de Rham Cohomology and Hodge Theory
Some Ideas

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1 *de Rham Cohomology*

Definition 1.1 The *exterior algebra* of $\mathcal{M}$ is a graded commutative algebra

$$\Lambda\mathcal{M} = \bigoplus_{p=0}^{n} \Lambda^p\mathcal{M}.$$

Definition 1.2 A p-form ω is called *closed* if $d\omega = 0$.

Denote the set of closed p-forms by $Z^p(\mathcal{M}, \mathbb{R})$.

Definition 1.3 A p-form ω is called *exact* if $\omega = d\eta$ for some $(p-1)$-form η.

Denote the set of exact p-forms by $B^p(\mathcal{M}, \mathbb{R})$.

Recall 1.4 Since $d^2 = 0$, exact p-forms are closed. So, the set of exact p-forms is a subset of the set of closed p-forms, that is $B^p(\mathcal{M}, \mathbb{R}) \subset Z^p(\mathcal{M}, \mathbb{R})$, but closed p-forms are not necessarily exact.

A closed differential form ω on a manifold $\mathcal{M}$ is *locally exact* when a neighbourhood exists around each point in $\mathcal{M}$ in which $\omega = d\eta$.

Lemma 1.5 (**Poincaré Lemma**) Any closed form on a manifold $\mathcal{M}$ is locally exact.

Definition 1.6 The *de Rham cohomology class* of $\mathcal{M}$ is defined as

$$H^p_{dR}(\mathcal{M}, \mathbb{R}) = \frac{Z^p(\mathcal{M}, \mathbb{R})}{B^p(\mathcal{M}, \mathbb{R})}.$$

Definition 1.7 The dimension of the de Rham cohomology is given by the p-th *Betti number*

$$b_p(\mathcal{M}) = \dim H^p_{dR}(\mathcal{M}, \mathbb{R}).$$

2 *Hodge Theory*

Theorem 2.1 (**de Rham Isomorphism Theorem**) Let $\mathcal{M}$ be a smooth manifold. Then the pth singular cohomology class of $\mathcal{M}$ is naturally isomorphic to the pth de Rham cohomology group

$$H^p(\mathcal{M}, \mathbb{R}) \simeq H^p_{dR}(\mathcal{M}, \mathbb{R}).$$

Definition 2.2 A differential form ω is called *harmonic* if it satisfies

$$\Delta\omega = 0$$

where $\Delta = dd^* + d^*d$ is the Laplacian.

Denote the space of harmonic p-forms on $\mathcal{M}$ by $\mathcal{H}^p(\mathcal{M})$.

Theorem 2.3 (Poincaré Duality)

$$\mathcal{H}^p(\mathcal{M}) \cong \mathcal{H}^{n-p}(\mathcal{M}).$$

In particular, $b_p(\mathcal{M}) = b_{n-p}(\mathcal{M})$ for every compact n-dimensional manifold $\mathcal{M}$.

Proof. The isomorphism is given by the Hodge $\star$ operator which maps harmonic p-forms to harmonic $(n-p)$-forms.

□

Definition 2.4 Suppose that E and E' are reflexive Banach spaces. A continuous linear map $T : E \to E'$ is said to be *Fredholm* if

- the kernel of T is finite-dimensional,
- the range of T is closed, and
- the cokernel of T is finite-dimensional .

Theorem 2.5 (Hodge's Theorem) Let $\mathcal{M}$ be a compact, oriented Riemannian manifold. Then every de Rham cohomology class on $\mathcal{M}$ contains a unique *harmonic representative* and

$$\mathcal{H}^p(\mathcal{M}) \cong H^p_{dR}(\mathcal{M}, \mathbb{R}).$$

This leads us to the following observation. Given the space of harmonic p-forms on $\mathcal{M}$,

$$\dim \mathcal{H}^p(\mathcal{M}) = b_p(\mathcal{M}).$$

Theorem 2.6 The space of harmonic p-forms on $\mathcal{M}$, $\mathcal{H}^p(\mathcal{M})$, is finite-dimensional.

Proof. The Laplacian, Δ, is an elliptic operator and is invertible.

□

2.1 *The Hodge Star Operator*

Let

$$(V, \langle ., . \rangle)$$

be an oriented 4-dimensional real inner product space. Then there exists a linear map

$$* : \Lambda^2 V \to \Lambda^2 V$$

known as the Hodge star operator defined by letting

$$(e_1, ..., e_4)$$

be an oriented orthonormal basis of V such that

$$*(e_i \wedge e_j) = e_k \wedge e_l$$

where (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$.

$*$ can be defined invariantly as

$$\phi \wedge *\psi = \langle \phi, \psi \rangle e_1 \wedge ... \wedge e_4.$$

As

$$** = I,$$

there is an eigenspace decomposition

$$\Lambda^2 V = \Lambda_+ V \oplus \Lambda_- V$$

where

$$\Lambda_{\pm}V = span\{e_1 \wedge e_2 \pm e_3 \wedge e_4, e_1 \wedge e_3 \pm e_4 \wedge e_2, e_1 \wedge e_4 \pm e_2 \wedge e_3\}.$$

Any element of ϕ can be expressed in the form

$$\phi = \lambda e_1 \wedge e_2 \pm \mu e_3 \wedge e_4$$

with respect to an oriented orthonormal basis $(e_1, ..., e_4)$ of V such that

$$*\phi = \pm\phi \Leftrightarrow \lambda = \pm\mu.$$

A form is called *self-dual* if it satisfied

$$*\phi = \phi.$$

Remark 2.7 *The Hodge star operator is conformally invariant.*