Bisection of Trapezoids in Elamite Mathematics

Nasser Heydari and Kazuo Muroi November 22, 2022

Abstract

The bisection of trapezoids by transversal lines has many examples in Babylonian mathematics. In this article, we study a similar problem in Elamite mathematics, inscribed on a clay tablet held in the collection of the Louvre Museum and thought to date from between 1894–1595 BC. We seek to demonstrate that this problem is different from typical Babylonian problems about bisecting trapezoids by transversal lines. We also identify some of the possible mathematical ideas underlying this problem and the innovative approach that might have motivated its design.

Acknowledgment. The authors wish to express their deep gratitude and respect to the memory of late Professor T. Taniguchi. He was the first to realize the creativity behind the problem examined in this article and made mention of it to the second author.

1 Introduction

This tablet is one of 26 excavated from Susa in southwest Iran by French archaeologists in 1933. The texts of all the Susa mathematical tablets (henceforth **SMT**) along with their interpretations were first published in 1961 (see [BR61]). In Elamite mathematics, two problems relating to the transversal bisectors of trapezoids are found in **SMT** No. 23 and four problems in **SMT** No. 26.

Bisection of trapezoids is often categorized among the inheritance problems found in Babylonian mathematics (see [Fri05-1, Fri07-1, Fri07-2, FA16, Høy02, Mur01-1, Oss18], for examples of such problems). In these problems, a trapezoidal land area is divided equally between two brothers. One of the many ways to achieve this task is to use a line parallel to the bases of the trapezoid (a transversal). This specific method of dividing trapezoids has been the source of great interest in Babylonian mathematical texts and the many problems regarding this issue found there. Recently, it has been suggested the Babylonians could compute the total distance of Jupiter's travel along the ecliptic during a certain interval of time from the area of a trapezoidal figure representing the planet's changing daily displacement along the ecliptic. Moreover, the time when Jupiter reaches half the total distance may be computed by bisecting the trapezoid

into two smaller ones of equal areas by a transversal bisector (see [Oss16, Oss18]). It should be noted that the reasons why Babylonians favored transversal bisectors remain unknown to us, but one may suggest both mathematical and nonmathematical rationales.

However, the fourth problem of **SMT No. 26** deals with this issue differently. The problem considers the use of transversal strips—not lines—to partition a trapezoid and tries to determine the transversal strip (if any) that bisects the trapezoid. Treating such a problem in this way involves finding the natural solutions to a quadratic equation whose solvability depends entirely on the values of the bases and the number of the strips. As discussed later, this problem rarely has a solution and finding a correct one requires checking many different values. It is a surprise that the Susa scribe of **SMT No. 26** seems to have successfully solved the problem.

This partly broken tablet¹ contains four problems each of which deals with the partition of a right trapezoid. The terminology of this text differs somewhat from other Susa mathematical texts in both its vocabulary and concise expression.

2 Bisection of Geometric Figures

Bisection of a geometric shape² usually involves the division of the shape into two congruent parts. For two dimensional figures this is done by a line while for three dimensional shapes a plane is used. The bisecting lines or planes are called *bisectors*. It should be noted that under the congruency condition the bisection of some figures might not be possible. Figure 1 shows the bisection of a few geometric figures with their congruent parts and bisectors. The pentagon in case (d) can not be bisected into congruent parts at all.

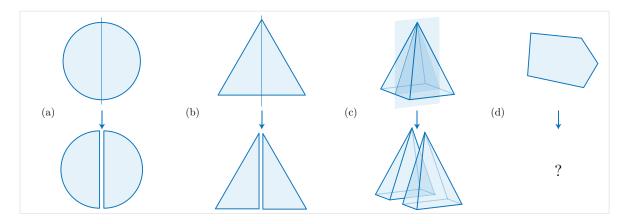


Figure 1: Bisection of different shapes

Now, let us replace the congruency condition in the definition of the bisection with

¹ The reader can see the new photos of this tablet on the website of the Louvre's collection. Please see https://collections.louvre.fr/ark:/53355/cl010186436 for obverse and reverse.

² We only consider shapes which have bounded areas or volumes and are made up of one piece such as polygons and polyhedra. In topology terms, these shapes are called *compact* and *connected* subsets of the plane and the 3-space. See [Rud76] for definitions of compactness and connectedness.

a new one which only considers the equality of areas or volumes of the obtained parts. This concept of bisection is more general than the previous one and from now on whenever we mention the bisection of a figure, we mean the process of dividing it by a line or a plane into two parts with equal areas or volumes. It is interesting that under this new condition any figure can be bisected in an infinite number of ways. For example, as is shown in Figure 2, for any given direction in the plane and any plane figure Λ , there is a line segment passing through it and dividing it into two equal parts.

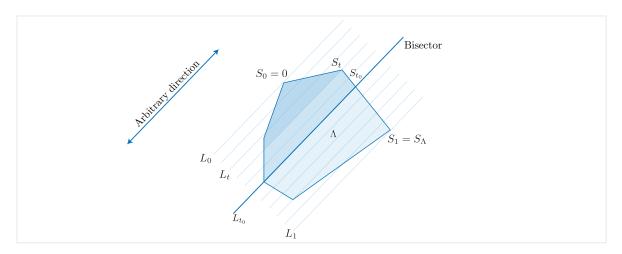


Figure 2: Bisection of a plane figure in an arbitrary direction

One explanation for the above-mentioned claim can be provided by the well-known Intermediate Value Theorem³ from mathematical analysis. The key idea here is to consider parallel lines in the fixed direction and define a function on real numbers $0 \le t \le 1$ which assigns to each number t the area S_t of the part of the figure Λ surrounded by two parallel lines L_0 and L_t (see Figure 2). Clearly, $S_0 = 0$ and $S_1 = S_{\Lambda}$. The Intermediate Value Theorem implies that there is a $0 < t_0 < 1$ for which

$$S_{t_0} = \frac{1}{2} S_{\Lambda},$$

meaning that the corresponding line L_{t_0} is a bisector for the figure Λ . We should note that although this method guarantees the existence of the bisector, determining the equation or the length of the bisecting line segment might not be so easy.

One problem in classic geometry is to compute the lengths of bisectors of polygons whose sides are known. For example, consider a triangle with sides a, b, c. Among all its bisectors, we can consider medians of its sides or the transversals parallel to its sides. In these two cases, we can easily compute the lengths of bisectors (see Figure 3). The length m of the median bisecting the side c of the triangle is obtained by

$$m = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2},$$

³ The theorem says that if f(x) is a continuous function such that $f(x_1) < y < f(x_2)$, then there exists a number x_0 between x_1 and x_2 for which $f(x_0) = y$. For a proof of this theorem and its applications, see [Rud76].

while a transversal line d parallel to the side c of the triangle has the length

$$d = \frac{\sqrt{2}c}{2}.$$

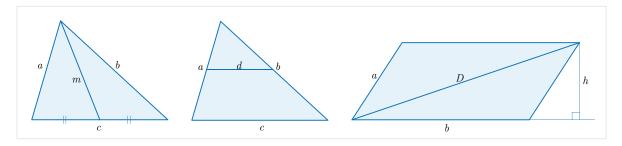


Figure 3: Bisectors of triangles and parallelograms

On the other hand, for a parallelogram with sides a, b and height h, any line passing through its centroid bisects the figure. For example, two of these bisectors are the diagonals of the parallelogram whose lengths can be computed with respect to a, b, h. If D is a diagonal of the parallelogram as in Figure 3, then

$$D = \sqrt{a^2 + b^2 + 2b\sqrt{a^2 - h^2}}.$$

3 Bisection of Trapezoids in Babylonian Mathematics

In comparison with other basic geometric figures, the trapezoid has been paid a great deal of attention in Babylonian mathematical tablets. In those texts, Babylonian scribes addressed many problems dealing with bisecting a trapezoid and the length of the corresponding bisector. Of infinitely many possible bisectors for a trapezoid, the Babylonian scribes seemed to have considered only the transversal line parallel to the two bases of the trapezoid (see [Fri05-1, Fri07-1, Fri07-2, FA16, Høy02, Mur01-1, Oss18], for examples of such problems).

One definite advantage of choosing the transversal bisector is that its length depends only on the length of two bases and we do not need to know the length of other sides or the height. In fact, as we see shortly, if the lengths of two bases of a trapezoid are a and b (see Figure 4), then the length of the transversal bisector d is given by

$$d = \sqrt{\frac{a^2 + b^2}{2}}. (1)$$

Convention. In Babylonian mathematical tablets, the trapezoids are usually drawn in a way that the bases are perpendicular to the horizontal direction. Because of this, the bases are called the upper and the lower widths and the other two sides are called the upper and the lower lengths (see Figure 4).

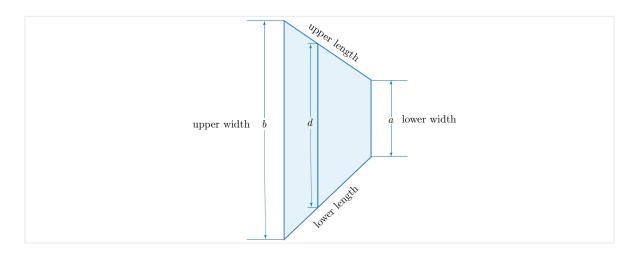


Figure 4: Transversal bisector of a trapezoid

It is clear that formula (1) is independent from height of the trapezoid. This is not true in general for other bisectors. For example, as is evident from Figure 5, the line segment connecting the midpoints of bases divides the trapezoid into two smaller trapezoids Λ_1 and Λ_2 with equal areas, because their bases and heights have the same lengths: $S_{\Lambda_1} = S_{\Lambda_2} = \frac{h}{4}(a+b)$. Note that the length of the dividing line d in Figure 5 can be computed by using the Pythagorean theorem as

$$d = \sqrt{h^2 + \left(\frac{a-b}{2}\right)^2} = \sqrt{c^2 - 3\left(\frac{a-b}{2}\right)^2}.$$
 (2)

In both formulas of (2), the value of d does not depend only on the length of bases.

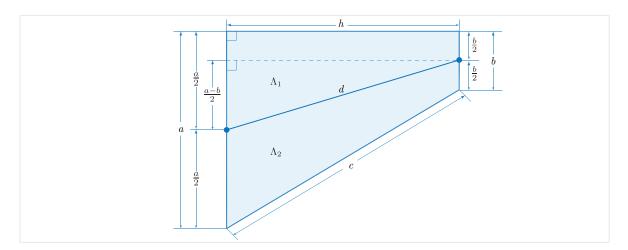


Figure 5: Bisection of a trapezoid using midpoints of bases

Another reason behind this Babylonian choice might have been that this division produces two trapezoids of the same kind (for example, right trapezoids⁴ are divided

⁴ A trapezoid having at least two right angles is called a right trapezoid.

into right trapezoids). Aside from mathematical advantages, social justice might also be reflected in this method which utilizes mathematical skills to solve real-life problems. Historically, division of an inheritance amongst members of a family has always been a delicate issue. An unsatisfactory outcome can start a bitter family feud. Consider a trapezoidal area of farmland as shown in Figure 6 and assume that a land surveyor is asked to divide it between two brothers equally. As is shown in the figure, three sides of the land are surrounded by roads and the fourth one is bounded by an irrigation canal.

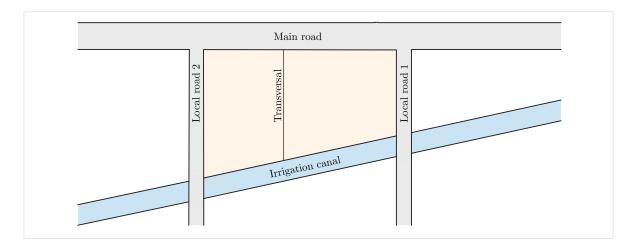


Figure 6: A trapezoidal area of farmland

In addition to dividing the land into two parts with equal areas, the land surveyor ought to include two important factors in his calculations: (1) direct access to the main road and (2) direct access to the irrigation canal. These two factors, even nowadays, are the most decisive ones in increasing the value of any farmland. The transversal bisecting method is the one giving these two advantages to both divided parts. As we can see from Figure 6, the transversal line bisects the farmland into two equal parts each of which can directly access both the irrigation canal and the main road. Moreover, the shape of the two parts are the same (both right trapezoids), a characteristic that other division methods (such a midpoint method) may not possess. By dividing the land in this way, the surveyor is able to kill two birds with one stone! He succeeds in his task both mathematically and also from a social justice point of view. It should be noted that in other methods, the owner of one part of the divided land might lose direct access to the main road or the irrigation canal.

Proof of the Babylonian formula

Now, we prove the Babylonian formula (1) for the bisecting transversal of a trapezoid. Consider a trapezoid ABCD the lengths of whose upper and lower widths (bases) are $\overline{AD} = a$ and $\overline{BC} = b$ respectively (see Figure 7). Suppose that this trapezoid is divided into two equal parts by the transversal line segment EF of length d which is parallel to the bases AD and BC. Consider the perpendicular lines from C to AD, the dotted

lines in Figure 7, and set

$$\begin{cases} \overline{CG} = h_1, \\ \overline{HG} = h_2 \\ \overline{CH} = h. \end{cases}$$

Clearly, $h = h_1 + h_2$ and

$$\begin{cases} S_{AEFD} = \frac{(a+d)h_2}{2}, \\ S_{EBCF} = \frac{(b+d)h_1}{2}. \end{cases}$$

Since $S_{AEFD} = \frac{1}{2} S_{ABCD}$, we have

$$\frac{(a+d)h_2}{2} = \frac{(a+b)h}{4}$$

which implies that

$$\frac{h_2}{h} = \frac{a+b}{2(a+d)}. (3)$$

Similarly, it follows from $S_{EBCF} = \frac{1}{2} S_{ABCD}$ that

$$\frac{h_1}{h} = \frac{a+b}{2(b+d)}. (4)$$

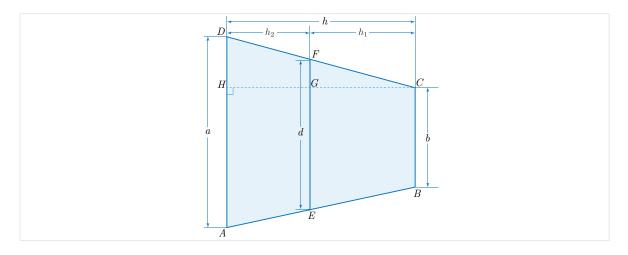


Figure 7: Bisecting transversal of a trapezoid

Now, by using (3) and (4) we can write

$$\frac{h_1 + h_2}{h} = 1$$

$$\Rightarrow \frac{h_1}{h} + \frac{h_2}{h} = 1$$

$$\Rightarrow \frac{a+b}{2(b+d)} + \frac{a+b}{2(a+d)} = 1$$

$$\Rightarrow \frac{a+b}{2} \left(\frac{1}{b+d} + \frac{1}{a+d}\right) = 1$$

$$\Rightarrow \frac{a+b+2d}{(a+d)(b+d)} = \frac{2}{a+b}$$

$$\Rightarrow a^2 + b^2 + 2ab + 2d(a+b) = 2d^2 + 2(a+b)d + 2ab$$

$$\Rightarrow 2d^2 = a^2 + b^2$$

$$\Rightarrow d = \sqrt{\frac{a^2 + b^2}{2}}$$

which completes the proof of (1).

4 Bisection of Trapezoids in the Susa Mathematical Texts

Of the six problems in the **SMT** regarding the bisection of trapezoids five of them might be termed standard problems whose calculations involve the transversal bisectors of trapezoids and the application of the Babylonian formula (1). However, the fourth problem in **SMT No. 26** is different from these typical problems and instead of a line, the Susa scribe has considered a transversal party wall to bisect a trapezoid. We now consider this fourth problem of **SMT No. 26** and seek to illuminate its mathematical significance.

4.1 Bisection of Trapezoids by Transversal Strips

Consider a trapezoid of bases a > b and height h. For any natural number n > 2, we can divide its height h into n equal parts of lengths $\frac{h}{n}$ by drawing n-1 vertical line segments of lengths $d_1, d_2, \ldots, d_{n-1}$, which are parallel to the two bases. Note that these transversal lines are labeled from left to right. In this case, we have partitioned our main trapezoid into n smaller trapezoids $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$. If we set $d_0 = a$ and $d_n = b$, then the bases of Λ_k are of lengths d_{k-1} and d_k , for all $k = 1, 2, \ldots, n$. Note also that the heights of all trapezoids Λ_k have the common value $\frac{h}{n}$ (see Figure 8).

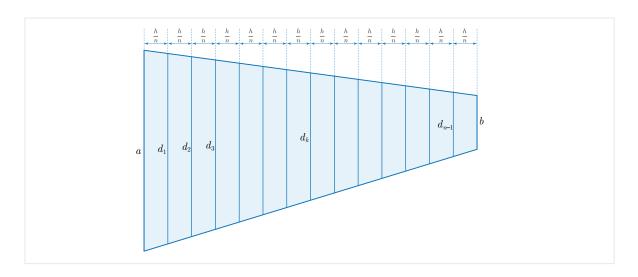


Figure 8: Partition of a trapezoid by transversal lines

It is easy to compute the length d_k of each transversal line (bases of the small right trapezoid Λ_k) by using the similarity of right triangles. This situation is shown in Figure 9 in which d_k is the common base of two trapezoids with heights $h_k = \frac{kh}{n}$ and $h'_k = h - \frac{kh}{n}$.

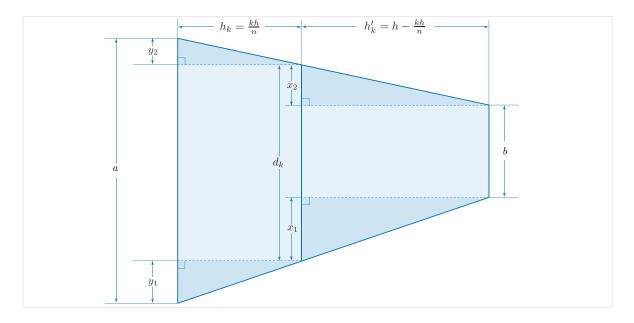


Figure 9: Similar triangles in a trapezoid divided by a transversal line

It is clear from Figure 9 that $d_k = x_1 + x_2 + b$ and $a = y_1 + y_2 + d_k$. Since the transversal d_k is parallel to bases, the two heavy-shaded right triangles in the lower part of Figure 9 are similar. So,

$$\frac{x_1}{h'_k} = \frac{y_1}{h_k} \Longrightarrow \frac{y_1}{\frac{kh}{n}} = \frac{x_1}{h - \frac{kh}{n}}$$

which easily implies that

$$y_1 = \left(\frac{k}{n-k}\right) x_1. \tag{5}$$

The same reasoning for the two right triangles in the upper part of Figure 9 also implies that

$$y_2 = \left(\frac{k}{n-k}\right) x_2. \tag{6}$$

Next, we can use (5) and (6) and write as follow:

$$d_k + y_1 + y_2 = a$$

$$\implies d_k + \left(\frac{k}{n-k}\right) x_1 + \left(\frac{k}{n-k}\right) x_2 = a$$

$$\implies d_k + \left(\frac{k}{n-k}\right) (x_1 + x_2) = a$$

$$\implies d_k + \left(\frac{k}{n-k}\right) (d_k - b) = a$$

$$\implies \left(\frac{n}{n-k}\right) d_k = a + \left(\frac{k}{n-k}\right) b$$

$$\implies d_k = \left(\frac{n-k}{n}\right) a + \left(\frac{k}{n}\right) b.$$

Thus, we get

$$d_k = \left(1 - \frac{k}{n}\right)a + \left(\frac{k}{n}\right)b, \quad \text{for} \quad 1 \le k \le n - 1. \tag{7}$$

We raise the question that under what conditions would one of the trapezoids Λ_k play the role of a transversal bisector? In another words, is it possible to choose a Λ_k such that the total areas of trapezoids on its left-hand side and those on its right-hand side are equal? This situation is shown in Figure 10 in which S_{k-1} is the total areas of trapezoids $\Lambda_1, \Lambda_2, \ldots, \Lambda_{k-1}$ and S'_k is that of trapezoids $\Lambda_{k+1}, \ldots, \Lambda_n$.

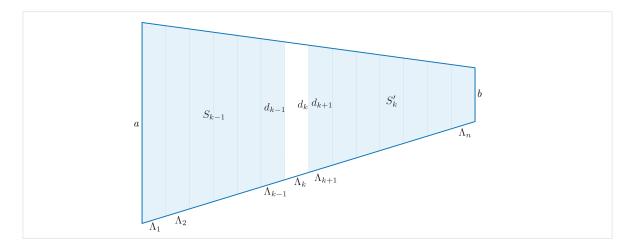


Figure 10: Grouping transversal strips

To answer this question, one needs to compute the values of S_k and S'_k and find the index k such that $S_{k-1} = S'_k$. It is clear that

$$S_k = \frac{(a+d_1)h}{2n} + \frac{(d_1+d_2)h}{2n} + \dots + \frac{(d_{k-1}+d_k)h}{2n}.$$

A simple calculation using (7) implies that

$$S_k = \frac{kh}{2n} \left(\left(2 - \frac{k}{n} \right) a + \left(\frac{k}{n} \right) b \right), \quad \text{for} \quad 1 \le k \le n - 1,$$
 (8)

which is exactly the area of a trapezoid with bases a, d_k and height $h_k = \frac{kh}{n}$ (see Figure 10). Also note that the total area of n - k small trapezoids $\Lambda_{k+1}, \Lambda_{k+2}, \ldots, \Lambda_n$ is

$$S'_{k} = \frac{h(a+b)}{2} - S_{k}, \quad \text{for} \quad 1 \le k \le n-1,$$
 (9)

which is the area of a trapezoid with bases d_k , b and height $h'_k = h - \frac{kh}{n}$.

Assume that one of these n small trapezoids, say Λ_{k_0} , is the bisecting party wall (note that $1 < k_0 < n$). So, the remaining n-1 small trapezoids belong to two groups:

- (1) the ones on the left-hand side of the party wall, i.e., $\Lambda_1, \Lambda_2, \ldots, \Lambda_{k_0-1}$, and
- (2) the ones on the right-hand side of the party wall, i.e, $\Lambda_{k_0+1}, \Lambda_{k_0+2}, \ldots, \Lambda_n$.

Since we require the total areas of these two groups to be equal, the problem boils down to finding the possible value of k_0 for which $S_{k_0-1} = S'_{k_0}$. It immediately follows from (9) that

$$S_{k_0} + S_{k_0 - 1} = \frac{h(a + b)}{2}.$$

By using (8) in the last equality and doing some calculations, we obtain the following quadratic equation with respect to the unknown variable k_0 :

$$2(a-b)k_0^2 - (4na - 2b + 2a)k_0 + n^2(a+b) + 2na + a - b = 0.$$
 (10)

Clearly, the solvability of the last quadratic equation deeply depends on the values of a, b and n. In fact, the discriminant Δ of this equation is

$$\Delta = 4(2n^2 - 1)a^2 + 4(2n^2 - 1)b^2 + 8ab \tag{11}$$

which is always positive, because n > 1. Since only natural values between 1 and n - 1 for k_0 are allowed, we definitely require the discriminant to be a perfect square which obviously is not true in general. Therefore, one may say that while this problem does not always have a solution, under what conditions might one find a correct solution?

Let a = br, where r > 1. In that case, the value of discriminant Δ and k_0 are

$$\Delta = 4b^2 \left((2n^2 - 1)(r^2 + 1) + 2r \right) \tag{12}$$

and

$$k_0 = \frac{(2n+1)r - 1 \pm \sqrt{(2n^2 - 1)(r^2 + 1) + 2r}}{2(r-1)}.$$
 (13)

For any fixed r > 1, we can try different values for n and see what happen. If one check values $r = 2, 3, 4, \ldots, 20$ and $n = 3, 4, 5, \ldots, 1000$, they will obtain the following answers:

a = rb	r	n	k_0
a = 2b	2	37	16
a = 3b	3	17	7
a = 3b	3	305	117
a = 4b	4	65	24
a = 5b	5	10	4
a = 6b	6	25	9
a = 8b	8	35	12
a = 9b	9	20	7
a = 12b	12	11	4
a = 13b	13	246	78
a = 15b	15	511	160
a = 17b	17	8	3
a = 17b	17	505	157
a = 18b	18	89	28

Table 1: Acceptable values for r and n

This table shows that among almost 20000 cases, only 14 provide us with correct answers. Even if r = 2, 3, ..., 211 and n = 3, 4, ..., 1000 which produce 209580 cases, there are only 32 such answers! In other words, this problem rarely has a solution and its solvability entirely depends on the values of $\frac{a}{b}$ and n.

5 Fourth Problem of SMT No. 26

Unlike the typical problems on trapezoids which deal with transversal bisectors, the fourth problem of SMT No. 26 considers a situation similar to the one we discussed in the previous section. The Susa scribe seems to divide an area of trapezoidal land between two brothers equally by using a party wall. This problem is treated in lines 1-17 on the reverse of this tablet. We give both the transliteration and the translation of this text and then explain its mathematical calculations in detail.

Transliteration

Obverse, Lines 1-15

- $\overline{(L1)}$ $\check{s}\grave{a}\cdots \overline{[\cdots]}\cdots \overline{[\cdots]}$
- (L2) 2,10 sag a[n-na] ugu 30 s[ag ki-ta 1,40 dirig]
- (L3) igi-3,45 uš du₈-ma 16 [a-na 1,40 il]
- (L4) 26,40 a-rá 2 53,20 *a-n*[*a* 4,30 íl 4]
- (L5) ta-ta-a-ar 2,10 sag an-n[a kú-kú-ma 4,41,40]

```
(L6) 4 i-na 4,41,40 kud-ma [41,40]
(L7) šumma (BAD) 50 dal murub<sub>4</sub> 30 nindan sag an(sic)-na(sic) 40(?) [...]
(L8) 50 dal murub<sub>4</sub> ugu 30 sag ki-<ta> mi-nam [dirig 20 dirig]
(L9) 1,20 a-rá 20 26,40 a-rá 2 5[3,20]
(L10) ta-ta-a-ar 50 dal murub<sub>4</sub> kú-[kú-ma 41,40]
(L11) íb-si<sub>8</sub>-bi ka-bi-is [1,4(?)]
(L12) sag-ki-gud 35 sa[g an-na] 5 [s]ag k[i-ta]
(L13) 2 šeš-meš mi-it-ha-ri-iš l[i-zu-zu]
(L14) 35 a-rá 35 20,25 [5 a-rá 5 25 a-na]
(L15) [20,25 §]i-ib-ma [20,25 bar] 10,25 [ib-si<sub>8</sub>-bi 25]
Reverse, Lines 1-17
(L1) [·······] 3[6(?) ···]
(L2) 1,40[sag an-na 20 sag] ki-ta 1 uš [....]
(L3) i-na l[i-bi u]š 1 kùš 6 šu-si é-[gar<sub>8</sub> dal-ba-na e-pu-uš]
(L4) íb-tag<sub>4</sub> a-na 2 šeš-meš [i-di-in]
(L5) 1,40 sag an-na ugu 20 sag ki-ta m[i-nam dirig 1,20 dirig]
(L6) 1,[20] a-na 6 é-gar<sub>8</sub> dal-<ba>-na íl 8 he-[pé-ma 4]
(L7) [ta]-ta-a-ar 1,40 sag an-na kú-[kú-ma 2,46,40]
(L8) [2]0 sag ki-ta kú-kú-ma 6,[40 a-na 2,46,40 şí-ib-ma]
(L9) 2,53,20 bar 2,53,20-da í[b-si<sub>8</sub>-bi ka-bi-is 1,12]
(L10) 1,12 a-na é-gar<sub>8</sub> dal-\langle ba \rangle-na a-na sag [šu-ku-un]
(L11) ta-ta-a-ar 1,40 sag an-<na> ugu [20 sag ki-ta]
(L12) 1,20 dirig 1,20 a-rá 6 8 bar 8 [4 4]
(L13) i-na 1,12 kud-ma 1,8 1,8 ù [20 ul-gar-ma 1,28]
(L14) bar-zu 1,12 a-na 6 é-gar<sub>8</sub> dal-<ba>-[na íl 7,12]
(L15) 7,12 ki ša é-gar<sub>8</sub> dal-\langle ba \rangle-na [\cdots \cdots]
(L16) 52,48 bar-zu 26,2[4 · · · · · · · ]
(L17) 1,40 \hat{u} 1,16 [ul-gar-ma 2,56 · · · · · · · ]
```

Translation

Obverse, Lines 1-15

- (L1) \cdots that of \cdots .
- (L2) 2,10 of the upper width exceeds 30 of the lower width by 1,40.
- (L3) Make the reciprocal of 3,45 of the length, and (the result is) 0;0,16. Multiply (it) by 1,40, (and the result is) 0;26,40.
- (L4) 2 times 0;26,40 is 0;53,20. Multiply (it) by 4,30,0, (and the result is) 4,0,0.
- (L5) You return. Square 2,10 of the upper width, (and the result is) 4,41,40.
- (L6) Subtract 4,0,0 from 4,41,40, and (the result is) 41,40. (Its square root is 50.)
- (L7) If the middle dividing line is 50, (and) the lower width 30 nindan (\approx 180m), \cdots
- (L8) How much 50 of the middle dividing line exceeds 30 of the lower width? It exceeds (30) by 20.
- (L9) 20 times 1,20 is 26,40. 2 times (26,40) is 53,20.
- (L10) You return. Square 50 of the middle dividing line, and (the result is) 41,40.
- (L11) Its square root is paced off. It is 1,4(?).
- (L12) A trapezoid. 35 is the upper width. 5 is the lower width.
- (L13) Two brothers ought to divide (it) equally.
- (L14) 35 times 35 is 20,25. 5 times 5 is 25.
- (L15) Add (25) to 20,25, and (the result is) 20,50. Halve (it). 10,25. Its square root is 25.

Reverse, Lines 1-17

- (L1) $\cdots \cdots 0;36(?) \cdots \cdots$
- (L2) 1;40 is the upper width. 0;20 is the lower width. 1 is the length. \cdots
- (L3) In the middle of the length I built a party wall, (whose thickness is) 1 kùš 6 šu-si (= 0;6 nindan).
- (L4) Give the remainder (of the trapezoid) to two brothers (equally).
- (L5) How much 1;40 of the upper width exceeds 0;20 of the lower width? It exceeds (0;20) by 1;20.
- (L6) Multiply 1;20 by 0;6 of the party wall, (and the result is) 0;8. Break (it in two), and (the result is) 0;4.
- (L7) You return. Square 1;40 of the upper width, and (the result is) 2;46,40.
- (L8) Square 0;20 of the lower width, and (the result is) 0;6,40. Add (it) to 2;46,40, and (the result is)
- (L9) 2;53,20. With half of 2;53,20, its square root is paced off. It is 1;12.
- (L10) Put down 1;12 for the party wall, for the width (of it).
- (L11-12) You return. 1;40 of the upper width exceeds 0;20 of the lower width by 1;20. 1;20 times 0;6 is 0;8. Half of 0;8 is 0;4.

- (L13) Subtract 0;4 from 1;12, and (the result is) 1;8. Add 1;8 and 0;20 together, and (the result is) 1;28.
 (L14) A half. Multiply 1;12 by 0;6 of the party wall, (and the result is) 0;7,12.
 (L15) 0;7,12 is the area of the party wall.
- (L16) 0.52.48. Half (of it) is 0.26.24. ...
- (L17) Add 1;40 and 1;16 together, and (the result is) $2;56. \cdots$

Mathematical Interpretation

Although parts of the text regarding the statement of this problem are lost, we can formulate the problem based on the calculations provided. Consider a right trapezoid ABCD with bases of lengths a=1;40, b=0;20 and height h=1 and divide it into two parts by a party wall (the trapezoid KMNL) of width $h_0=0;6$ such that the transversal line EF is in the middle of the left and the right edges LK and NM of the party wall respectively (see Figure 11). The problem now is how to determine the values of the edges and the middle line of the party wall KMNL provided that the two right trapezoids AKLD and MBCN have equal areas.

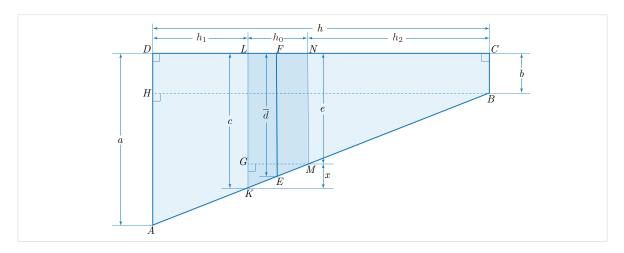


Figure 11: Bisection of a right trapezoid using a party wall

Let $\overline{LK} = c$, $\overline{NM} = e$, $\overline{FE} = \overline{d}$ and $\overline{GK} = x$. Clearly, x = c - e. Let d be the bisecting transversal of the trapezoid and \overline{d} be the middle line of the party wall (note that we only have shown \overline{d} in Figure 11). The scribe first computes the difference between the left and the right edges LK and NM of the party wall and then finds the length of the bisecting transversal line d in lines 5-9. Consider the perpendicular lines from M onto LK and from B onto AD with intersection points G and H respectively (see Figure 11). Since two right triangles $\triangle MGK$ and $\triangle ABH$ are similar, we have

$$\frac{\overline{GK}}{\overline{GM}} = \frac{\overline{AH}}{\overline{HB}}$$

or equivalently

$$\frac{x}{h_0} = \frac{a-b}{h}.$$

This implies that

$$x = \frac{h_0}{h}(a-b). \tag{14}$$

It is clear from Figure 11 that the difference between the lengths of the edges of the party wall and the middle line of the party wall is half the difference between two edges of the party wall. So this value is clearly $\frac{x}{2}$ and according to lines 5-6, we can use (14) to write

$$\frac{x}{2} = \frac{1}{2} \times \frac{h_0}{h} \times (a - b)$$

$$= (0; 30) \times \frac{(0; 6)}{1} \times (1; 40 - 0; 20)$$

$$= (0; 30) \times (0; 6) \times (1; 20)$$

$$= (0; 30) \times (0; 8)$$

$$= 0; 4.$$

Hence

$$\frac{x}{2} = 0;4$$
 (15)

or equivalently

$$x = 0; 8.$$
 (16)

On the other hand, according to lines 7-9, the value of bisecting transversal d, as usual, is obtained by the following calculations:

$$d = \sqrt{\frac{a^2 + b^2}{2}}$$

$$= \sqrt{\frac{(1;40)^2 + (0;20)^2}{2}}$$

$$= \sqrt{\frac{2;46,40 + 0;6,40}{2}}$$

$$= \sqrt{\frac{2;53,20}{2}}$$

$$= \sqrt{1;26,40}$$

$$= 1;12,6,39,41,30,\cdots$$

$$\approx 1;12.$$

From now on, the scribe appears to have assumed the value 1; 12 for \overline{d} and has utilized it in the rest of the calculations. So, we set

$$\overline{d} = 1; 12. \tag{17}$$

Note that \overline{d} is the mean value of two numbers c and e because by the similarity of two right triangles $\triangle KEE'$ and $\triangle EMM'$ in Figure 12, we have

$$\frac{\overline{KE'}}{\overline{EE'}} = \frac{\overline{EM'}}{\overline{MM'}},$$

since $\overline{KE'} = c - \overline{d}$, $\overline{EM'} = \overline{d} - e$ and $\overline{EE'} = \overline{MM'} = h'$, the last equality implies that

$$\frac{c - \overline{d}}{h'} = \frac{\overline{d} - e}{h'}$$

or

$$c - \overline{d} = \overline{d} - e.$$

This clearly gives the following formula

$$\overline{d} = \frac{c+e}{2}. (18)$$

Maybe this is the reason in line 10 the scribe calls the value $\overline{d} = 1$; 12 as the "width" of the party wall.

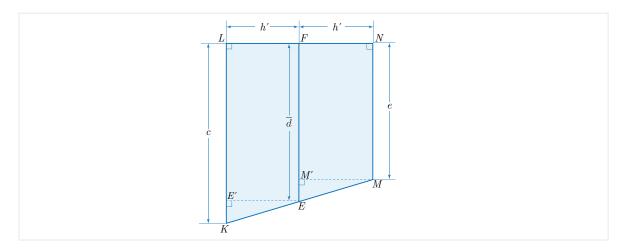


Figure 12: Dividing a party wall using a transversal line

The second part of the solution (lines 11-17) deals with finding the areas of two trapezoids ADLK and BCNM which is the main goal of the scribe. At the beginning of this part (lines 11-12) in our opinion the same calculations as in lines 5-6 are perhaps repeated. Moreover, the calculations in lines 13-16 are (to modern eyes) somewhat disorderly. In fact, lines 14 and 15 calculate the area of the base of the party wall, i.e., the trapezoid KMNL, and lines 13 and 16 the area of the trapezoid MBCN. For the area of the party wall KMNL, say S_0 , we note that since \overline{d} is the mean value of those of bases of the trapezoid KMNL and its height is $h_0 = 0$; 6, it follows from (17) and

(18) that

$$S_0 = \frac{h_0(c+e)}{2}$$
= $h_0 \overline{d}$
= $(0; 6) \times (1; 12)$
= $0; 7, 12$.

Thus

$$S_0 = 0; 7, 12. (19)$$

For the area of the trapezoid MBCN, we need to find the values of bases c, e and the height h_2 . It is obvious from Figure 12 that

$$\begin{cases}
c = \overline{d} + \frac{x}{2} \\
e = \overline{d} - \frac{x}{2}.
\end{cases}$$
(20)

So, it follows from (16), (17) and (20) that

$$c = 1; 12 + 0; 4 = 1; 16 (21)$$

and

$$e = 1; 12 - 0; 4 = 1; 8.$$
 (22)

Although there is no trace of values for the two heights h_1 and h_2 in the text, it seems that the scribe was aware that the correct values of these two heights are 0;18 and 0;36 respectively. In fact, he is implicitly using these numbers in his calculations. According to lines 13 and 15, it follows from (22) that

$$S_{MBCN} = \frac{1}{2} \times h_2 \times (e+b)$$

$$= \frac{1}{2} \times (0;36) \times (1;8+0;20)$$

$$= \frac{1}{2} \times (0;36) \times (1;28)$$

$$= \frac{1}{2} \times (0;52,48)$$

$$= 0;26,24.$$

Hence

$$S_{MBCN} = 0; 26, 24. (23)$$

In line 17 and the subsequent missing lines, the scribe might have computed the area of the other trapezoid AKLD by implicitly using the correct value $h_1 = 0$; 18. As a

matter of fact, it follows from (21) that

$$S_{AKLD} = \frac{1}{2} \times h_1 \times (c+a)$$

$$= \frac{1}{2} \times (0;18) \times (1;16+1;40)$$

$$= \frac{1}{2} \times (0;18) \times (2;56)$$

$$= \frac{1}{2} \times (0;52,48)$$

$$= 0;26,24$$

hence

$$S_{AKLD} = 0; 26, 24. (24)$$

Therefore, the equal share of each brother in land is 0; 26, 24.

6 Significance of SMT No. 26

Although one might think that the areas of the two trapezoids MBCN and AKLD given in (22) and (24) respectively are approximate values because the scribe has used an approximate value in (18) for \overline{d} , these area values surprisingly turn out to be precise! What is really happening here? Is this a just a lucky approximation or did the scribe already know that these are the correct values? In fact, from (19), (22) and (24), we suggest that the total sum of areas of the three parts is

$$S_{MBCN} + S_{AKLD} + S_0 = 2 \times (0; 26, 24) + 0; 7, 12$$

= 0; 52, 48 + 0; 7, 12
= 1,

which is exactly the area of the whole trapezoid ABCD. This suggests:

- (1) the scribe successfully accomplished his task of dividing land between two brothers equally by means of a party wall; and
- (2) he seemed to already know the exact value of \overline{d} , c, e and the areas of the two trapezoids MBCN and AKLD.

Another issue concerns the values of heights h_1 and h_2 of the two trapezoids MBCN and AKLD. How did the scribe know the correct values of the two heights h_1 and h_2 ? If we look into this problem, we notice that a = 5b, r = 5 and n = 10, which are the data in the sixth row of Table 1. In other words, the Susa scribe must have known that these numbers would provide exact areas for the two trapezoids MBCN and AKLD. But how did he know that? Our guess is that he might have designed the problem by using a similar approach to the one we discussed in section 5. However, since finding acceptable values for a, b and n might have been an arduous task in the sexagesimal numeral system, he might have utilized small values for n and r (for example less than

60) and then performed some calculation to find a correct answer. With some luck, he could have found the correct answers for n and r.

As he had to divide the height into n equal parts, the value of n must have been a regular⁵ number. By looking at the data in Table 1, those values for r that provide answers are 2,3,4,5,6,8,9 and the corresponding values for n are 37,17,305,65,10,25,35,20 among which only 20, 25 and 20 are regular numbers. Although, the acceptable pairs of (r, n) are (5, 10), (6, 25) and (9, 20), the first acceptable pair (5, 10) has the minimum value for n. One might reasonably anticipate this acceptable pair of values for r, n to be the obvious choice and surprisingly enough, these numbers are the very values that the Susa scribe has used in this problem! Using either the approach we discussed or through trial and error, the Susa scribe must have known these numbers in order to design this problem. Without using the correct choices for r and n, the calculations would be very complicated and the answers approximate not exact.

It is reasonable to posit that the Susa scribe of this text who designed such a beautiful and elegant problem employed an innovative approach by which means all the exact values for the required parts in question were obtained.

7 Conclusion

Like the Babylonians, the Susa scribes dealt with transversal bisectors of trapezoids and solved problems using the Babylonian formula for the transversal. However, the Susa scribe of SMT No. 26 dared to consider a more general case and instead of transversal lines he dealt with transversal strips. Although one may think treating such a problem is similar to a transversal line case, the reality is different and this problem requires a higher level of mathematical skill and experience. It seems that the Susa scribe was cognizant of the numbers that would lead to the solutions. These numbers could have been determined only by finding the natural solutions of a quadratic equation whose coefficients are with respect to the bases and the number of strips. In fact, the only way to find such solutions is to check a huge number of different choices for both the bases and the number of strips, which requires performing complicated calculations in the sexagesimal numeral system.

The design of mathematical problems requires deep knowledge and a great deal of experience on the part of a teacher. A beautiful and challenging problem can engage the imagination of students and encourage their creativity. The mathematical interpretation of the fourth problem in **SMT No. 26** that we have identified in this article displays just such a creative approach to both the design and the means of solving the problem. Among the many and varied characteristics attributable to the Susa scribes apparent from the **SMT**, we consider this characteristic to be by far the most striking one.

⁵ Any number in the form of $2^p 3^q 5^r$, where p, q, r are nonnegative integers, is called a regular number.

References

- [BR61] E. M. Bruins and M. Rutten, *Textes Mathématiques de Suse* [TMS], Librairie Orientaliste Paul Geuthner, Paris, 1961.
- [Fri05-1] J. Friberg, Unexpected Links Between Egyptian and Babylonian Mathematics, World Scientific, 2005.
- [Fri07-1] J. Friberg, A Remarkable Collection of Babylonian Mathematical Texts, Springer, 2007.
- [Fri07-2] J. Friberg, Amazing Traces of a Babylonian Origin in Greek Mathematics, World Scientific, 2007.
- [FA16] J. Friberg and F. Al-Rawi, New Mathematical Cuneiform Texts, Sources and Studies in the History of Mathematics and Physical Sciences, Springer, 2016.
- [Høy02] J. Høyrup, Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin, Springer, Studies and Sources in the History of Mathematics and Physical Sciences, New York, 2002.
- [Mur01-1] K. Muroi, Inheritance Problems in the Susa Mathematical Text No. 26, Historia Scientiarum Vol. 10-3 (2001), pp. 226–234.
- [Oss16] M. Ossendrijver, Ancient Babylonian Astronomers Calculated Jupiter's Position from the Area under a Time-Velocity Graph, Science Vol. 351, Issue 6272 (2016), pp. 482–484.
- [Oss18] M. Ossendrijver, Bisecting the Trapezoid: Tracing the Origins of a Babylonian Computation of Jupiter's Motion, Archive for History of Exact Sciences 72 (2018), pp.145–189.
- [Rud76] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, 1992.