Notes on Euler-Poincaré Characteristic and Gauss-Bonnet Theorem

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1 The Euler-Poincaré Characteristic

The number

$$\chi(X) = e(X) - k(X) + f(X)$$

is called the Euler-Poincaré characteristic where $e(X)$ is the number of vertices of X, $k(X)$ is the number of vertices of X and $f(X)$ is the number of faces of X. Let X be a tetrahedron in R^3 then

$$e(X) = 4$$

$$k(X) = 6$$

$$f(X) = 4$$

yielding

$$\chi(X) = 2.$$

Since the geometric realization of the tetrahedron is homeomorphic to S^2

$$\chi(S^2) = 2.$$

Let S be compact regular surface. Then the number

$$\int_S K\, dA$$

is independent of the Riemannian metric. This theorem is remarkable because the Gauss curvature K as a function on S depends significantly on the choice of the Riemannian metric.

Let S_1 and S_2 be compact regular surfaces with Riemmanian metrics. If S_1 and S_2 are diffeomorphic then

$$\int_{S_1} K\, dA = \int_{S_2} K\, dA.$$

The sphere and the torus can not be diffeomorphic since

$$\int_{torus} K\, dA = 0$$

and

$$\int_{S^2} K\,dA = 4\pi.$$

2 The Gauss-Bonnet Theorem

Let S be a compact regular surface with Riemannian metric. Let $K : S \to R$ be the Gauss curvature and dA the surface element. Let $\varphi : |X| \to S$ be a triangulation. Then

$$\int_S K\,dA = 2\pi\chi(X).$$