The Elamite Formula for The Area of a Regular Heptagon

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September 29, 2022

Abstract

In this article, we conduct a close study of the reverse of Susa Mathematical Text No. 2. In our discussion, we focus on the formula given in this text for the approximate area of a regular heptagon. We give a geometric explanation for the formula and show that this approximation is more accurate than other contemporaneous ones in Babylonian mathematics and even than the formula of Greek mathematician Heron who proved it almost 1800 years later. We also talk about the possible ways Susa scribes might have applied to construct this regular heptagon on a clay tablet.

1 Introduction

This tablet belongs to the 26 mathematical tablets excavated from Susa by French archaeologists in 1933. The texts of all Susa mathematical tablets (henceforth **SMT**) along with their interpretations were first published in 1961 (see [BR61]). Despite the great effort of authors, there are many mistakes and misinterpretations in this book and during last decades many scholars have tried to reexamine these mathematical texts including the second author of this article.[1]

The structure of this tablet is as follows[2] : there is a regular hexagon on the obverse of this tablet with some numbers, and on its reverse there is a regular heptagon with numbers and a formula for its area. The formula gives an instruction to find the area constant of a regular heptagon. Unfortunately, this formula has not been given enough attention by scholars and many of them have only mentioned it as an approximation for the area of a regular heptagon. However, we believe this formula deserves a special treatment because it uses geometric ideas to give a very good approximation for the

[1] Some texts of **SMT**have been reexamined in [Fri07-1, Fri07-2, FA16, Høy93, Høy99, Høy02, Høy02, Høy17, Mur92-1, Mur92-2, Mur92-3, Mur94-1, Mur94-2, Mur00-1, Mur01-1, Mur01-2, Mur03-2, Mur13, Mur16, Mur]. The authors of this paper are working on a book about Elamite mathematics in which all texts of **SMT** are reexamined. In this project, we are using the newly photographed and high-resolution photos of all **SMT** which we acquired from the Louvre Museum.

[2] The reader can see the new photos of this tablet on the website of the Louvre's collection. Please see `https://collections.louvre.fr/ark:/53355/cl010185652` for obverse and `https://collections.louvre.fr/ark:/53355/cl010185652`, for reverse.

area of a regular heptagon. Thus, we have decided to write this article specifically to discuss all the mathematical aspects of this formula.

It is also noteworthy that the dimensions of these two polygons on both sides of this tablet are accurate. Also, a part of the circumscribed circle is still visible on both sides of the tablet suggesting that the Susa scribe of this tablet has used a compass and straightedge to draw the figure by using a circumscribed circle.

Remark 1. By convention, we have used the sexagesimal numeral system to write numbers. In this system, the comma "," separates the double digits and the semi-colon ";" separates the non-negative and negative powers of 60. So, a number like $12, 23, 5; 13, 45, 9$ in the sexagesimal numeral system becomes

$$12 \times 60^2 + 23 \times 60^1 + 5 \times 60^0 + 13 \times 60^{-1} + 45 \times 60^{-2} + 9 \times 60^{-3}$$

in the decimal numeral system.

2 Areas of polygons in Babylonian and Elamite mathematics

Let $n \geq 3$ be a natural number. A *polygon* with n sides or an *n-gon* is a plane figure formed by a chain of n line segments v_1v_2, v_2v_3, $\cdots$, $v_{n-1}v_n$ and v_nv_1 connecting n points $v_1, v_2, \cdots, v_n$ and enclosing a bounded region of the plane. In such a case, one may call the bounded plane region, the boundary chain, or the two together as the polygon. The line segments v_iv_{i+1} are the *sides* and the points v_i are the *vertices* of the polygon. The *centroid* of a polygon is the arithmetic mean position of all the points in the figure.

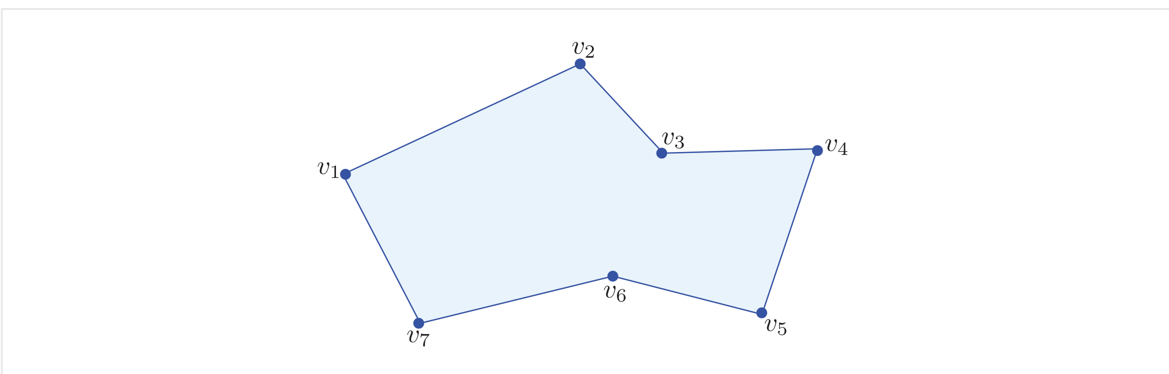

Figure 1: A polygon a with 7 sides and 7 vertices

A *regular polygon* is an n-gon which is equiangular (all of its inner angles have the same measure $\frac{360^\circ}{n}$) and equilateral (all of its sides are of the same length a). We use the notation $\Gamma_n(a)$ for a regular polygon with n sides of equal length a. Any regular polygon can be inscribed in a unique circle whose center coincides with the centroid of the regular polygon and passes through all vertices of the polygon. This unique circle with radius r is called the *circumscribed circle* of the regular polygon. Any straight line connecting a vertex of the regular polygon to its centroid is a *circumradius* of the regular polygon. The height from the center to any side is called the *apothem* of the polygon.

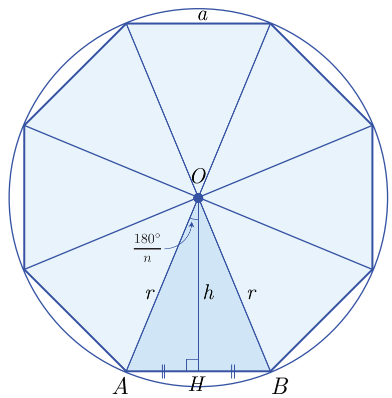

Figure 2: A regular polygon and its circumscribed circle

Elamite and Babylonian scribes were seemingly interested in computing the area of regular n-gons. This task seemed to be easy for them when $n = 3$ or $n = 4$. In fact, for $n = 4$ they had a square of side a whose area was simply computed as $S_{\Gamma_4(a)} = a^2$ while for $n = 3$, the polygon was an equilateral triangle with side a whose area could have been computed by the usual formula "half of base times height"[3] .

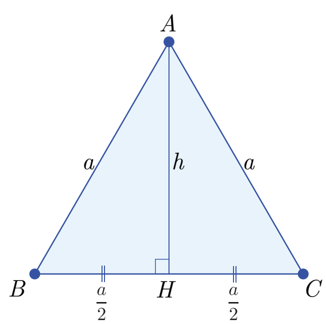

Figure 3: An equilateral triangle

It is believed that Babylonian scribes have used the Pythagorean theorem and the approximation $\sqrt{3} \approx \frac{7}{4}$ for an equilateral triangle $\Gamma_3(a)$ with side a to compute its approximate area. Because in that case, the height is obtained by the Pythagorean theorem as $h^2 = a^2 - \frac{a^2}{4}$ (see Figure **3**). So, $h = \frac{\sqrt{3}}{2}a \approx \frac{7}{8}a$ and thus the area $\frac{ah}{2}$ becomes

$$S_{\Gamma_3(a)} \approx \frac{7}{16}a^2. \tag{1}$$

[3] It is a well-known fact that ancient scribes used the known standard rules for computing the areas of basic figures: "length times width" for rectangles, "square of side" for squares, "half of base times height" for triangles, "product of diagonals" for rhombuses, and "half of height times the sum of bases" for trapezoids (see [Fri07-1], for example).

Although the area formulas for equilateral triangles and squares could have been easily obtained by scribes, the trouble would have started when they would increased the number of sides , i.e., for the cases $n \geq 5$. As has been discussed by many scholars, the ancient scribes might have been utilized the known standard method to compute the areas of polygons, namely using their circumscribed circle. Recall that to find the area of a regular n-gon with side a and circumradius r, we divide it into n equal isosceles triangles and try to find the area of this common isosceles triangle whose top angle is $\frac{360°}{n}$ (see Figure **2**). The general formula for the exact area of a regular polygon $\Gamma_n(a)$ is easily obtained as

$$S_{\Gamma_n(a)} = \frac{na^2}{4} \times \cot\left(\frac{180°}{n}\right). \tag{2}$$

3 SMT No. 2 and Elamite formula

In Figure **4**, we have reconstructed the reverse of **SMT No. 2** in which a regular heptagon $\Gamma_7(a)$: $ABCDEFG$ is inscribed in a circle with radius $r = 0;35$. If we connect the center of the circle to each vertex, we get seven equal isosceles triangles whose bases appears to have the same value $0;30$. In fact, by approximating the circumference of the circle $c_\Gamma = 2\pi r$ with that of the regular heptagon $c_{\Gamma_7} = 7a$ and using the Babylonian approximation $\pi \approx 3$, we have $7a \approx 6 \times (0;35)$ or $a = 0;30$.

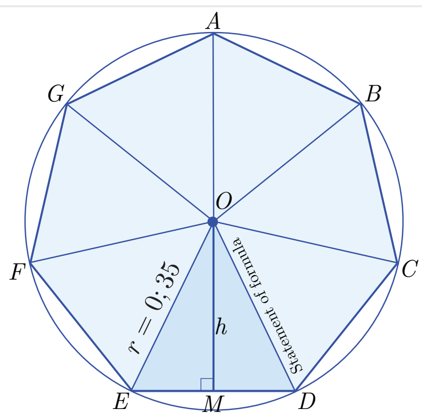

Figure 4: Reconstruction of the reverse of **SMT No. 2**

Unlike the case of regular hexagon on the obverse of this tablet in which the side a is given, here the circumradius r is known. In fact, the number $0;35$ is clearly recognizable as

35 **uš** "$0;35$ is the length"

over one of the legs of the isosceles triangle $\triangle OED$, namely $\overline{OE} = 0;35$.

Le us use this data and the Babylonian method to compute the area of the isosceles triangle $\triangle OED$ and the regular heptagon $\Gamma_7(0;30)$. To do so, consider the isosceles triangle $\triangle OED$ as shown in Figure **4**. Since the height OM bisects the base ED,

$\overline{EM} = \overline{DM} = \frac{0;30}{2} = 0;15$. By using the Pythagorean theorem in the right triangle $\triangle OME$, we have

$$\begin{aligned} h &= \sqrt{\overline{OE}^2 - \overline{EM}^2} \\ &\approx \sqrt{(0;35)^2 - \frac{(0;30)^2}{4}} \\ &= \sqrt{\frac{1}{4}\Big[4 \times (0;20,25) - 0;15\Big]} \\ &= \frac{1}{2}\sqrt{1;21,40 - 0;15} \\ &= \frac{1}{2}\sqrt{1;6,40}. \end{aligned}$$

Hence, the height h is

$$h \approx \frac{1}{2}\sqrt{1;6,40}, \tag{3}$$

To get the approximate value for h, we need to apply the approximate formula $\sqrt{1+x} \approx 1 + \frac{x}{2}$ to (3) which is the Babylonian method.[4] Thus, we can continue as follows:

$$\begin{aligned} h &\approx \frac{1}{2}\sqrt{1;6,40} \\ &= \frac{1}{2}\sqrt{1 + 0;6,40} \\ &\approx \frac{1}{2} \times \left(1 + \frac{0;6,40}{2}\right) \\ &= \frac{1}{2} \times (1;3,20) \\ &= 0;31,40. \end{aligned}$$

So, we get the following approximate value of the height h as

$$h \approx 0;31,40. \tag{4}$$

This easily implies the area of the isosceles triangle $\triangle OED$ as follows:

$$\begin{aligned} S_{\triangle OED} &= \frac{1}{2} \times \overline{ED} \times h \\ &\approx \frac{1}{2} \times (0;30) \times (0;31,40) \\ &= (0;15) \times (0;31,40) \\ &= 0;7,55. \end{aligned}$$

[4] It is a fact that Babylonians used the formula $\sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}$ for approximating irrational square roots like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and so on. See [FR98], for a discussion on this topic.

That is,

$$S_{\triangle OED} \approx 0;7,55. \tag{5}$$

Finally, we can compute the approximate area of the regular heptagon $\Gamma_7(0;30)$ by using (5):

$$\begin{aligned} S_{\Gamma_7(0;30)} &= 7 \times S_{\triangle OED} \\ &\approx 7 \times (0;7,55) \\ &= 0;55,25. \end{aligned}$$

Therefore, we get

$$S_{\Gamma_7(0;30)} \approx 0;55,25. \tag{6}$$

An approximate formula for the area of a general regular heptagon $\Gamma_7(a)$ of side a can be obtained from (6). In fact, since $a = 2a \times (0;30)$, the area of the regular heptagon $\Gamma_7(a)$ can be computed as follows:

$$\begin{aligned} S_{\Gamma_7(a)} &= (2a)^2 \times S_{\Gamma_7(0;30)} \\ &\approx 4 \times (0;55,25) \times a^2 \\ &= (3;41,40) \times a^2. \end{aligned}$$

If we ignore the second sexagesimal numeral 40, we get the following Babylonian approximate formula

$$S^B_{\Gamma_7(a)} \approx (3;41) \times a^2. \tag{7}$$

If $a = 1$, the approximate area becomes $3;41$ which is the very geometric coefficient $3;41$ listed in **SMT No. 3** as the constant of a regular heptagon. This confirms that (7) would have been used to approximate the area of a regular heptagon.

Although we do not know whether the previous numbers in (4), (5) and (6) have been written down on the missing part of the tablet, we can read a formula for the area of a regular heptagon below the missing part:

<u>Reverse</u>
(L1) [s]ag-7 *a-na* 4 *te-*[*ṣi*]*-ip-ma*
(L2) *ší-in-šé-ra-ti*
(L3) *ta-na-as-sà-ah-ma* a-šà

<u>Translation:</u>
"A (regular) heptagon. You multiply (the square of a side) by 4 and you subtract one-twelfth (of the result from the result itself), and (you see) the area."

In other words, to find the area of a regular heptagon $\Gamma_7(a)$ with side of length a one needs to multiply a by 4 and then subtract the twelfth of $4a$ from itself. If we translate this instruction in mathematical language and perform the related calculations, we get:

$$
\begin{aligned}
S^{E}_{\Gamma_7(a)} &= 4 \times a^2 - \frac{1}{12} \times \left(4 \times a^2\right) \\
&= \left(4 - \frac{4}{12}\right) \times a^2 \\
&= \left(\frac{48-4}{12}\right) \times a^2 \\
&= \frac{44}{12} a^2 \\
&= \frac{11}{3} a^2 \\
&= (3;40) \times a^2.
\end{aligned}
$$

This gives us another formula for the area of a regular heptagon as follows

$$S^{E}_{\Gamma_7(a)} = (3;40) \times a^2, \tag{8}$$

which from now on we call the *Elamite formula*.

Although the Elamite formula (8) seems to be only another approximate formula for the area of a regular heptagon, it provides us with a good approximate value which is more accurate than not only the Babylonian formula g (7) but also the Greek formula (see [Hea21], Page 328) due to Heron of Alexandria (circa 50 AD):

$$S^{H}_{\Gamma_7(a)} = \frac{43}{12} a^2 = (3;35) \times a^2. \tag{9}$$

To see the accuracy of these approximate formulas, we compute their error percentages. First, note that by (6) the accurate area of the regular heptagon $\Gamma_7(a)$ to four sexagesimal places is

$$
\begin{aligned}
S^{A}_{\Gamma_7(a)} &= \frac{7}{4} \times \cot\left(\frac{180^\circ}{7}\right) \times a^2 \\
&= (1;45) \times (2;4,35,28,37,17,\cdots) \times a^2 \\
&= (3;38,2,5,5,16,\cdots) \times a^2 \\
&\approx (3;38,2,5,5) \times a^2,
\end{aligned}
$$

then observe that

$$
\begin{aligned}
e^{H}_{7} &= \left| \frac{S^{A}_{\Gamma_7(a)} - S^{H}_{\Gamma_7(a)}}{S^{A}_{\Gamma_7(a)}} \right| \times 100\% \\
&\approx \frac{(3;38,2,5,5 - 3;35) \times a^2}{(3;38,2,5,5) \times a^2} \times 100\% \\
&= \frac{0;3,2,5,5}{3;38,2,5,5} \times 100\% \\
&\approx (0;0,50,6,25) \times 100\% \\
&\approx 1.39\%,
\end{aligned}
$$

and

$$
\begin{aligned}
e_7^B &= \left| \frac{S^A_{\Gamma_7(a)} - S^B_{\Gamma_7(a)}}{S^A_{\Gamma_7(a)}} \right| \times 100\% \\
&\approx \frac{(3;41 - 3;38,2,5,5) \times a^2}{(3;38,2,5,5) \times a^2} \times 100\% \\
&= \frac{0;2,57,54,55}{3;38,2,5,5} \times 100\% \\
&\approx (0;0,48,57,35) \times 100\% \\
&\approx 1.36\%,
\end{aligned}
$$

as well as

$$
\begin{aligned}
e_7^E &= \left| \frac{S^A_{\Gamma_7(a)} - S^E_{\Gamma_7(a)}}{S^A_{\Gamma_7(a)}} \right| \times 100\% \\
&\approx \frac{(3;40 - 3;38,2,5,5) \times a^2}{(3;38,2,5,5) \times a^2} \times 100\% \\
&= \frac{0;1,57,54,55}{3;38,2,5,5} \times 100\% \\
&\approx (0;0,32,26,55) \times 100\% \\
&\approx 0.9\%.
\end{aligned}
$$

It is clear that among three values e_7^H, e_7^B and e_7^E, the last one is the smallest error percentage proving that formula (8) produces the most accurate value for the area of a regular heptagon. In fact, the error is less than one percent which is remarkable and shows the impressive work of Susa scribes.

4 Geometric Explanations for Elamite formula

Unlike Heron's formula, we do not know how Susa scribes have come up with formula (8) on this tablet. It seems that Heron[5] utilized two approximations $r \approx \frac{8}{7}a$ and $\sqrt{23} \approx \frac{43}{9}$ for deriving formula (9):

$$
\begin{aligned}
S^H_{\Gamma_7(a)} &\approx 7 \times \frac{a}{2} \times \sqrt{\frac{64a^2}{49} - \frac{a^2}{4}} \\
&= \frac{7a^2}{2}\sqrt{\frac{9 \times 23}{4 \times 49}} \\
&= \frac{3\sqrt{23}}{4}a^2 \\
&\approx \frac{3}{4} \times \frac{43}{9} \times a^2 \\
&= \frac{43}{12}a^2.
\end{aligned}
$$

[5] For more details, see [Hea21].

It is interesting that Heron has used the approximation $a \approx \frac{7}{8}r$ to construct a regular heptagon of side a inscribed in a circle of radius r. In other words, he has used the apothem of the inscribed regular hexagon as the side of the regular heptagon (see Figure **14**). As we will see in §**6**, this choice turns out to be a pretty good approximation.

Although Susa scribes have not explained how they have found their formula, one may use geometric explanations to obtain the Elamite formula for the area of a regular heptagon with side a. We try to give our explanation.

According to the statement of the formula, one first needs to consider the area $4a^2$ and then subtract $\frac{1}{12}(4a^2)$ from it in order to find the approximate formula of a regular heptagon. This suggests that we first need to consider a geometric figure with area $4a^2$ which can be easily divided into 12 equal parts. While there are many different geometric shapes whose area is $4a^2$, the second condition can lead us to choose one of the following basic figures, i.e.,
1) a square with side $2a$; or
2) a rectangle with sides a and $4a$.

The main advantage of these choices is that we can easily divide them into equal parts by using vertical and horizontal lines. As is shown in Figure **5**, a square and a rectangle are divided in 12 equal small rectangles $R_1, R_2, \ldots, R_{12}$ by horizontal and vertical lines. In each case, if one-twelfth of the shape is removed, the remaining part is claimed to be an approximation for the area of the regular heptagon with side a. We have removed the last rectangle R_{12} in each figure.

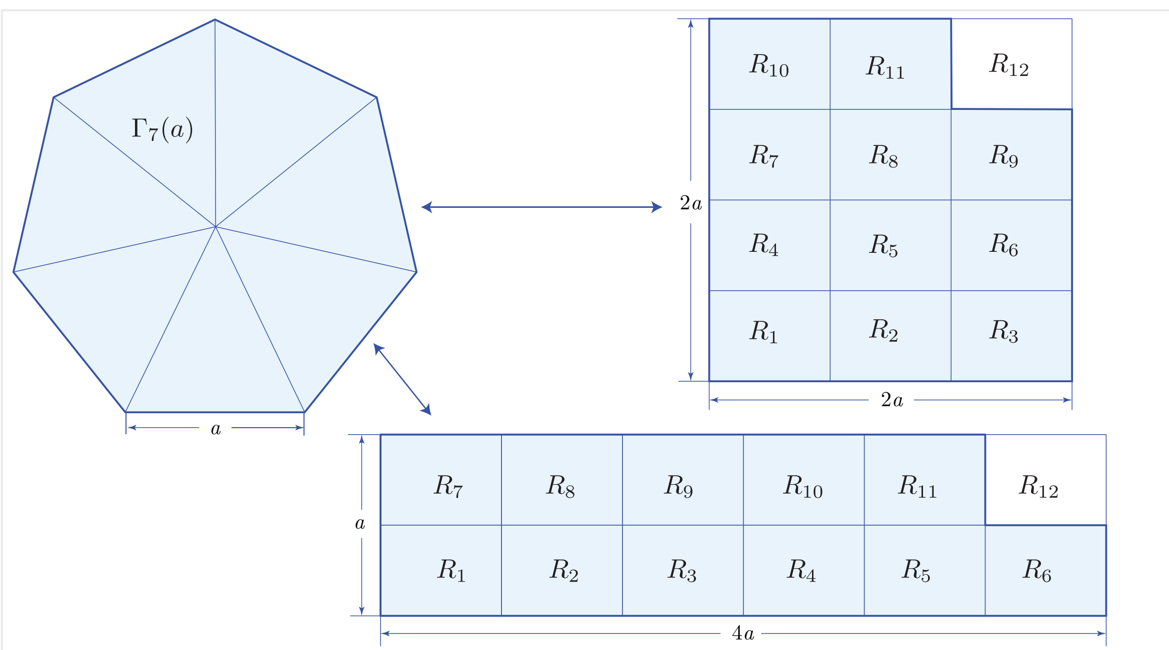

Figure 5: Two approximations for a regular heptagon

We use a geometrical approach, which we would like to call *cut and paste*, to verify the Elamite formula. This method is very useful when we want to show that two figures have the same (or almost the same) area. We usually have an original figure and a goal figure whose area approximates that of the original figure. The main idea of this method is to cut the original figure into smaller pieces and then to paste them together in a way so as to (almost) fill up the goal figure. The most obvious example for this method might be an isosceles triangle and a rectangle. If we cut the isosceles triangle along its height, we get two equal right triangles. By attaching these two right triangles along their hypotenuses, we obtain a rectangle with the same area as the original isosceles triangle. In the current problem, the original figure is a regular heptagon and the goal figure is either a square without a corner or a rectangle without a corner (see Figure **5**).

Start with a regular heptagon $\Gamma_7(a)$ and connect its center to each vertex by straight lines. Then slice off the regular heptagon $\Gamma_7(a)$ along those lines to get seven equal isosceles triangles whose bases are of length a. Now, cut off two or four of these isosceles triangles along their heights to get four or eight equal right triangles. So we obtain 9 or 11 triangles as shown in Figure **6**.

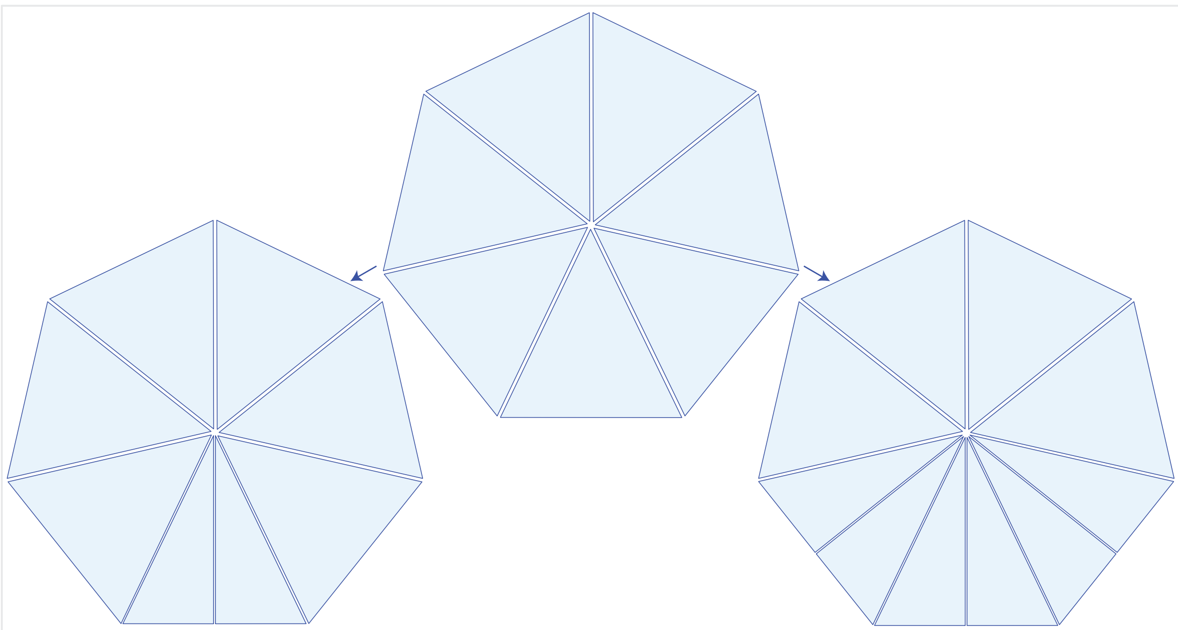

Figure 6: Cutting off a regular heptagon in two ways

Now, consider a grid pattern of squares whose sides are of length $\frac{a}{12}$.[6] We consider our figures in this grid pattern. For each figure, we try to lay out the obtained triangles on the grid so as to fill up the figure (see Figure **7**).[7] For the square without corner, there are five isosceles triangles and four right triangles, while for the rectangle without corners we have used three isosceles triangles and eight right triangles. Note that in each case a part of the colored area goes out of the red hexagon $ABCDEF$ and another part of it is not covered by colored area.

[6] In fact, any multiple of 12 works and the bigger the number, the better the approximation.

[7] In this picture we have scaled the dimensions to help the reader clearly see what is happening.

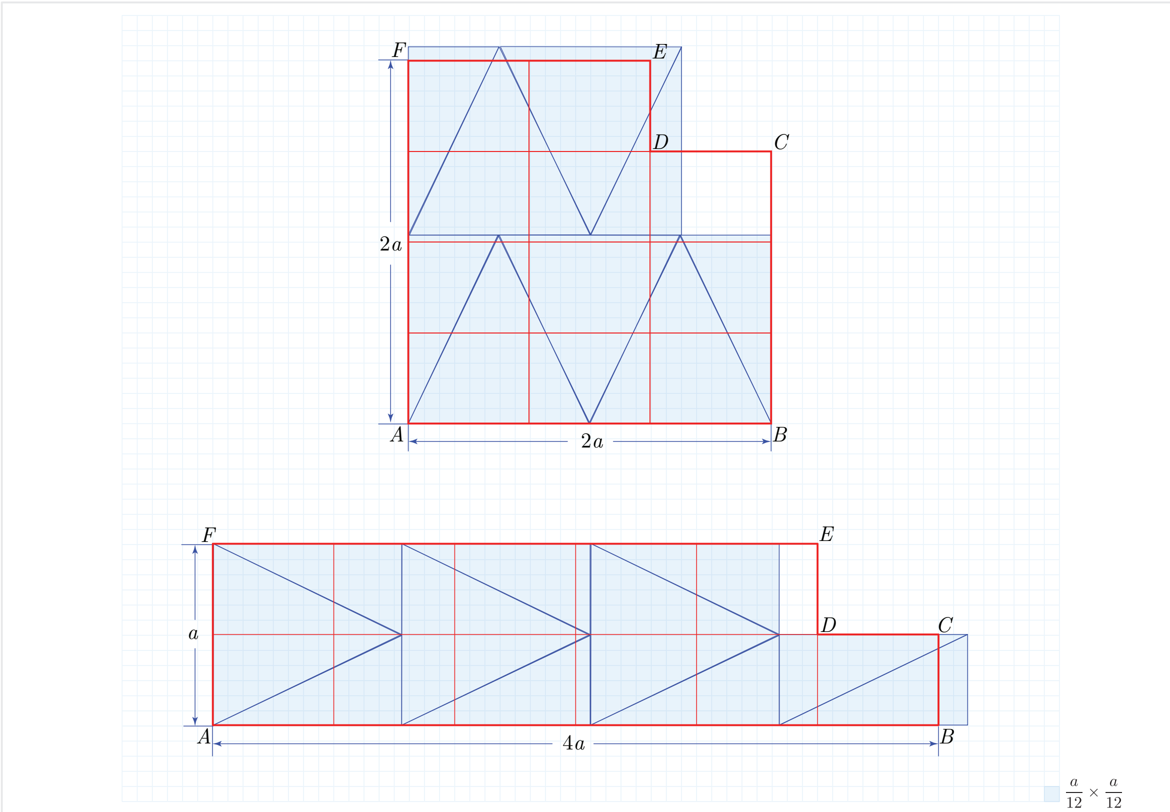

Figure 7: Square and rectangular layouts of triangular pieces

Finally, we cut the small colored squares outside the hexagon $ABCDEF$ and set them out in the blank region inside it. In fact, we can easily count the colored and blank small squares and rectangles in each case.

Case 1. Square layout

According to the upper layout in Figure **7**, we count as follows:
i) complete small colored squares: 12
ii) almost complete small colored squares: 18
iii) complete blank squares: 30
iv) almost blank half squares: 6 (= 3 almost complete blank squares).
So there are totally 30 colored squares and 33 blank ones in this case. After arranging the colored ones inside the blank region, there remains three blank small squares.

Case 2. Rectangular layout

According to the lower layout in Figure **7**, we count as follows:
i) almost complete small colored squares: 12
ii) complete blank squares: 12
iii) almost blank half squares: 6 (= 3 almost complete blank squares).
So there are totally 12 colored squares and 15 blank ones in this case. After arranging the colored ones inside the blank region, there remains three blank small squares.

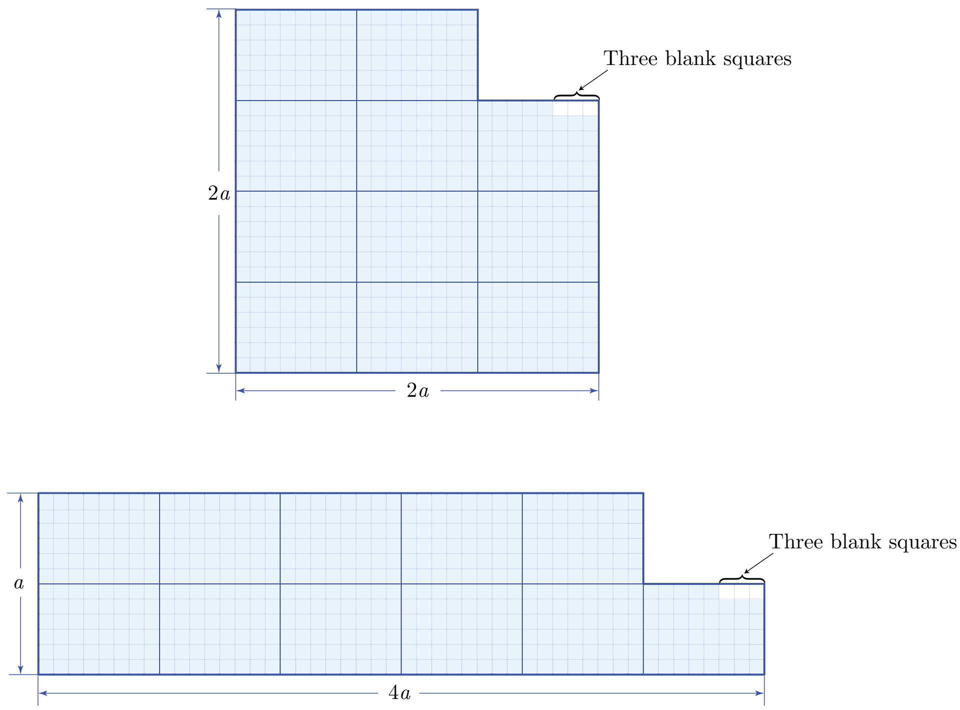

Figure 8: Two layouts of a regular heptagon

In both case, there are three blank small square of side $\frac{a}{12}$ left and their total areas is (see Figure 8)

$$3 \times \left(\frac{a}{12}\right)^2 = \frac{3a^2}{144} = \frac{a^2}{48}.$$

Therefore, we have

$$S_{\Gamma_7(a)} = S_{ABCDEF} - \frac{1}{48}a^2 \approx S_{ABCDEF}.$$

Note that the error in this approximation is

$$\left|S_{\Gamma_7(a)} - S_{ABCDEF}\right| = \frac{1}{48}a^2$$

and the error percentage is

$$\frac{\left|S_{\Gamma_7(a)} - S_{ABCDEF}\right|}{S_{ABCDEF}} \times 100\% = \left(\frac{\frac{1}{48}a^2}{\frac{11}{3}a^2}\right) \times 100\% = \frac{3}{528} \times 100\% \approx 0.57\%,$$

which is surprisingly small.

By comparing Figure 7 with Figure 8, the reader may realize why Susa scribes have used Elamite formula (8) to approximate the area of a regular heptagon. It should be emphasized that we do not know what is the real reasoning behind the Elamite formula and the previous geometric explanations are our take. We believe that there might have been a geometric intuition behind this formula confirming that Susa scribes must have had a great deal of mathematical experience and skills. This geometric intuition is one of the major factors enabled them to come up with such a beautiful and accurate formula.

Remark 2. The reader should note that the geometric explanation we gave is one of possible explanations and there are definitely many other ways to justify this formula. For instance, Figure **9** shows another visual representation for this approximation.

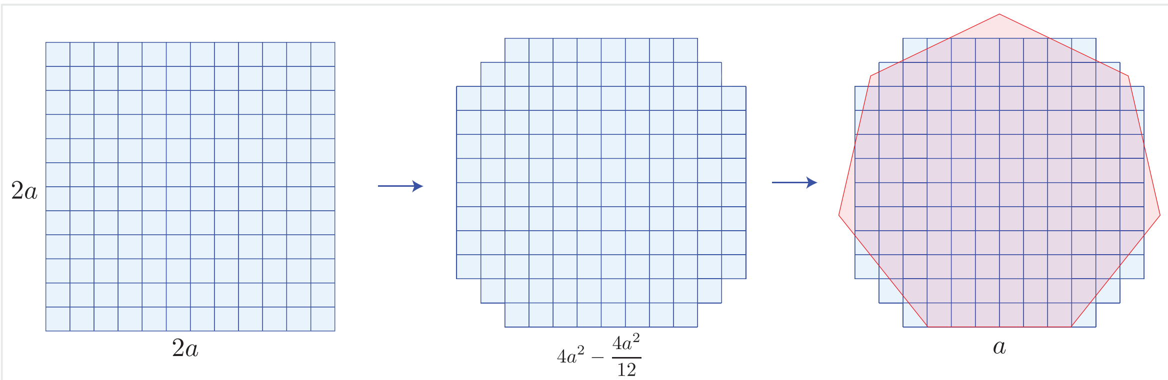

Figure 9: Another visual representation for Elamite formula

5 Significance of Elamite formula

From the history of mathematics point of view, this formula might be of great importance showing that Elamite scribes not only had a great deal of mathematical skills but also were masters of geometry. It seems that geometry have played a major role in the lives of Elamite people, specially Elamite artisans, as wide application of geometric patterns attest in the Elamite art.[8] Unfortunately, this social and cultural characteristic of Elamite civilization and in particular Susa scribes has not been fully appreciated by other scholars who merely consider Susa mathematical tablets as a part of Babylonian mathematics. Elamite civilization as an independent ancient entity in southwestern Iran has a very long history and its origin dates more or less to that of Sumerians in Mesopotamia. Like other ancient civilizations in the Near East, Elamites had their own culture, customs, traditions and writing systems one of which is recently argued to be the first known phonetic writing system in history.[9] . Although Elamites were living as a western foreign neighbor to Mesopotamian civilizations (Sumer, Akkad, Babylonia, and

[8] The reader can consult the book "Art of Elam" in which there are many colored plates representing Elamite artifacts [Álv20].

[9] The reader can consult the newly published paper *The Decipherment of Linear Elamite Writing* [DTKBM] whose authors have deciphered this ancient inscription for the first time. This writing system writing was used in southern Iran in the late third/early second millennium BC (ca. 2300–1880 BC).

Assyria) for almost three millennia and had cultural, social, and literature interactions with them, all the attested features of this civilization lead us to study its mathematics independently and give them credit for their great mathematical contributions to the ancient world.

The Elamite formula for the area of a regular heptagon shows high mathematical skills of Susa scribes in computing areas of geometric figures, however other scholars seem to have not appreciated its significance in their interpretations completely. For example, in [Fri07-1, Fri07-2, PAT92, FR98], this formula is considered to produce only another Babylonian constant for the area of a regular heptagon. In [BR61], they only give the transliteration of the formula without any further information. Most scholars think that the constant $3;40$ is an approximate value of the more familiar constant $3;41$ and they do not consider it an independent approximation and ignore the fact that it can be derived from a geometric idea. The main reason to defy this opinion is that the Susa scribe of this tablet has given an instruction on the tablet that produces this constant. Additionally, the discussion above clearly demonstrates the impressive accuracy of this approximate formula and the geometric beauty behind it, which is not clear at the first sight. By studying geometric patterns in the Elamite art and the mathematical skills of Susa scribes to compute the areas of complicated figures (specially a list of geometric constants given in **SMT No. 3**), we eventually came to the logical conclusion that this formula was independently derived by using geometric skills and mathematical experience.

6 Construction of regular polygons

To construct a regular n-gon by compass and straightedge, it is usually very helpful to use a circle, because in this method we just need to divide the circumference of the circle into n equal arcs. After doing that, we connect the adjacent points to get n equal chords of the circle which are the sides of the required regular n-gon.

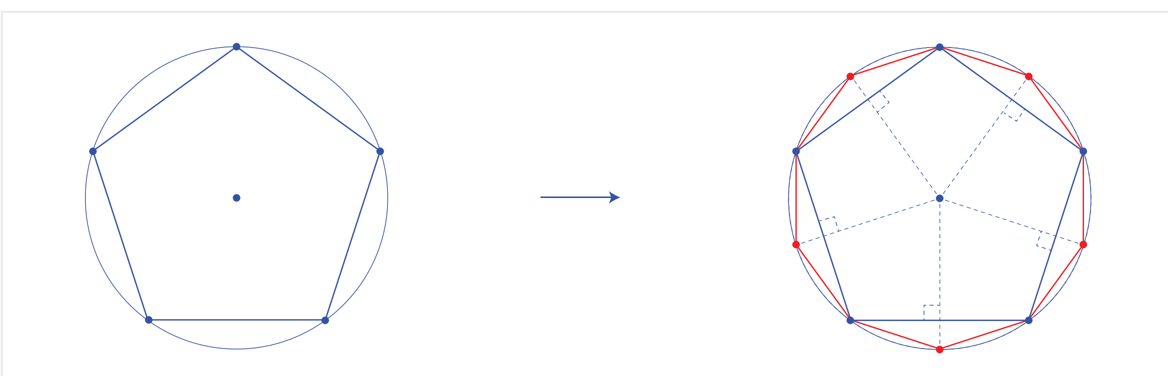

Figure 10: Construction of a regular $2n$-gon

We can construct new regular polygons out of given ones. There are two basic methods to do so. Firstly, when we have a regular n-gon, we can easily find the midpoint of each chord by using straight lines passing through the center and perpendicular to each side of n-gon. These new midpoints along with the vertices of the main n-gon can be used to construct a regular $2n$-gon (see Figure **10**). By repeating this process, one can construct any $2^m n$-gon for any natural number m.

For the second method, let n and m be two coprime natural numbers where $n > m > 2$. Also assume that we have constructed a regular n-gon and a regular m-gon in the same circumscribed circle. We can use these two polygons to construct a regular nm-gon as follows:

(1) First construct a regular n-gon with vertices $v_1, v_2, \cdots, v_n$.

(2) Construct a regular m-gon with vertices $w_1, w_2, \cdots, w_m$ where $w_1 = v_1$, i.e., they have a common vertex.

(3) Repeat step 2 with $v_2, \cdots, v_n$ as the first vertex each time.

(4) Mark the mn vertices obtained and connect the adjacent ones to form a regular mn-gon.

The key point in the second method is that the obtained vertices in each step never coincide with the previous ones and we end up with exactly nm points separating equal arcs of the circle (see Figure **11**).

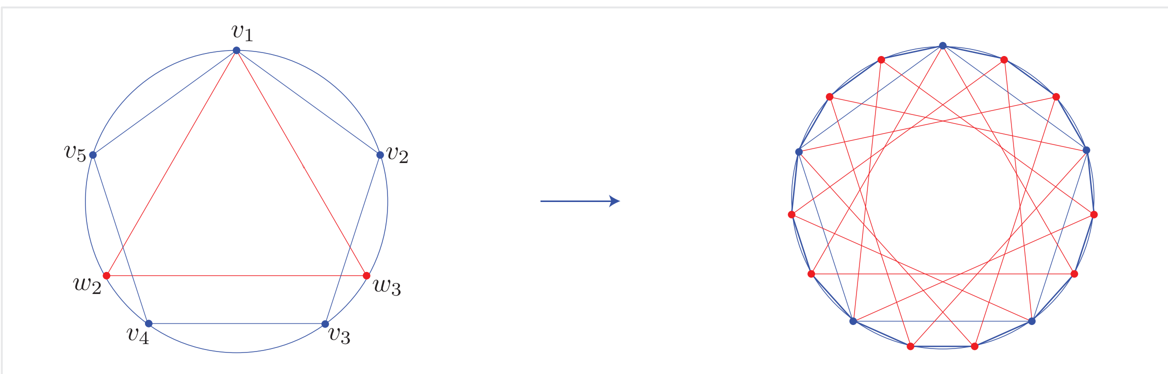

Figure 11: Construction of a regular mn-gon

Among all the basic regular polygons, the construction of the regular hexagon is probably the easiest one, because its side and the radius of its circumscribed circle are equal. This unique characteristic enables us to use the same circle to divide the circumference of the circumscribed circle into 6 equal arcs. We can do as follows: choose an arbitrary point on a circle of radius r. Then put the compass to distance r in the point and draw an arc intersecting the circumscribed circle in a new point. Repeat this process five more times to get six points and connect adjacent points to get a regular hexagon.

It is interesting that the construction of a regular hexagon plays an important role in constructing other regular n-gons. In fact, by connecting odd or even-labeled vertices we always get a regular 3-gon (equilateral triangle). We can also use a regular hexagon to construct a regular 4-gon (square) if we use the first and third vertices and the midpoints of the second and fifth arcs of the circumscribed circle (see Figure **12**).

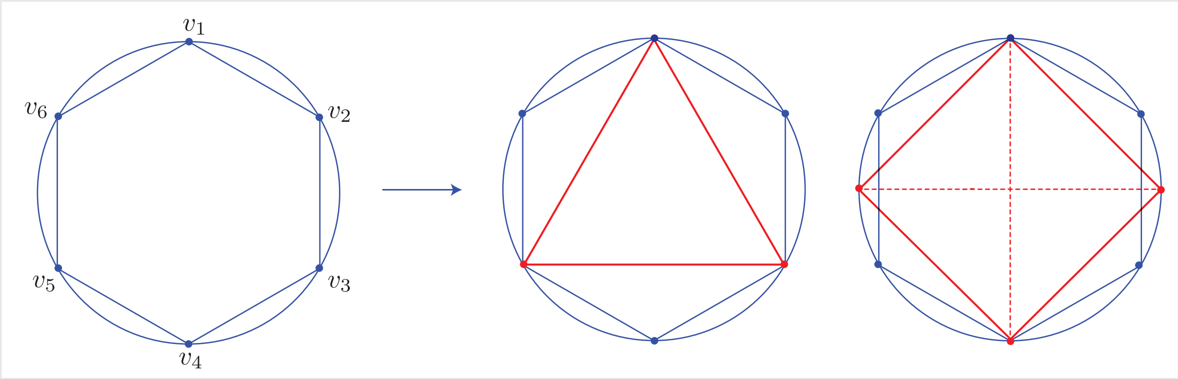

Figure 12: Construction of a regular 3-gon and 4-gon using a regular hexagon

A regular pentagon is also constructible using a compass and straightedge, however it is not as easy as hexagon case. The main part in this process is to construct the Golden ratio $\frac{\sqrt{5}-1}{2}$. Greek mathematicians gave different methods for this problem among which Ptolemy's method seems rather simple. In this method, we start with a circle of radius r and consider the diagonal AB and the point C which is the midpoint of the upper arc AB. Then we connect C to the midpoint of OB, say D. Name the intersection point of the circle centered at D with radius CD with radius AO as E. It is easy to show that

$$\frac{\overline{OE}}{\overline{OB}} = \frac{\overline{EB}}{\overline{OE}} = \frac{\sqrt{5}-1}{2}$$

and the line segment CE is the side of the inscribed pentagon. By using a compass, we can mark five points on the circle separating five equal arcs and construct a regular pentagon (see Figure **13**).

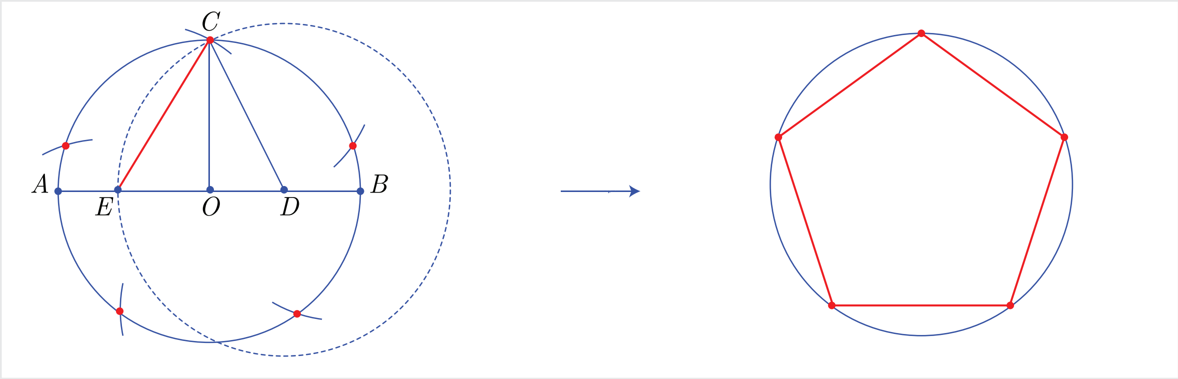

Figure 13: Ptolemy's construction of a regular pentagon

So far, we can construct any regular n-gon inscribed in a given circle for all *regular numbers* $n = 2^p 3^q 5^r \geq 3$, where p, q, r are non-negative integers. The first number that is not in this list is the prime number 7. That might be the reason that the construction of a regular heptagon has been one of the most interesting problems in the history of mathematics and many mathematicians have tried to tackle it. This problem had obsessed mathematicians for a long time until the German mathematician Gauss showed that it is impossible to construct a heptagon with a compass and a straightedge. So, all the constructions before Gauss were approximations, although they were considered accurate for a long time. As usual, Greek mathematicians such as Archimedes and Heron have played a major role in developing this theory and many others including Muslim mathematicians followed them to give their own constructions[10] . Although Archimedes's method is not simple[11] , the construction of Heron is easy and practical. He uses a regular hexagon inscribed in a circle of radius r and takes the apothem of the regular hexagon as the side of the regular heptagon (see Figure **14**). The length of this line segment is $\frac{\sqrt{3}}{2}r$ and the error of the approximation is around 2%. Heron uses the approximation $\sqrt{3} = \frac{7}{4}$ to compute the side of the regular heptagon as $a = \frac{7}{8}r$.

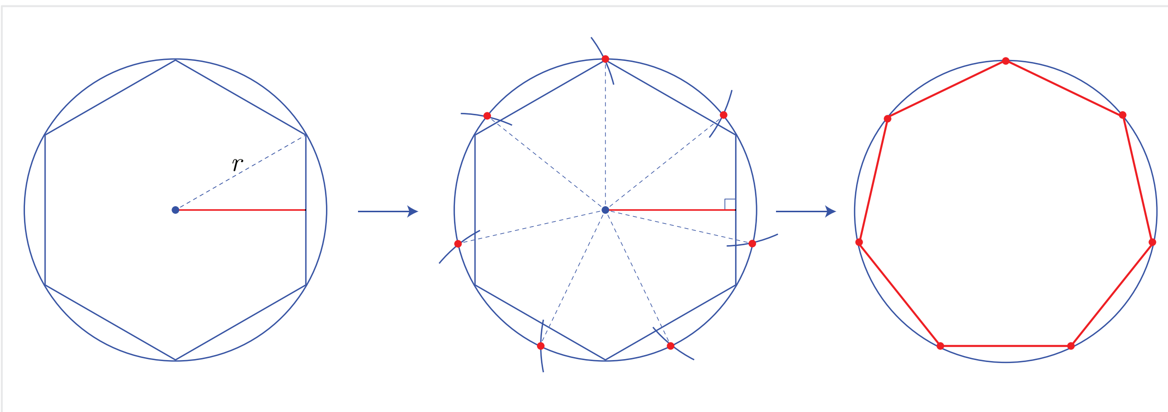

Figure 14: Heron's construction of a regular heptagon

Besides mathematicians, artists also tried to solve this problem one of whom was the German artist Albrecht Dürer. In one of his books on geometry, he gave different ways to construct regular polygons. The method for the regular heptagon is given in Figure **15**. It is interesting that he has used an inscribed regular hexagon to construct his regular heptagon. He uses the odd-labeled vertices v_1, v_2, v_3 of hexagon to construct an equilateral triangle. Then he draws the line passing through the center O and the vertex v_2 of the hexagon intersecting the side between vertices v_1, v_3 at a point, say A. He finally uses the line segment between A and the vertex v_1 as the side of the required regular heptagon. This method is similar to Heron's construction, because the side of regular heptagon here is also the common height of the six equilateral triangles forming the regular hexagon.

[10] See [Hog84], for a history and list of such constructions.
[11] See [Hol10], for a modern explanation of Archimedes's construction.

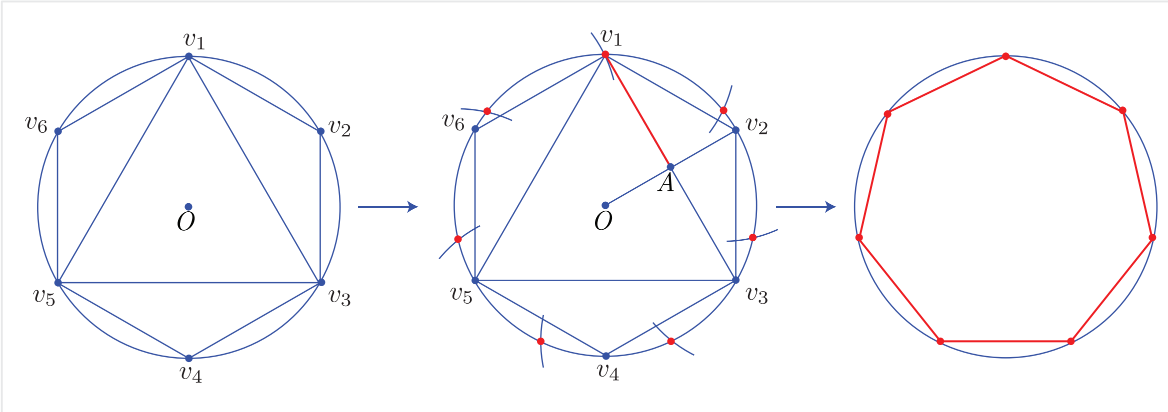

Figure 15: Dürer's construction of a regular heptagon

It is clear that the Susa scribe of this tablet has used approximate methods to construct the regular polygons on this clay tablet. As we saw, he uses the approximations to compute the area of regular polygons. In the case of hexagon, fortunately the approximate value of the side a and the radius r of the circumscribed circle are the same. So he could have simply taken advantage of this fact and assumed the radius of the circumscribed circle to be the very side of the regular hexagon in order to construct the figure. In other words, although he has used an approximation, he has obtained an accurate construction.

In the heptagon case, the radius of the circumscribed circle r was approximated by $r \approx \frac{7}{6}a$, where a is the side of the regular heptagon. If he would have tried to use this approximate value as the radius of the circumscribed circle and done a similar process to the one in the hexagon case, as we can see in Figure **16**, the last arc centered at G would have intersected the circle in a point H not coincide with the initial point A and there would be a small gap between H and A. The cause of this small gap is the approximation $r \approx \frac{7}{6}a$. In fact, in the isosceles triangle $\triangle OAB$ in Figure **16**, we have $\sin\left(\frac{\alpha}{2}\right) = \frac{3}{7}$ implying that the central angle of the arc AB is $\alpha = 2 \times \arcsin\left(\frac{3}{7}\right) \approx 50.76°$. So the total of all central angles is $7\alpha \approx 355.32°$, which is $4.68°$ less than a complete rotation $360°$.

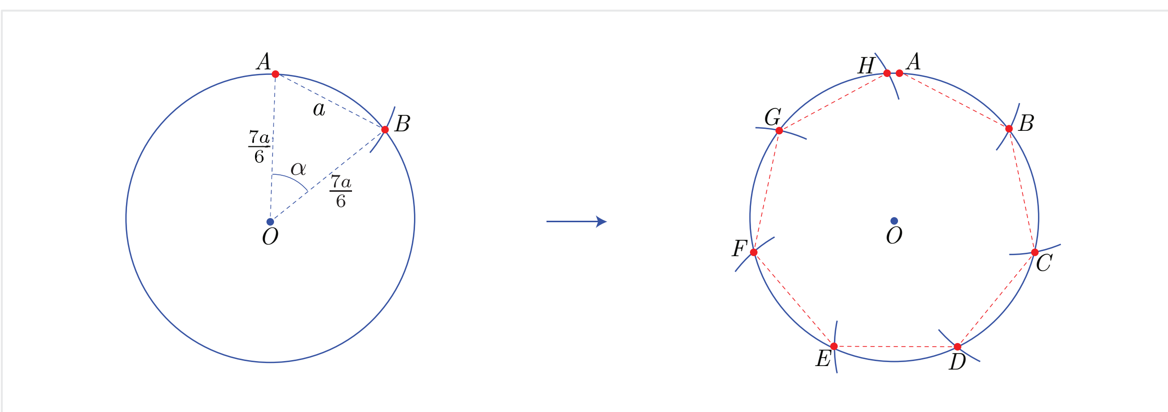

Figure 16: Approximate construction of a regular heptagon

To complete the figure and get a heptagon, one could connect G to either A or the midpoint of arc HA. In both cases, by connecting the adjacent points of the seven obtained points, we get an approximation of a regular heptagon (see Figure **16**). This might have been the way Susa scribes have constructed the regular heptagon on the reverse of this tablet. It should be noted that the error of this method is negligible when one constructs this figure on a small clay tablet with dimensions 12cm $\times$ 12cm!

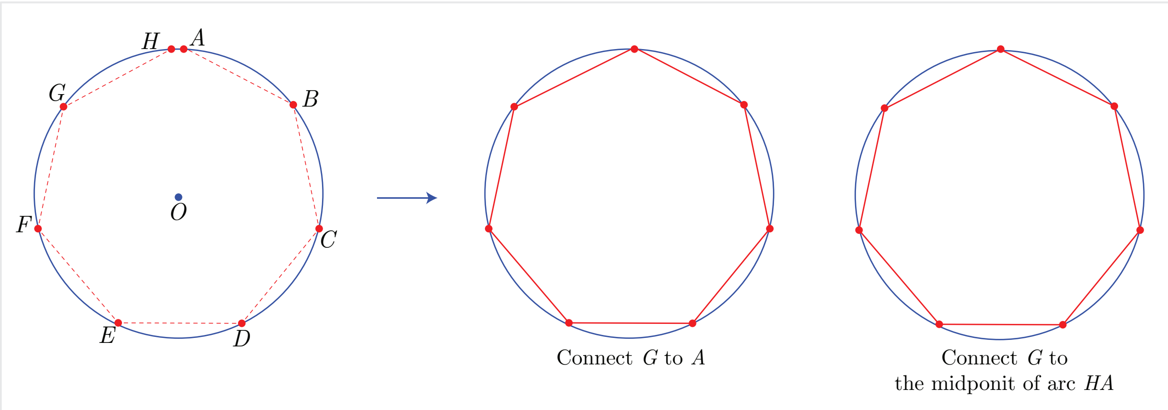

Figure 17: Possible Elamite construction of a regular heptagon

7 Conclusion

The numerical data and the figures on this tablet as well as the related mathematical calculations reveal that Susa scribes might have applied a standard method to compute the approximate areas of certain polygons. Besides the standard formulas, they seemed to have used a simple and interesting formula for the area of a regular heptagon which surprisingly gives a more accurate approximate value in comparison to other ancient formulas including Heron's formula appearing in the history of mathematics almost 1800 years later. Although the significance of this achievement has not been appreciated by most scholars, we hope this article sheds light on the importance of this formula and convinces mathematical historians to reconsider its position in the history of mathematics.

On the other hand, the geometrical figures and their rather accurate dimensions on this tablet suggest that Susa scribes were well familiar with regular polygons and able to construct them on clay tablets with accuracy. This characteristic is also acknowledged by other scholars, for example, in [Fri07-1, Fri07-2, Høy02]. Not only they apparently had the ability to divide a circle into n almost equal sectors for regular numbers, but also could have done the task for irregular numbers such as $n = 7, 11, 14$.[12] Although they had to use approximate values in irregular cases, it seems that they had practical skills to carry out this task very well.

[12] In an ongoing book project on Elamite mathematics, the authors have studied and categorized more than 300 geometric patterns bearing on Elamite artifacts and artworks. Many of these patterns contain specific designs whose structures involve dividing a circle into equal arcs.

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