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PURE MATHEMATICS | RESEARCH ARTICLE

The k -nacci triangle and applications

Kantaphon Kuhapatanakul^{1*} and Pornpawee Anantakitpaisal¹

Abstract: A generalization of the classical Fibonacci numbers F_n is the k -generalized Fibonacci numbers $F_n^{(k)}$ for $n \geq 2 - k$ whose first k terms are 0, ..., 0, 1 and each term afterward is the sum of the preceding k terms. In this article, we first introduce the k -nacci triangle to derive an explicit formula of the n th k -generalized Fibonacci number. Second, we also introduce the k -generalized Pascal triangle for deriving the formula of the k -generalized Fibonacci numbers.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics

Keywords: k -generalized Fibonacci numbers; Fibonacci numbers; Pascal triangle; k -nacci triangle; binomial coefficient

AMS Mathematics subject classifications: 11B37; 11B39

1. Introduction

For fixed $k \geq 2$, the k -generalized Fibonacci sequence or, for simplicity, the k -nacci sequence $\{F_n^{(k)}\}_{n \geq 2-k}$ is defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)},$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-1)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$.

Such a sequence is also called k -step Fibonacci sequence or the Fibonacci k -sequence. Clearly for $k = 2$, we obtain the well-known Fibonacci numbers $F_n^{(2)} = F_n$, for $k = 3$, the tribonacci numbers $F_n^{(3)} = T_n$, for $k = 4$, the tetranacci numbers $F_n^{(4)} = T_n$, and for $k = 5$, the pentanacci numbers $F_n^{(5)} = P_n$.

In general case, the first $k + 1$ non-zero terms in $F_n^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1, F_2^{(k)} = 1, F_3^{(k)} = 2, F_4^{(k)} = 4, \dots, F_{k+1}^{(k)} = 2^{k-1},$$

while the next term in the above sequence is $F_{k+2}^{(k)} = 2^k - 1$.

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PUBLIC INTEREST STATEMENT

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art. Pascals triangle has been explored for links to the Fibonacci sequence as well as to generalized sequences. In this paper, the authors give some connections of the coefficients in the multinomial expansion with the generalized Fibonacci numbers. They also construct the k -generalized Pascal triangle to derive the formula of the n th k -generalized Fibonacci numbers.

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art (e.g. see Debnart, 2011; Koshy, 2001; Vajda, 1989).

It is well-known that the Fibonacci number F_n can be derived by the summing of elements on the rising diagonal lines in Pascal's triangle

$$F_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \quad (n \geq 0), \quad (1.1)$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x , see Koshy (2001, chapter 12).

Wong and Maddocks (1975) generalized the Pascal's triangle and showed that sums of elements on the rising diagonal lines in their triangle give the tribonacci number T_n (some authors called this triangle that the tribonacci triangle, e.g. see Alladi and Hoggatt (1977), Kuhapatanakul (2012)). There is yet another triangular array that yields the various tribonacci numbers. Feinberg (1964) used the trinomial expansions of $(1+x+x^2)^n$ for $n \geq 0$ and showed that the rising diagonal sums of this trinomial coefficient array also yield the tribonacci numbers. Kuhapatanakul and Sukruan (2014) have shown the n -tribonacci triangle similar to Pascal's triangle and derived an explicit formula for the tribonacci numbers. The n -tribonacci triangle is an array of the shape

$$\begin{array}{cccccccc} & & & & \binom{0}{0}_n & & & \\ & & & & \binom{1}{0}_n & & \binom{1}{1}_n & \\ & & & \binom{2}{0}_n & \binom{2}{1}_n & & \binom{2}{2}_n & \\ & & \binom{3}{0}_n & \binom{3}{1}_n & \binom{3}{2}_n & & \binom{3}{3}_n & \\ & \binom{4}{0}_n & \binom{4}{1}_n & \binom{4}{2}_n & \binom{4}{3}_n & & \binom{4}{4}_n & \\ \binom{5}{0}_n & \binom{5}{1}_n & \binom{5}{2}_n & \binom{5}{3}_n & \binom{5}{4}_n & & \binom{5}{5}_n & \\ \vdots & & & & & & & \vdots \end{array}$$

$$\text{where } \binom{i}{j}_n = \binom{i}{j} \binom{n-i-j-1}{i}.$$

They showed that the sums of all elements in the n -tribonacci triangle give the n th tribonacci number. The expansion for the tribonacci number T_n in terms of binomial coefficients, see Kuhapatanakul (2012), Kuhapatanakul and Sukruan (2014), Shannon (1977), as the following

$$T_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{i}{j} \binom{n-i-j}{i}.$$

Philippou and Muwafi showed in (1982) that the $F_n^{(k)}$ can be written in the form

$$F_n^{(k)} = \sum_{\substack{0 \leq r_1, \dots, r_k \leq n \\ r_1 + 2r_2 + \dots + kr_k = n}} \frac{(r_1 + \dots + r_k)!}{r_1! \dots r_k!}.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be the roots of $x^k - x^{k-1} - \dots - x - 1 = 0$. The following "Binet-like" formula for $F_n^{(k)}$ appears in Dresden and Du (2014):

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}.$$

In this article, we extend the result of Kuhapatanakul and Sukruan (2014) on the n -tribonacci triangle to the k -nacci triangle for n , and derive the n th k -generalized Fibonacci numbers. We also construct the k -generalized Pascal's triangle to derive the n th k -generalized Fibonacci numbers.

2. The k -nacci triangle

Define the symbol $C_{i,j}^q$ as the coefficient of x^j in the multinomial expansion of $(1 + x + x^2 + \dots + x^q)^i$ for $q \geq 1$ and $i, j \geq 0$, i.e.

$$(1 + x + x^2 + \dots + x^q)^i = \sum_{j=0}^{qi} C_{i,j}^q x^j, \quad (2.1)$$

with $C_{i,j}^1 = \binom{i}{j}$ is the *binomial coefficient* and $C_{i,j}^q = 0$ for $j > qi$. Using the classical binomial coefficient, we get that

$$C_{i,j}^q = \sum_{i_1+i_2+\dots+i_q=j} \binom{i}{i_1} \binom{i_1}{i_2} \dots \binom{i_{q-1}}{i_q}, \quad (2.2)$$

or

$$C_{i,j}^q = \sum_{i_1+2i_2+\dots+qi_q=j} \binom{i}{i_1} \binom{i-i_1}{i_2} \dots \binom{i-i_1-\dots-i_{q-1}}{i_q}. \quad (2.3)$$

We give the arrays of the coefficients of x^j in the multinomial expansion for $q = 2, 3, 4$ to show in the Figures 1–3, respectively, see also Sloane (2011) as A027907, A008287, A035343.

Figure 1. The trinomial coefficients array.

$i \setminus x^j$	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	95	126	151	126	95	50	21	6	1

Figure 2. The quadrinomial coefficients array.

$i \setminus x^j$	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
0	1												
1	1	1	1	1									
2	1	2	3	4	3	2	1						
3	1	3	6	10	12	12	10	6	3	1			
4	1	4	10	20	31	40	44	40	31	20	10	4	1
5	1	5	15	35	65	101	135	155	135	101	65	35	...
6	1	6	21	56	120	216	336	456	546	580	546	...	

Figure 3. The pentanomial coefficients array.

$i \backslash x^j$	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	
0	1														
1	1	1	1	1	1										
2	1	2	3	4	5	4	3	2	1						
3	1	3	6	10	15	18	19	18	15	10	6	3	1		
4	1	4	10	20	35	52	68	80	85	80	68	52	35	20	...
5	1	5	15	35	70	121	185	...							

Some well-known properties of the multinomial coefficient arrays:

(1) Every row is symmetric about a vertical line through the middle, i.e.

$$C_{i,j}^q = C_{i,q-i-j}^q$$

(2) Any interior number in each row, the exception of the first two rows, can be obtained from the preceding row, i.e.

$$C_{i,j}^q = \sum_{l=0}^q C_{i-1,j-l}^q$$

Next, we introduce the k -nacci triangle for a positive integer n .

Definition 1 Let n and $k \geq 3$ be positive integers. The k -nacci triangle for n is an array that each element in row i and column j is products of $C_{i,j}^{k-2}$ and $\binom{n-i-j-1}{i}$ as shown (put $t = k - 2$):

$i \setminus j$	0	1	...	t	...	$2t$...	it
0	$C_{0,0}^t \binom{n-1}{0}$							
1	$C_{1,0}^t \binom{n-2}{1}$	$C_{1,1}^t \binom{n-3}{1}$...	$C_{1,t}^t \binom{n-k}{1}$				
2	$C_{2,0}^t \binom{n-3}{2}$	$C_{2,1}^t \binom{n-4}{2}$...		$C_{2,2t}^t \binom{n-2k+1}{2}$		
3	$C_{3,0}^t \binom{n-4}{3}$	$C_{3,1}^t \binom{n-5}{3}$...					
\vdots		\vdots						
i	$C_{i,0}^t \binom{n-i-1}{i}$	$C_{i,1}^t \binom{n-i-2}{i}$...				$C_{i,it}^t \binom{n-ik+i-1}{i}$
\vdots		\vdots						

For clarity, we give the examples of the k -nacci triangle for $k = 3, 4, 5$.

$$\begin{array}{ccccccccccc}
 1 \binom{n-1}{0} & & & & & & & & & & \\
 1 \binom{n-2}{1} & 1 \binom{n-3}{1} & & & & & & & & & \\
 1 \binom{n-3}{2} & 2 \binom{n-4}{2} & 1 \binom{n-5}{2} & & & & & & & & \\
 1 \binom{n-4}{3} & 3 \binom{n-5}{3} & 3 \binom{n-6}{3} & 1 \binom{n-7}{3} & & & & & & & \\
 1 \binom{n-5}{4} & 4 \binom{n-6}{4} & 6 \binom{n-7}{4} & 4 \binom{n-8}{4} & 1 \binom{n-9}{4} & & & & & & \\
 1 \binom{n-6}{5} & 5 \binom{n-7}{5} & 10 \binom{n-8}{5} & 10 \binom{n-9}{5} & 5 \binom{n-10}{5} & 1 \binom{n-11}{5} & & & & & \\
 1 \binom{n-7}{6} & 6 \binom{n-8}{6} & 15 \binom{n-9}{6} & 20 \binom{n-10}{6} & 15 \binom{n-11}{6} & 6 \binom{n-12}{6} & 1 \binom{n-13}{6} & & & & \\
 & \vdots & & \vdots & & & & & & &
 \end{array}$$

The 3-nacci triangle for n

$$\begin{array}{cccccccc}
 1 \binom{n-1}{0} & & & & & & & \\
 1 \binom{n-2}{1} & 1 \binom{n-3}{1} & 1 \binom{n-4}{1} & & & & & \\
 1 \binom{n-3}{2} & 2 \binom{n-4}{2} & 3 \binom{n-5}{2} & 2 \binom{n-6}{2} & 1 \binom{n-7}{2} & & & \\
 1 \binom{n-4}{3} & 3 \binom{n-5}{3} & 6 \binom{n-6}{3} & 7 \binom{n-7}{3} & 6 \binom{n-8}{3} & 3 \binom{n-9}{3} & 1 \binom{n-10}{3} & \\
 1 \binom{n-5}{4} & 4 \binom{n-6}{4} & 10 \binom{n-7}{4} & 16 \binom{n-8}{4} & 19 \binom{n-9}{4} & 16 \binom{n-10}{4} & 10 \binom{n-11}{4} & \dots \\
 1 \binom{n-6}{5} & 5 \binom{n-7}{5} & 15 \binom{n-8}{5} & 30 \binom{n-9}{5} & 45 \binom{n-10}{5} & 51 \binom{n-11}{5} & 45 \binom{n-12}{5} & \dots \\
 1 \binom{n-7}{6} & 6 \binom{n-8}{6} & 21 \binom{n-9}{6} & 50 \binom{n-10}{6} & \dots & & & \\
 & \vdots & & & & & & \vdots
 \end{array}$$

The 4-nacci triangle for n

$$\begin{array}{cccccccc}
 1 \binom{n-1}{0} & & & & & & & \\
 1 \binom{n-2}{1} & 1 \binom{n-3}{1} & 1 \binom{n-4}{1} & 1 \binom{n-5}{1} & & & & \\
 1 \binom{n-3}{2} & 2 \binom{n-4}{2} & 3 \binom{n-5}{2} & 4 \binom{n-6}{2} & 3 \binom{n-7}{2} & 2 \binom{n-8}{2} & 1 \binom{n-9}{2} & \\
 1 \binom{n-4}{3} & 3 \binom{n-5}{3} & 6 \binom{n-6}{3} & 10 \binom{n-7}{3} & 12 \binom{n-8}{3} & 12 \binom{n-9}{3} & 10 \binom{n-10}{3} & \dots \\
 1 \binom{n-5}{4} & 4 \binom{n-6}{4} & 10 \binom{n-7}{4} & 20 \binom{n-8}{4} & 31 \binom{n-9}{4} & 40 \binom{n-10}{4} & 44 \binom{n-11}{4} & \dots \\
 1 \binom{n-6}{5} & 5 \binom{n-7}{5} & 15 \binom{n-8}{5} & 35 \binom{n-9}{5} & 65 \binom{n-10}{5} & 101 \binom{n-11}{5} & 135 \binom{n-12}{5} & \dots \\
 1 \binom{n-7}{6} & 6 \binom{n-8}{6} & 21 \binom{n-9}{6} & 56 \binom{n-10}{6} & \dots & & & \\
 & \vdots & & & & & & \vdots
 \end{array}$$

The 5-nacci triangle for n

Note that the 3-nacci triangle for n is the n -tribonacci triangle which is defined by Kuhapatanakul and Sukruan, see Kuhapatanakul and Sukruan (2014).

It is interesting that sums of all elements in the k -nacci triangle for n give the n th k -nacci number. We will give some examples.

Example 1

(1) In the 4-nacci triangle for n

(a) Substituting $n = 10$, we get

$$\begin{array}{cccccc}
 1 \binom{9}{0} & & & & & \\
 1 \binom{8}{1} & 1 \binom{7}{1} & 1 \binom{6}{1} & & & \\
 1 \binom{7}{2} & 2 \binom{6}{2} & 3 \binom{5}{2} & 2 \binom{4}{2} & 1 \binom{3}{2} & \\
 1 \binom{6}{3} & 3 \binom{5}{3} & 6 \binom{4}{3} & 7 \binom{3}{3} & & \\
 1 \binom{5}{4} & 4 \binom{4}{4} & & & & \\
 & & & & &
 \end{array}$$

and then sums of all elements is equal to 208, which is the 10th tetranacci number \mathcal{T}_{10} .

(b) Substituting $n = 15$, we get

$$\begin{array}{cccccccc}
 1 \binom{14}{0} & & & & & & & \\
 1 \binom{13}{1} & 1 \binom{12}{1} & 1 \binom{11}{1} & & & & & \\
 1 \binom{12}{2} & 2 \binom{11}{2} & 3 \binom{10}{2} & 2 \binom{9}{2} & 1 \binom{8}{2} & & & \\
 1 \binom{11}{3} & 3 \binom{10}{3} & 6 \binom{9}{3} & 7 \binom{8}{3} & 6 \binom{7}{3} & 3 \binom{6}{3} & 1 \binom{5}{3} & \\
 1 \binom{10}{4} & 4 \binom{9}{4} & 10 \binom{8}{4} & 16 \binom{7}{4} & 19 \binom{6}{4} & 16 \binom{5}{4} & 10 \binom{4}{4} & \\
 1 \binom{9}{5} & 5 \binom{8}{5} & 15 \binom{7}{5} & 30 \binom{6}{5} & 45 \binom{5}{5} & & & \\
 1 \binom{8}{6} & 6 \binom{7}{6} & 21 \binom{6}{6} & & & & & \\
 1 \binom{7}{7} & & & & & & &
 \end{array}$$

and sums of all elements is equal to 5600, which is the 15th tetranacci number \mathcal{T}_{15} .

Example 2

(2) In the 5-nacci triangle for n

(a) Substituting $n = 10$, we get

$$\begin{array}{ccccccc}
 1 \binom{9}{0} & & & & & & \\
 1 \binom{8}{1} & 1 \binom{7}{1} & 1 \binom{6}{1} & 1 \binom{5}{1} & & & \\
 1 \binom{7}{2} & 2 \binom{6}{2} & 3 \binom{5}{2} & 4 \binom{4}{2} & 3 \binom{3}{2} & 2 \binom{2}{2} & \\
 1 \binom{6}{3} & 3 \binom{5}{3} & 6 \binom{4}{3} & 10 \binom{3}{3} & & & \\
 1 \binom{5}{4} & 4 \binom{4}{4} & & & & &
 \end{array}$$

and sums of all elements is equal to 236, which is the 10th pentanacci number \mathcal{P}_{10} .

(b) Substituting $n = 15$, we get

$$\begin{array}{cccccccc}
 1 \binom{14}{0} & & & & & & & \\
 1 \binom{13}{1} & 1 \binom{12}{1} & 1 \binom{11}{1} & 1 \binom{10}{1} & & & & \\
 1 \binom{12}{2} & 2 \binom{11}{2} & 3 \binom{10}{2} & 4 \binom{9}{2} & 3 \binom{8}{2} & 2 \binom{7}{2} & & \\
 1 \binom{11}{3} & 3 \binom{10}{3} & 6 \binom{9}{3} & 10 \binom{8}{3} & 12 \binom{7}{3} & 6 \binom{6}{3} & 3 \binom{5}{3} & 1 \binom{4}{3} \\
 1 \binom{10}{4} & 4 \binom{9}{4} & 10 \binom{8}{4} & 20 \binom{7}{4} & 31 \binom{6}{4} & 40 \binom{5}{4} & 44 \binom{4}{4} & \\
 1 \binom{9}{5} & 5 \binom{8}{5} & 15 \binom{7}{5} & 35 \binom{6}{5} & 65 \binom{5}{5} & & & \\
 1 \binom{8}{6} & 6 \binom{7}{6} & 21 \binom{6}{6} & & & & & \\
 1 \binom{7}{7} & & & & & & &
 \end{array}$$

and sums of all elements is equal to 6930, which is the 15th pentanacci number P_{15} .

We will in fact prove that the sums of all elements in the k -nacci triangle for n give the n th k -nacci number.

Let $k \geq 3$. Denote $S_n(i)$ as the sums of elements in the i th row of k -nacci triangle for a positive integer n , that is,

$$S_n(i) = \sum_{j=0}^{(k-2)i} C_{i,j}^{k-2} \binom{n-i-j-1}{i}. \quad (2.4)$$

The $S_n(i)$ can write in the recurrent relation.

LEMMA 1 Let n, i be positive integers. Then

$$S_{n+k}(i) = S_{n+k-1}(i) + S_{n+k-2}(i-1) + \cdots + S_n(i-1). \quad (2.5)$$

Proof Set $C_{i,j} := C_{i,j}^{k-2}$ and we see that $C_{i,j} = 0$ when $j > (k-1)(i-1)$. We have

$$\begin{aligned} S_{n+k}(i) &= \sum_{j=0}^{(k-2)i} C_{i,j} \binom{n+k-i-j-1}{i} \\ &= \sum_{j=0}^{(k-2)i} C_{i,j} \binom{n+k-i-j-2}{i} + \sum_{j=0}^{(k-2)i} C_{i,j} \binom{n+k-i-j-2}{i-1} \\ &= S_{n+k-1}(i) + \sum_{j=0}^{(k-2)i} C_{i-1,j} \binom{n+k-i-j-2}{i-1} + \sum_{j=0}^{(k-2)i} C_{i-1,j-1} \binom{n+k-i-j-2}{i-1} \\ &\quad + \sum_{j=0}^{(k-2)i} C_{i-1,j-2} \binom{n+k-i-j-2}{i-1} + \cdots + \sum_{j=0}^{(k-2)i} C_{i-1,j-(k-2)} \binom{n+k-i-j-2}{i-1} \\ &= S_{n+k-1}(i) + \sum_{j=0}^{(k-2)i} C_{i-1,j} \binom{n+k-3-(i-1)-j}{i-1} + \sum_{j=0}^{(k-2)i-1} C_{i-1,j} \binom{n+k-i-j-3}{i-1} \\ &\quad + \sum_{j=0}^{(k-2)i-2} C_{i-1,j} \binom{n+k-i-j-4}{i-1} + \cdots + \sum_{j=0}^{(k-2)i-(k-2)} C_{i-1,j} \binom{n-i-j}{i-1} \\ &= S_{n+k-1}(i) + S_{n+k-2}(i-1) + \sum_{j=0}^{(k-2)i-1} C_{i-1,j} \binom{(n+k-4)-(i-1)-j}{i-1} \\ &\quad + \sum_{j=0}^{(k-2)i-2} C_{i-1,j} \binom{(n+k-5)-(i-1)-j}{i-1} + \cdots + \sum_{j=0}^{(k-2)(i-1)} C_{i-1,j} \binom{n-(i-1)-j-1}{i-1} \\ &= S_{n+k-1}(i) + S_{n+k-2}(i-1) + \cdots + S_n(i-1), \end{aligned}$$

as desired. \square

Now, we state an explicit formula for $F_n^{(k)}$ by summing all row sums of the k -nacci triangle for n and prove the following theorem.

THEOREM 1 Let $k \geq 3$ and $S_n(i)$ as defined in this section. Then

$$F_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/2 \rfloor} S_{n+1}(i). \quad (2.6)$$

Proof We will prove by induction on n . We see that

$$F_1^{(k)} = 1 = S_1(0), \quad F_2^{(k)} = 1 = S_2(0), \quad F_3^{(k)} = 2 = 1 + 1 = S_3(0) + S_3(1),$$

$$\text{and } F_k^{(k)} = S_k(0) + S_k(1) + \cdots + S_k(\lfloor (k-1)/2 \rfloor).$$

Assume that (2.6) holds for all integers $n = 0, 1, 2, \dots, k-1$. By the inductive hypothesis and Lemma 1, we get

$$\begin{aligned} F_{n+1}^{(k)} &= F_n^{(k)} + F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \\ &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} S_{n-1}(i) + \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} S_{n-2}(i) + \cdots + \sum_{i=0}^{\lfloor (n-k-1)/2 \rfloor} S_{n-k-1}(i) \\ &= S_{n-1}(0) + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} S_{n-1}(i) + \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} S_{n-2}(i) + \cdots + \sum_{i=0}^{\lfloor (n-k-1)/2 \rfloor} S_{n-k-1}(i) \\ &= S_{n-1}(0) + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} S_{n-1}(i) + \sum_{i=1}^{\lfloor n/2 \rfloor} S_{n-2}(i) + \cdots + \sum_{i=1}^{\lfloor (n-k+1)/2 \rfloor} S_{n-k-1}(i-1) \\ &= S_n(0) + \sum_{i=1}^{\lfloor n/2 \rfloor} (S_{n-1}(i) + S_{n-2}(i-1) + \cdots + S_{n-k-1}(i-1)) \\ &= S_n(0) + \sum_{i=1}^{\lfloor n/2 \rfloor} S_n(i) \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} S_{n+1}(i). \end{aligned}$$

Thus, the proof is complete. □

We can rewrite Equation (2.6) in terms of binomial coefficients using Equation (2.4).

$$F_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{(k-2)i} C_{i,j}^{k-2} \binom{n-i-j}{i}.$$

3. The k -generalized Pascal's triangle

Throughout the section, the integer $k \geq 2$ will be fixed.

Definition 2 Let n, i be integers with $n \geq 0$. Define

$$C_k(n, i) = C_k(n-1, i) + \sum_{j=1}^{k-1} C_k(n-j, i-1), \quad (3.1)$$

where $C_k(n, 0) = C_k(n, n) = 1$, and $C_k(n, i) = 0$ for $i < 0$ or $i > n$.

Definition 3 Denote the k -generalized Pascal's triangle as follows:

$$\begin{array}{ccccccc} & & & & & & C_k(0, 0) \\ & & & & & & \\ & & & & & & C_k(1, 0) & C_k(1, 1) \\ & & & & & & C_k(2, 0) & C_k(2, 1) & C_k(2, 2) \\ & & & & & & C_k(3, 0) & C_k(3, 1) & C_k(3, 2) & C_k(3, 3) \\ & & & & & & C_k(4, 0) & C_k(4, 1) & C_k(4, 2) & C_k(4, 3) & C_k(4, 4) \\ & & & & & & C_k(5, 0) & C_k(5, 1) & C_k(5, 2) & C_k(5, 3) & C_k(5, 4) & C_k(5, 5) \\ & & & & & & C_k(6, 0) & C_k(6, 1) & C_k(6, 2) & C_k(6, 3) & C_k(6, 4) & C_k(6, 5) & C_k(6, 6) \\ & & & & & & \vdots & & & & \vdots & & \\ & & & & & & & & & & & & \end{array}$$

Well-known examples of $C_k(n, i)$ for $k = 2, 3$ are wrote in terms of binomial coefficients

$$C_2(n, i) = \binom{n}{i} \text{ and } C_3(n, i) = \sum_{j=0}^i \binom{i}{j} \binom{n-j}{i}.$$

It is to see that the 2-generalized Pascal's triangle is the classical Pascal's triangle and the 3-generalized Pascal's triangle is the generalized Pascal's triangle which is defined by Wong and Maddocks (1975).

For convenience, we arrange the elements of the k -generalized Pascal's triangle to form a left-justified triangular array as follows:

	0	1	2	3	4	5	...	n
0	$C_k(0, 0)$							
1	$C_k(1, 0)$	$C_k(1, 1)$						
2	$C_k(2, 0)$	$C_k(2, 1)$	$C_k(2, 2)$					
3	$C_k(3, 0)$	$C_k(3, 1)$	$C_k(3, 2)$	$C_k(3, 3)$				
4	$C_k(4, 0)$	$C_k(4, 1)$	$C_k(4, 2)$	$C_k(4, 3)$	$C_k(4, 4)$			
5	$C_k(5, 0)$	$C_k(5, 1)$	$C_k(5, 2)$	$C_k(5, 3)$	$C_k(5, 4)$	$C_k(5, 5)$		
\vdots		\vdots						
n	$C_k(n, 0)$	$C_k(n, 1)$	$C_k(n, 2)$...		$C_k(n, n)$	
\vdots		\vdots						

The k -generalized Pascal's triangle

For clarity, we also give the following examples of the k -generalized Pascal's triangles for $k = 4, 5$.

	0	1	2	3	4	5	6	7	8	...
0	1									
1	1	1								
2	1	3	1							
3	1	6	5	1						
4	1	9	15	7	1					
5	1	12	33	28	9	1				
6	1	15	60	81	45	11	1			
7	1	18	96	189	161	66	13	1		
8	1	21	141	378	459	281	91	15	1	
\vdots				\vdots						

The 4-generalized Pascal's triangle

	0	1	2	3	4	5	6	7	8	...
0	1									
1	1	1								
2	1	3	1							
3	1	6	5	1						
4	1	10	15	7	1					
5	1	14	35	28	9	1				
6	1	18	68	84	45	11	1			
7	1	22	116	207	165	66	13	1		
8	1	26	180	441	491	286	91	15	1	
\vdots				\vdots						

The 5-generalized Pascal's triangle

Observe that the sum of elements on each rising diagonal line in The 4, 5-generalized Pascal's triangles give the 4-generalized Fibonacci numbers $F_n^{(4)}$ and the 5-generalized Fibonacci numbers $F_n^{(5)}$, respectively. The following table provides further information:

n	$F_n^{(4)}$	$F_n^{(5)}$
1	1	1
2	1	1
3	$1 + 1 = 2$	$1 + 1 = 2$
4	$1 + 3 = 4$	$1 + 3 = 4$
5	$1 + 6 + 1 = 8$	$1 + 6 + 1 = 8$
6	$1 + 9 + 5 = 15$	$1 + 10 + 5 = 16$
7	$1 + 12 + 15 + 1 = 29$	$1 + 14 + 15 + 1 = 31$
8	$1 + 15 + 33 + 7 = 56$	$1 + 18 + 35 + 7 = 61$
9	$1 + 18 + 60 + 28 + 1 = 108$	$1 + 22 + 68 + 28 + 1 = 120$

We conjecture that the sums of elements on each rising diagonal line in the k -generalized Pascal's triangle gives the k -generalized Fibonacci number.

THEOREM 2 Let $k \geq 2$ be fixed, and let n be non-negative integer. Then

$$F_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/2 \rfloor} C_k(n-i, i). \quad (3.2)$$

Proof For $k = 2$, $C_2(n, i)$ is a binomial coefficient, the Equation (3.2) becomes the Equation (1.1). Suppose $k \geq 3$. We will prove this result by induction on n . It is easy to see that, for $1 \leq m \leq k-1$,

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} C_k(m-i-1, i) = 2^{m-2} = F_m^{(k)}.$$

Now, we assume Equation (3.2) holds for $n > 1$ and prove that it holds for $n+1$. Using the definition of $F_n^{(k)}$ and the inductive hypothesis, we get that

$$\begin{aligned} F_{n+2}^{(k)} &= F_{n+1}^{(k)} + F_n^{(k)} + \cdots + F_{n-k+2}^{(k)} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} C_k(n-i, i) + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} C_k(n-i-1, i) + \cdots + \sum_{i=0}^{\lfloor (n-k+1)/2 \rfloor} C_k(n-i-k+1, i) \\ &= C_k(n, 0) + \sum_{i=1}^{\lfloor n/2 \rfloor} C_k(n-i, i) + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} C_k(n-i-1, i) + \cdots + \sum_{i=1}^{\lfloor (n-k+3)/2 \rfloor} C_k(n-i-k+1, i) \\ &= C_k(n, 0) + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} C_k(n-i+1, i) \\ &= \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} C_k(n-i+1, i). \end{aligned}$$

Thus, the proof of result is complete. \square

Next, we give an alternative definition of $C_k(n, i)$ in terms of binomial coefficients. We begin provide the following lemma which will be used in the proof.

LEMMA 2 Let n be non-negative integer. Then

$$\begin{aligned} (i) \quad & \sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l-1}{i} \\ &= \sum_{j=0}^i \sum_{l=0}^{j+1} \binom{i}{j} \binom{j+1}{l} \binom{n-j-l}{i} \end{aligned}$$

$$(ii) \sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j+1}{l} \binom{n-j-l}{i} + \sum_{j=0}^{i+1} \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l-1}{i} \\ = \sum_{j=0}^{i+1} \sum_{l=0}^j \binom{i+1}{j} \binom{j}{l} \binom{n-j-l}{i}.$$

Proof Using the Pascal's identity, we get the parts (i) and (ii) that

$$\sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l-1}{i} \\ = \sum_{j=0}^i \binom{i}{j} \binom{n-j}{i} + \sum_{j=0}^i \sum_{l=1}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} \\ + \sum_{j=0}^i \sum_{l=1}^j \binom{i}{j} \binom{j}{l-1} \binom{n-j-l}{i} + \sum_{j=0}^i \binom{i}{j} \binom{n-2j-1}{i} \\ = \sum_{j=0}^i \binom{i}{j} \binom{n-j}{i} + \sum_{j=0}^i \sum_{l=1}^j \binom{i}{j} \binom{j+1}{l} \binom{n-j-l}{i} + \sum_{j=0}^i \binom{i}{j} \binom{n-2j-1}{i} \\ = \sum_{j=0}^{i+1} \sum_{l=0}^j \binom{i}{j} \binom{j+1}{l} \binom{n-j-l}{i},$$

and \square

$$\sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{j=0}^{i+1} \sum_{l=0}^j \binom{i}{j} \binom{j+1}{l} \binom{n-j-l-1}{i} \\ = \binom{n}{i} + \sum_{j=1}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{j=0}^{i-1} \sum_{l=0}^{j+1} \binom{i}{j} \binom{j+1}{l} \binom{n-j-l-1}{i} \\ + \sum_{l=0}^{i+1} \binom{i+1}{l} \binom{n-i-l-1}{i} \\ = \binom{n}{i} + \sum_{j=1}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{j=1}^i \sum_{l=0}^j \binom{i}{j-1} \binom{j}{l} \binom{n-j-l}{i} \\ + \sum_{l=0}^{i+1} \binom{i+1}{l} \binom{n-i-l-1}{i} \\ = \binom{n}{i} + \sum_{j=1}^i \sum_{l=0}^j \binom{i+1}{j} \binom{j}{l} \binom{n-j-l}{i} + \sum_{l=0}^{i+1} \binom{i+1}{l} \binom{n-i-l-1}{i} \\ = \sum_{j=0}^{i+1} \sum_{l=0}^j \binom{i+1}{j} \binom{j}{l} \binom{n-j-l}{i}.$$

THEOREM 3 Let $k \geq 3$ be fixed, and let n, i be integers with $n \geq 0$. Then

$$C_k(n, i) = \sum_{i_1=0}^i \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{n-i_1-\cdots-i_{k-2}}{i}. \quad (3.3)$$

Proof For $n = 0$, it is to see that Equation (3.3) holds. Assume Equation (3.3) is true for $n \geq 0$ and $0 \leq i \leq n$. We will show (3.3) holds for $n + 1$. Set $N = n - i_1 - \cdots - i_{k-2}$, using Lemma 2(i), we get

$$C_k(n - k + 3, i - 1) + C_k(n - k + 2, i - 1) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-2}=0}^{i_{k-3}+1} \binom{i-1}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}+1}{i_{k-2}} \binom{N-k+3}{i-1}.$$

Using Lemma 2(ii) and the above equation, we get

$$C_k(n-k+4, i-1) + C_k(n-k+3, i-1) + C_k(n-k+2, i-1) \\ = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-3}=0}^{i_{k-4}+1} \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i-1}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-4}+1}{i_{k-3}} \binom{i_{k-3}}{i_{k-2}} \binom{N-k+4}{i-1}.$$

Repeating Lemma 2(ii), we obtain

$$C_k(n, i-1) + \cdots + C_k(n-k+3, i-1) + C_k(n-k+2, i-1) \\ = \sum_{i_1=0}^i \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{N}{i-1}.$$

By the recurrence (3.1) and Pascal's identity, we obtain

$$C_k(n+1, i) = C_k(n, i) + C_k(n, i-1) + \cdots + C_k(n-k+2, i-1) \\ = \sum_{i_1=0}^i \cdots \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i}{i_1} \cdots \binom{i_{k-2}}{i_{k-3}} \binom{N}{i} + \sum_{i_1=0}^i \cdots \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i}{i_1} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{N}{i-1} \\ = \sum_{i_1=0}^i \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-2}=0}^{i_{k-3}} \binom{i}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{N+1}{i}.$$

This makes Equation (3.3) holds for $n+1$. Therefore, Equation (3.3) is true. \square

Now we present some examples of the explicit formulas for the tribonacci, tetranacci, and pentanacci numbers.

$$\begin{aligned} \text{(i)} \quad F_{n+1}^{(3)} = T_{n+1} &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^i \binom{i}{j} \binom{n-i-j}{i}; \\ \text{(ii)} \quad F_{n+1}^{(4)} = \tau_{n+1} &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^i \sum_{l=0}^j \binom{i}{j} \binom{j}{l} \binom{n-i-j-l}{i}; \\ \text{(iii)} \quad F_{n+1}^{(5)} = \mathcal{P}_{n+1} &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^i \sum_{l=0}^j \sum_{m=0}^l \binom{i}{j} \binom{j}{l} \binom{l}{m} \binom{n-i-j-l-m}{i}. \end{aligned}$$

We can rewrite $C_k(n, i)$ in the only one summation as follows

$$C_k(n, i) = \sum_{i_1+i_2+\cdots+i_{k-2} \leq n-i} \binom{i}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{n-i_1-\cdots-i_{k-2}}{i}.$$

Hence, we write the identity (3.2) in the binomial coefficients.

COROLLARY 1 For fixed $k \geq 2$, we have

$$F_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{i_1+\cdots+i_{k-2} \leq n-i} \binom{i}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{k-3}}{i_{k-2}} \binom{n-i-i_1-\cdots-i_{k-2}}{i}.$$

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