Representations of $SU(2)$ and the 3-Sphere

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An arbitrary representation of the group $SU(2)$ is given by the set of three generators T_k satisfying the Lie algebra

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

with

$$\epsilon_{123} = 1.$$

The element of the group is given by the matrix

$$U = e^{i\mathbf{T}\cdot\omega}$$

where in the fundamental representation

$$T_k = \frac{1}{2}\sigma_k, \quad k = 1, 2, 3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

being the standard Pauli matrices which satisfy

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k.$$

The vector ω has components ω_k in a given coordinate frame.
Geometrically, the matrices U are generators of spinor rotations in 3-dimensional space $\mathbb{R}^3$ and the parameters ω_k are the corresponding angles of rotation.
The Euler parametrization of an arbitrary matrix of $SU(2)$ transformation is defined in terms of the three angles θ, ψ and ϕ as

$$U(\phi, \theta, \psi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{\frac{i}{2}(\psi+\phi)} & \sin\frac{\theta}{2}e^{-\frac{i}{2}(\psi-\phi)} \\ -\sin\frac{\theta}{2}e^{\frac{i}{2}(\psi-\phi)} & \cos\frac{\theta}{2}e^{-\frac{i}{2}(\psi+\phi)} \end{pmatrix}.$$

Thus, the group manifold is isomorphic to S^3.