerical

Find the divergence and curl of a vector field $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

$$\text{Div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= 0$$

Divergence theorem

The **divergence theorem** says that when **you** add up all the little bits of outward flow in a volume using a triple integral of **divergence**, it gives the total outward flow from that volume, as measured by the flux through its surface.

Volume integral of the divergence of the vector field A over any volume is equal to the surface integral of vector A over the surface which enclosing the volume

$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot \mathbf{dS} \qquad \text{[Divergence Theorem]}$$

Stokes' Theorem

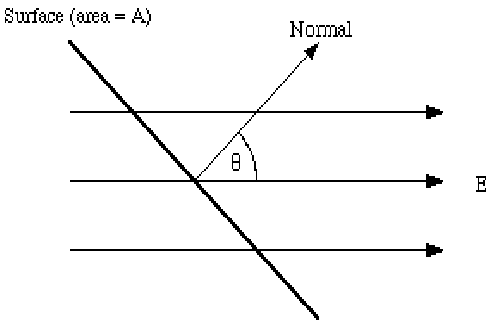
The **Stoke's theorem** states that “the surface integral of the **curl of a function** over a surface bounded by a closed surface is equal to the **line integral of the particular vector function around that surface.”**

Surface integral of curl of vector over any surface is equal to the line integral of vector over the curve bounding the surface.

$$\int (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dL} \qquad \text{[Stokes' Theorem]}$$

Electric flux

- **S** is an arbitrary surface held in an electric field **E**. Field lines will pass through it as shown. An electric flux of electric field **E** through a surface S is a measure of the number of lines passing through the surface. Theoretically infinite number of lines would be passing through the surface S.
- To calculate flux the component of field **E** perpendicular to the surface is considered. If θ is the angle between field direction and normal drawn to the surface of area A then the perpendicular component of **E** will be **E**cosθ



Electric flux [ϕ] through a surface is the product of the surface area 'A' and magnitude of the normal component of the electric field **E**,

$\phi = EAcos\theta$ = Dot product of Field **E** and Area **A**

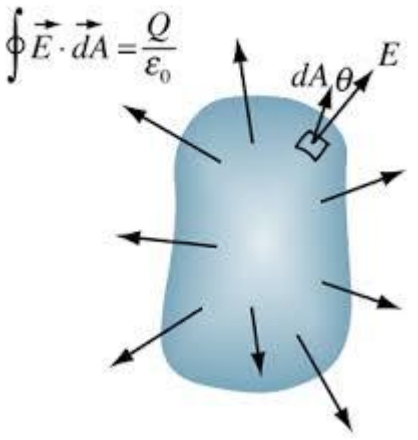
In general, for arbitrary surface S, the total flux is calculated by taking surface integral,

$$\phi_E = \int_S \boldsymbol{E.da}$$

'**E'** is the electric field and '**da**' is the infinitesimal surface element normal to electric field.

Gauss's Law in Electrostatics

- Gauss's law is an important law and in fact it is the converse of Coulomb's law. By Coulomb's law we can calculate electric field intensity for a given charge. Gauss's law enables us to determine the charge if field '**E**' or flux 'ϕ_E' is known.



- According to Gauss's law, the total flux 'ϕ_E' through any closed surface is equal to $1/\varepsilon_0$ times the net charge 'Q' enclosed by the surface. Thus,

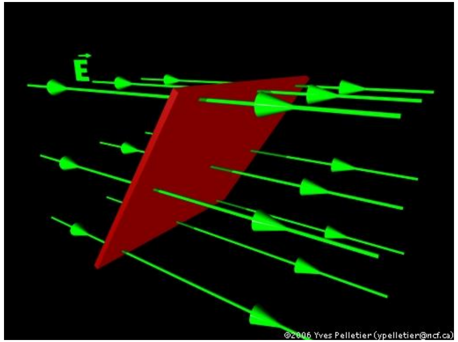
$$\phi_E = \int_s E.da = \frac{q}{\varepsilon_0}$$

Gauss law can also be stated as, ***the surface integral of the normal component the electric field over any closed surface is equal to the total charge enclosed by the surface divided by the permittivity of free space.***

Gauss's Law in Electrostatics

Using divergence theorem the surface integral can be converted into volume integral as,

$$\oint_s \boldsymbol{E}.\boldsymbol{da} = \int_V (\boldsymbol{\nabla}.\boldsymbol{E})\, d\tau$$



If $Q_{encl.}$ is given in terms of charge density ρ, then we have,

$$Q_{encl.} = \int_V \rho d\tau$$

Putting this value we can write Gauss's law as

$$\int_V (\boldsymbol{\nabla}.\mathbf{E}) d\tau = \int_V \left(\frac{\rho}{\varepsilon_0}\right) d\tau$$

As for any volume, the integrands should be equal and hence we have,

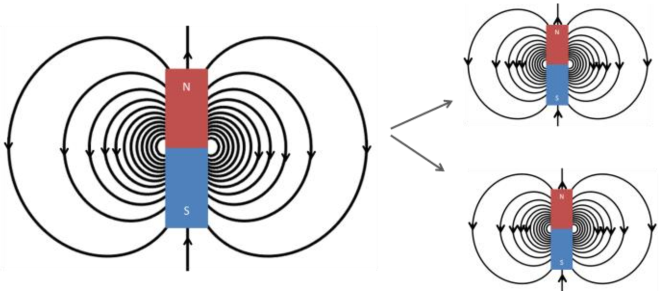
$$\nabla.\boldsymbol{E} = \frac{\boldsymbol{\rho}}{\boldsymbol{\varepsilon_0}}$$

The above equation is known as differential form of Gauss's law. Further, If **D** is electric flux density then, we can write, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ And the Gauss Law becomes,

$$\boldsymbol{\nabla}.\boldsymbol{D} = \boldsymbol{\rho}$$

Where ϵ_0 is permittivity of free space.

Gauss's Law in Magnetism



- In magnetic field, the magnetic line are closed and total outgoing flux is zero. Thus we can write, Gauss Law as, $\oint \boldsymbol{B}.\boldsymbol{da} = 0$

- This is called as Gauss's Law in magnetism in integral form. Using Gauss Divergence theorem we can write the above equation as,

- $\oint_s \boldsymbol{B}.\boldsymbol{da} = \int_V (\nabla.\boldsymbol{B})\, d\tau = 0$ Or, $\nabla.\boldsymbol{B} = 0$

- This is known as Differential form of Gauss's Law in magnetism. Gauss law in magnetism states that, " ***Magnetic monopoles do not exist and total magnetic flux through a closed surface is zero.***"

Faraday's Law and Lenz's Law

"Whenever there is a change in the magnetic flux linked with a circuit an emf proportional to the rate of change of magnetic flux is induced in the circuit." Mathematically, $\varepsilon \propto \frac{d\phi_B}{dt}$

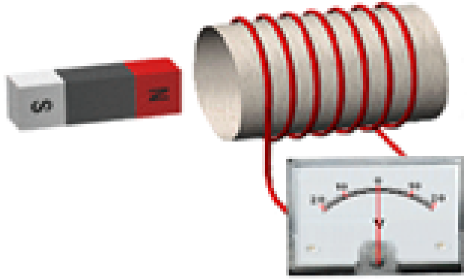
According to Lenz's law, *"Induced emf opposes the change in magnetic flux producing it."* Including Lenz's law, induced emf becomes, $\varepsilon = -\frac{d\phi_B}{dt}$

Using definition of emf we can write, $\varepsilon = \oint \boldsymbol{E}.\boldsymbol{dl} = -\frac{d\phi_B}{dt}$

Magnetic flux is given by the surface integral of magnetic induction **B** over a surface S,

$\phi_B = \int \boldsymbol{B}.\boldsymbol{da}$ Using this value Faraday's law becomes,

$$\varepsilon = \oint \boldsymbol{E}.\boldsymbol{dl} = -\frac{d}{dt}\int \boldsymbol{B}.\boldsymbol{da} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{da}$$



$$\varepsilon = \oint \boldsymbol{E}.\boldsymbol{dl} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{da}$$

This is Faraday's law in integral form.

Faraday's Law in differential form

- Faraday’s law in differential form can be obtained by converting line integral into surface integral using Stokes’s theorem as,

$$\mathcal{E} = \oint \boldsymbol{E}.\boldsymbol{dl} = \oint_S (\nabla \times \boldsymbol{E}).\boldsymbol{da} \quad Or,$$

$$\oint_S (\nabla \times \boldsymbol{E}).\boldsymbol{da} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{da}$$

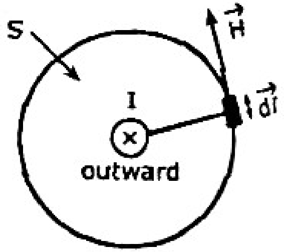
- Since integrals are equal the integrands must be equal and hence,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

- This is Faraday’s law in differential form. ***It is seen that for moving charges the curl of associated electric field is not zero. It is equal to the rate of change in magnetic flux. Obviously, LHS above is induced emf.***

Ampere's circuital Law

Ampere's circuital law is about the current passing through a loop and magnetic field created around the loop. According to Ampere's law, ***"The line integral of magnetic field intensity H around a closed path is exactly equal to the direct current enclosed by the path."*** Mathematically,



$$\oint_l H.dl = I$$

The law is useful to determine H when the current distribution is symmetrical. Since, $B = \mu_0 H$ we can write, the above the expression as,

$$\oint_l B.dl = \mu_0 I$$

This known as Ampere's circuital law.

Ampere's circuital Law

Ampere's circuital law is, $\oint_l H.dl = I$. Converting line integral into surface integral And replacing current in terms of current density we can write, the law as,

$\oint_s \nabla \times H.da = \oint_s J_f.da$ or simply, $\nabla \times H = J_f$ **Equation --A**

This is true only for steady current. For changing currents it should be modified.

For instance taking divergence of the above equation violets the equation of continuity as divergence of curl is always zero. That is,

$$\nabla.(\nabla \times H) = \nabla.J_f = 0$$

Because, Equation of continuity says, $\nabla.J_f = -\frac{\partial \rho}{\partial t}$. So modify equation A as,

$\nabla \times H = J_f + J_d$ **So that its divergence is** $\nabla.(\nabla \times H) = \nabla.J_f + \nabla.J_d = 0$

Here, J_f and J_d are free and bound charge current densities respectively. We have,

$$\nabla.J_d = -\nabla.J_f = \frac{\partial \rho}{\partial t}$$

Gauss's law says, $\nabla.D = \rho$ so the equation above becomes, $\nabla.J_d = \frac{\partial}{\partial t}(\nabla.D) = \nabla.\frac{\partial D}{\partial t}$ And, $J_d = \frac{\partial D}{\partial t}$ With this, the Ampere's Law becomes,

$$\nabla \times H = J_f + \frac{\partial D}{\partial t}$$

Maxwell’s Equations

- Maxwell’s equations summarize all known laws of electromagnetism. The integral form of the Maxwell’s equations depicts under laying physical laws on the other hand differential forms are useful in solving problems.

No.	Differential Form	Integral Form	Underlying Law
01	$\nabla . \boldsymbol{D} = \rho_f$	$\oint \boldsymbol{D}.\boldsymbol{da} = \int_v \rho\, dv$	Gauss's Law for Electrostatics
02	$\nabla . \boldsymbol{B} = 0$	$\oint \boldsymbol{B}.\boldsymbol{da} = 0$	Gauss's Law in Magnetism
03	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint \boldsymbol{E}.\boldsymbol{dl} = -\int \frac{\partial B}{\partial t}.\boldsymbol{da}$	Faraday's Law
04	$\nabla \times \boldsymbol{H} = \boldsymbol{J_f} + \frac{\partial \boldsymbol{D}}{\partial t}$	$\oint \boldsymbol{H}.\boldsymbol{dl} = \oint \left(J_f + \frac{\partial D}{\partial t}\right).\boldsymbol{da}$	Ampere's Law

Maxwell's Equations in free space or in Vacuum

- In vacuum there is no charge.
- No charge means no currents enclosed

No.	Differential Form	Underlying Law
01	$\nabla . D = 0$	Gauss's Law for Electrostatics
02	$\nabla . B = 0$	Gauss's Law in Magnetism
03	$\nabla \times E = -\frac{\partial B}{\partial t}$	Faraday's Law
04	$\nabla \times H = \frac{\partial D}{\partial t}$	Ampere's Law

In medium

No.	Differential Form	Underlying Law
01	$\nabla . D = \rho_f$	Gauss's Law for Electrostatics
02	$\nabla . B = 0$	Gauss's Law in Magnetism
03	$\nabla \times E = -\frac{\partial B}{\partial t}$	Faraday's Law
04	$\nabla \times H = J_f + \frac{\partial D}{\partial t}$	Ampere's Law

free space

No.	Differential Form	Underlying Law
01	$\nabla . D = 0$	Gauss's Law for Electrostatics
02	$\nabla . B = 0$	Gauss's Law in Magnetism
03	$\nabla \times E = -\frac{\partial B}{\partial t}$	Faraday's Law
04	$\nabla \times H = \frac{\partial D}{\partial t}$	Ampere's Law

Numerical

If the magnetic field $\vec{H} = (3x\cos\beta + 6y\sin\alpha)\,\hat{k}$, find the current density $\vec{J}$ for steady fields.

Solution :

Ampere's circuital law is $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$

For steady fields $\dfrac{\partial \vec{D}}{\partial t} = 0.$

$$\vec{J} = \vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta + 6y\sin\alpha) \end{vmatrix}$$

$$= \hat{i}\,\frac{\partial}{\partial y}(3x\cos\beta + 6y\sin\alpha) - \hat{j}\,\frac{\partial}{\partial x}(3x\cos\beta + 6y\sin\alpha)$$

$$= 6\sin\alpha\,\hat{i} - 3\cos\beta\,\hat{j}$$

Numerical

If $\vec{D} = 10x\,\hat{i} - 4y\,\hat{j} + Cz\,\hat{k}$, where C is a constant, find the value of C using Gauss' law for a charge free region.

Solution :

Gauss' law : $\vec{\nabla} \cdot \vec{D} = \rho_v$ where ρ_v = volume charge density.

In a charge free region $\rho_v = 0$. Hence,

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\therefore \quad \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (10x\,\hat{i} - 4y\,\hat{j} + Cz\,\hat{k}) = 0$$

$$\frac{\partial}{\partial x}(10x) + \frac{\partial}{\partial y}(-4y) + \frac{\partial}{\partial z}(Cz) = 0$$

$$10 - 4 + C = 0$$

Ans. : $C = -6$

Numerical

Show that $\vec{F} = (x+i)\,\hat{i} + (x+z)\,\hat{j} + (y-z)\,\hat{k}$ represents a conservative field.

Solution :

For a conservative field, $\vec{\nabla} \times \vec{F} = 0$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (x+z) & (y-z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(y-z) - \frac{\partial}{\partial z}(x+z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x+y) - \frac{\partial}{\partial x}(y-z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial y}(x+y)\right]$$

$$= \hat{i}\,(1-1) + \hat{j}\,(0) + \hat{k}\,(1-1)$$

$$= 0$$

Hence, $\vec{F}$ is a conservative field.