Swanson Model:

beyond the PT-symmetry phase.

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1. A brief review of related previous works.
2. no-standard hofp algebras and the Swanson Model.
3. Swanson Model: regions in the parameter model-space.
4. Swanson Model: spectrum and generalized eigenfunctions.

Some time ago...

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Nonstandard *q*-deformed realizations of the harmonic oscillator

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The boson expansion method is applied to nd the spectrum of a *q*-deformed harmonic oscillator. We use two di erent boson expansions, each of them including a deformation parameter, de ned in terms of exponential and logarithmic functionals. The resulting Hamiltonians are bilinear forms of the transformed operators. Physical e ects resulting from the deformation of the generators of the algebra are studied by comparing known nite-range potentials and the e ective potentials obtained for each of the considered Hamiltonians.

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Nonstandard q-deformed harmonic oscillator.

Weyl Algebra

$$\{a^\dagger, a, N, \mathbf{1}\}:$$

$$\begin{aligned}
[N, a] &= -a \\
[N, a^\dagger] &= a^\dagger \\
[a, a^\dagger] &= \mathbf{1} \\
[\mathbf{1}, \cdot] &= 0
\end{aligned}$$

$$C = N - \frac{1}{2}\{a^\dagger, a\}$$

Hopf Algebra: $U_\lambda^n(h_4)$

$$\{A_+, A_-, N, \mathbf{M}\}:$$

$$\begin{aligned}
[N, A_-] &= -A_- \\
[N, A_+] &= -\left(\mathrm{e}^{\lambda A_+} - 1\right)/\lambda \\
[A_-, A_+] &= \mathbf{M}\mathrm{e}^{\lambda A_+} \\
[\mathbf{M}, \cdot] &= 0
\end{aligned}$$

$$C_\lambda = NM - \frac{1}{2}\left\{\frac{\mathrm{e}^{\lambda A_+} - 1}{\lambda}, A_-\right\}$$

Ballesteros A., Herranz F. J., Nieto L. M. and Negro J., J. Phys. A: Math. Gen. **33**, 4859(2000)

Nonstandard q-deformed harmonic oscillator.

Exponential boson mapping

$$\begin{aligned} A_+ &= a^\dagger \\ A_- &= e^{\lambda a^\dagger} a \\ N &= \frac{e^{\lambda A_+} - 1}{\lambda} a \end{aligned}$$

Logarithmic boson mapping

$$\begin{aligned} A_+ &= \frac{1}{\lambda} \ln\left(\frac{1}{1-\lambda a^\dagger}\right) a^\dagger \\ A_- &= a \\ N &= a^\dagger a \end{aligned}$$

Nonstandard q-deformed harmonic oscillator.

$$
\begin{aligned}
p &= \mathbf{i} \sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)
\end{aligned}
$$

$$
\begin{aligned}
P &= \mathbf{i} \sqrt{\tfrac{m\hbar\omega}{2}}(A_+ - A_-) \quad \Rightarrow P = p - \mathbf{i} m\omega\Theta_\lambda \\
X &= \sqrt{\tfrac{\hbar}{2m\omega}}(A_+ + A_-) \quad \Rightarrow X = x + \Theta_\lambda
\end{aligned}
$$

Exponential Mapping

$$\Theta_\lambda = \sqrt{\tfrac{\hbar}{2m\omega}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!}\, a^{\dagger\,\lambda} a$$

Logarithmic Mapping

$$\Theta_\lambda = -\sqrt{\tfrac{\hbar}{2m\omega}} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{k!}\, a^{\dagger\,\lambda}$$

Nonstandard q-deformed harmonic oscillator.

$$\boxed{H = \frac{P^2}{2m}} = \frac{\hbar\omega}{4}\left(A_+A_- - (A_+^2 + A_-^2)\right)$$

$$H = \boxed{\frac{p^2}{2m} - \frac{1}{2}m\omega^2\Theta_\lambda^2 - \mathbf{i}\frac{\omega}{2}[p, \Theta_\lambda]}$$

Nonstandard q-deformed harmonic oscillator.

Exponential boson mapping

$$
\begin{aligned}
A_+ &= a^\dagger \\
A_- &\approx a + \lambda a^\dagger a + \frac{\lambda^2}{2} a^{\dagger 2} a
\end{aligned}
$$

Logarithmic boson mapping

$$
\begin{aligned}
A_+ &\approx a^\dagger + \frac{\lambda}{2} a^{\dagger 2} + \frac{\lambda^2}{3} a^{\dagger 3} a \\
A_- &= a
\end{aligned}
$$

Nonstandard q-deformed harmonic oscillator.

$$\frac{H_{exp}}{(\hbar\omega/2)} \approx p^2 + \frac{\lambda}{\sqrt{2}}\left(\mathbf{i}p + q - \mathbf{i}p(p^2+q^2)\right) + \frac{\lambda^2}{2}\left(p^2 + \frac{5}{4}(p^2+q^2) - p^2\,q^2 - \frac{3}{4}p^4 - \frac{1}{4}q^4 - \mathbf{i}\frac{1}{2}p(p^2+q^2+2)q\right),$$

Local gauge transformation:

$$\Psi(q) = \mathrm{e}^{\mathbf{i}\alpha(q)}\phi(q), \qquad \alpha(q) = -\mathbf{i}\frac{\lambda}{6\sqrt{2}}q(3-q^2)$$

$$-\frac{\hbar^2}{2m}\varphi''(q) + \boxed{\frac{\hbar\omega}{16}\lambda^2(1-q^2)^2}\varphi(q) = E\varphi(q).$$

Nonstandard q-deformed harmonic oscillator.

$$
\begin{aligned}
\frac{H_{log}}{(\hbar\omega/2)} \approx\ & p^2 + \frac{\lambda}{\sqrt{2}}\left(ip - \frac{1}{2}q - ip(ip - \frac{1}{2}q)q - i\frac{1}{2}p^3\right) \\
& + \frac{\lambda^2}{8}\left(-\frac{11}{4} + \frac{15}{2}p^2 - \frac{7}{2}q^2 + \frac{11}{2}p^2q^2 - \frac{19}{12}p^4 - \frac{1}{4}q^4\right. \\
& \left. + ip(-5p^2 + \frac{7}{3}q^2 + 11)q\right).
\end{aligned}
$$

Local gauge transformation:

$$\Psi(q) = \mathrm{e}^{\mathbf{i}\alpha(q)}\phi(q), \qquad \alpha(q) = \mathbf{i}\,\frac{\lambda}{12\sqrt{2}}q(6+q^2)$$

$$-\frac{\hbar^2}{2m}\varphi''(q) + \boxed{\frac{\hbar\omega}{64}\lambda^2(2-q^2)^2}\varphi(q) = E\varphi(q)$$

Nonstandard q-deformed harmonic oscillator.

$$H = \frac{\hbar\omega}{4}\left(\eta A_+ A_- + \zeta(A_+^2 + A_-^2)\right)$$

$$\begin{aligned} H &= g_+(\eta,\zeta)\frac{p^2}{2m} + \zeta\frac{1}{2}m\omega^2\Theta_\lambda^2 - i\, g_+(\eta,\zeta)\frac{w}{2}\{p,\Theta_\lambda\} \\ &+ g_-(\eta,\zeta)\frac{m\omega^2}{2}(x^2 + \{x,\Theta_\lambda\}) \\ &- \frac{\omega\eta}{8}\left(\hbar - m\omega[x,\Theta_\lambda] + i\,[p,\Theta_\lambda]\right), \end{aligned}$$

$g_\pm(\eta,\zeta) = (\eta \pm 2\zeta)/4.$

Nonstandard q-deformed harmonic oscillator.

Exponential boson mapping

$$
\begin{aligned}
A_+ &= a^\dagger \\
A_- &\approx a + \lambda a^\dagger a + \frac{\lambda^2}{2} a^{\dagger 2} a
\end{aligned}
$$

Logarithmic boson mapping

$$
\begin{aligned}
A_+ &\approx a^\dagger + \frac{\lambda}{2} a^{\dagger 2} + \frac{\lambda^2}{3} a^{\dagger 3} a \\
A_- &= a
\end{aligned}
$$

Nonstandard q-deformed harmonic oscillator.

$$\begin{aligned}
\frac{H_{exp}}{(\hbar\omega/4)} \approx\ & \frac{\eta}{2}(p^2+q^2-1)+\zeta(q^2-p^2) \\
& +\frac{\lambda}{\sqrt{2}}\Big[\frac{\eta}{2}(3+3ip-q+p^2q-ip(p^2+q^2)) \\
& \qquad +\zeta(3-2q+p^2q+ipq^2+ip^3)\Big] \\
& +\frac{\lambda^2}{2}\Big[\frac{\eta}{4}(-3+6p^2+2ip(3-p^2-q^2)q-p^4+q^4) \\
& \qquad -\zeta\left(1+2(p^2+q^2)-2p^2q^2-p^4-q^4-4ipq\right)\Big]
\end{aligned}$$

Nonstandard q-deformed harmonic oscillator.

$$\Psi(q) = \mathrm{e}^{\mathbf{i}\alpha(q)}\phi(q), \qquad \alpha(q) = -\mathbf{i}\frac{\lambda}{6\sqrt{2}} q \left(3\frac{(34\zeta-\eta)}{(2\zeta-\eta)} - q^2\right)$$

Local gauge transformation:

$$-\frac{\hbar^2}{8m}(\eta - 2\zeta)\varphi''(q) + V(q)\varphi(q) = E\varphi(q).$$

$$\frac{V(q)}{(\hbar\omega/4)} = -\frac{\eta}{2} - \frac{(4\zeta-\eta)^2\lambda^2}{16(2\zeta-\eta)} - \frac{(\zeta+\eta)}{\sqrt{2}}\lambda q + \frac{q^2}{8}(4(\eta+\zeta) + \lambda^2(4\zeta-\eta)) + \frac{2\zeta+\eta}{2\sqrt{2}}\lambda q^3 - \frac{(2\zeta-\eta)}{16}\lambda^2 q^4.$$

Nonstandard q-deformed harmonic oscillator.

$$
\begin{aligned}
H'_{log} \approx\; & \frac{\hbar\omega}{4}\left\{\frac{\eta}{2}(p^2+q^2-1)+\zeta(q^2-p^2)\right. \\
& +\frac{\lambda}{2\sqrt{2}}\left[\frac{\eta}{2}(3+3ip-q+p^2q-ip(p^2+q^2))\right. \\
& \left.+3\zeta(1-ip+q-p^2q-ipq^2+i\frac{1}{3}p^3)\right] \\
& +\frac{\lambda^2}{12}\left[\eta(-3+6p^2+2ip(3-p^2-q^2)q-p^4+q^4)\right. \\
& -\frac{11}{4}\zeta\left(-3+6(p^2-q^2)+6p^2q^2-p^4-q^4\right. \\
& \left.\left.\left.+4ip(3-p^2+q^2)q)\right]\right\}.
\end{aligned}
$$

Nonstandard q-deformed harmonic oscillator.

$$\Psi(q) = \mathrm{e}^{\mathbf{i}\alpha(q)}\phi(q), \qquad \alpha(q) = -\mathbf{i}\frac{(\eta+6\zeta)\lambda}{12\sqrt{2}(2\zeta-\eta)} q(3-q^2)$$

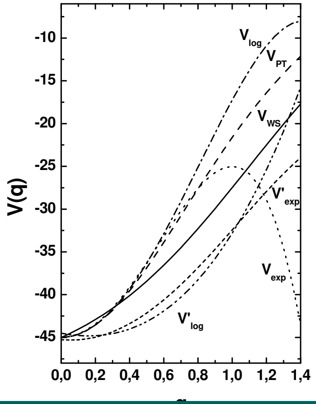
Local gauge transformation:

$$-\frac{\hbar^2}{8m}(\eta - 2\zeta)\varphi''(q) + V(q)\varphi(q) = E\varphi(q).$$

$$\frac{V(q)}{(\hbar\omega/4)} = \boxed{\begin{aligned} & -\frac{\eta}{2} - \frac{(6\zeta+\eta)^2}{64(2\zeta-\eta)}\lambda^2 - \frac{\eta}{2\sqrt{2}}\lambda q \\ & + \left(\zeta + \frac{\eta}{2} + \frac{(6\zeta+\eta)^2}{32(2\zeta-\eta)}\lambda^2\right) q^2 \\ & + \frac{2\zeta+\eta}{4\sqrt{2}}\lambda q^3 - \frac{(6\zeta+\eta)^2}{64(2\zeta-\eta)}\lambda^2 q^4 \end{aligned}}$$

Nonstandard q-deformed harmonic oscillator.

Results



Woods-Saxon Potential:

$$V(r) = \frac{V_0}{1 + e^{\frac{r-R}{a_0}}},$$

Poeschl-Teller Potential:

$$V_{PT}(r) = \frac{V_0}{\left(\cosh\left(\frac{r}{R}\right)\right)^2}.$$

Nonstandard q-deformed harmonic oscillator.

1. Nonstandard q-deformed oscillator $\Rightarrow$ non-hermitian hamitonian.
2. Local gauge transformation $\Rightarrow$ $V(q) = \sum c_n q^n$

More recently...

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Squeezed states from a quantum deformed oscillator Hamiltonian

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ABSTRACT

The spectrum and the time evolution of a system, which is modeled by a non-hermitian quantum deformed oscillator Hamiltonian, is analyzed. The proposed Hamiltonian is constructed from a non-standard realization of the algebra of Heisenberg. We show that, for certain values of the coupling constants and for a range of values of the deformation parameter, the deformed Hamiltonian is a pseudo-hermitic Hamiltonian. We explore the conditions under which the Hamiltonian is similar to a Swanson Hamiltonian. Also, we show that the lowest eigenstate of the system is a squeezed state. We study the time evolution of the system, for different initial states, by computing the corresponding Wigner functions.

Swanson model and deformed harmonic oscillator...

Non-standard deformed Hamiltonian

$$H_\lambda = \eta\{A_+, A_-\} + \zeta(A_+^2 + A_-^2).$$

$$\begin{aligned}
[N, A_-] &= -A_- \\
[N, A_+] &= -\left(\mathrm{e}^{\lambda A_+} - 1\right)/\lambda \\
[A_-, A_+] &= \mathbf{M}\mathrm{e}^{\lambda A_+} \\
[\mathbf{M}, \cdot] &= 0
\end{aligned}$$

$$\begin{aligned}
A_+ &= a^\dagger, \\
A_- &= \delta e^{\lambda a^\dagger} a + \delta\beta z e^{\lambda a^\dagger}, \\
N &= \frac{e^{\lambda a^\dagger} - 1}{\lambda} a + \beta \frac{e^{\lambda a^\dagger} + 1}{2} \\
M &= \delta \mathbf{I},
\end{aligned}$$

Swanson model and deformed harmonic oscillator...

Boson realization of H_λ

$$H_\lambda = (\eta-\zeta)(p^2 - i\ \{p,\Theta_\lambda\}) + \zeta\Theta_\lambda^2 + (\eta+\zeta)(x^2 + \{x,\Theta_\lambda\}).$$

$$\begin{aligned} p &= \frac{i}{\sqrt{2}}(a^\dagger - a), \\ x &= \frac{1}{\sqrt{2}}(a^\dagger + a), \end{aligned} \qquad \Theta_\lambda = \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}{a^\dagger}^k a.$$

$O(\lambda^3)$: $H \approx H_0 + H_{res}$

$$\begin{aligned} H_0 &= -\lambda^2\zeta + 2(\eta+\zeta\lambda^2)(a^\dagger a + 1/2) + \lambda(\eta a^\dagger + \zeta a) \\ &\quad +\zeta a^2 + (\zeta + \eta\lambda^2/2)a^{\dagger 2}, \\ H_{res} &= 2\lambda(\eta a^{\dagger 2}a + \zeta a^\dagger a^2) + \lambda^2(2\zeta a^{\dagger 2}a^2 + \eta a^{\dagger 3}a). \end{aligned}$$

Swanson model and deformed harmonic oscillator...

Quadratic Hamiltonian H_0 (Swanson Model)

$$H_0 = \omega(\widetilde{a}^\dagger \widetilde{a} + \frac{1}{2}) + \alpha \widetilde{a}^2 + \beta(\widetilde{a}^\dagger)^2 + H_{00},$$

$$\begin{aligned} H_0 &= V H_{SW} V^{-1}, \quad V = \exp\left(-\frac{a_1 - a_0}{2} x\right) \exp\left(-i\frac{a_1 + a_0}{2} p\right), \\ H_{Sw} &= \omega\left(a^\dagger a + \frac{1}{2}\right) + \alpha a^2 + \beta {a^\dagger}^2 - H_{00}. \end{aligned}$$

$$\omega = 2\eta + 2\zeta\lambda^2, \qquad \alpha = \zeta, \quad \beta = \frac{\lambda^2\eta}{2} + \zeta, \qquad H_{00} = -5\zeta\lambda^2/4,$$

$$a_0 = -\frac{2\eta^2\lambda - 2\zeta^2\lambda}{2(2\eta^2 - 2\zeta^2 + 3\eta\zeta\lambda^2)}, \quad a_1 = -\frac{1}{2(2\eta^2 - 2\zeta^2 + 3\eta\zeta\lambda^2)}, \quad .$$

Swanson model.

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Swanson model and deformed harmonic oscillator...

$h = h^\dagger$,
$h = W H_{SW} W^{-1}, W = \exp\left(-\frac{\alpha-\beta}{4} p^2\right) \Rightarrow h = \Upsilon H_0 \Upsilon^{-1}, \Upsilon = W V^{-1}$.

$$h = \gamma\{a^\dagger, a\} + \varrho({a^\dagger}^2 + a^2) - H_{00},$$

$$\gamma = \frac{1}{4}\left(\omega + \alpha + \beta + \frac{\sqrt{\omega^2 - 4\alpha\beta}}{\omega + \alpha + \beta}\right),$$
$$\varrho = \frac{1}{4}\left(\omega + \alpha + \beta - \frac{\sqrt{\omega^2 - 4\alpha\beta}}{\omega + \alpha + \beta}\right).$$

Swanson model and deformed harmonic oscillator...

Eigenvalues and Eigenstates

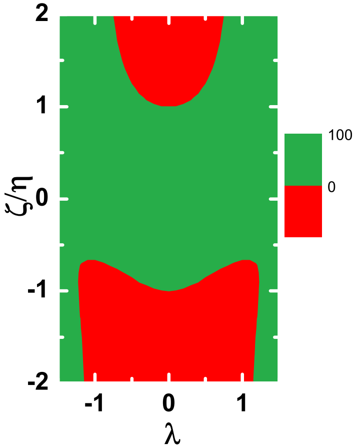
$$H_0 = H_{00} + \Omega\,\left(\tilde{c}\tilde{d} + \frac{1}{2}\right), \quad \Omega = \sqrt{D}, \quad D = \omega^2 - 4\alpha\beta.$$

$$\begin{aligned} \tilde{d} &= g_4\,\tilde{a} - g_2\,\tilde{a}^\dagger, \\ \tilde{c} &= -g_3\,\tilde{a} + g_1\,\tilde{a}^\dagger, \qquad \left[\tilde{d}, \tilde{c}\right] = 1, \quad c \neq d^\dagger. \end{aligned}$$

$$H_0|\widetilde{\Phi}\rangle = E_n|\widetilde{\Phi}\rangle, \quad E_n = \left(n + \frac{1}{2}\right)\Omega, \qquad |\widetilde{\Phi}\rangle = \frac{1}{\sqrt{n!}}\tilde{c}^n|0_{\tilde{d}}\rangle.$$

$$|0_{\tilde{d}}\rangle = \mathcal{N}_d\ \exp\left(\tau\tilde{a}^{\dagger 2}\right)|0\rangle, \quad \tilde{a}|0\rangle = 0, \quad \tau = (\Omega - \omega)/(4\alpha).$$

Swanson model and deformed harmonic oscillator...



$$\frac{3\lambda^2 - 4r}{4\left(1-\lambda^4\right)} < \frac{\zeta}{\eta} < \frac{3\lambda^2 + 4r}{4\left(1-\lambda^4\right)}$$

$$r = \sqrt{1 - \frac{7\lambda^4}{16}}$$

Swanson model and deformed harmonic oscillator...

Pseudo-hermitian Hamiltonian.

$$h = \Upsilon H \Upsilon^{-1}, \qquad h^\dagger = {\Upsilon^{-1}}^\dagger H^\dagger \Upsilon^\dagger, \qquad h = h^\dagger \Rightarrow \quad H^\dagger U = UH, \quad U = \Upsilon^\dagger \Upsilon$$

Bi-orthogonality.

$$\begin{array}{lll} H|\widetilde{\psi}_{\rangle} = & E_n|\widetilde{\psi}\rangle, & E_n \in \mathbb{R} \\ H^\dagger|\overline{\psi}_m\rangle = & E_m|\overline{\psi}_m\rangle, & |\overline{\psi}_m\rangle = U|\widetilde{\psi}_m\rangle, \quad \langle\overline{\psi}_m|\widetilde{\psi}_n\rangle = \delta_{nm}. \end{array}$$

Swanson model and deformed harmonic oscillator...

Inner Product

$$\langle\, .|.\,\rangle_U : \mathcal{H}\times\mathcal{H}\to \boldsymbol{C}, \quad \langle\psi|\phi\rangle_U := \langle\psi U|\phi\rangle,$$

$$\langle\widetilde{\psi}_\alpha|\widetilde{\psi}_\beta\rangle_U = \langle\overline{\psi}_\alpha|\widetilde{\psi}_\beta\rangle = \delta_{\alpha\beta}, \quad I = \sum_\alpha |\widetilde{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| = \sum_\alpha |\overline{\psi}_\alpha\rangle\langle\widetilde{\psi}_\alpha|$$

Mean Value Observables, $o = o^\dagger$.

$$\hat{O} = \Upsilon^{-1}\hat{o}\Upsilon, \quad o = o^\dagger.$$

$$\langle\widetilde{\psi}|\hat{O}|\widetilde{\phi}\rangle_U = \langle\widetilde{\psi}|U\hat{O}|\widetilde{\phi}\rangle = \langle\widetilde{\psi}|\Upsilon^\dagger\hat{o}\Upsilon|\widetilde{\phi}\rangle.$$

Swanson model and deformed harmonic oscillator...

Time Evolution.

$$|\widetilde{I}\rangle = \sum_k c_k\, |\widetilde{\Phi}_k\rangle \Rightarrow \qquad |\widetilde{I}(t)\rangle = \mathrm{e}^{-iHt}|\widetilde{I}\rangle = \sum_k c_k\, \mathrm{e}^{-iE_k t}\, |\widetilde{\Phi}_k\rangle,$$

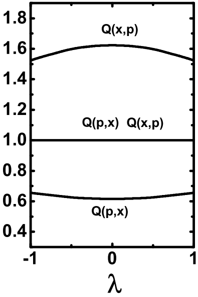
$$|\bar{I}\rangle = U|\widetilde{I}\rangle \sum_k c_k\, |\overline{\Phi}_k\rangle \Rightarrow \quad |\bar{I}(t)\rangle = \mathrm{e}^{-iH^\dagger t}|\bar{I}\rangle = \sum_k c_k\, \mathrm{e}^{-iE_k t}\, |\overline{\Phi}_k\rangle.$$

$$\begin{aligned} \langle \bar{I}(t)|O|\widetilde{I}(t)\rangle &= \langle \widetilde{I}(0)|U\mathrm{e}^{iHt}O\mathrm{e}^{-iHt}|\widetilde{I}(0)\rangle = \\ &= \sum_{n,m} c_n c_m^* e^{i(E_m-E_n)t}\langle \widetilde{\Phi}_m|\Upsilon^\dagger o \Upsilon|\widetilde{\Phi}_n\rangle. \end{aligned}$$

Swanson model and deformed harmonic oscillator...

Uncertainty Relations and Squeezing.

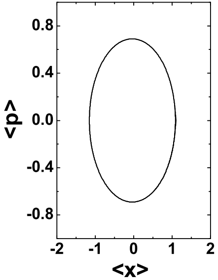
$$\Delta_U^2(\hat{O}) \;=\; \langle\widetilde{\Phi}_n|\Upsilon^\dagger\,\hat{o}^2\,\Upsilon|\widetilde{\Phi}_n\rangle - (\langle\widetilde{\Phi}_n|\Upsilon^\dagger\,\hat{o}\,\Upsilon|\widetilde{\Phi}_n\rangle)^2.$$

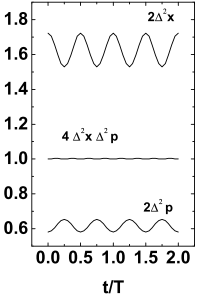


$$\begin{aligned} X &= \Upsilon^{-1} x \Upsilon \\ P &= \Upsilon^{-1} p \Upsilon \\ Q(x,p) &= 2\Delta_U^2 \hat{\mathrm{X}}, \\ Q(p,x) &= 2\Delta_U^2 \hat{\mathrm{P}} \end{aligned}$$

Swanson model and deformed harmonic oscillator...

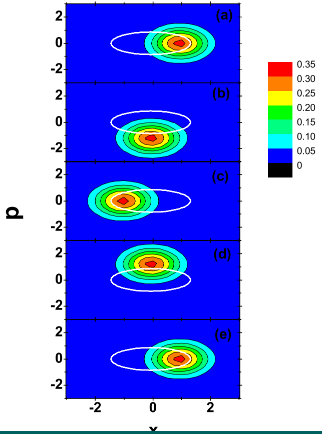
Time Evolution: $|I(0)\rangle = |GZ\rangle = \mathcal{N} \sum \frac{z_\gamma^n}{\sqrt{\rho_n}} \mathrm{e}^{i\gamma E_n} |\widetilde{\Phi}_n\rangle$.





Swanson model and deformed harmonic oscillator...

Wigner Function.



$$W(x,p,t) = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}py}\,\langle \widetilde{I}(t)|\Upsilon^{\dagger}\,|x-\frac{y}{2}\rangle\,\langle x+\frac{y}{2}\,|\Upsilon|\widetilde{I}(t)\rangle\,\mathrm{d}y,$$

$$\int W(x,p,t)\,\mathrm{d}x\,\mathrm{d}p = 1.$$

Swanson model and deformed harmonic oscillator...

General Hamiltonian.

$$
\begin{aligned}
H &= \frac{\zeta\lambda^2}{2} + (\eta - \zeta\lambda^2)(x^2+p^2) + \zeta(x^2-p^2) \\
&+ \frac{\eta\lambda^2}{4}(x^2-p^2)(x^2+p^2) - \mathrm{i}\frac{\eta\lambda^2}{4}\{x,p\}(x^2+p^2) \\
&+ \frac{\lambda\eta}{\sqrt{2}}(x-\mathrm{i}\,p)(x^2+p^2) + \frac{\lambda\zeta}{\sqrt{2}}(x^2+p^2)(x+\mathrm{i}\,p) \\
&+ \zeta\frac{\lambda^2}{2}(x^2+p^2)(x^2+p^2)
\end{aligned}
$$

Swanson model and deformed harmonic oscillator...

Pseudo-hermitian hamiltonian.

$$h = \Upsilon H \Upsilon^{-1}, \quad \Upsilon = \mathrm{e}^{-F(p)} \mathrm{e}^{-G(x)},$$

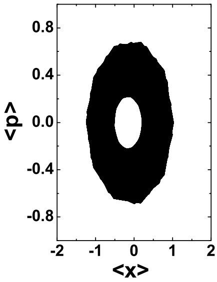
$$\begin{aligned} G(x) &= g_1(\theta,\lambda)x^2 + g_2(\theta,\lambda)x^3 + g_3(\theta,\lambda)x^4, \\ F(p) &= f_1(\theta,\lambda)p^2 - \mathbf{i} f_2(\theta,\lambda)p^3 + f_3(\theta,\lambda)p^4. \end{aligned}$$

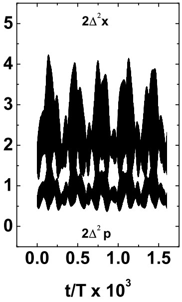
$h = \Upsilon H \Upsilon^{-1}$:

$$\begin{aligned} h &= h_0(\theta,\lambda) + h_1(\theta,\lambda)p^2 + h_2(\theta,\lambda)x^2 + h_3(\theta,\lambda)p^4 + \\ & h_4(\theta,\lambda)x^4 + h_5(\theta,\lambda)\{x^2,p^2\} + h_6(\theta,\lambda)\{x,p^2\} + \\ & h_7(\theta,\lambda)x + h_8(\theta,\lambda)x^3, \end{aligned}$$

Swanson model and deformed harmonic oscillator...

Time Evolution: $|I(0)\rangle = |GZ\rangle = \mathcal{N}\sum \frac{z_\gamma^n}{\sqrt{\rho_n}} \mathrm{e}^{i\gamma E_n}|\widetilde{\Phi}_n\rangle.$





and now? ...

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Dynamics of f nite dimensional non-hermitian systems with indef nite metric

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ABSTRACT

We discuss the time evolution of physical f nite dimensional systems which are modelled by non-hermitian Hamiltonians. We address both general non-hermitian Hamiltonians and pseudo-hermitian ones. We apply the theory of Krein Spaces to construct metric operators and well-def ned inner products. As an application, we study the stationary behavior of dissipative one axis twisting Hamiltonians. We discuss the effect of decoherence under different coupling schemes.

Figura

PT-symmetry phase ...

Squeezing Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right) + \hbar\alpha \,(a^2 + \; {a^\dagger}^2),$$

Swanson Hamiltonian

$$H(\omega, \alpha, \beta) = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right) + \hbar\alpha \; a^2 + \hbar\beta \; {a^\dagger}^2.$$

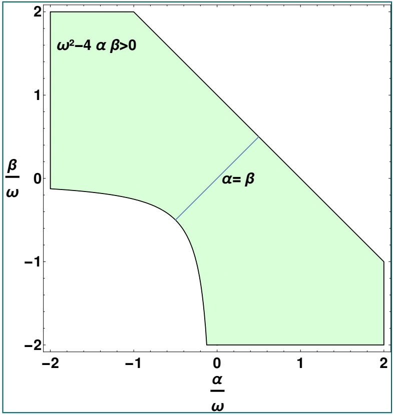


Figura: PT-symmetry phase

Beyond PT-symmetry phase...

$$\begin{aligned} \hat{x} &= \frac{b_0}{\sqrt{2}}\left(a^\dagger + a\right) \\ \hat{p} &= \frac{\mathbf{i}\hbar}{\sqrt{2b_0}}\left(a^\dagger - a\right) \end{aligned}$$

$H(\omega, \alpha, \beta)$ & $H_c(\omega, \alpha, \beta)$:

$$\begin{aligned} H(\omega, \alpha, \beta) &= \frac{1}{2}\hbar(\omega + \alpha + \beta)\left(\frac{\hat{x}}{b_0}\right)^2 + \hbar\frac{(\alpha - \beta)}{2}\left(2\,\hat{x}\,\frac{\mathbf{i}}{\hbar}\hat{p} + 1\right) \\ &\quad + \frac{1}{2}\hbar(\omega - \alpha - \beta)\left(\frac{b_0\,\hat{p}}{\hbar}\right)^2 \end{aligned}$$

$H_c(\omega, \alpha, \beta) = H(\omega, \beta, \alpha)$.

Beyond PT-symmetry phase...

$H = \frac{1}{2m}\, \hat{P}^2 \phi(X) + \frac{k}{2}\, \hat{X}^2 \phi(X)$

$$
\begin{aligned}
\hat{P} &= \left(\hat{p} + \mathbf{i}\hbar \frac{\alpha - \beta}{(\omega - \alpha - \beta) b_0^2} \hat{x}\right), \\
\hat{X} &= \hat{x},
\end{aligned}
$$

$$
\begin{aligned}
m = m(\omega, \alpha, \beta, b_0) &= \frac{\hbar}{(\omega - \alpha - \beta) b_0^2} \\
\Omega = \Omega(\omega, \alpha, \beta) &= \sqrt{\omega^2 - 4\alpha\beta} = |\Omega| \mathrm{e}^{\mathbf{i}\phi}, \\
k &= m\, \Omega^2
\end{aligned}
$$

Beyond PT-symmetry phase...

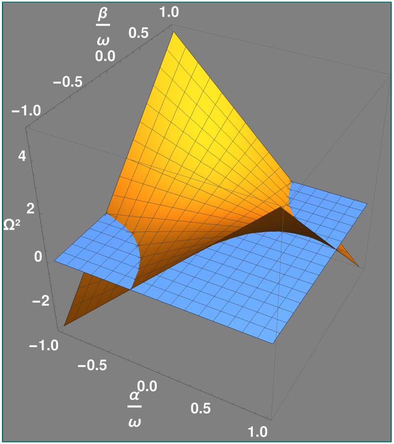


Figura: $\Omega^2 = \omega^2(1 - 4\frac{\alpha\beta}{\omega^2})$

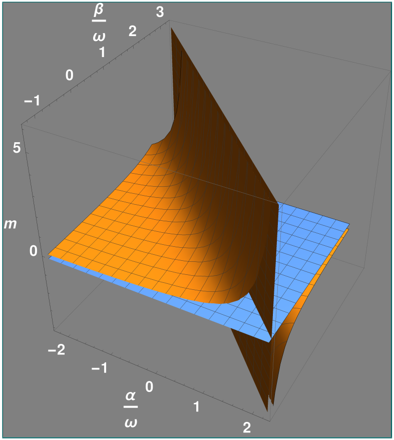


Figura: $m = \frac{\hbar}{b_0^2\omega} \frac{1}{(1-\frac{\alpha}{\omega}-\frac{\beta}{\omega})}$

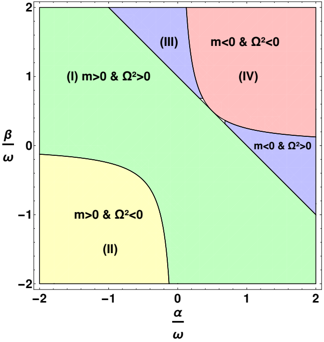
Beyond PT-symmetry phase...

-Region I: $m > 0$ and $\Omega^2 > 0$

-Region III: $m < 0$ and $\Omega^2 > 0$

-Region II: $m > 0$ and $\Omega^2 < 0$

-Region IV: $m < 0$ and $\Omega^2 < 0$



Beyond PT-symmetry phase...

Defining...

$$\sigma = \left(\frac{m\Omega}{\hbar}\right)^{1/2} b_0 = \mathrm{e}^{\mathbf{i}\gamma^{\pm}}|\sigma|.$$

$$\Omega = \mathrm{e}^{\mathbf{i}\phi}|\Omega|.$$

	sg(m)	sg(Ω^2)	γ	ϕ
I	+	+	0	0
			$\pi/2$	π
III	-	+	$\pi/2$	0
			0	π
II	+	-	$\pi/4$	$\pi/2$
			$-\pi/4$	$-\pi/2$
IV	-	-	$-\pi/4$	$\pi/2$
			$\pi/4$	$-\pi/2$

$$\Omega = \mathrm{e}^{\mathbf{i}\phi}|\Omega|$$

Beyond PT-symmetry phase.

Region II: Parabolic Barrier

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Region III: Harmonic Oscillator with effective negative mass

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F. Di Mei et al., Phys. Lett. **116**, (2016)153902.

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Rigged Hilbert Space...

Gelfand Triplet...

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}$$

Φ dense in $\mathcal{H}$

$\mathcal{H}$: Hilbert Sapce

$\Phi^{\times} = \{F \mid F : \Phi \to \mathbb{C},\ F(\phi) = \langle \phi | F \rangle\}$

Φ depends on the problem, i.e. $\mathcal{D}$ or $\mathcal{S}$.

Rigged Hilbert Space.

**Example: $f_n^{\pm}$ generalized functions.
D. Chruciski, J. Math. Phys. 44 (2003)3718.**

$$H^{\times} = -\frac{\zeta}{2}(\hat{u}\hat{v} + \hat{v}\hat{u}) = \mathbf{i}\zeta \left(u\frac{\mathrm{d}}{\mathrm{du}} + \frac{1}{2} \right), \; [\hat{u}, \hat{v}] = \mathbf{i}$$

$$\hat{u} f_0^- = 0 \qquad \hat{v} f_0^+ = 0, \qquad H^{\times} f_0^{\pm} = \pm\mathbf{i}\frac{\zeta}{2} f_0^{\pm}$$

$$f_0^-(u) = \delta(u) \qquad f_0^+(u) = 1.$$

$$f_n^- = \frac{(-\mathbf{i})^n}{\sqrt{n!}} \hat{v}^n f_0^- \qquad f_n^+ = \frac{1}{\sqrt{n!}} \hat{u}^n f_0^+, \qquad H^{\times} f_n^{\pm} = \pm\mathbf{i}\zeta(n + \tfrac{1}{2}) f_n^{\pm}$$

$$f_n^-(u) = \frac{(-\mathbf{i})^n}{\sqrt{n!}} \delta^{(n)}(u) \qquad f_n^+(u) = \frac{(1)^n}{\sqrt{n!}} u^n.$$

Rigged Hilbert Space.

Properties of $f_n^{\pm}$

$$\int_{-\infty}^{\infty} f_n^{+}(u) f_m(u) \mathrm{du} = \delta_{nm}$$

$$\sum_{n=0}^{\infty} f_n^{+}(u) f_n(u') = \delta(u - u').$$

For $\phi \in \mathcal{D}$:

$$|\phi\rangle = \sum_{n=0}^{\infty} |f_n^{+}\rangle\langle f_n^{-}|\phi\rangle.$$

Rigged Hilbert Space.

Continuous Spectrum:

$$u\frac{\mathrm{d}}{\mathrm{du}}\Psi_{\pm}^{E} = -\left(\mathbf{i}\frac{E}{\zeta}+\frac{1}{2}\right)\Psi_{\pm}^{E},$$

$$\Psi_{\pm}^{E}(u) = \frac{1}{\sqrt{2\pi\zeta}}u_{\pm}^{-\left(\mathbf{i}\frac{E}{\zeta}+\frac{1}{2}\right)}$$

with the distribution:

$$s_{+}^{\lambda} = \begin{cases} s^{\lambda} & s \geq 0 \\ 0 & s < 0 \end{cases}$$

$$s_{-}^{\lambda} = \begin{cases} 0 & s \geq 0 \\ |s|^{\lambda} & s < 0 \end{cases}$$

$$\int_{-\infty}^{\infty}\Psi_{\pm}^{E_1}(u)^{*}\Psi_{\pm}^{E_2}(u)\mathrm{du} = \delta(E_1 - E_2)$$

$$\int_{-\infty}^{\infty}\Psi_{\pm}^{E}(u)^{*}\Psi_{\pm}^{E}(u')\mathrm{dE} = \delta(u-u').$$

$$\phi(u) = \sum_{\pm}\int\Psi_{\pm}^{E}(u)\langle\Psi_{\pm}^{E}(u)|\phi\rangle\mathrm{dE}.$$

Beyond PT-symmetry phase...

Gauge Transformation.

$$H^{\times}\ \phi(x) = E\ \phi(x). \qquad H_c^{\times}\ \psi(x) = E_c\ \psi(x).$$

$$\phi(x) = \mathrm{e}^{\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\ f(x), \qquad \psi(x) = \mathrm{e}^{\frac{\beta-\alpha}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\ g(x),$$

$$\hat{y} \ = \ y(x,\gamma) = |\sigma|\mathrm{e}^{\mathbf{i}\gamma}\frac{x}{b_0}, \qquad \hat{p}_y \ = \ \mathbf{i}\frac{\mathrm{d}}{\mathrm{dy}}.$$

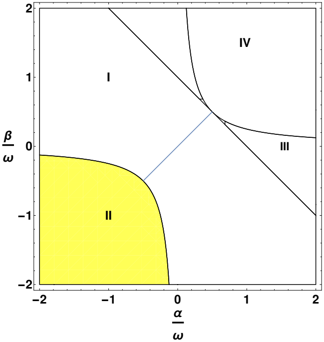
$$\hbar\Omega\frac{1}{2}\left(\hat{p}_y^2 + \hat{y}^2\right) f(y) = Ef(y), \qquad \hbar\Omega\frac{1}{2}\left(\hat{p}_y^2 + \hat{y}^2\right) g(y) = E_c g(y),$$

Beyond PT-symmetry phase...

Region II.

$$\Omega = \pm \mathbf{i}|\Omega|$$

$$\sigma = b_0\sqrt{\frac{m\Omega}{\hbar}} = e^{\pm \mathbf{i}\pi/4}|\sigma| \Rightarrow \gamma = \pm\pi/4.$$



Beyond PT-symmetry phase...

Swanson model

$$\frac{1}{2}\left(\hat{p}_y^2+\hat{y}^2\right) f(y) = E\, f(y), \qquad \frac{1}{2}\left(\hat{p}_y^2+\hat{y}^2\right) g(y) = E_c\, g(y),$$

$$u = u(\gamma) = \frac{y-\mathbf{i}p_y}{\sqrt{2}}, \qquad v = v(\gamma) = \frac{y+\mathbf{i}p_y}{\sqrt{2}}$$

$$\overline{u} = u(-\gamma) = \frac{\overline{y}-\mathbf{i}\hat{p}_{\overline{y}}}{\sqrt{2}}, \qquad \overline{v} = v(-\gamma) = \frac{\overline{y}+\mathbf{i}\hat{p}_{\overline{y}}}{\sqrt{2}}$$

$$\overline{u}^* = v, \quad \overline{v}^* = u.$$

$$\frac{\hbar\Omega}{2}\left(uv+vu\right) f(u) = E\, f(u), \qquad \frac{\hbar\Omega}{2}\left(uv+vu\right) g(u) = E_c\, g(u).$$

Beyond PT-symmetry phase...

$$\frac{\hbar\Omega}{2}\,(uv+vu)\,f = E\ f$$

$\gamma = \pi/4$:

$$\boxed{\begin{array}{llll} f_n^+(u) = \frac{1}{\sqrt{n!}}u^n & & E_n = & \mathbf{i}E_0[n] \\ f_n^-(u) = \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(u) & & E_n = & -\mathbf{i}E_0[n] \end{array}}$$

$\gamma = -\pi/4$:

$$\begin{array}{llll} f_n^+(\overline{u}) = \frac{1}{\sqrt{n!}}\overline{u}^n & & E_n = & -\mathbf{i}E_0[n] \\ f_n^-(\overline{u}) = \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(\overline{u}) & & E_n = & \mathbf{i}E_0[n] \end{array}$$

$$\frac{\hbar\Omega}{2}\,(uv+vu)\,g = E^c\ g.$$

$\gamma = \pi/4$:

$$\begin{array}{llll} g_n^+(v) = \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(v) & & E_n^c = & \mathbf{i}E_0[n] \\ g_n^-(v) = \frac{1}{\sqrt{n!}}v^n & & E_n^c = & -\,\mathbf{i}E_0[n] \end{array}$$

$\gamma = -\pi/4$:

$$\boxed{\begin{array}{llll} g_n^+(\overline{v}) = \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(\overline{v}^n) & & E_n = & -\mathbf{i}E_0[n] \\ g_n^-(\overline{v}) = \frac{1}{\sqrt{n!}}\overline{v}^n & & E_n = & \mathbf{i}E_0[n] \end{array}}$$

$$\langle g_m^{\pm}(u)|f_n^{\pm}(u)\rangle = \delta_{nm}$$

$$\langle g_m^{\pm}(v)|f_n^{\pm}(v)\rangle = \delta_{nm}$$

Beyond PT-symmetry phase.

$$S(y,u) = \tfrac{1}{2}y^2 - \sqrt{2}yu + \tfrac{1}{2}u^2, \qquad S_c(y_c,v_c) = \tfrac{1}{2}y_c^2 - \sqrt{2}y_c v_c + \tfrac{1}{2}v_c^2.$$

$$\frac{\partial S}{\partial y} = p_y, \quad \frac{\partial S}{\partial u} = -v, \qquad \frac{\partial S_c}{\partial y_c} = p_{y_c}, \quad \frac{\partial S_c}{\partial v_c} = -u_c.$$

$$f_n^+(y) = \mathcal{C}\int_\Gamma f_n^+(u)\mathrm{e}^{S(y,u)}\mathrm{d}u, \qquad g_n^-(\overline{y}_c) = \mathcal{C}\int_\Gamma g_n^-(\overline{v}_c)\mathrm{e}^{-S_c(\overline{y}_c,\overline{v}_c)}\mathrm{d}\overline{v}_c,$$

$$g_n^+(y) = f_n^+(y)^*, \qquad f_n^-(y) = g_n^-(y)^*.$$

$$\mathcal{C}\int_\Gamma \mathrm{e}^{S(y,u)-S_c(\overline{y}'_c,\overline{v}_c)^*}\mathrm{d}u = \mathcal{C}\,\delta(y-y'),\ \overline{v}_c = v_c(-\gamma).$$

Beyond PT-symmetry phase...

$$E_n^{\pm} = \pm \mathbf{i}\hbar\,|\Omega|\left(n+\frac{1}{2}\right), \qquad E^{c\pm}_{n} = \mp \mathbf{i}\hbar\,|\Omega|\left(n+\frac{1}{2}\right),$$

$$y(x,\gamma) = \mathrm{e}^{\mathbf{i}\gamma^{\pm}}\,\frac{x}{b_0}|\sigma|, \qquad y_c(x,\gamma) = \mathrm{e}^{-\mathbf{i}\gamma^{\pm}}\,\frac{x}{b_0}|\sigma|,$$

$$f_n^{\pm}(y) \propto 2^{-\frac{n}{2}}\mathrm{e}^{-\frac{y^2}{2}|\sigma|^2}H_n(y), \qquad g_n^{\pm}(y_c) \propto 2^{-\frac{n}{2}}\mathrm{e}^{-\frac{y_c^2}{2}}\,H_n(y_c),$$

$$\langle \Psi_n^{\pm}|\phi_m^{\pm}\rangle = \delta_{mn}, \qquad \sum_{\substack{n=0\\ \sigma=\pm}}^{\infty} \psi_n^{\sigma}(x)^*\phi_m^{\sigma}(x') = \delta(x-x').$$

$$\phi_m^{\pm}(x) = \mathrm{e}^{\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\,f_m^{\pm}(x), \qquad \psi_n^{\pm}(x) = \mathrm{e}^{\frac{\beta-\alpha}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\,g_n^{\pm}(x)$$

Beyond PT-symmetry phase...

Region II: Continuous Spectrum.

$$f^{\nu}_{\pm}(u) = u^{\nu}_{\pm},\ f^{\nu}_{\pm}(\overline{u})^* = v^{\nu}_{c\pm}, \qquad g^{\nu}_{\pm}(\overline{v}_c)^* = u_{\pm}^{-(\nu+1)},\ g^{\nu}_{\pm}(v_c) = v_{c\pm}^{-(\nu+1)}.$$

$$\langle g^{\nu'}_{\pm}(\overline{v}_c)|f^{\nu}_{\pm}(u)\rangle = \delta_{\nu\nu'} \qquad \nu = -\mathbf{i}\frac{E}{\hbar|\Omega|}, \quad \nu^* = -(\nu+1).$$

$$\begin{aligned} f^{E}_{+}(y) &= \mathcal{C}\int_{\Gamma} u^{\nu}\mathrm{e}^{S(y,u)}\mathrm{du} \\ &= \mathcal{C}\Gamma(\nu+1)\ D_{-\nu-1}(-\mathbf{i}\sqrt{2}y) \\ f^{E}_{-}(y) &= \mathcal{C}\Gamma(\nu+1)\ D_{-\nu-1}(\mathbf{i}\sqrt{2}y) \end{aligned} \qquad \begin{aligned} g^{E}_{+}(\overline{y}_c) &= \mathcal{C}\int_{\Gamma} \overline{v}_c^{\nu}\mathrm{e}^{-S(\overline{y}_c,\overline{v}_c)}\mathrm{d}\overline{v}_c \\ &= \mathcal{C}\ \Gamma(\nu+1)\ D_{-\nu-1}(-\sqrt{2}\overline{y}_c) \\ g^{E}_{-}(\overline{y}_c) &= \mathcal{C}\ \Gamma(\nu+1)\ D_{-\nu-1}(\sqrt{2}\overline{y}_c), \end{aligned}$$

$$\int_{-\infty}^{\infty} g^{E}_{\pm}(y^*(x))^*\, f^{E'}_{\pm}(y(x))\mathrm{dx} = \delta(E-E'),$$

$$\int_{-\infty}^{\infty} g^{E}_{\pm}(y^*(x))^*\, f^{E}_{\pm}(y(x))\mathrm{dE} = \delta(x-x'),$$

Beyond PT-symmetry phase...

Region II: Continuous Spectrum.

$$f^E_\pm(y) \propto \Gamma(\nu+1)\, D_{-\nu-1}(\mp \mathbf{i}\sqrt{2}y), \quad g^E_\pm(y^*)^* \propto \Gamma(-\nu)\, D_\nu(\mp\sqrt{2}y)$$

Poles of $f^E_\pm \rightarrow E_n = -\mathbf{i}\hbar\Omega(n+1/2),\ f^+_n \rightarrow \mathrm{e}^{-\mathbf{i}E_n t} = \mathrm{e}^{-\hbar\Omega(n+1/2)t}$
Poles of $g^{E\,*}_\pm \rightarrow E_n = +\mathbf{i}\hbar\Omega(n+1/2),\ f^-_n \rightarrow \mathrm{e}^{-\mathbf{i}E_n t} = \mathrm{e}^{\hbar\Omega(n+1/2)t}$
(Poles $\Gamma(\lambda),\ \lambda = -n$.)

$$\Phi_- = \{\phi_- | \phi_- = \langle\phi|g^E_\pm\rangle \in \mathcal{H}_-\},\ \ \Phi_+ = \{\phi_+ | \phi_+ = \langle\phi|f^E_\pm\rangle \in \mathcal{H}_+\}$$

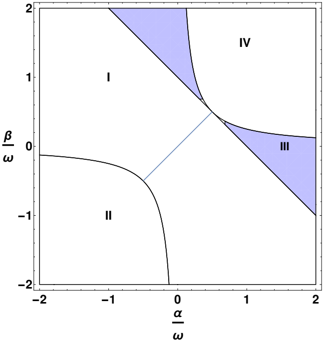
$$\begin{aligned} \phi_-(x,t) &= \sum_n \mathrm{e}^{+\hbar\Omega(n+1/2)t}\langle\phi^-|f_n^{\rangle^* f_n^+(x)} \\ \phi_+(x,t) &= \sum_n \mathrm{e}^{-\hbar\Omega(n+1/2)t}\langle\phi^+|f^+_n\rangle^* f^-_n(x) \end{aligned}$$

Beyond PT-symmetry phase...

Region III:

$$\Omega = \pm|\Omega|, \quad m = -|m|$$

$$\sigma = b_0\sqrt{\frac{m\Omega}{\hbar}} = e^{\mathrm{i}\gamma}|\sigma| \Rightarrow \gamma = 0,\ \pi/2.$$



Beyond PT-symmetry phase...

$\gamma = 0$:

$$
\begin{aligned}
E_n &= -\hbar|\Omega|\left(n+\frac{1}{2}\right) \\
f_n^+(y) &\propto \mathrm{e}^{-\frac{y^2}{2}} H_n(y) \\
g_n^+(y) &= f_n^+(y)^*.
\end{aligned}
$$

$\gamma = \pi/2$:

$$
\begin{aligned}
E_n &= \hbar|\Omega|\left(n+\frac{1}{2}\right) \\
f_n^-(y) &\propto \mathrm{e}^{-\frac{y^2}{2}} H_n(y) \\
g_n^-(y) &= f_n^-(y)^*.
\end{aligned}
$$

$$\phi_m^{\pm}(x) = \mathrm{e}^{\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}} f_m^{\pm}(x), \qquad \psi_n^{\pm}(x) = \mathrm{e}^{\frac{\beta-\alpha}{\omega-\alpha-\beta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}} g_n^{\pm}(x)$$

Beyond PT-symmetry phase...

$\omega = \alpha + \beta,\ \alpha. \neq \beta$.

$$H^{\times}(\theta) = \hbar(\alpha+\beta)\left(\frac{\hat{x}}{b_0}\right)^2 + \hbar\frac{(\alpha-\beta)}{2}\left(2\,\hat{x}\,\frac{\mathbf{i}}{\hbar}\hat{p} + 1\right),$$

$$H^{\times}_{c}(\theta) = \hbar(\alpha+\beta)\left(\frac{\hat{x}}{b_0}\right)^2 + \hbar\frac{(\beta-\alpha)}{2}\left(2\,\hat{x}\,\frac{\mathbf{i}}{\hbar}\hat{p} + 1\right).$$

$$\phi(x) = \mathrm{e}^{-\frac{\hat{x}^2}{4b_0^2}\frac{\alpha+\beta}{\alpha-\beta}}\,x^{-\frac{1}{2}+\frac{E}{\hbar(\alpha-\beta)}} \qquad \psi(x) = \mathrm{e}^{\frac{\hat{x}^2}{4b_0^2}\frac{\alpha+\beta}{\alpha-\beta}}\,x^{-\frac{1}{2}-\frac{E_c}{\hbar(\alpha-\beta)}},$$

Beyond PT-symmetry phase...

Free particle: $k = 0$ ($\Omega^2 = \omega^2 - 4\alpha\beta = 0$).

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = E\,\phi(x),$$

the wave function can be written as $\phi(x) = A\mathrm{e}^{ikx} + A\mathrm{e}^{-ikx}$, with $k = \sqrt{\frac{2\varepsilon}{\hbar(\omega-\alpha-\beta)b_0^2}}$.

Open questions

1. Metric operators
2. Exceptional Points?
3. Time Evolution
4. Work is in progress concerning Complex Scaling Method

Thanks!

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