

Legendre curves on 3-dimensional C_{12} -Manifolds

Gherici Beldjilali, Benaoumeur Bayour and Habib Bouzir

Abstract. Legendre curves play a very important and special role in geometry and topology of almost contact manifolds. There are certain results known for Legendre curves in 3-dimensional normal almost contact manifolds. The aim of this paper is to study Legendre curves of three-dimensional C_{12} -manifolds which are non-normal almost contact manifolds and classifying all biharmonic Legendre curves in these manifolds.

1 Introduction

In [8], D. Chinea and C. Gonzalez have defined 12 classes of almost contact metric manifolds. In dimension 3, these manifolds are reduced to five classes: $|C|$ class of cosymplectic manifolds, C_5 class of β -Kenmotsu manifolds, C_6 class of α -Sasakian manifolds, C_9 -manifolds and C_{12} -manifolds.

Only the last two classes can never be normal. For this reason, all work concerning curves on almost contact metric manifolds focuses on the first three classes. For example, Legendre curves on contact manifolds have been studied by C. Baikoussis and D.E. Blair in the paper [1]. M. Belkhefha et al. [3] have investigated Legendre curves in Riemannian and Lorentzian manifolds. Also, Legendre curves have been studied on Kenmotsu manifolds in [11], on α -Sasaki manifolds in [9], on quasi-Sasakian manifolds in [12], on Trans-Sasaki manifolds in [10] and others.

C_{12} -manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzalez [8] in 1990. Recently, these manifolds were studied

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and most of its properties were deduced in [6], especially [4]. As C_{12} -manifolds differs from the above almost contact metric manifolds, at least in the normality property, in the present paper, we will show that the results related to Legendre curves in 3-dimentional C_{12} -manifolds are fundamentally different from the results in other almost contact metric manifolds.

The present paper is organized as follows: After the introduction, we recall some required preliminaries on almost contact geometry and Frenet curves in general Riemannian geometry in Section 3. In Section 4, we give a very brief review of 3-dimentional C_{12} -manifolds with concrete example. The next section is focused on the study of Legendre curves on 3-dimentional C_{12} -manifolds. In the last section, we demonstrate a nice property of biharmonic Legendre curves in 3-dimentional C_{12} -manifolds.

2 Almost contact manifold

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exists on M a $(1, 1)$ -tensor field φ , a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \quad (2.1)$$

for any vector fields X, Y on M .

In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

The fundamental 2-form ϕ is defined by

$$\phi(X, Y) = g(X, \varphi Y).$$

It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0, \quad (2.2)$$

for any X, Y on M , where N_φ denotes the Nijenhuis torsion of φ , given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \quad (2.3)$$

For more background on almost contact metric manifolds, we recommend the references [2, 5, 13].

3 Legendre curves

Let (M, g) be a 3-dimensional Riemannian manifold with Levi-Civita connection ∇ and $\gamma : I \rightarrow M$ parameterized by the arc length. γ is said to be a Frenet curve if there exists an orthonormal frame $\{E_1 = \dot{\gamma}, E_2, E_3\}$ along γ such that

$$\begin{cases} \nabla_{\dot{\gamma}} E_1 = \kappa E_2, \\ \nabla_{\dot{\gamma}} E_2 = -\kappa E_1 + \tau E_3, \\ \nabla_{\dot{\gamma}} E_3 = -\tau E_2. \end{cases} \quad (3.1)$$

The curvature κ is defined by the formula

$$\kappa = |\nabla_{\dot{\gamma}} \dot{\gamma}|. \quad (3.2)$$

The second unit vector field E_2 is thus obtained by

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa E_2. \quad (3.3)$$

Next, the torsion τ and the third unit vector field E_3 are defined by the formulas

$$\tau = |\nabla_{\dot{\gamma}} E_2 + \kappa E_1| \quad \text{and} \quad \nabla_{\dot{\gamma}} E_2 + \kappa E_1 = \tau E_3. \quad (3.4)$$

A Frenet curve $\gamma : I \rightarrow M$ in an almost contact metric manifold is said to be a Legendre curve [1], if it is an integral curve of the contact distribution $\mathcal{D} = \ker \eta$. Formally, it is also said that a Frenet curve γ in an almost contact metric manifold is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$.

4 Three dimensional C_{12} -manifold

In the classification of D. Chinea and C. Gonzalez [8] of almost contact metric manifolds there is a class C_{12} -manifolds which can be integrable but never normal. In this classification, C_{12} -manifolds are defined by

$$(\nabla_X \phi)(Y, Z) = \eta(X)\eta(Z)(\nabla_\xi \eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi \eta)\varphi Z. \quad (4.1)$$

In [4] and [6], The $(2n + 1)$ -dimensional C_{12} -manifolds is characterized by:

$$(\nabla_X \varphi)Y = \eta(X)(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi), \quad (4.2)$$

for any X and Y vector fields on M , where $\omega = -(\nabla_\xi \xi)^\flat = -\nabla_\xi \eta$ and ψ is the vector field given by

$$\omega(X) = g(X, \psi) = -g(X, \nabla_\xi \xi),$$

for all X vector field on M .

Moreover, in [4] the $(2n + 1)$ -dimensional C_{12} -manifolds is also characterized by

$$d\eta = \omega \wedge \eta \quad d\phi = 0 \quad \text{and} \quad N_\varphi = 0. \quad (4.3)$$

Here, we emphasize that the almost C_{12} -manifolds is defined by the following:

Defintion 4.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact manifold. M is called almost C_{12} -manifold if there exists a one-form ω which satisfies

$$d\eta = \omega \wedge \eta \quad \text{and} \quad d\phi = 0.$$

In addition, if $N_\varphi = 0$ we say that M is a C_{12} -manifold.

In [4], the authors studied the 3-dimensional unit C_{12} -manifold i.e. the case where ω is closed and ψ is a unit vector field. We will deal here with the general case, i.e. ψ is not necessarily unitary. For that, taking $V = e^{-\rho}\psi$ where $e^\rho = |\psi|$, we get immediately that $\{\xi, V, \varphi V\}$ is an orthonormal frame. We refer to this basis as Fundamental basis.

Using this frame, one can get the following:

Proposition 4.2. For any C_{12} -manifold, for all vector field X on M we have

- 1) $\nabla_X \xi = -e^\rho \eta(X)V$
- 2) $\nabla_\xi V = e^\rho \xi$
- 3) $\nabla_V V = \varphi V(\rho)\varphi V$
- 4) $\nabla_\xi \varphi V = 0$
- 5) $\nabla_V \varphi V = -\varphi V(\rho)V$.

Proof. For the first, using (4.2) for $Y = \xi$ we get

$$\begin{aligned} (\nabla_X \varphi)\xi &= \eta(X)\varphi\psi \\ &= e^\rho \eta(X)\varphi V, \end{aligned}$$

knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$ and applying φ we obtain

$$\begin{aligned} \nabla_X \xi &= e^\rho \eta(X)\varphi^2 V \\ &= -e^\rho \eta(X)V. \end{aligned}$$

For the second, we have

$$\begin{aligned} 2d\omega(\xi, X) = 0 \Leftrightarrow g(\nabla_\xi \psi, X) &= g(\nabla_X \psi, \xi) \\ &= -g(\psi, \nabla_X \xi) \\ &= e^{2\rho} \eta(X), \end{aligned}$$

which gives $\nabla_\xi \psi = e^{2\rho} \xi$ and then

$$\begin{aligned} \nabla_\xi V &= \nabla_\xi (e^{-\rho} \psi) \\ &= -\xi(\rho)V + e^\rho \xi. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\xi(\rho) &= \frac{1}{2}e^{-2\rho}\xi(e^{2\rho}) \\ &= \frac{1}{2}e^{-2\rho}\xi(g(\psi, \psi)) \\ &= e^{-2\rho}g(\nabla_\xi\psi, \psi) = 0,\end{aligned}$$

then,

$$\nabla_\xi V = e^\rho \xi.$$

For $\nabla_V V$, we have

$$\begin{aligned}2d\omega(\psi, X) = 0 \Leftrightarrow g(\nabla_\psi\psi, X) &= g(\nabla_X\psi, \psi) \\ &= \frac{1}{2}Xg(\psi, \psi) \\ &= e^{2\rho}g(\text{grad}\rho, X),\end{aligned}$$

i.e. $\nabla_\psi\psi = e^{2\rho}\text{grad}\rho$ which gives $\nabla_V V = \text{grad}\rho - V(\rho)V$.

Also, we have

$$\begin{aligned}\text{grad}\rho &= \xi(\rho)\xi + V(\rho)V + \varphi V(\rho)\varphi V \\ &= V(\rho)V + \varphi V(\rho)\varphi V,\end{aligned}$$

then,

$$\nabla_V V = \varphi V(\rho)\varphi V.$$

For the rest, just use the formula $\nabla_X\varphi Y = (\nabla_X\varphi)Y + \varphi\nabla_X Y$ noting that $(\nabla_V\varphi)X = (\nabla_{\varphi V}\varphi)X = 0$. \square

It remains to count $\nabla_{\varphi V} V$ and $\nabla_{\varphi V}\varphi V$. For that, we have the following lemma

Lemma 4.3. *For any 3-dimensional C_{12} -manifold, we have*

$$1) \nabla_{\varphi V} V = (-e^\rho + \text{div}V)\varphi V,$$

$$2) \nabla_{\varphi V}\varphi V = (e^\rho - \text{div}V)V.$$

Proof. Since $\{\xi, V, \varphi V\}$ is an orthonormal frame then,

$$\nabla_{\varphi V} V = a \xi + b V + c \varphi V,$$

Using Proposition 4.2, we have

$$a = g(\nabla_{\varphi V} V, \xi) = -g(V, \nabla_{\varphi V} \xi) = 0$$

and $b = g(\nabla_{\varphi V} V, V) = 0$. for get c we have

$$\begin{aligned}\text{div}V &= g(\nabla_\xi V, \xi) + g(\nabla_{\varphi V}\psi, \varphi V) \\ &= e^\rho + g(\nabla_{\varphi\psi}\psi, \varphi\psi) \Leftrightarrow g(\nabla_{\varphi V} V, \varphi V) = -e^\rho + \text{div}V,\end{aligned}$$

then,

$$\nabla_{\varphi V} V = (-e^\rho + \operatorname{div} V) \varphi V.$$

Applying φ with (4.2), we obtain

$$\nabla_{\varphi V} \varphi V = (e^\rho - \operatorname{div} V) V.$$

□

According to the Proposition 4.2 and Lemma 4.3, the 3-dimensional C_{12} -manifold is completely controllable. That is:

Corollary 4.4. *For any C_{12} -manifold, we have*

$$\begin{aligned} \nabla_\xi \xi &= -e^\rho V, & \nabla_\xi V &= e^\rho \xi, & \nabla_\xi \varphi V &= 0, \\ \nabla_V \xi &= 0, & \nabla_V V &= \varphi V(\rho) \varphi V, & \nabla_V \varphi V &= -\varphi V(\rho) V, \\ \nabla_{\varphi V} \xi &= 0, & \nabla_{\varphi V} V &= (-e^\rho + \operatorname{div} V) \varphi V, & \nabla_{\varphi V} \varphi V &= (e^\rho - \operatorname{div} V) V. \end{aligned}$$

Example 4.5. *We denote the Cartesian coordinates in a 3-dimensional Euclidean space $M = \mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by*

$$g = e^{2y} \begin{pmatrix} 1 + \alpha^2 & 0 & -1 \\ 0 & \alpha^2 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

where $\alpha = \alpha(x, y) \geq 0$ every where is a function on \mathbb{R}^3 . Further, we define an almost contact metric (φ, ξ, η) on \mathbb{R}^3 by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \xi = e^{-y} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = e^y(-1, 0, 1).$$

The fundamental 1-form η and the 2-form ϕ have the forms,

$$\eta = e^y(dz - dx) \quad \text{and} \quad \phi = -2\alpha^2 e^{2y} dx \wedge dy,$$

and hence

$$\begin{aligned} d\eta &= e^y(dx \wedge dy + dy \wedge dz) = dy \wedge \eta, \\ d\phi &= 0. \end{aligned}$$

By a direct computation the non trivial components of $N_{kj}^{(1)i}$ are given by

$$N_{12}^{(1)3} = 1, \quad N_{23}^{(1)3} = 1.$$

i.e. $N^{(1)} \neq 0$. But, $\forall i, j, k \in \{1, 2, 3\}$

$$(N_\varphi)_{kj}^i = 0,$$

implying that (φ, ξ, η) becomes integrable non normal. We have $\omega = dy$ i.e. $d\omega = 0$ and knowing that ω is the g -dual of ψ i.e. $\omega(X) = g(X, \psi)$, we have immediately that

$$\psi = \frac{e^{-2y}}{\alpha^2} \frac{\partial}{\partial y}. \quad (4.4)$$

Thus, $(\varphi, \xi, \psi, \eta, \omega, g)$ is a 1-parameter family of C_{12} -structure on \mathbb{R}^3 . Notice that

$$|\psi|^2 = \omega(\psi) = g(\psi, \psi) = \frac{e^{-2y}}{\alpha^2}$$

implies $V = \frac{e^{-y}}{\alpha} \frac{\partial}{\partial y}$ is a unit vector field, then

$$\left\{ \xi = e^{-y} \frac{\partial}{\partial z}, \quad V = \frac{e^{-y}}{\alpha} \frac{\partial}{\partial y}, \quad \varphi V = \frac{e^{-y}}{\alpha} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) \right\}$$

form an orthonormal basis. To verify result in formula (4.2), the components of the Levi-Civita connection corresponding to g are given by:

$$\begin{aligned} \nabla_\xi \xi &= -\frac{e^{-y}}{\alpha} V, & \nabla_\xi V &= \frac{e^{-y}}{\alpha} \xi, & \nabla_\xi \varphi V &= 0, \\ \nabla_V \xi &= 0, & \nabla_V V &= -\frac{e^{-y}}{\alpha^2} \alpha_1 \varphi V, & \nabla_V \varphi V &= -\varphi \nabla_V V, \\ \nabla_{\varphi V} \xi &= 0, & \nabla_{\varphi V} V &= \frac{e^{-y}}{\alpha^2} (\alpha + \alpha_2) \varphi V, & \nabla_{\varphi V} \varphi V &= \varphi \nabla_{\varphi V} V, \end{aligned}$$

where $\alpha_2 = \frac{\partial \alpha}{\partial y}$. Then, one can easily check that for all $i, j \in \{1, 2, 3\}$

$$\begin{aligned} (\nabla_{e_i} \varphi) e_j &= \nabla_{e_i} \varphi e_j - \varphi \nabla_{e_i} e_j \\ &= \eta(e_i) (\omega(\varphi e_j) \xi + \eta(e_j) \varphi \psi). \end{aligned}$$

Through the rest of this paper $(M, \varphi, \xi, \psi, \eta, \omega, g)$ always denotes a 3-dimensional C_{12} -manifold and $\{\xi, V, \varphi V\}$ it's fundamental frame.

5 Legendre curves on 3-dimensional C_{12} -manifold

Let $\gamma(s)$ be a Legendre curve in a 3-dimensional C_{12} -manifold M . Let us compute the equations of motion for the associated Frenet frame is $\{\dot{\gamma}, \varphi \dot{\gamma}, \xi\}$.

Differentiating $\eta(\dot{\gamma}) = 0$ along γ we get

$$g(\nabla_{\dot{\gamma}} \dot{\gamma}, \xi) = -g(\nabla_{\dot{\gamma}} \xi, \dot{\gamma}),$$

with the help of Proposition 4.2, we get $g(\nabla_{\dot{\gamma}} \dot{\gamma}, \xi) = 0$, then,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa \varphi \dot{\gamma}, \quad (5.1)$$

where $\kappa = |g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\dot{\gamma})\varphi\dot{\gamma}|$.

For $\nabla_{\dot{\gamma}}\varphi\dot{\gamma}$, using (4.2) we obtain

$$\begin{aligned}\nabla_{\dot{\gamma}}\varphi\dot{\gamma} &= (\nabla_{\dot{\gamma}}\varphi)\dot{\gamma} + \varphi\nabla_{\dot{\gamma}}\dot{\gamma} \\ &= \kappa\varphi^2\dot{\gamma} \\ &= -\kappa\dot{\gamma}.\end{aligned}\tag{5.2}$$

Using Proposition 4.2, we get

$$\nabla_{\dot{\gamma}}\xi = 0.\tag{5.3}$$

We compare the equations (5.1)-(5.3) and (3.1), we conclude the following:

Proposition 5.1. *In a 3-dimensional C_{12} -manifold, a Legendre curve is a plane Frenet curve.*

Conversely, suppose that γ is a plane Frenet curve in a 3-dimensional C_{12} -manifold. That is $\tau = 0$ and the equations of motion (3.1) becomes

$$\begin{cases} \nabla_{\dot{\gamma}}E_1 = \kappa E_2, \\ \nabla_{\dot{\gamma}}E_2 = -\kappa E_1, \\ \nabla_{\dot{\gamma}}E_3 = 0. \end{cases}\tag{5.4}$$

Putting $\eta(\dot{\gamma}) = \sigma$ then, since $\dot{\gamma}$ is different from ξ and both of them are unit vectors we observe that $0 \leq |g(\xi, \dot{\gamma})| = \sigma < 1$ on M . We note that $\dot{\gamma}$ is not collinear with ξ . So, it can be verified that the vector fields

$$E_1 = \dot{\gamma}, \quad E_2 = \frac{\varphi\dot{\gamma}}{\sqrt{1-\sigma^2}}, \quad E_3 = \frac{\xi - \sigma\dot{\gamma}}{\sqrt{1-\sigma^2}},\tag{5.5}$$

form an orthonormal frame along γ and consequently, we can write

$$\psi = \omega(\dot{\gamma})E_1 + \frac{\omega\varphi\dot{\gamma}}{\sqrt{1-\sigma^2}}E_2 - \frac{\sigma\omega(\dot{\gamma})}{\sqrt{1-\sigma^2}}E_3.\tag{5.6}$$

From (5.5) and (5.6), one can get

$$\begin{aligned}\nabla_{\dot{\gamma}}E_3 &= \nabla_{\dot{\gamma}}\left(\frac{\xi - \sigma\dot{\gamma}}{\sqrt{1-\sigma^2}}\right) \\ &= \frac{-1}{\sqrt{1-\sigma^2}}\left(\dot{\sigma} + \sigma\omega(\dot{\gamma})\right)E_1 - \frac{\sigma}{1-\sigma^2}\left(\omega(\varphi\dot{\gamma}) + \kappa\sqrt{1-\sigma^2}\right)E_2 \\ &\quad + \frac{\sigma}{1-\sigma^2}\left(\dot{\sigma} + \sigma\omega(\dot{\gamma})\right)E_3.\end{aligned}$$

Since $\nabla_{\dot{\gamma}}E_3 = 0$ then, we get

$$\begin{cases} \dot{\sigma} + \sigma\omega(\dot{\gamma}) = 0 \\ \sigma(\omega(\varphi\dot{\gamma}) + \kappa\sqrt{1-\sigma^2}) = 0. \end{cases}\tag{5.7}$$

This establishes the following theorem:

Theorem 5.2. *Let M be a 3-dimensional C_{12} -manifold and $\gamma : I \rightarrow M$ a plane Frenet curve in M (i.e. $\tau = 0$), set $\sigma = \eta(\dot{\gamma})$. If at a certain point $t_0 \in I$, $\sigma(t_0) = 0$, then γ is a Legendre curve.*

6 Biharmonic curves on 3-dimensional C_{12} -manifold

The purpose of this last section is to study the curves $\gamma : I \rightarrow M$ which are biharmonic, i.e. which satisfy $\Delta^2 \gamma = 0$ where Δ is the Laplacian of I i.e. $\Delta = -\frac{d^2}{ds^2}$ with s is the arclength parameter.

in [7], Chen defined a biharmonic submanifold $N \subset \mathbb{E}^n$ of the Euclidean space as its mean curvature vector field H satisfies $\Delta H = 0$.

We note $H = \nabla_{\dot{\gamma}} \dot{\gamma}$, then

$$\Delta H = -\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} H = 0.$$

Let γ be a Legendre curve 3-dimensional C_{12} -manifold parametrized by arc length and its associated Frenet frame is $\{\dot{\gamma}, \varphi \dot{\gamma}, \xi\}$. So, we have

$$H = \nabla_{\dot{\gamma}} \dot{\gamma} = \kappa \varphi \dot{\gamma}.$$

With the help of (4.2), we get

$$\nabla_{\dot{\gamma}} H = \dot{\kappa} \varphi \dot{\gamma} - \kappa^2 \dot{\gamma},$$

then,

$$\begin{aligned} \Delta H &= -\nabla_{\dot{\gamma}} (\dot{\kappa} \varphi \dot{\gamma} - \kappa^2 \dot{\gamma}) \\ &= 3\dot{\kappa} \kappa \dot{\gamma} - (\ddot{\kappa} - \kappa^3) \varphi \dot{\gamma}, \end{aligned}$$

therefore, $\Delta H = 0$ if and only if

$$\begin{cases} \dot{\kappa} \kappa = 0 \\ \ddot{\kappa} - \kappa^3 = 0. \end{cases} \quad (6.1)$$

Hence, $\kappa = 0$ and we have the following result:

Theorem 6.1. *All biharmonic Legendre curves γ in a 3-dimensional C_{12} -manifold are straight lines, i.e. totally geodesic.*

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