(1)

Hopfion Solution

The Hopf fibration is a mapping $h: S^3 \to S^2$ such ~~that~~

$$h(a,b,c,d) = \left(a^2+b^2-c^2-d^2,\ 2(ad+bc),\ 2(bd-ac)\right)$$

where $a^2+b^2+c^2+d^2 = 1$.

The domain of this map is a 3-sphere.

The range of this map is a 2-sphere.

Quick Check:

$$\sqrt{h \cdot h} = \sqrt{(a^2+b^2+c^2+d^2)^2}$$

$$= a^2+b^2+c^2+d^2$$

$$= 1$$

Thus, the range is indeed a 2-sphere.

We can alternatively define this map using complex numbers.

First identify the 3-sphere with $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$ and 2-sphere with $\mathbb{C}\times\mathbb{R}$:

$$S^3 = \{(z_0, z_1) \in \mathbb{C}\times\mathbb{C} \mid |z_0|^2 + |z_1|^2 = 1\}$$

and

$$S^2 = \{(z, x) \in \mathbb{C}\times\mathbb{R} \mid |z|^2 + x^2 = 1\}$$

We can then define the Hopf map using the notation

$$h_{\mathbb{C}}(z_0, z_1) = (\underbrace{2 z_0 \bar{z}_1}_{\text{complex}}, \underbrace{|z_0|^2 - |z_1|^2}_{\text{real}})$$

<u>Quick check</u>: mapping lies on the 2-sphere.

$$\begin{aligned} |h_{\mathbb{C}}(z_0, z_1)| &= |z_0|^4 + 2|z_0|^2|z_1|^2 + |z_1|^4 \\ &= (|z_0|^2 + |z_1|^2)^2 \\ &= 1. \end{aligned}$$

It does.

③

Consider a point in S^3 such that

$$(w_0, w_1) = (\lambda z_0, \lambda z_1)$$

In this instance we have that

$$h_{\mathbb{C}}(w_0, w_1) = (2\lambda z_0 \bar{\lambda} \bar{z}_1, |\lambda|^2(|z_0|^2 + |z_1|^2))$$

$$= |\lambda|^2 h_{\mathbb{C}}(z_0, z_1)$$

and the two points in S^3 map to the same point in S^2 as long as

$$|\lambda|^2 = 1.$$

Then we can say that $\lambda = e^{i\beta}$ where β ranges from 0 to 2π.

This allows us to observe that the preimage of a point on S^2 is a great circle on S^3 passing through the point (z_0, z_1). The circles are the fibers that make up the fibration and they can be parametrized as follows

$$z_0 = e^{i(\beta + \frac{\phi}{2})} \sin\left(\frac{\theta}{2}\right)$$

$$z_1 = e^{i(\beta - \frac{\phi}{2})} \cos\left(\frac{\theta}{2}\right)$$

To see this, we let $z_0 = r_0 e^{\beta_0}$

and $z_1 = r_1 e^{\beta_1}$

Then we have that

$$h_{\mathbb{C}}(z_0, z_1) = (2 z_0 \bar{z}_1, |z_0|^2 - |z_1|^2)$$
$$= (2 r_0 r_1 e^{i(\beta_0 - \beta_1)}, r_0^2 - r_1^2)$$

and this must map to a point on S^2 so we have the relations

$$x = 2 r_0 r_1 \cos(\beta_0 - \beta_1) \qquad = \sin\theta \cos\phi$$

$$y = 2 r_0 r_1 \sin(\beta_0 - \beta_1) \qquad = \sin\theta \sin\phi$$

$$z = r_0^2 - r_1^2 \qquad = \cos\theta$$

We immediately deduce that

$$\phi = \beta_0 - \beta_1$$

and we go on to find that

$$(r_0^2 - r_1^2) + (r_0^2 + r_1^2) = 1 + \cos\theta$$

$$\Rightarrow \quad r_0^2 = \frac{1 + \cos\theta}{2}$$

$$= \cos^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad r_0 = \cos\left(\frac{\theta}{2}\right)$$

Similarly, we obtain

$$r_1 = \sin\left(\frac{\theta}{2}\right)$$

so that

$$z_0 = \cos\left(\frac{\theta}{2}\right) e^{i\chi_0}$$

and

$$z_1 = \sin\left(\frac{\theta}{2}\right) e^{i(\chi_0 - \phi)}$$

Now define

$$\lambda = e^{i(\chi - \chi_0 + \frac{\phi}{2})}$$

where χ ranges from 0 to 2π.

This leads to

$$w_0 = \lambda z_0 = \cos\left(\frac{\theta}{2}\right) e^{i(\chi + \frac{\phi}{2})}$$

and

$$w_1 = \lambda z_1 = \sin\left(\frac{\theta}{2}\right) e^{i(\chi - \frac{\phi}{2})}$$

(6)

It turns out that a solution to Maxwell's equations can be constructed such that the electric, magnetic and Poynting fields each independently have the structure of the Hopf fibration.

Returning to Bateman's construction, we define the parameters for such a solution. These turn out to be

$$\alpha = -\frac{d}{b}$$

$$\beta = -\frac{ia}{2b}$$

$$f = \frac{1}{\alpha^2}$$

$$g = \beta$$

where

$$a = x - iy$$

$$b = t - i - z$$

$$d = r^2 - (t-i)^2$$

Now

$$\nabla\alpha = (\partial_x\alpha, \partial_y\alpha, \partial_z\alpha)$$

and

$$\nabla\beta = (\partial_x\beta, \partial_y\beta, \partial_z\beta)$$

with

$$\partial_x\alpha = -\frac{b\partial_x d - d\,\partial_x b}{b^2}$$

$$= \frac{-2bx}{b^2}$$

$$= -\frac{2x}{b}$$

Similarly,

$$\partial_y\alpha = -\frac{2y}{b}$$

and

$$\partial_z\alpha = -\frac{2bz - d}{b^2}$$

So

$$\nabla\alpha = -\frac{1}{b^2}(2bx, 2by, 2bz + d)$$

and

$$\nabla\beta = -\frac{1}{2b^2}(ib, b, ia)$$

8

$$\nabla\alpha \times \nabla\beta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x\alpha & \partial_y\alpha & \partial_z\alpha \\ \partial_x\beta & \partial_y\beta & \partial_z\beta \end{vmatrix}$$

$$= \frac{1}{2b^4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2bx & 2by & 2bz+d \\ ib & b & ia \end{vmatrix}$$

$$= \frac{1}{2b^3}\Big[\hat{i}\,(2iay - 2bz - d) - \hat{j}\,(2iax - 2ibz - id) + \hat{k}\,(2bx - 2iby)\Big]$$

$$= \frac{1}{2b^3}\Big[(b^2-a^2)\hat{i} + i(a^2+b^2)\hat{j} + 2ab\hat{k}\Big]$$

and we also have that

$$h(\alpha,\beta) = \partial_\alpha f\,\partial_\beta g - \partial_\beta f\,\partial_\alpha g$$

$$= -\frac{2}{\alpha^3}$$

$$= \frac{2b^3}{d^3}$$

and we determine that

$$F = h(\alpha,\beta)\,(\nabla\alpha \times \nabla\beta) = \frac{1}{d^3}\left(b^2-a^2,\ i\,(a^2+b^2),\ 2ab\right)$$

⑨

The normalized Poynting field:

$$F \cdot F^* = \frac{1}{|d|^6}\Big[(b^2-a^2)(\bar{b}^2-\bar{a}^2)+(a^2+b^2)(\bar{a}^2+\bar{b}^2)$$
$$+4|a|^2|b|^2\Big]$$

$$= \frac{2}{|d|^6}\left(|a|^4+2|a|^2|b|^2+|b|^4\right)$$

$$= \frac{2}{|d|^6}\left(|a|^2+|b|^2\right)^2$$

and

$$F \times F^* = \frac{1}{|d|^6}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b^2-a^2 & i(a^2+b^2) & 2ab \\ \bar{b}^2-\bar{a}^2 & -i(\bar{a}^2+\bar{b}^2) & 2\bar{a}\bar{b} \end{vmatrix}$$

yielding

$$\frac{F \times F^*}{F \cdot F^*} = \frac{1}{|a|^2+|b|^2}\Big(i(a\bar{b}+\bar{a}b),\ a\bar{b}-\bar{a}b,$$
$$i(|a|^2-|b|^2)\Big)$$

So, the normalized Poynting field for the Hopfian solution is

$$\hat{S} = \frac{E \times B}{|E \times B|}$$

$$= - \frac{1}{|a|^2 + |b|^2} \left(a\bar{b} + \bar{a} b,\ i(a\bar{b} - \bar{a} b),\ |a|^2 - |b|^2 \right)$$

Substituting

$$a = x - iy$$

and

$$b = t - i - z$$

gives

$$\hat{S} = \frac{1}{1 + x^2 + y^2 + (z-t)^2} \Big(2(x(z-t) - y),\ 2(x + y(z-t)),$$
$$1 + (z-t) - y^2 - x^2 \Big)$$