# Differential Geometry
# Key Definitions

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**Definition 0.1** A *submersion* is a smooth map $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, such that the differential is surjective for every $x \in \mathcal{M}$.

**Definition 0.2** A *fibre bundle* $\xi$ is a quadruple $(E, \mathcal{M}, \mathcal{F}, \pi)$, where

(i) $E$ is called the *total space* of the fibre bundle ;

(ii) $\mathcal{M}$ is called the *base* of the fibre bundle ;

(iii) $\mathcal{F}$ is called the *fibre* ;

(iv) $\pi : E \to \mathcal{M}$ is a *submersion* with $\pi^{-1}(x) = \mathcal{F}$ ,

such that there exist an open covering $\{U_i\}$ of the base $\mathcal{M}$ and diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathcal{F}$ such that $\pi_1 \circ \phi_i = \pi$, where $\pi_1$ denotes the projection onto the first coordinate.

Let $\mathcal{M}$ be a smooth manifold and let $G$ be a Lie group.

**Definition 0.3** A *section* of the fibre bundle is a differentiable map $\sigma : \mathcal{M} \to E$ such that $\pi \circ \sigma = id_{\mathcal{M}}$. The space of all sections of $E$ is denoted by $\Gamma(E)$.

**Definition 0.4** A *cocycle* of $G$ in $\mathcal{M}$ is an open covering $\{U_i\}$ of $\mathcal{M}$ together with a family of differentiable maps $\gamma_{ij} : U_i \cap U_j \to G$ such that $\gamma_{ij} \cdot \gamma_{jk} = \gamma_{ik}$ for all $i, j, k$. In particular, $\gamma_{ii} = e$ the identity element in $G$.

**Definition 0.5** A *vector bundle* of rank $k$ over $\mathcal{M}$ is a pair $(E, \pi)$, where $E$ is a smooth manifold and $\pi : E \to \mathcal{M}$ is a submersion such that

(i) each fibre $E_x = \pi^{-1}(x)$ has a structure of $k$-dimensional real vector space;

(ii) for every $x \in \mathcal{M}$ there exists an open neighbourhood $U$ of $x$ such that $\pi^{-1}(U) \cong U \times V$, where $V$ is a fibre of $E$.

**Recall 0.6** A group $G$ which has a smooth manifold structure such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are smooth is called a *Lie group*.

**Recall 0.7** A group action is called *free* if for all $m \in \mathcal{M}$, $gm = m \Rightarrow g = e$ where $e$ is the identity element in $G$.

**Recall 0.8** A group action is called *transitive* if for every pair of elements $x, y \in \mathcal{M}$ there is a group element $g$ such that $gx = y$.

**Recall 0.9** A *right group action* of a Lie group $G$ on a manifold $\mathcal{M}$ is a smooth map $\mathcal{M} \times G \to \mathcal{M}$, such that

(i) $me = m, \quad \forall\ m \in \mathcal{M}$ ;

(ii) $m(gh) = (mg)h, \quad \forall\ m \in \mathcal{M},\ g,\ h \in G$.

Let $G$ be a Lie group and let $\mathcal{M}$ be a smooth manifold.

**Definition 0.10** A *G-principal bundle* $P$ is a fibre bundle $P$ with a right group action of a Lie group $G$ on the fibres such that $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$ and such that the action of $G$ is free and transitive on the fibres.