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Bateman's Construction

Maxwell's Eqs.

$$\left.\begin{aligned}
\nabla \times B &= \partial_t E \\
\nabla \times E &= -\partial_t B \\
\nabla \cdot B &= 0 \\
\nabla \cdot E &= 0
\end{aligned}\right\} \quad c = 1$$

Construct a new field, the Riemann-Silberstein Vector

$$F = E + iB$$

with $i = \sqrt{-1}$.

Maxwell's equations can now be written as

$$\begin{aligned}
\nabla \times F &= \nabla \times E + i(\nabla \times B) \\
&= -\partial_t B + i\partial_t E \\
&= i\partial_t (E + iB) \\
&= i\partial_t F
\end{aligned}$$

and

$$\nabla \cdot F = 0$$

②

The next step is to find two complex scalar functions

$$\alpha(x, y, z, t)$$

and

$$\beta(x, y, z, t)$$

such that

$$\nabla\alpha \times \nabla\beta = i\left(\partial_t\alpha \nabla\beta - \partial_t\beta \nabla\alpha\right)$$

This condition is equivalent to the following conditions

- $(\partial_x\alpha)^2 + (\partial_y\alpha)^2 + (\partial_z\alpha)^2 - (\partial_t\alpha)^2 = (\nabla\alpha)^2 - (\partial_t\alpha)^2$
$$= 0$$

- $(\partial_x\beta)^2 + (\partial_y\beta)^2 + (\partial_z\beta)^2 - (\partial_t\beta)^2 = (\nabla\beta)^2 - (\partial_t\beta)^2$
$$= 0$$

- $\partial_x\alpha\,\partial_x\beta + \partial_y\alpha\,\partial_y\beta + \partial_z\alpha\,\partial_z\beta - \partial_t\alpha\,\partial_t\beta = \nabla\alpha\cdot\nabla\beta - \partial_t\alpha\,\partial_t\beta$
$$= 0$$

③

To see how these conditions are equivalent consider the x- component of $\nabla\alpha\times\nabla\beta$ as follows

$$(\nabla\alpha\times\nabla\beta)_x = \partial_y\alpha\,\partial_z\beta - \partial_y\beta\,\partial_z\alpha$$

Squaring this term we get

$$(\partial_y\alpha\,\partial_z\beta - \partial_y\beta\,\partial_z\alpha)^2 = (\partial_y\alpha^2 + \partial_z\alpha^2)(\partial_y\beta^2 + \partial_z\beta^2) - (\partial_y\alpha\,\partial_y\beta + \partial_z\alpha\,\partial_z\beta)^2$$

$$= (\partial_t\alpha^2 - \partial_x\alpha^2)(\partial_t\beta^2 - \partial_x\beta^2) - (\partial_t\alpha\,\partial_t\beta - \partial_x\alpha\,\partial_x\beta)^2$$

$$\Rightarrow (\nabla\alpha\times\nabla\beta)^2_x = -(\partial_t\alpha\,\partial_x\beta - \partial_t\beta\,\partial_x\alpha)^2$$

$$\Rightarrow (\nabla\alpha\times\nabla\beta)_x = i(\partial_t\alpha\,\partial_x\beta - \partial_t\beta\,\partial_x\alpha)$$

and so

$$\left.\begin{aligned}(\nabla\alpha\times\nabla\beta)_x &= i(\partial_t\alpha\,\partial_x\beta - \partial_t\beta\,\partial_x\alpha)\\ (\nabla\alpha\times\nabla\beta)_y &= i(\partial_t\alpha\,\partial_y\beta - \partial_t\beta\,\partial_y\alpha)\\ (\nabla\alpha\times\nabla\beta)_z &= i(\partial_t\alpha\,\partial_z\beta - \partial_t\beta\,\partial_z\alpha)\end{aligned}\right\}\ (*)$$

$(*)$ establishes $\nabla\alpha\times\nabla\beta = i(\partial_t\alpha\,\nabla\beta - \partial_t\beta\,\nabla\alpha)$

We now consider

$$\nabla \times (\nabla\alpha \times \nabla\beta) \quad \text{and} \quad \underbrace{\nabla \cdot (\nabla\alpha \times \nabla\beta)}_{=0}$$

since $\nabla \times \nabla\alpha = \nabla \times \nabla\beta$
$= 0$

$$\nabla \times (\nabla\alpha \times \nabla\beta) = i\partial_t [\nabla\alpha \times \nabla\beta]$$

We see that $\nabla\alpha \times \nabla\beta$ possesses the same property as F and therefore

$$\boxed{F = \nabla\alpha \times \nabla\beta}$$

Since $F \cdot F = 0$
$= E \cdot E - B \cdot B + 2i(E \cdot B)$

we conclude

$$E \cdot E - B \cdot B = 0 \Rightarrow E \cdot E = B \cdot B$$

and

$$E \cdot B = 0$$

Normalized Poynting Field

$$\frac{E \times B}{|E \times B|} = i \left(\frac{F \times F^*}{F \cdot F^*} \right)$$

$$= i \frac{(\nabla \alpha \times \nabla \beta) \times (\nabla \alpha^* \times \nabla \beta^*)}{(\nabla \alpha \times \nabla \beta) \cdot (\nabla \alpha^* \times \nabla \beta^*)}$$

Example: Elliptically Polarized Plane Wave

Let

$\alpha(x, y, z, t) = z - t$

$\beta(x, y, z, t) = x + iy$

$f(\alpha, \beta) = e^{i\alpha}$

$g(\alpha, \beta) = \beta$

We see that

$$(\nabla \alpha)^2 = 1^2 = 1 \qquad (\partial_t \alpha)^2 = (-1)^2 = 1$$

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$(\nabla\beta)^2 = 1^2 + i^2$
$= 0$

$(\partial_t \beta)^2 = 0$

$\nabla\alpha \cdot \nabla\beta = (0,0,1)\cdot(1,i,0)$
$= \underline{\underline{0}}$

$\partial_t\alpha\,\partial_t\beta = 1.0$
$= \underline{\underline{0}}$

So α, β satisfy the conditions.

Now

$$F = \nabla f(\alpha,\beta) \times \nabla g(\alpha,\beta)$$
$$= \nabla\left(e^{i(z-t)}\right) \times \nabla(x+iy)$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & ie^{i(z-t)} \\ 1 & i & 0 \end{vmatrix}$$
$$= e^{i(z-t)}(\hat{i} + i\hat{j})$$
$$= E + iB$$

$$\Rightarrow E = \cos(z-t)\,\hat{i} - \sin(z-t)\,\hat{j}$$
$$B = \sin(z-t)\,\hat{i} + \cos(z-t)\,\hat{j}$$

⑦

The normalized Poynting field is computed as follows:

$$F \times F^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{i(z-t)} & ie^{i(z-t)} & 0 \\ e^{-i(z-t)} & -ie^{-i(z-t)} & 0 \end{vmatrix}$$

$$= \hat{k}(-i-i)$$

$$= \underline{\underline{-2i\hat{k}}}$$

and

$$F \cdot F^* = e^{(z-t)(i-i)} + i(-i)e^{(z-t)(i-i)}$$

$$= \underline{\underline{2}}$$

We obtain

$$i\left(\frac{-2i\hat{k}}{2}\right) = \hat{k}$$

as the normalized Poynting field.