SIMPLE CONTINUED FRACTIONS AN APPROACH FOR HIGH SCHOOL STUDENTS

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ABSTRACT

This paper studies the work of the French mathematician This paper introduces high school students to continued fractions and develops basic properties of Finite Continued Fractions and Infinite Continued Fractions. This also includes computation of the quadratic number with a given periodic continued fraction, conjugate quadratic numbers, and approximation of reals and convergent of continued fractions. Through the study of continued fractions, you will gain increased insight into the proper relationships of number systems that are emphasized in today's modern mathematics courses. I believe that you will find continued fractions fun to work with.

Keywords mathematics · continued · fractions · finite · infinite · problems

1 Introduction

In mathematics, a continued fraction is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part another reciprocal, and so on. In a finite fraction, the iteration is terminated after finitely many steps by using an integer in lieu of another continued fraction. In contrast, an infinite continued fraction is an infinite expression. In either case, all integers in the sequence, other than the first, must be positive. The integers a_i called the coefficients or terms of the continued fraction.

It is generally assumed that the numerator of all the fractions is 1. If arbitrary values and/or functions are used in place of one or more numerators or integers in the denominators, the resulting expression is a generalized continued fraction. When it is necessary to distinguish the first form generalized fractions, the former may be called simple or regular continued fraction or said to be in a canonical form.

Continued fractions have a number of remarkable properties related to the Euclidean algorithm for integers or real numbers. Every rational number $\frac{p}{q}$ has two closely related expressions as a finite continued fraction, whose coefficients $a + i$ can be determined by applying the Euclidean algorithm to (p, q). The numerical value of an infinite continued fraction is irrational; it is defined from its infinite sequence of integers as the limit of a sequence of values for finite continued fractions. Each finite continued fraction of the sequence is obtained by using a finite prefix of the infinite continued fraction's defining sequence of integers. Moreover, every irrational number a is the value of a unique infinite regular continued fraction, whose coefficients can be found using the non-terminating version of the Euclidean algorithm applied to the incommensurable values α and 1. This way of expressing real numbers (rational and irrational) is called their continued fraction representation. Pettofrezzo and Byrkit [1970]

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2 Generalized continued fraction

The term "continued fraction" is used to refer to a class of expressions in which a generalized continued fraction of the form.

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a3 + \cdots}}}$$

(and the terms may be integers, reals, complexes, or functions of these) are the most general variety Rockett et al. [1992].

Wallis first used the term "continued fraction" in his Arithmetica infinitorum of 1653 Havil [2003], , although other sources list the publication date as 1655 or 1656. An archaic word for a continued fraction is anthyphairetic ratio.

3 Simple Continued Fractions

A simple continued fraction is a special case of a generalized continued fraction for which the partial numerators are equal to unity, i.e., $b_n = 1$ for all $n = 1, 2, ...$. A simple continued fraction is therefore an expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a3 + \cdots}}}$$

When used without qualification, the term "continued fraction" is often used to mean "simple continued fraction" or, more specifically, regular (i.e., a simple continued fraction whose partial denominators $a_0, a_1, ...$ are positive integers;Rockett et al. [1992]). The number of terms can either be finite or infinite. A more convenient way to denote continued fractions such as the one above would be to denote it by: $N = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, \cdots]$

Finite Continued Fractions

A finite continued fraction is an expression such as the one shown above which could end. Every rational number can be equated to a finite continued fraction. The only skill needed would be the division of fractions.

$$\frac{47}{17} = 2 + \frac{13}{17} = 2 + \frac{1}{\frac{17}{13}} = 2 + \cfrac{1}{1 + \frac{4}{13}} = 2 + \cfrac{1}{1 + \frac{1}{\frac{13}{4}}} = 2 + \cfrac{1}{1 + \frac{1}{3 + \frac{1}{4}}}$$

In the short form, $\frac{47}{17} = [2; 1, 3, 4]$

Infinite Continued Fractions

Unlike the finite continued fractions, the chain of fractions never ends in an infinite continued fraction. Every irrational number can be equated to an infinite continued fraction. This fact was discovered and proven by the Swiss Mathematician, Leonhard Euler (1707-1783). Some of Euler's infinite continued fractions are as we will see below:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$$

A way to summarise this expression is to let x be the value of the continued fraction.

$$x = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \Leftrightarrow x = 1 + \frac{1}{1 + x}$$

Solving quadratic equations with continued fractions

In mathematics, a quadratic equation is a polynomial equation of the second degree. The general form is $ax^2+bx+c = 0$, where $a \neq 0$.

The quadratic equation on a number x can be solved using the well-known quadratic formula, which can be derived by completing the square. That formula always gives the roots of the quadratic equation, but the solutions are expressed in a form that often involves a quadratic irrational number, which is an algebraic fraction that can be evaluated as a decimal fraction only by applying an additional root extraction algorithm.

If the roots are real, there is an alternative technique that obtains a rational approximation to one of the roots by manipulating the equation directly. The method works in many cases, and long ago it stimulated further development of the analytical theory of continued fractions. Wall [1948]

Joseph-Louis Lagrange (1736-1813) proved that the continued fraction expansion of a real number x is ultimately periodic, i.e,

$$x = [a_0, ...a_k, b_1, ..., b_h, b_1, ..., b_h]$$

if and only if x is a quadratic number, that is, x is the root of quadratic polynomial with rational coefficients. In such case, we use the shorter notation

$$x = [a_0, ...a_k, b_1, \bar{...}, b_h],$$

in a ways similar to how it is done for repeating decimals.

Example

Here is a simple example to illustrate the solution of a quadratic equation using continued fractions. We begin with the equation $x^2 = 2$ and manipulate it directly. Subtracting one from both sides we obtain $x^2 - 1 = 1$. This is easily factored into $(x + 1)(x - 1) = 1$ from which we obtain $(x - 1) = \frac{1}{1+x}$ and finally $x = 1 + \frac{1}{1+x}$. Now comes the crucial step. We substitute this expression for x back into itself, recursively, to obtain

$$x = 1 + \cfrac{1}{1 + \left(1 + \cfrac{1}{1+x}\right)} = 1 + \cfrac{1}{2 + \cfrac{1}{1+x}}.$$

But now we can make the same recursive substitution again, and again, and again, pushing the unknown quantity x as far down and to the right as we please, and obtaining in the limit the infinite continued fraction

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}} = \sqrt{2}.$$

Hence the continued fraction expansion of $\sqrt{2}$ is given by

$$\sqrt{2} = [1, 2, 2, 2, ...] = [1, \bar{2}]$$

.

By applying the fundamental recurrence formulas, we may easily compute the successive convergent of this continued fraction to be 1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, ..., where each successive convergent is formed by taking the numerator plus the denominator of the preceding term as the denominator in the next term, then adding in the preceding denominator to form the new numerator. This sequence of denominators is a particular Lucas sequence known as the Pell numbers.

Pell's equation

Continued fractions play an essential role in the solution of Pell's equation. For example, for positive integers p and q, and non-square n, it is true that if $p^2 - nq^2 = \pm 1$, then $\frac{p}{q}$ is a convergent of the regular continued fraction for $\sqrt{n}$. The converse holds if the period of the regular continued fraction for$\sqrt{n}$ is 1, and in general the period describes which convergent give solutions to Pell's equation. Niven et al. [1991]

Task 1

Let's consider the continued fraction below

A continued fraction is a fraction whose numerator is an integer and whose denominator is an integer added to a fraction whose numerator is an integer and whose denominator is an integer added to a fraction, and so on.

$$1+\cfrac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}}},$$

First, we will create a formula that expresses the above expression as a sequence. Second, we will calculate the first ten terms of the sequence that have created and represent it graphically. We will also try to calculate the 200th term of the sequence and finally, Finally, we will study the "connection" that continued fractions have with the quadratic equation.

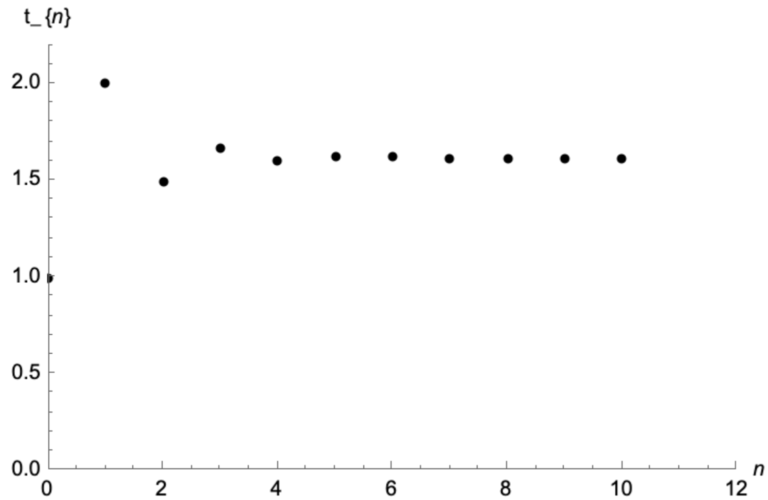
Solution 1 *We can write this "infinite fraction" as a sequence of terms, t_n, where*
$t_0 = 1$
$t_1 = 1 + 1$
$t_2 = 1 + \frac{1}{1+1}$
$t_3 = 1 + \frac{1}{1+\frac{1}{1+1}}$
...

- *So, if we determinate a generalized formula for t_{n+1} in terms of t_n, we have the following formula*

$$t_{n+1} = 1 + \frac{1}{t_n}$$

- *Now, let's compute the decimal equivalents of the first 10 terms.*
  - $t_0 = 1.000$
  - $t_1 = 1 + 1 = 2.000$
  - $t_2 = 1 + \frac{1}{1+1} = 1 + \frac{1}{2} = \frac{3}{2} = 1.500$
  - $t_3 = 1 + \frac{1}{1+\frac{1}{1+1}} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3} = 1.666$
  - $t_4 = 1 + \frac{1}{\frac{5}{3}} = 1 + \frac{3}{5} = \frac{8}{5} = 1.600$
  - $t_5 = 1 + \frac{1}{\frac{8}{5}} = 1 + \frac{5}{8} = \frac{13}{8} = 1.625$
  - $t_6 = 1 + \frac{1}{\frac{13}{8}} = 1 + \frac{8}{13} = \frac{21}{13} \simeq 1.615$
  - $t_7 = 1 + \frac{1}{\frac{21}{13}} = 1 + \frac{13}{21} = \frac{34}{21} \simeq 1.619$
  - $t_8 = 1 + \frac{1}{\frac{34}{21}} = 1 + \frac{21}{34} = \frac{55}{34} \simeq 1.618$
  - $t_9 = 1 + \frac{1}{\frac{55}{34}} = 1 + \frac{34}{55} = \frac{89}{55} \simeq 1.618$
  - $t_{10} = 1 + \frac{1}{\frac{89}{55}} = 1 + \frac{55}{89} = \frac{144}{89} \simeq 1.618$

As the graph below shows, from the 8^{th} *term onwards, all terms converge to 1.618*



We are able to notice that, when n gets very large the terms tend to be 1.618.

- *Now,If we want to determine for example the 200th term, we will notice that our main problem is that we have to calculate all the previous terms until we reach the 200th, which is also time-consuming. Another important problem is that from the table above, we could assume that since the terms from the 7th onwards are at 1.618, the 200th term also tends to 1.618. But this is not proof.*

However, this sequence is not just a regular sequence, it is the Fibonacci sequence. And the Fibonacci sequence is a recursive sequence, which is a sequence in which a general term is defined as a fraction of one or more of the previous term.

We will prove that the sequence of successive Fibonacci numbers $\frac{t_{n+1}}{t_n}$ *converges to the golden ratio* $\varphi = \frac{1+\sqrt{5}}{2} \simeq 1.618$

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \varphi$$

Observe that $t_{10} \simeq 1.618$ *is already pretty close to* φ*.*

It also satisfies the property

$$\varphi = 1 + \frac{1}{\varphi} \tag{1}$$

Consider the sequence $R_n = \frac{t_{n+1}}{t_n}$*, for* $n = 1, 2, 4, ...$ *By the definition of Fibonacci numbers, we get*

$$R_n = \frac{t_{n+1}}{t_n} = \frac{t_n - t_{n-1}}{t_n} = 1 + \frac{1}{R_{n-1}} \tag{2}$$

From (1) and (2) we can deduce that for all $n = 1, 2, 3, ...,$ *we have*

$$
\begin{aligned}
|R_n - \varphi| &= \left| \left(1 + \frac{1}{R_{n-1}}\right) - \left(1 + \frac{1}{\varphi}\right) \right| \\
&= \left| \frac{1}{R_{n-1}} - \frac{1}{\varphi} \right| \\
&= \left| \frac{\varphi - R_{n-1}}{R_{n-1}\varphi} \right| \\
&\leq \frac{1}{\varphi} \left| R_{n-1} - \varphi \right| \\
&\leq \left(\frac{1}{\varphi}\right)^{n-1} \left| R_1 - \varphi \right|
\end{aligned}
$$

Since $0 < \frac{1}{\varphi} < 1$,*then*

$$\lim_{n\to\infty} \left(\frac{1}{\varphi}\right)^{n-1} = 0 \Rightarrow \lim_{n\to\infty} |R_n - \varphi| = 0$$

Therefore,

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \varphi$$

So the 200th term we are looking for also tends to 1.618

- *So, it looks as if this sequence of consecutive ratios of consecutive of the Fibonacci sequence is traveling towards the golden ratio.*
  We can say that:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}} = \frac{1 + \sqrt{5}}{2}$$

That is a conjecture that we are making based on this inductive reasoning that we are doing right here.

Now, we will give deductive proof that that is actually in fact true. Here we have the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

so forth goes infinitely far down like this. Now, if we can say let's let

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

then, it must be true that $x = 1 + \frac{1}{x}$, *because of the property that infinity has, we can add another level to it any time we want. It will not change it.*
If we multiply both sides by x, *we get* $x^2 = x + 1$. *Here, we have a quadratic equation and we put it in standard form*

$$x^2 - x - 1 = 0$$

We use the quadratic formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 + \sqrt{5}}{2} \simeq 1.618$$

Since x *is a string of positive terms right here, it is not going to be* $\frac{1-\sqrt{5}}{2}$.
The unique positive real number

$$x = \frac{1 + \sqrt{5}}{2}$$

it is known as the Golden Ratio. The Golden Ratio makes its appearance in many different contexts, from Mathematics to Arts Huntley [1970]. From $x = 1 + \frac{1}{x}$ it is clear that the continued fraction expansion of x is

$$x = [1, 1, 1, ...] = [\bar{1}]$$

The simplest infinite continued fraction.

Task 2

Let's consider another continued fraction

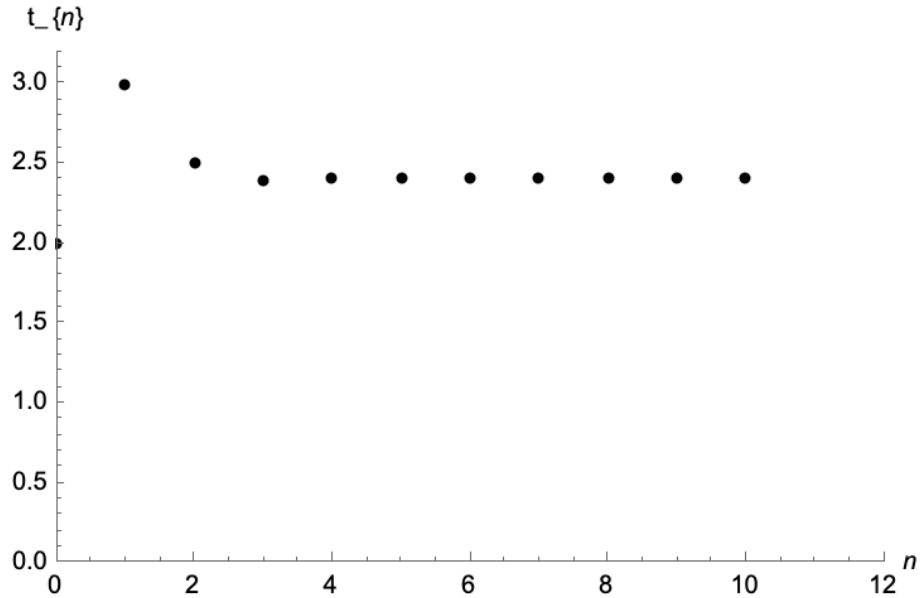
$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{\cdots}}}}}$$

We will follow the same steps in this problem as in task 1.

Solution 2 *We can write this "infinite fraction" as a sequence of terms, t_n, where*
$t_0 = 2$
$t_1 = 2 + 1$
$t_2 = 2 + \frac{1}{2}$
$t_3 = 2 + \frac{1}{2+\frac{1}{2}}$
...

- *So, if we determinate a generalized formula for t_{n+1} in terms of t_n, we have the following formula $t_{n+1} = 2 + \frac{1}{t_n}$*
- *Now, let's compute the decimal equivalents of the first 10 terms.*
  - $t_0 = 2.000$
  - $t_1 = 2 + 1 = 3.000$
  - $t_2 = 2 + \frac{1}{2} = \frac{5}{2} = 2.500$
  - $t_3 = 2 + \frac{1}{\frac{5}{2}} = 2 + \frac{2}{5} = \frac{12}{5} = 2.400$
  - $t_4 = 2 + \frac{1}{\frac{12}{5}} = 2 + \frac{5}{12} = \frac{29}{12} \simeq 2.416$
  - $t_5 = 2 + \frac{1}{\frac{29}{12}} = 2 + \frac{12}{29} = \frac{70}{29} \simeq 2.414$
  - $t_6 = 2 + \frac{1}{\frac{70}{29}} = 2 + \frac{29}{70} = \frac{169}{70} \simeq 2.414$
  - $t_7 = 2 + \frac{1}{\frac{169}{70}} = 2 + \frac{70}{169} = \frac{408}{169} \simeq 2.414$
  - $t_8 = 2 + \frac{1}{\frac{408}{169}} = 2 + \frac{169}{408} = \frac{985}{408} \simeq 2.414$
  - $t_9 = 2 + \frac{1}{\frac{985}{408}} = 2 + \frac{408}{985} = \frac{2378}{985} \simeq 2.414$
  - $t_{10} = 2 + \frac{1}{\frac{2378}{985}} = 2 + \frac{985}{2378} = \frac{5741}{2378} \simeq 2.414$

As the graph below shows, from the 5^{th} *term onwards, all terms converge to 2.414*



We are able to notice that, when n gets very large the terms tend to be 2.414.

- *If we want to determine for example the 200th term, we will notice that our main problem is that we have to calculate all the previous terms until we reach the 200th, which is also time-consuming. Another important problem is that from the table above, we could assume that since the terms from the 4th onwards are at 2.414, the 200th term also tends to 2.414. But this is not proof. Just like problem 1 in the same way we prove that the 200th term also tends to 2.414*
- *So, we can say let's let:*

$$x = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\cdots}}}}$$

then, it must be true that $x = 1 + \frac{1}{x}$*, because of the property that infinity has, we can add another level to it any time we want. It will not change it.*
If we multiply both sides by x*, we get* $x^2 = 2x + 1$*. Here, we have a quadratic equation and we put it in standard form*

$$x^2 - 2x - 1 = 0$$

We use the quadratic formula

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2} \simeq 2.414$$

Since x *is a string of positive terms right here, it is not going to be* $1 - \sqrt{2}$*.*

Task 3

Now consider the general continued fraction

$$\kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\cdots}}}}}$$

As we have already shown continued fractions are most conveniently applied to solve the general quadratic equation expressed in the form of a monic polynomial $x^2 + bx + c = 0$.

In our case, we have $x^2 - kx - 1 = 0$ which can always be obtained by dividing the original equation by its leading coefficient. Starting from this monic equation we see that

$$x^2 - \kappa x = 1 \Leftrightarrow x - \kappa = \frac{1}{x} \Leftrightarrow x = \kappa + \frac{1}{x}$$

But now we can apply the last equation to itself recursively to obtain

$$x = \kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\kappa + \frac{1}{\cdots}}}}}$$

If this infinite continued fraction converges at all, it must converge to one of the roots of the monic polynomial $x^2 + \kappa x + 1 = 0$. That depends on both the coefficient κ and the value of the discriminant, $b^2 - 4ac$.

In general, by applying a result obtained by Euler in 1748 it can be shown that the continued fraction solution to the general monic quadratic equation with real coefficients

$$x^2 + bx + c = 0$$

which can always be obtained by dividing the original equation by its leading coefficient. Starting from this monic equation we see that

$$x^2 + bx = -c \Leftrightarrow x + b = -\frac{c}{x} \Leftrightarrow x = -b - \frac{c}{x}$$

But now we can apply the last equation to itself recursively to obtain

$$x = -b - \frac{c}{-b - \frac{c}{-b - \frac{c}{-b - \frac{c}{-b - \frac{c}{\cdots}}}}}$$

either converges or diverges depending on both the coefficient b and the value of the discriminant, $b^2 - 4ac$.

If $b = 0$ the general continued fraction solution is totally divergent; the convergent alternate between 0 and ∞ . If $b \neq 0$ we distinguish three cases.

1. If the discriminant is negative, the fraction diverges by oscillation, which means that its convergent wander around in a regular or even chaotic fashion, never approaching a finite limit.
2. If the discriminant is zero, the fraction converges to the single root of multiplicity two.
3. If the discriminant is positive the equation has two real roots, and the continued fraction converges to the larger (in absolute value) of these. The rate of convergence depends on the absolute value of the ratio between the two roots: the farther that ratio is from unity, the more quickly the continued fraction converges.

4 The history of continued fractions

1. 300 BCE Euclid's Elements contains an algorithm for the greatest common divisor, whose modern version generates a continued fraction as the sequence of quotients of successive Euclidean divisions that occur in it.
2. 499 The Aryabhatiya contains the solution of indeterminate equations using continued fractions.
3. 1572 Rafael Bombelli, L'Algebra Opera – method for the extraction of square roots which is related to continued fractions.
4. 1613 Pietro Cataldi, Trattato del modo di trovar la ratice quadra delli numeri—first notation for continued fractions. Cataldi represented a continued fraction as $a_0 \& \frac{n_1}{d_1.} \& \frac{n_2}{d_2.} \& \frac{n_3}{d_3.}$ with dots indicating where the following fractions went.
5. 1695 John Wallis, Opera Mathematica – introduction of the term "continued fraction".
6. 1737 Leonhard Euler, De fractionibus continuis dissertation — Provided the first then-comprehensive account of the properties of continued fractions and included the first proof that the number e is irrational. Sandifer [2006]

7. 1748 Euler, Introductio in analsin infinitorum. Vol.I, Chapter 18 – proved the equivalence of a certain form of continued fraction and a generalized infinite series, proves that every rational number can be written as a finite continued fraction, and proved that the continued fraction of an irrational number is infinite. Euler [1748]
8. 1761 Johann Lambert – gave the first proof of the irrationality of π using a continued fraction for $\tan(x)$.
9. 1768 Joseph-Louis Lagrange – provide the general solution to Pell's equation using continued fraction's like Bombelli's
10. 1770 Lagrange – proved that quadratic irrationals expand to periodic continued fractions.
11. 1770 Lagrange – proved that quadratic irrationals expand to periodic continued fractions.
12. 1813 Carl Friedrich Gauss, Werke, Vol.3, pp. 134-138 – derived a very general complex-valued continued fraction via a clever identity involving the hypergeometric function.
13. 1892 Henri Pade defined Pade approximant.
14. 1892 Henri Pade defined Pade approximant.
15. 1972 Bill Gosper – First exact algorithms for continued fraction arithmetic.

5 Conclusion

To summarize, with this paper we first familiarize ourselves with the concept of continued fractions and more specifically simple continued fractions through the definitions we have given. We also studied three concrete problems of continued fractions, which we solved with the knowledge you have acquired so far in your studies, through which we observed the direct correlation of continued fractions with sequences and with quadratic equations. The two primary problems were numerical problems in which we observed that after one term and afterward all other terms converge to a certain number. In the last problem, we solved the generalized pattern of simple continued fractions and showed the connection they have with quadratic equations. We saw that infinite continuous fractions represent irrational numbers and conversely, any irrational can be represented in this way. Finally, we provide a historical review of how the concept of continued fractions has evolved over the years.

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